



A note on the almost 1-tough graph

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Abstract. A graph G is almost 1-tough if $c(G - S) \leq |S| + 1$ for any $S \subseteq V(G)$, where $c(G - S)$ is the number of components of $G - S$. Let $F : V(G) \rightarrow \{\{1\}, \{0, 2\}\}$ be a set-valued function and $F^{-1}(1) := \{v \in V(G) : f(v) = \{1\}\}$. A spanning subgraph H of G is called an F -factor if $d_H(v) \in F(v)$ for all $v \in V(G)$. It is interesting to know whether a graph is almost 1-tough or a graph has an F -factor. In this note, we establish a lower bound on the size (resp. the spectral radius) to ensure a graph to be almost 1-tough. This also provide a sufficient condition for the existence of an F -factor for which every $F : V(G) \rightarrow \{\{1\}, \{0, 2\}\}$ with $|F^{-1}(1)|$ even in a graph.

1. Introduction

All graphs considered in this paper are simple. Let $G = (V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. Its order is $|V(G)|$, denoted by n , and its size is $|E(G)|$, denoted by $e(G)$. For $v \in V(G)$, let $d_G(v)$ and $N_G(v)$ (or simply $d(v)$ and $N(v)$) be the degree and the set of neighbors of v , respectively. The minimum degree of G is denoted by $\delta(G)$ (or simply δ). For a subset $S \subseteq V(G)$, we use $G[S]$ and $G - S$ to denote the subgraphs of G induced by S and $V(G) \setminus S$, respectively. Let $c(G)$ be the number of components of G . For two vertex-disjoint graphs G_1 and G_2 , let $G_1 \cup G_2$ be the disjoint union of G_1 and G_2 , and $G_1 \vee G_2$ be the graph obtained from $G_1 \cup G_2$ by connecting all the vertices between $V(G_1)$ and $V(G_2)$. Let $A(G)$ be the adjacency matrix of G , and $\rho(G)$ be the spectral radius of G (the largest eigenvalue of $A(G)$).

The *toughness* $\tau(G)$ of a non-complete graph G was defined in [3] as

$$\tau(G) = \min \left\{ \frac{|S|}{c(G - S)} : S \subseteq V(G) \text{ and } c(G - S) \geq 2 \right\},$$

where the minimum is taken over all proper subsets $S \subseteq V(G)$ and $c(G - S)$ is the number of components of $G - S$. For convention, $t(K_n) = +\infty$ and non-complete graph G is *t-tough* if $\tau(G) \geq t$. A graph G is *almost 1-tough* [5] if $c(G - S) \leq |S| + 1$ for any $S \subseteq V(G)$. In [6], Lu and Kano gave a characterization for a (resp.

2020 *Mathematics Subject Classification*. Primary 05C50.

Keywords. almost 1-tough, size, spectral radius, F -factor.

Received: 21 March 2025; Revised: 10 July 2025; Accepted: 07 August 2025

Communicated by Paola Bonacini

The research is partially supported by NSFC (No.12361069).

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almost) 1-tough graph in terms of graph factors. Later, Lu, Wang and Jiang [5] showed that it is NP-hard to recognize almost 1-tough cubic graphs. For any $S \subseteq V(G)$, note that $c(G - S) \leq |S| + 1$ is also a well-known necessary condition for the traceability of a graph. Inspired by the work of Lu and Kano [6], it is interesting to find some sufficient conditions for a graph to be almost 1-tough. For this purpose, we first present a size condition to guarantee a graph G to be almost 1-tough and construct the corresponding extremal graph to show that our condition is best possible.

Theorem 1.1. *Let G be a connected graph of order $n \geq \max\{6\delta + 6, \frac{1}{6}(\delta^2 + 13\delta + 7)\}$ with the minimum degree $\delta \geq 2$. If*

$$e(G) > \binom{n - \delta - 1}{2} + (\delta + 1)\delta,$$

then G is almost 1-tough.

Remark 1.2. *Note that $e(K_\delta \vee (K_{n-2\delta-1} \cup (\delta + 1)K_1)) = \binom{n-\delta-1}{2} + (\delta + 1)\delta$ and $K_\delta \vee (K_{n-2\delta-1} \cup (\delta + 1)K_1)$ is not almost 1-tough. This indicates that the bound on $e(G)$ in Theorem 1.1 is best possible.*

Furthermore, we also give the following spectral radius condition for a graph to be almost 1-tough.

Theorem 1.3. *Let G be a connected graph of order $n \geq \max\{5\delta + 3, \frac{3\delta^2}{5} + \delta\}$ with the minimum degree δ . If $\rho(G) \geq \rho(K_\delta \vee (K_{n-2\delta-1} \cup (\delta + 1)K_1))$, then G is almost 1-tough unless $G \cong K_\delta \vee (K_{n-2\delta-1} \cup (\delta + 1)K_1)$.*

For a graph G , let $F : V(G) \rightarrow \{\{1\}, \{0, 2\}\}$ be a set-valued function. So $F(v) \in \{\{1\}, \{0, 2\}\}$ for each $v \in V(G)$. Let $F^{-1}(1) := \{v \in V(G) : F(v) = \{1\}\}$. A spanning subgraph H of G is called an F -factor if $d_H(v) \in F(v)$ for all $v \in V(G)$. Such F -factor is also called a $\{\{1\}, \{0, 2\}\}$ -factor of G . It is clear that if G has an F -factor, then $|F^{-1}(1)|$ must be even by the Handshaking Lemma. So if $|F^{-1}(1)|$ is odd, then G has no F -factor. In [6], Lu and Kano gave a characterization of a graph having an F -factor.

Theorem 1.4 ([6]). *Let G be a connected graph. G has an F -factor for every $F : V(G) \rightarrow \{\{1\}, \{0, 2\}\}$ with $|F^{-1}(1)|$ even if and only if G is almost 1-tough.*

By combining Theorem 1.3 with Theorem 1.4, we establish a relationship between the existence of an F -factor in a connected graph G and its spectral radius.

Theorem 1.5. *Let G be a connected graph of order $n \geq \max\{5\delta + 3, \frac{3\delta^2}{5} + \delta\}$, where δ is the minimum degree of G . If $e(G) \geq e(K_\delta \vee (K_{n-2\delta-1} \cup (\delta + 1)K_1))$ or $\rho(G) \geq \rho(K_\delta \vee (K_{n-2\delta-1} \cup (\delta + 1)K_1))$, then G has an F -factor for every $F : V(G) \rightarrow \{\{1\}, \{0, 2\}\}$ with $|F^{-1}(1)|$ even unless $G = K_\delta \vee (K_{n-2\delta-1} \cup (\delta + 1)K_1)$.*

Moreover, by Theorem 1.5, we may have the following size (or spectral radius) version for a result in [6] due to Lu and Kano.

Theorem 1.6. *Let G be a connected graph of order $n \geq \max\{5\delta + 3, \frac{3\delta^2}{5} + \delta\}$, where δ is the minimum degree of G . If $e(G) > e(K_\delta \vee (K_{n-2\delta-1} \cup (\delta + 1)K_1))$ or $\rho(G) > \rho(K_\delta \vee (K_{n-2\delta-1} \cup (\delta + 1)K_1))$, then for any red-blue coloring of $V(G)$ such that the number of red vertices is even, G has vertex-disjoint paths whose end-vertices are exactly the same as the red vertices of G . In particular, all the internal vertices of the paths are blue.*

2. Proof of Theorem 1.1

Before giving the proof of Theorem 1.1, we need the following lemma.

Lemma 2.1 ([2]). *Let $n = \sum_{i=1}^t n_i + s$. If $n_1 \geq n_2 \geq \dots \geq n_t \geq 1$ and $n_1 < n - s - t + 1$, then*

$$e(K_s \vee (K_{n_1} \cup K_{n_2} \cup \dots \cup K_{n_t})) \leq e(K_s \vee (K_{n-s-t+1} \cup (t-1)K_1)).$$

Now, we give the proof of Theorem 1.1.

Proof of Theorem 1.1: Let G be a graph satisfying the conditions in Theorem 1.1. We assume that G is not an almost 1-tough graph, and show that $e(G) \leq \binom{n-\delta-1}{2} + (\delta+1)\delta$. By the assumption, there exists a subset $S \subseteq V(G)$ such that $c(G-S) \geq |S| + 2$. Let $|S| = s$. We first assert that S is not an empty set. If $S = \emptyset$, then $1 = c(G-S) \geq 2$, a contradiction. Hence $|S| = s \geq 1$. Note that G is a spanning subgraph of $G_1 = K_s \vee (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_{s+2}})$, where $n_1, n_2, \dots, n_{s+2}$ are positive integers with $n_1 \geq n_2 \geq \cdots \geq n_{s+2} \geq 1$ and $n = \sum_{i=1}^{s+2} n_i + s$. Therefore, we have

$$e(G) \leq e(G_1).$$

We now consider the following two cases.

Case 1. $s \leq \delta$.

Let $G_2 = K_s \vee (K_{n-s-(s+1)(\delta+1-s)} \cup (s+1)K_{\delta+1-s})$. Since G is a spanning subgraph of G_1 , we have $\delta(G_1) \geq \delta$ and $n_{s+2} \geq \delta+1-s$. If $n_i \geq \delta+2-s$ for some integers $i \in \{2, 3, \dots, s+2\}$, let $G_3 = K_s \vee (K_{n_1+1} \cup K_{n_2} \cup \cdots \cup K_{n_{i-1}} \cup \cdots \cup K_{n_{s+2}})$. Obviously,

$$e(G_3) = e(G_1) - (n_i - 1) + n_1 = e(G_1) + (n_1 - n_i) + 1 > e(G_1).$$

We proceed the above procedure until $n_2 = n_3 = \cdots = n_{s+2} = \delta+1-s$ in G_1 , this is exactly the graph G_2 . Then $e(G_1) \leq e(G_2)$. Thus, we have

$$e(G) \leq e(G_1) \leq e(G_2) = e(K_s \vee (K_{n-s-(s+1)(\delta+1-s)} \cup (s+1)K_{\delta+1-s})),$$

Note that $e(G_2) = \binom{n-(s+1)(\delta+1-s)}{2} + s(s+1)(\delta+1-s) + (s+1)\binom{\delta+1-s}{2}$. By a directed computation, we have

$$\binom{n-\delta-1}{2} + \delta(\delta+1) - e(G_2) = \frac{\delta-s}{2}\varphi(s),$$

where $\varphi(s) = s^3 - (\delta+1)s^2 - 3s(\delta+1) + 2ns + \delta + 1$. Note that $s \geq 1$ and $\delta \geq 2$. Then

$$\varphi'(s) = 3s^2 - 2s(\delta+1) + 2n - 3\delta - 3.$$

The symmetry axis of $\varphi'(s)$ is $\frac{\delta+1}{3} \in [1, \delta)$. Thus,

$$\varphi'(s) \geq \varphi'\left(\frac{\delta+1}{3}\right) = \frac{6n - (\delta+4)\delta - 7\delta - 10}{3} \geq \frac{1}{3}(2\delta - 3) > 0$$

by $n \geq \frac{1}{6}(\delta^2 + 13\delta + 7)$ and $\delta \geq 2$. It follows that $\varphi(s)$ is increasing in $1 \leq s \leq \delta$. Thus,

$$\varphi(s) \geq \varphi(1) = 2(n-1) - 3\delta \geq \frac{1}{3}(\delta^2 + \delta + 1) + \delta > 0$$

for $s \in [1, \delta]$ and $\delta \geq 2$, and we have

$$e(G) \leq e(G_2) \leq \binom{n-\delta-1}{2} + \delta(\delta+1),$$

as desired.

Case 2. $s \geq \delta + 1$.

By Lemma 2.1, we have

$$e(G) \leq e(G_1) \leq e(K_s \vee (K_{n-2s-1} \cup (s+1)K_1)).$$

Let $G_4 = K_s \vee (K_{n-2s-1} \cup (s+1)K_1)$. Note that $e(G_4) = \binom{n-s-1}{2} + s(s+1)$. By a directed computation, we have

$$\binom{n-\delta-1}{2} + \delta(\delta+1) - e(G_4) = \frac{s-\delta}{2}\phi(n),$$

where $\phi(n) = 2n - 3(s + \delta) - 5$.

If $s \geq 3\delta + 1$, then it follows from $n \geq s + c(G - S) \geq 2s + 2$ that

$$\phi(n) \geq \phi(2s + 2) \geq 4(s + 1) - 3(s + \delta) - 5 = s - 3\delta - 1 \geq 0.$$

It follows that $e(G_4) \leq \binom{n-\delta-1}{2} + (\delta + 1)\delta$.

If $\delta + 1 \leq s < 3\delta + 1$, then it follows from $n \geq 6\delta + 6$ that

$$\phi(n) = 2n - 3(s + \delta) - 5 > 12(\delta + 1) - 12\delta - 8 > 0.$$

It follows that $e(G_4) < \binom{n-\delta-1}{2} + (\delta + 1)\delta$.

In view of Cases 1 and 2, we have $e(G) \leq \binom{n-\delta-1}{2} + (\delta + 1)\delta$, as desired. The proof is completed. $\square$

3. Proof of Theorem 1.3

Recall that $A(G)$ is a non-negative and real symmetric matrix. Then there is a non-negative unit eigenvector $\mathbf{x}$ of $A(G)$ corresponding to $\rho(G)$ such that

$$\rho(G) = \mathbf{x}^T A(G) \mathbf{x} = 2 \sum_{uv \in E(G)} x(u)x(v),$$

where $x(u)$ is the entry of $\mathbf{x}$ corresponding to the vertex u . We call such eigenvector $\mathbf{x}$ the *Perron vector* of $A(G)$. Clearly, the Perron vector of $A(G)$ satisfies the *eigenvalue equation* $A(G)\mathbf{x} = \rho(G)\mathbf{x}$, that is

$$A(G)x(u) = \sum_{v \in N(u)} x(v).$$

Lemma 3.1 ([1]). *If H is a spanning subgraph of a graph G , then $\rho(H) \leq \rho(G)$, with equality if and only if $G = H$. Moreover, if H is a proper subgraph of G , then $\rho(H) < \rho(G)$.*

Lemma 3.2 ([4]). *Let $n = \sum_{i=1}^t n_i + s$. If $n_1 \geq n_2 \geq \dots \geq n_t \geq p$ and $n_1 < n - s - p(t - 1)$, then*

$$\rho(K_s \vee (K_{n_1} \cup K_{n_2} \cup \dots \cup K_{n_t})) < \rho(K_s \vee (K_{n-s-p(t-1)} \cup (t-1)K_p)).$$

Consider an $n \times n$ real symmetric matrix

$$M = \begin{bmatrix} M_{1,1} & M_{1,2} & \cdots & M_{1,m} \\ M_{2,1} & M_{2,2} & \cdots & M_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ M_{m,1} & M_{m,2} & \cdots & M_{m,m} \end{bmatrix}$$

whose rows and columns are partitioned according to a partitioning $X_1, X_2, \dots, X_m$ of $\{1, 2, \dots, n\}$. The *quotient matrix* $\mathcal{B}$ of the matrix M is the $m \times m$ matrix whose entries are the average row sums of the blocks $M_{i,j}$ of M . The partition is *equitable* if each block $M_{i,j}$ of M has constant row (and column) sum.

Lemma 3.3 ([7]). *Let M be a square matrix with an equitable partition π and let M_π be the corresponding quotient matrix. Then every eigenvalue of M_π is an eigenvalue of M . Furthermore, if M is nonnegative and irreducible, then the largest eigenvalues of M and M_π are equal.*

Now, we are in a position to present the proof of Theorem 1.3.

Proof of Theorem 1.3: Let G be a graph satisfying the conditions in Theorem 1.3. It is easy to check that $K_\delta \vee (K_{n-2\delta-1} \cup (\delta + 1)K_1)$ is not an almost 1-tough graph. From now on, we assume that $G \neq K_\delta \vee (K_{n-2\delta-1} \cup (\delta + 1)K_1)$. We shall prove the contrapositive of the theorem. That means, we assume that G is a connected

graph of order n which is not an almost 1-tough graph, and show that $\rho(G) < \rho(K_\delta \vee (K_{n-2\delta-1} \cup (\delta+1)K_1))$. Thus, there exists a subset $S \subseteq V(G)$ such that $c(G-S) \geq |S| + 2$. Let $|S| = s$. We first assert that S is not an empty set. If $S = \emptyset$, then $1 = c(G-S) \geq 2$, a contradiction. Hence $|S| = s \geq 1$. Then G is a spanning subgraph of $G_1 = K_s \vee (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_{s+2}})$ for some positive integers $n_1 \geq n_2 \geq \cdots \geq n_{s+2} \geq 1$ and $\sum_{i=1}^{s+2} n_i = n - s$. Combining this with Lemma 3.1 implies that

$$\rho(G) \leq \rho(G_1), \quad (1)$$

with equality if and only if $G = G_1$.

We now consider the following three cases.

Case 1. $s = \delta$.

By Lemma 3.2, we have $\rho(G_1) \leq \rho(K_\delta \vee (K_{n-2\delta-1} \cup (\delta+1)K_1))$ with equality holding if and only if $G_1 = K_\delta \vee (K_{n-2\delta-1} \cup (\delta+1)K_1)$. Combining this with (1), we conclude that $\rho(G) \leq \rho(K_\delta \vee (K_{n-2\delta-1} \cup (\delta+1)K_1))$, where the equality holds if and only if $G = K_\delta \vee (K_{n-2\delta-1} \cup (\delta+1)K_1)$. Since $G \neq K_\delta \vee (K_{n-2\delta-1} \cup (\delta+1)K_1)$, we have $\rho(G) < \rho(K_\delta \vee (K_{n-2\delta-1} \cup (\delta+1)K_1))$, as desired.

Case 2. $s \geq \delta + 1$.

Let $G_2 = K_s \vee (K_{n-2s-1} \cup (s+1)K_1)$. By Lemma 3.2, we have $\rho(G_1) \leq \rho(G_2)$, with equality if and only if $(n_1, n_2, \dots, n_{s+2}) = (n-2s-1, 1, \dots, 1)$. Therefore, it follows that

$$\rho(G) \leq \rho(G_1) \leq \rho(G_2). \quad (2)$$

The quotient matrix of $A(G_2)$ corresponding to the partition $V(G_2) = V(K_s) \cup V(K_{n-2s-1}) \cup V((s+1)K_1)$ is given as

$$\mathcal{B}_1^s = \begin{bmatrix} s-1 & n-2s-1 & s+1 \\ s & n-2s-2 & 0 \\ s & 0 & 0 \end{bmatrix}.$$

By a direct calculation, the characteristic polynomial of $\mathcal{B}_1^s$ is

$$\varphi(\mathcal{B}_1^s, x) = x^3 - (n-s-3)x^2 - (n+s^2-2)x + s^2n + sn - 2s^3 - 4s^2 - 2s.$$

Clearly, the vertex partition of G_2 is equitable. By Lemma 3.3, the largest root of $\varphi(\mathcal{B}_1^s, x) = 0$ equals the spectral radius of $A(G_2)$.

Let $\theta_1 = \rho(G_2) \geq \theta_2 \geq \theta_3$ be the three roots of $\varphi(\mathcal{B}_1^s, x) = 0$. We assert that $\theta_2 \leq n-2\delta-4$. In fact, let $\mathcal{D} = \text{diag}(s, n-2s-1, s+1)$. Therefore,

$$\begin{aligned} & \mathcal{D}^{\frac{1}{2}} \mathcal{B}_1^s \mathcal{D}^{-\frac{1}{2}} \\ &= \begin{bmatrix} \sqrt{s} & 0 & 0 \\ 0 & \sqrt{n-2s-1} & 0 \\ 0 & 0 & \sqrt{s+1} \end{bmatrix} \begin{bmatrix} s-1 & n-2s-1 & s+1 \\ s & n-2s-2 & 0 \\ s & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{s}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{n-2s-1}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{s+1}} \end{bmatrix} \\ &= \begin{bmatrix} (s-1)\sqrt{s} & (n-2s-1)\sqrt{s} & (s+1)\sqrt{s} \\ s\sqrt{n-2s-1} & (n-2s-2)\sqrt{n-2s-1} & 0 \\ s\sqrt{s+1} & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{s}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{n-2s-1}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{s+1}} \end{bmatrix} \\ &= \begin{bmatrix} s-1 & \sqrt{s(n-2s-1)} & \sqrt{s(s+1)} \\ \sqrt{s(n-2s-1)} & n-2s-2 & 0 \\ \sqrt{s(s+1)} & 0 & 0 \end{bmatrix}. \end{aligned}$$

It is easy to check that $\mathcal{D}^{\frac{1}{2}} \mathcal{B}_1^s \mathcal{D}^{-\frac{1}{2}}$ is symmetric, and also contains

$$\begin{bmatrix} n-2s-2 & 0 \\ 0 & 0 \end{bmatrix}$$

as its principal submatrix. Obviously, the spectrum of $\mathcal{D}^{\frac{1}{2}}\mathcal{B}_1^s\mathcal{D}^{-\frac{1}{2}}$ is the same as that of $\mathcal{B}_1^s$. By the Cauchy Interlacing Theorem, we obtain

$$\theta_2 \leq n - 2s - 2 \leq n - 2(\delta + 1) - 2 = n - 2\delta - 4.$$

Also, note that $A(K_\delta \vee (K_{n-2\delta-1} \cup (\delta + 1)K_1))$ has the equitable quotient matrix $\mathcal{B}_1^\delta$, which is obtained by replacing s with δ in $\mathcal{B}_1^s$. Thus, we have

$$\varphi(\mathcal{B}_1^\delta, x) = x^3 - (n - \delta - 3)x^2 - (n + \delta^2 - 2)x + \delta^2n + \delta n - 2\delta^3 - 4\delta^2 - 2\delta.$$

According to Lemma 3.3, the largest root, say ϑ , of $\varphi(\mathcal{B}_1^\delta, x) = 0$ equals $\rho(K_\delta \vee (K_{n-2\delta-1} \cup (\delta + 1)K_1))$. Notice that $K_{n-\delta-1}$ is a proper subgraph of $K_\delta \vee (K_{n-2\delta-1} \cup (\delta + 1)K_1)$, it follows from Lemma 3.1 that

$$\vartheta = \rho(K_\delta \vee (K_{n-2\delta-1} \cup (\delta + 1)K_1)) > \rho(K_{n-\delta-1}) = n - \delta - 2 > n - 2\delta - 4 \geq \theta_2.$$

We are to verify $\varphi(\mathcal{B}_1^s, \vartheta) = \varphi(\mathcal{B}_1^s, \vartheta) - \varphi(\mathcal{B}_1^\delta, \vartheta) > 0$. By a simple computation, we obtain

$$\varphi(\mathcal{B}_1^s, \vartheta) = \varphi(\mathcal{B}_1^s, \vartheta) - \varphi(\mathcal{B}_1^\delta, \vartheta) = (s - \delta)f(\vartheta),$$

where $f(\vartheta) = \vartheta^2 - (s + \delta)\vartheta + (n - 2(s + 1))(s + 1) - 2\delta s + n\delta - 2\delta^2 - 4\delta$. Then the symmetry axis of $f(\vartheta)$ is $\vartheta = \frac{s+\delta}{2}$, which implies that $f(\vartheta)$ is increasing in the interval $[\frac{s+\delta}{2}, +\infty)$. It is easy to check that $n \geq s + c(G - S) \geq 2s + 2$ and $\frac{s+\delta}{2} < n - \delta - 2 < \vartheta$. Therefore, $s \leq \frac{n-2}{2}$ and

$$\begin{aligned} f(\vartheta) &> f(n - \delta - 2) \\ &= n^2 - (2\delta + 3)n - 2s^2 - (\delta + 2)s + 2\delta + 2 \\ &\geq n^2 - (2\delta + 3)n - 2\left(\frac{n-2}{2}\right)^2 - (\delta + 2)\left(\frac{n-2}{2}\right) + 2\delta + 2 \quad \left(\text{As } s \leq \frac{n-2}{2}\right) \\ &= \frac{1}{2}(n^2 - (5\delta + 4)n + 6\delta + 4) \\ &\geq \frac{\delta + 1}{2} \quad (\text{As } n \geq 5\delta + 3) \\ &> 0. \end{aligned}$$

Hence we have

$$\varphi(\mathcal{B}_1^s, \vartheta) = \varphi(\mathcal{B}_1^s, \vartheta) - \varphi(\mathcal{B}_1^\delta, \vartheta) = (s - \delta)f(\vartheta) > 0.$$

Since $\vartheta = \rho(K_\delta \vee (K_{n-2\delta-1} \cup (\delta + 1)K_1)) > n - 2\delta - 4 \geq \theta_2$, we conclude that $\rho(G_2) = \theta_1 < \vartheta = \rho(K_\delta \vee (K_{n-2\delta-1} \cup (\delta + 1)K_1))$. From (2), we have

$$\rho(G) \leq \rho(G_1) \leq \rho(G_2) < \rho(K_\delta \vee (K_{n-2\delta-1} \cup (\delta + 1)K_1)),$$

as desired.

Case 3. $s \leq \delta - 1$.

Since G is a spanning subgraph of G_1 , $\delta(G_1) \geq \delta$. On the other hand, $\delta(G_1) \leq n_{s+2} - 1 + s$. Thus $n_{s+2} \geq \delta + 1 - s$. Let $G_3 = K_s \vee (K_{n-s-(\delta+1-s)(s+1)} \cup (s+1)K_{\delta+1-s})$. By Lemma 3.2, we have $\rho(G_1) \leq \rho(G_3)$, with equality if and only if $(n_1, n_2, \dots, n_{s+2}) = (n - s - (\delta + 1 - s)(s + 1), \delta + 1 - s, \dots, \delta + 1 - s)$. Therefore, it follows that

$$\rho(G) \leq \rho(G_1) \leq \rho(G_3). \quad (3)$$

If $s = 1$, then $G_3 = K_1 \vee (K_{n-2\delta-1} \cup 2K_\delta)$. Suppose $\rho(G_3) \geq n - 2\delta$. Let $\rho = \rho(G_3)$ and let $\mathbf{x}$ be the Perron vector of $A(G_3)$ with respect to ρ . By symmetry, $\mathbf{x}$ takes the same values x_1 , x_2 and x_3 on the vertices of $K_{n-2\delta-1}$, K_δ and K_1 , respectively. Then, by $A(G_3)\mathbf{x} = \rho\mathbf{x}$, we have

$$\rho x_1 = (n - 2\delta - 2)x_1 + x_3, \quad (4)$$

$$\rho x_2 = (\delta - 1)x_2 + x_3, \quad (5)$$

$$\rho x_3 = (n - 2\delta - 1)x_1 + 2\delta x_2. \quad (6)$$

From (4) and (5), we obtain

$$x_1 = \frac{x_3}{\rho - (n - 2\delta - 2)}, \quad (7)$$

$$x_2 = \frac{x_3}{\rho - \delta + 1}. \quad (8)$$

Since $n \geq 5\delta + 3$ and $\delta > s$, we have

$$\rho \geq n - 2\delta \geq 5\delta + 3 - 2\delta = 3(\delta + 1) > \delta + 1.$$

Putting (7) and (8) into (6), we get

$$\begin{aligned} \rho &= \frac{n - 2\delta - 1}{\rho - (n - 2\delta - 2)} + \frac{2\delta}{\rho - \delta + 1} < \frac{n - 2\delta - 1}{2} + \frac{2\delta}{2} = \frac{n - 1}{2} \\ &= n - \frac{n + 1}{2} \leq n - \frac{5\delta + 4}{2} < n - 2\delta \leq \rho, \end{aligned}$$

which is impossible. Thus $\rho(G_3) < n - 2\delta$, it follows that

$$\rho(G_3) < n - 2\delta = n - \delta - 2 - (\delta - 2) \leq n - \delta - 2.$$

Since $K_\delta \vee (K_{n-2\delta-1} \cup (\delta + 1)K_1)$ contains $K_{n-\delta-1}$ as a proper spanning subgraph, we have $\rho(K_\delta \vee (K_{n-2\delta-1} \cup (\delta + 1)K_1)) > \rho(K_{n-\delta-1}) = n - \delta - 2$. Therefore, it follows that

$$\rho(G) \leq \rho(G_1) \leq \rho(G_3) < n - \delta - 2 < \rho(K_\delta \vee (K_{n-2\delta-1} \cup (\delta + 1)K_1)),$$

as desired. Thus, we consider $s \geq 2$ in the following.

On the other hand, the quotient matrix of $A(G_3)$ corresponding to the partition $V(G_3) = V(K_s) \cup V(K_{n-s-(\delta+1-s)(s+1)}) \cup V((s+1)K_{\delta+1-s})$ is given as

$$\mathcal{B}_2^s = \begin{bmatrix} s-1 & n-s-(\delta+1-s)(s+1) & (s+1)(\delta+1-s) \\ s & n-s-(\delta+1-s)(s+1)-1 & 0 \\ s & 0 & \delta-s \end{bmatrix}.$$

By a direct calculation, the characteristic polynomial of $\mathcal{B}_1^s$ is

$$\begin{aligned} \varphi(\mathcal{B}_2^s, x) &= x^3 - (s^2 - (\delta + 1)s + n - 3)x^2 + ((\delta - 1)(s^2 + n) - (n + \delta^2 - \delta - 2)s - \delta^2 - 2\delta + 2)x \\ &\quad - (s(n + (s - 2)(s + 1) + 1))s^2 + (2s^4 + s^3 + (n - 2)(s^2 + s + 1))\delta \\ &\quad - (s^2 + s + 1)(s + 1)\delta^2 \end{aligned}$$

Clearly, the vertex partition of G_3 is equitable. By Lemma 3.3, the largest root, say μ , of $\varphi(\mathcal{B}_2^s, x) = 0$ equals the spectral radius of $A(G_3)$. Recall that $\varphi(\mathcal{B}_1^\delta, x)$ is the characteristic polynomial of $K_\delta \vee (K_{n-2\delta-1} \cup (\delta + 1)K_1)$ and $\vartheta = \rho(K_\delta \vee (K_{n-2\delta-1} \cup (\delta + 1)K_1))$ is the largest root of $\varphi(\mathcal{B}_1^\delta, x) = 0$. By plugging the value ϑ into x of $\varphi(\mathcal{B}_2^s, x) - \varphi(\mathcal{B}_1^\delta, x)$, we obtain

$$\varphi(\mathcal{B}_2^s, \vartheta) = \varphi(\mathcal{B}_2^s, \vartheta) - \varphi(\mathcal{B}_1^\delta, \vartheta) = (\delta - s)g(\vartheta),$$

where $g(\vartheta) = (s - 1)\vartheta^2 + (n - (\delta - 1)s - 2)\vartheta + s^4 - (\delta + 1)s^3 + (n - 2\delta - 2)s^2 + s - (n - 3)\delta + 2\delta^2$. Then the symmetry axis of $f(\vartheta)$ is $\vartheta = -\frac{n-(\delta-1)s-2}{2(s-1)}$, which implies that $g(\vartheta)$ is increasing in the interval $[-\frac{n-(\delta-1)s-2}{2(s-1)}, +\infty)$. Since $2 \leq s \leq \delta - 1$ and $n \geq (s + 2)(\delta + 1 - s) + s$, we have

$$-\frac{n - (\delta - 1)s - 2}{2(s - 1)} < n - \delta - 2 < \vartheta,$$

which implies that

$$g(\vartheta) > g(n - \delta - 2) = n^2s + (s(s - 3) - 3\delta s)n + s^4 - (\delta + 1)s^3 - (2\delta + 2)s^2 + (2\delta^2 + 5\delta + 3)s + \delta^2 + \delta.$$

Note that

$$\frac{(3\delta - s + 3)s}{2s} = \frac{3\delta}{2} - \frac{1}{2}(s - 3) \leq \frac{3\delta + 1}{2} < \frac{3}{5}\delta^2 + \delta \leq n.$$

Therefore,

$$\begin{aligned} g(\vartheta) &> g(n - \delta - 2) \\ &= n^2s + (s(s - 3) - 3\delta s)n + s^4 - (\delta + 1)s^3 - (2\delta + 2)s^2 + (2\delta^2 + 5\delta + 3)s + \delta^2 + \delta \\ &\geq \left(\frac{3}{5}\delta^2 + \delta\right)^2 s + (s(s - 3) - 3\delta s)\left(\frac{3}{5}\delta^2 + \delta\right) + s^4 - (\delta + 1)s^3 - (2\delta + 2)s^2 \\ &\quad + (2\delta^2 + 5\delta + 3)s + \delta^2 + \delta \\ &= \delta + \frac{9\delta^4s}{25} - \frac{3\delta^3s}{5} + \frac{1}{5}\delta^2(3s(s - 3) + 5) - \delta s(s^2 + s - 2) + s(s(s - 2)(s + 1) + 3) \\ &\triangleq \phi(\delta) + s(s(s - 2)(s + 1) + 3), \end{aligned}$$

where $\phi(\delta) = \delta + \frac{9\delta^4s}{25} - \frac{3\delta^3s}{5} + \frac{1}{5}\delta^2(3s(s - 3) + 5) - \delta s(s^2 + s - 2)$. By a simple computation of the second derivative, we have

$$\phi''(\delta) = 2 + \frac{108\delta^2s}{25} - \frac{18\delta s}{5} + \frac{6}{5}s(s - 3) \geq 2 + \frac{234s(s + 1)}{25} + \frac{6}{5}s(s - 3) = 2 + \frac{24}{25}s(11s + 6) > 0,$$

which implies that

$$\begin{aligned} \phi'(\delta) &\geq \phi'(s + 1) = 1 + 2(s + 1) - \frac{9s(s + 1)^2}{5} + \frac{36s(s + 1)^3}{25} + \frac{6}{5}s(s + 1)(s - 3) - s(s^2 + s - 2) \\ &= \frac{1}{25}(36s^4 + 68s^3 - 67s^2 + s + 75) > 0. \end{aligned}$$

It follows that $\phi(\delta)$ is increasing in the interval $[s + 1, +\infty)$. Therefore, we have

$$\begin{aligned} g(\vartheta) &> g(n - \delta - 2) \geq \phi(\delta) + s(s(s - 2)(s + 1) + 3) \\ &\geq \phi(s + 1) + s(s(s - 2)(s + 1) + 3) \\ &= 2 + \frac{1}{25}s(9s^4 + 36s^3 - 81s^2 - 84s + 149) \\ &> 0. \end{aligned}$$

Thus, we have

$$\varphi(\mathcal{B}_2^s, \vartheta) = \varphi(\mathcal{B}_2^s, \vartheta) - \varphi(\mathcal{B}_1^\delta, \vartheta) = (\delta - s)g(\vartheta) > 0.$$

By a simple computation of the first derivative, we have

$$\varphi'(\mathcal{B}_2^s, x) = 3x^2 - 2(s^2 - (\delta + 1)s + n - 3)x + (\delta - 1)(s^2 + n) - (n + \delta^2 - \delta - 2)s - \delta^2 - 2\delta + 2.$$

Notice that $n \geq (s + 2)(\delta + 1 - s) + s$ and $\frac{s^2 - (\delta + 1)s + n - 3}{3} < n - \delta - 2$, and so $\varphi'(\mathcal{B}_2^s, x)$ is increasing in the interval

$[n - \delta - 2, +\infty)$. When $x > n - \delta - 2$, we have

$$\begin{aligned}
 & \varphi'(\mathcal{B}_2^s, x) > \varphi'(\mathcal{B}_2^s, n - \delta - 2) \\
 & = n^2 - (2s^2 - (2\delta + 1)s + 3\delta + 3)n + (3s^2 - 5s + 4)\delta + (3s - 2)(s - \delta^2) + 2 \\
 & \geq \left(\frac{3}{5}\delta^2 + \delta\right)^2 - (2s^2 - (2\delta + 1)s + 3\delta + 3)\left(\frac{3}{5}\delta^2 + \delta\right) + (3s^2 - 5s + 4)\delta + (3s - 2)(s - \delta^2) + 2 \\
 & = 2 + \delta + \frac{9}{25}\delta^4 + \delta(s - 4)s + \frac{3}{5}\delta^3(2s - 1) + s(3s - 2) - \frac{1}{5}\delta^2(6s^2 + 2s + 9) \\
 & \geq 2 + \delta + \frac{9}{25}\delta^4 + \delta(s - 4)s + \frac{1}{5}\delta^2(3(s + 1)(2s - 1) - 6s^2 - 2s - 9) + s(3s - 2) \\
 & = 2 + \delta + \frac{9}{25}\delta^4 + \delta(s - 4)s + \frac{1}{5}\delta^2(s - 12) + s(3s - 2) \\
 & \geq 2 + \delta + \frac{\delta^2}{25}(9\delta^2 - 60) + \delta(s(s - 4) + \frac{1}{5}s(s + 1)) + s(3s - 2) \\
 & \geq 2 + \delta + \frac{\delta}{25}[(s + 1)(9(s + 1)^2 - 60) + 30s^2 - 95s] + s(3s - 2) \\
 & \geq 2 + \frac{18}{25}\delta + s(3s - 2) \\
 & \geq 3s^2 - \frac{32}{25}s + \frac{68}{25} \\
 & > 0.
 \end{aligned}$$

Therefore, we have $\rho(G_3) < \vartheta = \rho(K_\delta \vee (K_{n-2\delta-1} \cup (\delta + 1)K_1))$. This together with (3), we then have

$$\rho(G) \leq \rho(G_1) \leq \rho(G_3) < \rho(K_\delta \vee (K_{n-2\delta-1} \cup (\delta + 1)K_1)),$$

as desired.

In view of Cases 1, 2 and 3, the proof is completed. $\square$

Acknowledgements The authors would like to thank the anonymous referees for their constructive corrections and valuable comments on this paper, which have considerably improved the presentation of this paper.

Data Availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors have no relevant financial or non-financial interests to disclose.

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