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Perimeter of an ascent sequence

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Abstract. We present an explicit formula for the generating function of the number of ascent sequences of length n according to the half perimeter. As consequence, we present an explicit formula for the generating function of the number of ascent sequences of length n according to the length of the initial longest increasing subword.

1. Introduction

An *ascent*, short for *ascent index*, in an integer sequence $s_1s_2\cdots s_m$ is an index $1 \leq j \leq m-1$ such that $s_j < s_{j+1}$. An *ascent sequence* $a = a_1a_2\cdots a_n$ is a sequence of non-negative integers that satisfies $a_1 = 1$ and $1 \leq a_i \leq \mathrm{asc}(a_1a_2\cdots a_{i-1}) + 2$ for $1 < i \leq n$, where $\mathrm{asc}(a_1a_2\cdots a_k)$ is the number of ascents in the sequence $a_1a_2\cdots a_k$. For example, the sequence $a = 12134315$ is an ascent sequence, whereas 12114 is not. Bousquet-Mèlou, Claesson, Dukes, and Kitaev [4] connected ascent sequences to $(2+2)$-free posets. Since that discovery, ascent sequences have been explored in numerous papers, revealing connections to other combinatorial structures (see, for instance, [7–11, 14–16]).

For any ascent sequence $a = a_1a_2\cdots a_m$, we associate a *bargraph* which is a column convex polyomino on a planar lattice grid $\mathbb{Z}^2$, where the lower edge lies on the x-axis and the i-th column contains exactly a_i 1×1 square cells, for $i = 1, 2, \ldots, m$. For example, Figure 1 represents the bargraph of the ascent sequence 1121324512.

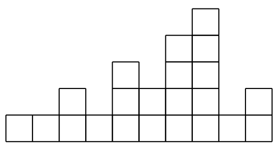

Figure 1: Bargraph of the ascent sequence 1121324512.

The perimeter is the number of edges on the boundary of the bargraph. For instance, the perimeter of the ascent sequence 1121324512 is 36. The enumeration of bargraphs according to their area and perimeter has been well studied and is a basic problem in combinatorics (for instance, see [1–3, 5, 17, 18]). In particular,

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in [5] (see also [13]), it was shown that the generating function for the number of bargraphs according to half the perimeter (i.e., semi-perimeter) is given by

$$\frac{1-2x-x^2-\sqrt{(1-2x-x^2)^2-4x^3}}{2x}.$$

In this note, we state a similar result in the case of ascent sequences. More precisely, we find an explicit formula for the generating function of the number ascent sequences of length n according to the half perimeter, see Theorems 2.2-2.3. As consequence, we present an explicit formula for the generating function of the number of ascent sequences of length n according to the length of the initial longest increasing subword, see Theorem 2.6.

2. Proofs

Let $P(x,q)$ be the generating function of the number of ascent sequences $a_1a_2\cdots a_m$ according to m (counted by x), and the half the perimeter of a (counted by q). In order to study this generating function, we define $P(x,q;a_1a_2\cdots a_s)$ to be the generating function of the number of ascent sequences $a = a_1a_2\cdots a_sa_{s+1}\cdots a_m$ according to the same variable trackers x and q of each a but such that the leftmost part of a is $a_1a_2\cdots a_s$.

Lemma 2.1. *For all $m \geq i \geq 1$,*

$$P(x,q;12\cdots m) = x^mq^{2m} + \sum_{j=1}^{m} P(x,q;12\cdots mj) + P(x,q;12\cdots(m+1)),$$

$$P(x,q;12\cdots mi) = x^{m+1}q^{2m+1} + xq\sum_{j=1}^{i} P(x,q;12\cdots mj) + \sum_{j=i+1}^{m+1} q^{j-i-1}P(x,q;12\cdots(m+1)j).$$

Proof. By considering the next letter (if exits) in all ascent sequences with the leftmost part either $12\cdots m$ or $12\cdots mi$, we obtain

$$P(x,q;12\cdots m) = x^mq^{2m} + \sum_{j=1}^{m} P(x,q;12\cdots mj) + P(x,q;12\cdots(m+1))$$

and

$$\begin{aligned}P(x,q;12\cdots mi) &= x^{m+1}q^{2m+1} + \sum_{j=1}^{i} P(x,q;12\cdots mij) + \sum_{j=i+1}^{m+1} P(x,q;12\cdots mij)\\ &= x^{m+1}q^{2m+1} + xq\sum_{j=1}^{i} P(x,q;12\cdots mj) + \sum_{j=i+1}^{m+1} q^{j-i-1}P(x,q;12\cdots(m+1)j),\end{aligned}$$

as claimed. □

Define

$$A(v) = A(x,q;v) = \sum_{m\geq1} P(x,q;12\cdots m)v^{m-1} \text{ and } B(v,u) = B(x,q;v,u) = \sum_{m\geq1}\sum_{i=1}^{m} P(x,q;12\cdots mi)v^{m-1}u^{i-1}.$$

By translating the recurrence relations in Lemma 2.1 in terms of generating functions $A(v)$ and $B(v,u)$, we obtain

$$\begin{aligned}
A(v) &= \sum_{m\geq 1} P(x,q;12\cdots m)v^{m-1}\\
&= \sum_{m\geq 1} x^m q^{2m} v^{m-1} + \sum_{m\geq 1}\sum_{j=1}^{m} P(x,q;12\cdots mj)v^{m-1} + \sum_{m\geq 1} P(x,q;12\cdots(m+1))v^{m-1}\\
&= \frac{xq^2}{1-xq^2v} + B(v,1) + \frac{1}{v}\sum_{m\geq 2} P(x,q;12\cdots m)v^{m-1}\\
&= \frac{xq^2}{1-xq^2v} + B(x,q;v,1) + \frac{1}{v}(A(x,q;v)-A(x,q;0))
\end{aligned}$$

and

$$\begin{aligned}
B(v,u) &= \sum_{m\geq 1}\sum_{i=1}^{m} P(x,q;12\cdots mi)v^{m-1}u^{i-1}\\
&= \sum_{m\geq 1}\sum_{i=1}^{m} x^{m+1}q^{2m+1}v^{m-1}u^{i-1} + xq\sum_{m\geq 1}\sum_{i=1}^{m}\sum_{j=1}^{i} P(x,q;12\cdots mj)v^{m-1}u^{i-1}\\
&\quad + \sum_{m\geq 1}\sum_{i=1}^{m}\sum_{j=i+1}^{m+1} q^{j-i-1}P(x,q;12\cdots(m+1)j)v^{m-1}u^{i-1}\\
&= \frac{q^3x^2}{(1-xq^2uv)(1-xq^2v)} + xq\sum_{m\geq 1}\sum_{j=1}^{m}\sum_{i=j}^{m} P(x,q;12\cdots mj)v^{m-1}u^{i-1}\\
&\quad + \sum_{m\geq 1}\sum_{j=1}^{m+1}\sum_{i=1}^{j-1} q^{j-i-1}P(x,q;12\cdots(m+1)j)v^{m-1}u^{i-1}\\
&= \frac{q^3x^2}{(1-xq^2uv)(1-xq^2v)} + xq\sum_{m\geq 1}\sum_{j=1}^{m} P(x,q;12\cdots mj)v^{m-1}\frac{u^{j-1}-u^m}{1-u}\\
&\quad + \sum_{m\geq 2}\sum_{j=1}^{m} P(x,q;12\cdots mj)v^{m-2}\frac{q^{j-1}-u^{j-1}}{q-u}\\
&= \frac{q^3x^2}{(1-xq^2uv)(1-xq^2v)} + \frac{xq}{1-u}(B(v,u)-uB(vu,1)) + \frac{1}{v(q-u)}(B(v,q)-B(v,u)),
\end{aligned}$$

which leads to the following equations

$$A(x,q;v) = \frac{xq^2}{1-xq^2v} + B(x,q;v,1) + \frac{1}{v}(A(x,q;v)-A(x,q;0)), \tag{1}$$

$$\begin{aligned}
B(x,q;v,u) &= \frac{q^3x^2}{(1-xq^2uv)(1-xq^2v)} + \frac{xq}{1-u}(B(x,q;v,u)-uB(x,q;vu,1))\\
&\quad + \frac{1}{v(q-u)}(B(x,q;v,q)-B(x,q;v,u)).
\end{aligned} \tag{2}$$

Let $u_\pm$ be the roots of the kernel equation $1 = \frac{xq}{1-u} - \frac{1}{v(q-u)}$, namely,

$$u_\pm = \frac{1+(1+q-qx)v \pm \sqrt{(1+(1+q-qx)v)^2 - 4v(1+qv-q^2vx)}}{2v}.$$

By substituting $u = u_+$ and $u = u_-$, we obtain

$$B(x,q;v,q) = \frac{qxu_\pm v(q-u_\pm)}{1-u_\pm}B(x,q;u_\pm v,1) - \frac{(q-u_\pm)vq^3x^2}{(1-q^2u_\pm vx)(1-q^2vx)},$$

which implies that the generating function $B(x,q;v,1)$, satisfies

$$\frac{u_+(q-u_+)}{1-u_+}B(x,q;u_+v,1) - \frac{u_-(q-u_-)}{1-u_-}B(x,q;u_-v,1) = \frac{q^2x(u_+-u_-)(1-q^3vx)}{(1-q^2vx)(1-q^2u_+vx)(1-q^2u_-vx)}. \tag{3}$$

Now by taking (1) with $v = 1$, we obtain the following result.

Theorem 2.2. *The generating function for the number of ascent sequences of length m and half the perimeter is given by*

$$A(x,q;0) = \frac{xq^2}{1-xq^2} + B(x,q;1,1),$$

where the generating function $B(x,q;v,1)$ satisfies (3).

Note that Equation (3) can be solved by iterating as follows. Define

$$\begin{aligned}\alpha(v) &= \frac{u_+(q-u_+)}{1-u_+},\\ \beta(v) &= \frac{u_-(q-u_-)}{1-u_-},\\ \gamma(v) &= \frac{q^2x(u_+-u_-)(1-q^3vx)}{(1-q^2vx)(1-q^2u_+vx)(1-q^2u_-vx)}.\end{aligned}$$

Then (3) can be written as

$$B(x,q;v,1) = \frac{\alpha(\frac{v}{u_-})}{\beta(\frac{v}{u_-})}B(x,q;\frac{vu_+}{u_-},1) - \frac{\gamma(\frac{v}{u_-})}{\beta(\frac{v}{u_-})}.$$

By the fact that $1/u_- = \frac{vu_+}{1+qv-q^2vx}$, we obtain

$$B(x,q;v,1) = \frac{\alpha(\frac{v^2u_+}{1+q-q^2vx})}{\beta(\frac{v^2u_+}{1+q-q^2vx})}B(x,q;\frac{v^2u_+^2}{1+q-q^2vx},1) - \frac{\gamma(\frac{v^2u_+}{1+q-q^2vx})}{\beta(\frac{v^2u_+}{1+q-q^2vx})}. \tag{4}$$

Note that

$$\begin{aligned}\frac{\alpha(\frac{v^2u_+}{1+q-q^2vx})}{\beta(\frac{v^2u_+}{1+q-q^2vx})} &= \frac{(qv+1)qx}{(qv-v+1)^2} - \frac{vq^2(2q^2v^2-q^2v-2qv^2+3qv-q-2v+2)x^2}{(qv-v+1)^4} + \cdots,\\ \frac{v^2u_+^2}{1+q-q^2vx} &= \frac{(qv+1)^2}{q+1} + \frac{2vq(q^2v-qv-1)(qv+1)x}{(q+1)(qv-v+1)} + \cdots,\\ \frac{\gamma(\frac{v^2u_+}{1+q-q^2vx})}{\beta(\frac{v^2u_+}{1+q-q^2vx})} &= q^3x^2 + \frac{q^4(q^3v^2(2v+1)-q^2v(4v^2-3v-2)+q(2v^3-5v^2+v+1)+v^2-2v)x^3}{(qv-v+1)^2} + \cdots.\end{aligned}$$

Thus, there exist two power series $F(x,v)$ and $G(x,q)$ such that (4) can be written as

$$B(x,q;v,1) = xF(x,v)B(x,q;\frac{v^2u_+^2}{1+q-q^2vx},1) + G(x,v).$$

By iterating (4) an infinite number of times under the condition $|x| < 1$, we obtain an explicit formula for the generating function $B(x, q; v, 1)$. More precisely, let $\rho(v) = \frac{v^2u_+}{1+q-q^2vx}$ and $\psi_j(v) = \psi_{j-1}(\frac{v^2u_+^2}{1+q-q^2vx})$ with $\psi_0(v) = v$. Thus,

$$B(x, q; v, 1) = \frac{\alpha(\rho(v))}{\beta(\rho(v))}B(x, q; \psi_1(v), 1) - \frac{\gamma(\rho(v))}{\beta(\rho(v))}.$$

Thus, by iterating an infinity number of times, we obtain

$$B(x, q; v, 1) = -\sum_{j\geq 0}\frac{\gamma(\rho(\psi_j(v)))\prod_{i=0}^{j-1}\alpha(\rho(\psi_i(v)))}{\prod_{i=0}^{j}\beta(\rho(\psi_i(v)))}.$$

By Theorem 2.2, we obtain the following result.

Theorem 2.3. *The generating function for the number of ascent sequences of length m and half the perimeter is given by*

$$A(x, q; 0) = \frac{xq^2}{1 - xq^2} - \sum_{j\geq 0}\frac{\gamma(\rho(\psi_j(1)))\prod_{i=0}^{j-1}\alpha(\rho(\psi_i(1)))}{\prod_{i=0}^{j}\beta(\rho(\psi_i(1)))}.$$

By applying this formula, we see that the expansion of the generating function $B(x, q; v, 1)$ is $q^3x^2 + q^4(2qv + q + 1)x^3 + q^5(q^3v + 3q^2v^2 + 2q^2v + q^2 + 3qv + 3q + 1)x^4 + q^6(q^5v^2 + 2q^4v^2 + 4q^3v^3 + 2q^4v + 3q^3v^2 + 7q^3v + 6q^2v^2 + 2q^3 + 8q^2v + 6q^2 + 4qv + 6q + 1)x^5 + q^7(q^7v^3 + 2q^6v^3 + 3q^6v^2 + 3q^5v^3 + 5q^4v^4 + q^6v + 11q^5v^2 + 4q^4v^3 + 7q^5v + 15q^4v^2 + 10q^3v^3 + q^5 + 21q^4v + 15q^3v^2 + 5q^4 + 30q^3v + 10q^2v^2 + 16q^3 + 20q^2v + 20q^2 + 5qv + 10q + 1)x^6 + \cdots$. By Theorem 2.3, we see that the expansion of the generating function for the number of ascent sequences of length m and the half the perimeter is $q^2x + q^3(q + 1)x^2 + q^4(q^2 + 3q + 1)x^3 + \mathbf{q^5(2q^3 + 6q^2 + 6q + 1)}x^4 + q^6(q^5 + 5q^4 + 16q^3 + 20q^2 + 10q + 1)x^5 + q^7(q^7 + 6q^6 + 23q^5 + 50q^4 + 71q^3 + 50q^2 + 15q + 1)x^6 + \cdots$. The boldface coefficient describes the generating function for the number of ascent sequences of length 4 according to the half perimeter, see Figure 2.

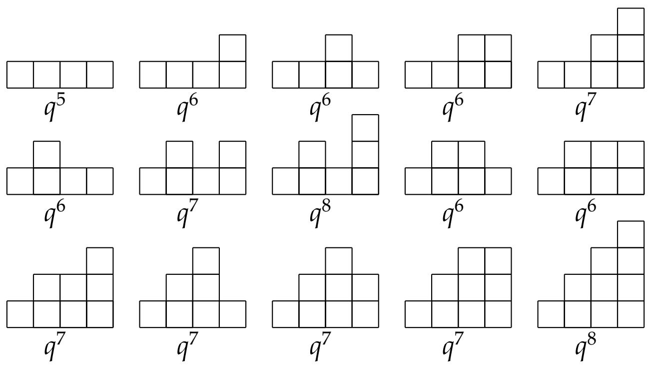

Figure 2: Bargraph of the ascent sequence of length four.

Note that by (3) and expression of $B(x, 1; v, 1)$, we obtain an explicit formula for $B(x, 1; v, u)$, where we need this in our next step.

Our next goal is to find the generating function $A(x, 1; v)$. At first, we study the (1)-(2) with $q = 1$:

$$A(x, 1; v) = \frac{x}{1 - xv} + B(x, 1; v, 1) + \frac{1}{v}(A(x, 1; v) - A(x, 1; 0)), \tag{5}$$

$$\begin{aligned}B(x, 1; v, u) = {} & \frac{x^2}{(1 - xuv)(1 - xv)} + \frac{x}{1 - u}(B(x, 1; v, u) - uB(x, 1; vu, 1)) \\ & + \frac{1}{v(1 - u)}(B(x, 1; v, 1) - B(x, 1; v, u)).\end{aligned} \tag{6}$$

By substituting $v = \frac{1}{-1+u+x}$ into (6), we obtain

$$-\frac{x^2(u+x-1)}{(x-1)(1-u)} - \frac{ux}{u+x-1}B(x,1;\frac{u}{u+x-1},1) + B(x,1;\frac{1}{u+x-1},1) = 0,$$

which implies

$$B(x,1;v,1) = x(1+(1-x)v)B(x,1;1+(1-x)v,1) + \frac{x^2}{(1-x)(1-xv)}.$$

Define $b_m(v) = \frac{1-(1-x)^m}{x} + (1-x)^m v$. Then by iterating this equation an infinite number of times, we obtain

$$B(x,1;v,1) = \frac{x^2}{1-x}\sum_{j\geq 0}\frac{x^j}{(1-x)^j(1-xv)}\prod_{i=1}^{j}\left(\frac{1-(1-x)^i}{x} + (1-x)^i v\right).$$

Hence, by (5) with $v = 1$, we have

$$A(x,1;0) = \frac{x}{1-x} + B(x,1;1,1) = \frac{x}{1-x} + \frac{x^2}{1-x}\sum_{j\geq 1}\frac{\prod_{i=1}^{j-1}(1-(1-x)^{i+1})}{(1-x)^j}.$$

as expected, see [4]. Thus, by expression of $B(x,1;v,1)$ and (5), we have the following result.

Theorem 2.4. *The generating function $A(x,1;v)$ is given by*

$$\begin{aligned}A(x,1;v) &= \frac{x}{(1-x)(1-xv)} - \frac{x^2}{(1-x)(1-v)(1-xv)}\sum_{j\geq 0}\frac{x^j}{(1-x)^j}\prod_{i=0}^{j}\left(\frac{1-(1-x)^i}{x} + (1-x)^i v\right)\\ &\quad + \frac{x}{(1-x)(1-v)}\sum_{j\geq 1}\frac{\prod_{i=0}^{j-1}(1-(1-x)^{i+1})}{(1-x)^j}\\ &= x + (2+v)x^2 + (v^2+3v+5)x^3 + (v^3+4v^2+10v+15)x^4\\ &\quad + (v^4+5v^3+17v^2+38v+53)x^5 + (v^5+6v^4+26v^3+80v^2+164v+217)x^6 + \cdots.\end{aligned}$$

For any ascent sequence $a = a_1a_2\cdots a_n$, we define $inc(a)$ to be the maximal j, $1 \leq j \leq n$, such that $a_1 < a_2 < \cdots < a_j$. By (5) and Lemma 2.1, we have the following result.

Corollary 2.5. *Define*

$$F(x,v) = \sum_{n\geq 1}\sum_{a \text{ ascent sequence of length } n} x^n(1+v+v^2+\cdots+v^{inc(a)-1}).$$

Then,

$$F(x,v) = \sum_{m\geq 1}P(x,1;12\cdots m)v^{m-1} = A(x,1;v).$$

For instance, there are five ascent sequences of length 3, namely, 111, 112, 121, 122, 123, where $inc(111) = 1$, $inc(112) = 1$, $inc(121) = 2$, $inc(122) = 2$, and $inc(123) = 3$. Thus, the coefficient of x^3 is the generating function $F(x,v)$ is $1+1+1+v+1+v+1+v+v^2 = 5+3v+v^2$, which equals the coefficient of x^3 in $A(x,1;v)$, see Theorem 2.3.

Moreover, by the definitions, we have

$$\begin{aligned}G(x,v) &= \sum_{n\geq 1}\sum_{a \text{ ascent sequence of length } n} x^n v^{inc(a)-1} = \sum_{m\geq 1}(P(x,q;12\cdots m) - P(x,q;12\cdots m+1))v^{m-1}\\ &= A(x,1;v) - \frac{1}{v}(A(x,1;v) - A(x,1;0)).\end{aligned}$$

Thus, by Theorem 2.4, we obtain the following result.

Theorem 2.6. *The generating function $G(x, v)$ is given by*

$$G(x, v) = \frac{x}{1 - xv} + \frac{x}{1 - xv} \sum_{j \geq 1} \frac{x^j}{(1 - x)^j} \prod_{i=1}^{j-1} \left(\frac{1 - (1 - x)^i}{x} + (1 - x)^i v \right).$$

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References

[1] A. Bacher, *Average site perimeter of directed animals on the two-dimensional lattices*, Discrete Math. **312** (2012), 1038–1058.
[2] A. Blecher, C. Brennan, and A. Knopfmacher, *The site-perimeter in permutations*, Util. Math., in press.
[3] A. Blecher, C. Brennan, A. Knopfmacher, and T. Mansour, *The perimeter of words*, Discrete Math. **340**:10 (2017), 2456-2465.
[4] M. Bousquet-Mèlou, A. Claesson, M. Dukes, and S. Kitaev, (2 + 2)*-free posets, ascent sequences and pattern avoiding permutations*, J. Combin. Theory Ser. A **117**(7) (2010), 884–909.
[5] M. Bousquet-Mélou and A. Rechnitzer, *The site-perimeter of bargraphs*, Adv. Appl. Math. **31** (2003), 86–112.
[6] C. Banderier, M. Bousquet-Mélou, A. Denise, P. Flajolet, D. Gardy, and D. GouyouBeauchamps, *Generating functions for generating trees*, Discrete Math. **246**(1–3) (2000), 29–55.
[7] D. Callan, T. Mansour, and M. Shattuck, *Restricted ascent sequences and Catalan numbers*, Appl. Anal. Discrete Math. **8** (2014), 288–303.
[8] D. Callan and T. Mansour, *Ascent sequences avoiding a triple of 3-letter patterns and Fibonacci numbers*, Art of Discrete and Appl. Math. **7** (2024), #P2.10
[9] M. Dukes and R. Parviainen, *Ascent sequences and upper triangular matrices containing non-negative integers*, Electron. J. Combin. **17**(1) (2010), #R53.
[10] M. Dukes, J. Remmel, S. Kitaev, and E. Steingrímsson, *Enumerating* (2 + 2)*-free posets by indistinguishable elements*, J. Comb. **2**(1) (2011), 139–163.
[11] P. Duncan and E. Steingrímsson, *Pattern avoidance in ascent sequences*, Electron. J. Combin. **18**(1) (2011), #P226.
[12] P. Flajolet and R. Sedgewick, *Analytic Combinatorics* , Cambridge University Press, Cambridge, 2009.
[13] S. Feretiĉ, *A perimeter enumeration of column-convex polyominoes*, Discrete Math. Theoret. Comput. Sci. **9** (2007), 57–84
[14] S. Kitaev and J. Remmel, *Enumerating* (2 + 2)*-free posets by the number of minimal elements and other statistics*, Discrete Appl. Math. **159** (2011), 2098–2108.
[15] T. Mansour and M. Shattuck, *Some enumerative results related to ascent sequences*, Discrete Math. **315** (2014), 29–41.
[16] T. Mansour and M. Shattuck, *Ascent sequences and Fibonacci numbers*, FILOMAT **29**:4 (2015), 703–712.
[17] T. Mansour, *The perimeter and the site-perimeter of set partitions*, Electron. J. Combin. **26**(2) (2019), #P2.30.
[18] T. Mansour and A. Sh. Shabani, *The perimeter of* 2*-compositions*, Quaest. Math. **43**:11 (2020), 1639–1644.