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Fuzzy approximation theorems via statistical deferred Nörlund Riemann integrability

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Abstract. This study introduces and investigates the concepts of deferred Nörlund statistical Riemann integrability and statistical deferred Nörlund Riemann summability for sequences of fuzzy number-valued functions. It begins by establishing an inclusion result that clarifies the relationship between these newly proposed notions. Subsequently, new fuzzy Korovkin-type theorems are developed using the three fundamental algebraic test functions 1, x, and x^2, based on our proposed means. To demonstrate the practical significance of these results, an example is presented involving a fuzzy positive linear operator associated with Bernstein polynomials. Additionally, the convergence behavior of these operators is illustrated graphically using MATLAB.

1. Introduction and Motivation

Let $\mathcal{E} = \{\gamma : \mathbb{R} \to [0,1]\}$ denote the set of functions that satisfy the following conditions:

(i) Normality: There exists $t_0 \in \mathbb{R}$ such that $\gamma(t_0) = 1$,

(ii) Fuzzy Convexity: γ is a fuzzy convex function,

(iii) Upper Semi-Continuity: γ is upper semi-continuous,

(iv) Compact Support: The set $[\gamma]^0 = \overline{\{t \in \mathbb{R} \quad \text{and} \quad \gamma(t) > 0\}}$ is compact.

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A function $\gamma \in \mathcal{E}$ satisfying these properties is called a fuzzy number, and $\mathcal{E}$ represents the fuzzy number space.

Let $\gamma \in \mathcal{E}$, and define $[\gamma]^{\Lambda} = \{t \in \mathbb{R} : X(t) \geqq \Lambda\}$, which represents a closed and bounded interval for $\Lambda \in [0,1]$.

Let us now review some fundamental properties of fuzzy numbers.

Let $\gamma, \beta \in \mathcal{E}$, $\Lambda \in [0,1]$ and $\lambda \in \mathbb{R}$. Then

(i) $(\gamma + \beta)(x) = \sup_{x=t+s} \min\{\gamma(t), \beta(s)\}$

(ii) $k\gamma(x) = \gamma(x/k) \quad (k \neq 0)$

(iii) $0\gamma(x) = \bar{0}$, where

$$\bar{a}(x) = \begin{cases} 1 & (x = a) \\ 0 & (\text{otherwise}) \end{cases}$$

(iv) $[\gamma + \beta]^{\Lambda} = [\gamma]^{\Lambda} + [\beta]^{\Lambda} = [\gamma^{-}_{\Lambda} + \beta^{-}_{\Lambda}, \gamma^{+}_{\Lambda} + \beta^{+}_{\Lambda}]$

(v) $[k\gamma]^{\Lambda} = k[u]^{\Lambda} = [k\gamma^{-}_{\Lambda}, k\gamma^{+}_{\Lambda}]$ for $(k \geqq 0)$

(vi) $\gamma \leqq \beta \Longleftrightarrow [\gamma]^{\Lambda} \leqq [\beta]^{\Lambda}$

The metric $\mathcal{D}$ is defined as $\mathcal{D} : \mathcal{E} \times \mathcal{E} \to \mathbb{R}_{+}$, given by

$$\mathcal{D}(\gamma, \beta) = \sup_{0 \leqq \Lambda \leqq 1} \max\{|\gamma^{\Lambda}_{-} - \beta^{\Lambda}_{-}|, |\gamma^{\Lambda}_{+} - \beta^{\Lambda}_{+}|\},$$

where the metric space $d(\mathcal{E}, \mathcal{D})$ is complete (see [40]).

Let $\mathcal{D}^{*}(\tilde{f}, \tilde{g})$ denote the distance between the functions $\tilde{f}$ and $\tilde{g}$ defined as follows:

$$\mathcal{D}^{*}(\tilde{f}, \tilde{g}) = \sup_{a \leqq t \leqq b} \sup_{0 \leqq \Lambda \leqq 1} \max\{|\tilde{f}^{\Lambda}_{-} - \tilde{g}^{\Lambda}_{-}|, |\tilde{f}^{\Lambda}_{+} - \tilde{g}^{\Lambda}_{+}|\}.$$

The study of convergence in sequence spaces is a fundamental and significant area in real and functional analysis. Over time, advancements in this field have led to the development of statistical convergence, a concept more general than traditional convergence. This elegant idea was independently introduced by Fast [10] and Steinhaus [38]. Today, statistical convergence remains an active and dynamic area of research, attracting the attention of numerous scholars. It finds applications in diverse fields of pure and applied mathematics, including machine learning, soft computing, number theory, measure theory, and probability theory. For recent works in this direction, see [2], [3], [5], [6], [7], [11], [14], [15], [16], [17], [18], [19], [21], [31], [32], [34], and [41].

Let $\mathfrak{I}^{*} \subseteq \mathbb{N}$, and define $\mathfrak{I}^{*}_{k} = \{\xi : \xi \leqq k \quad \text{and} \quad \xi \in \mathfrak{I}^{*}\}$. The natural density $d(\mathfrak{I}^{*})$ of $\mathfrak{I}^{*}$ is given by

$$d(\mathfrak{I}^{*}) = \lim_{k \to \infty} \frac{|\mathfrak{I}^{*}_{k}|}{k} = \rho,$$

where ρ is a real and finite number, and $|\mathfrak{I}^{*}_{k}|$ represents the cardinality of $\mathfrak{I}^{*}_{k}$.

A sequence (η_k) is said to be statistically convergent to a fuzzy number ℓ if, for every $\epsilon > 0$, the set

$$\mathfrak{I}^{*}_{\epsilon} = \{\xi : \xi \in \mathbb{N} \quad \text{and} \quad \mathcal{D}(\eta_{\xi}, \ell) \geqq \epsilon\}$$

has a natural density of zero (see [29]). This implies that, for each $\epsilon > 0$,

$$d(\mathfrak{I}_\epsilon^*) = \lim_{k\to\infty} \frac{|\mathfrak{I}_\epsilon^*|}{k} = 0.$$

We express this as

$$\text{stat} \lim_{k\to\infty} \mathcal{D}(\eta_\xi, \ell) = 0.$$

Let $[a, b] \subset \mathcal{E}$. For each $k \in \mathbb{N}$, there exists a fuzzy number-valued function $\tilde{g}_k : [a, b] \to \mathcal{E}$. As a result, $(\tilde{g}_k)$ constitutes a sequence of fuzzy number-valued functions defined on $[a, b]$.

The Riemann sum for a sequence $(\tilde{g}_i)$ of fuzzy number-valued functions associated to a tagged partition $\dot{\mathcal{P}}$ is defined as

$$\delta(\tilde{g}_i; \dot{\mathcal{P}}) := \sum_{i=1}^{k} \tilde{g}(t_i)\mathcal{D}(y_i, y_{i-1}),$$

where t_i represents the tags, and $\mathcal{D}(y_i, y_{i-1})$ denotes the difference between consecutive partition points.

Next, we revisit the concept of Riemann integrability for a sequence of fuzzy number-valued functions defined on the interval $[a, b]$.

A sequence $(\tilde{g}_k)_{k\in\mathbb{N}}$ of fuzzy number-valued functions is said to be Riemann integrable to a fuzzy number-valued function $\tilde{g}$ on $[a, b]$ if, for every $\epsilon > 0$, there exists $\sigma_\epsilon > 0$ such that for any tagged partition $\dot{\mathcal{P}}$ of $[a, b]$ with $|\dot{\mathcal{P}}| < \sigma_\epsilon$, the following inequality holds:

$$\mathcal{D}(\delta(\tilde{g}_k; \dot{\mathcal{P}}), \tilde{g}) < \epsilon.$$

We now present the definition of statistical convergence for Riemann integrable fuzzy number-valued functions.

A sequence $(\tilde{g}_k)_{k\in\mathbb{N}}$ of fuzzy number-valued functions is said to be statistically Riemann integrable to a fuzzy number-valued function $\tilde{g}$ on the interval $[a, b]$ if, for every $\epsilon > 0$ and for each $t \in [a, b]$, there exists a threshold $\sigma_\epsilon > 0$ such that for any tagged partition $\dot{\mathcal{P}}$ of $[a, b]$ satisfying $|\dot{\mathcal{P}}| < \sigma_\epsilon$, the set

$$\mathfrak{I}^*{}_\epsilon = \{\xi : \xi \in \mathbb{N} \quad \text{and} \quad \mathcal{D}(\delta(\tilde{g}_\xi; \dot{\mathcal{P}}), \tilde{g}) \geqq \epsilon\}$$

has zero natural density. In other words, for every $\epsilon > 0$,

$$d(\mathfrak{I}^*{}_\epsilon) = \lim_{k\to\infty} \frac{|\mathfrak{I}^*{}_\epsilon|}{k} = 0.$$

This condition is expressed as

$$\text{stat}_{\text{Rie}} \lim_{k\to\infty} \mathcal{D}(\delta(\tilde{g}_k; \dot{\mathcal{P}}), \tilde{g}) = 0.$$

The following example illustrates that every Riemann integrable fuzzy number-valued function is also statistically Riemann integrable. However, the reverse implication does not necessarily hold.

Example 1.1. *Let $(\tilde{g}_k)_{k\in\mathbb{N}} : [0, 1] \to \mathbb{R}$ be a sequence of fuzzy number-valued functions defined as follows:*

$$(\tilde{g}_k)_{k\in\mathbb{N}} = \begin{cases} \frac{1}{2} & (x \in \mathbb{Q} \cap [0, 1];\ k = j^2,\ j \in \mathbb{N}) \\ \\ \frac{k}{k+1} & (\text{otherwise}). \end{cases} \tag{1}$$

It is easy to see that the sequence $(\tilde{g}_k)$ is statistically Riemann integrable to 1 on the interval $[0,1]$. However, in the traditional sense, it is not Riemann integrable over $[0,1]$.

Inspired by the previous studies, we introduce the concepts of deferred Nörlund statistical Riemann integrability and statistical deferred Nörlund Riemann summability for sequences of fuzzy number-valued functions. First, we establish an inclusion theorem to highlight the relationship between these novel and valuable concepts. Building upon these foundational ideas, we develop new versions of fuzzy Korovkin-type theorems utilizing three algebraic test functions 1, x and x^2, based on our proposed means. Finally, we present an example involving a fuzzy positive linear operator associated with Bernstein polynomials to demonstrate the practical significance of our findings. Additionally, we use MATLAB to illustrate the convergence behavior of these operators graphically.

2. Deferred Nörlund Statistical Riemann Integrability

Let (ϕ_k) and (φ_k) be sequences in $\mathbb{Z}^{0+}$ such that $\phi_k < \varphi_k$ and $\lim_{k\to\infty}\varphi_k = +\infty$. Additionally, let (p_k) be a sequence of non-negative real numbers, where

$$P_k = \sum_{\mu=\phi_k+1}^{\varphi_k} p_{\varphi_k-\mu}.$$

The deferred Nörlund summability mean for the Riemann sum of a sequence of fuzzy number-valued functions, $\delta(\tilde{g}_k;\dot{\mathcal{P}})$, with respect to a tagged partition $\dot{\mathcal{P}}$, is defined as

$$\mathcal{N}(\delta(\tilde{g}_k;\dot{\mathcal{P}})) = \frac{1}{P_k}\sum_{\lambda=\phi_k+1}^{\varphi_k} p_{\varphi_k-\lambda}\, \delta(\tilde{g}_\lambda;\dot{\mathcal{P}}). \tag{2}$$

We introduce the notions of statistical Riemann integrability and statistical Riemann summability for a sequence of fuzzy number-valued functions based on the deferred Nörlund mean.

Definition 2.1. *Let (ϕ_k) and (φ_k) be sequences in $\mathbb{Z}^{0+}$, and let (p_k) be a sequence of non-negative real numbers. A sequence $(\tilde{g}_k)_{k\in\mathbb{N}}$ of fuzzy number-valued functions is said to be deferred Nörlund statistically Riemann integrable ($DNFR_{\text{stat}}$) to a fuzzy number-valued function $\tilde{g}$ on $[a,b]$ if, for every $\epsilon > 0$, there exists $\sigma_\epsilon > 0$ such that for any tagged partition $\dot{\mathcal{P}}$ of $[a,b]$ satisfying $|\dot{\mathcal{P}}| < \sigma_\epsilon$, the set*

$$\{\xi : \xi \leqq P_k \quad \text{and} \quad p_\xi \mathcal{D}(\delta(\tilde{g}_\xi;\dot{\mathcal{P}}),\tilde{g}) \geqq \epsilon\}$$

has zero natural density. That is, for each $\epsilon > 0$,

$$\lim_{k\to\infty}\frac{|\{\xi : \xi \leqq P_k \quad \text{and} \quad p_\xi \mathcal{D}(\delta(\tilde{g}_\xi;\dot{\mathcal{P}}),\tilde{g}) \geqq \epsilon\}|}{P_k} = 0.$$

This condition is expressed as

$$DNFR_{\text{stat}} \lim_{k\to\infty} \mathcal{D}(\delta(\tilde{g}_k;\dot{\mathcal{P}}),\tilde{g}) = 0.$$

Example 2.2. *We consider the interval $[0,1]$ and define a sequence of fuzzy number-valued functions $\tilde{g}(x)$. For simplicity. Let's define*

$$\tilde{g}(x) = \frac{\sin(kx)}{k},\ x \in [0,1].$$

We take a tagged partition $\dot{\mathcal{P}}$ consisting of equidistant points

$$y_i = \frac{i}{n},\ (i = 0,1,...,n)$$

with midpoints as tags

$$t_i = \frac{y_i + y_{i-1}}{2}.$$

The Riemann sum is given by

$$\delta(\tilde{g}_k;\dot{\mathcal{P}}) := \sum_{i=1}^{n} \tilde{g}_k(t_i)\mathcal{D}(y_i, y_{i-1}),$$

For our case, since the partition is uniform, $\mathcal{D}(y_i, y_{i-1}) = \frac{1}{n}$, *leading to*

$$\delta(\tilde{g}_k;\dot{\mathcal{P}}) := \sum_{i=1}^{n} \frac{\sin(kt_i)}{k} \cdot \frac{1}{n}.$$

Consequently, it is not Riemann integrable in the usual sense.
Next, in view of the Deferred Nörlund Summability,

$$P_k = \sum_{\mu=\phi_k+1}^{\varphi_k} p_{\varphi_k-\mu}, \text{ with } p_k = \frac{1}{k+1},$$

the mean is given by:

$$\mathcal{N}(\delta(\tilde{g}_k;\dot{\mathcal{P}})) = \frac{1}{P_k} \sum_{\lambda=\phi_k+1}^{\varphi_k} p_{\varphi_k-\lambda}\, \delta(\tilde{g}_\lambda;\dot{\mathcal{P}}).$$

Hence,

$$DNFR_{\text{stat}} \lim_{k\to\infty} \mathcal{D}(\delta(\tilde{g}_k;\dot{\mathcal{P}}), \tilde{g}) = 0.$$

That is, it has deferred Riemann sum 0*. Figure 1 illustrates the convergence behavior of the function under usual Riemann sum as well as deferred Nörlund Riemann sum.*

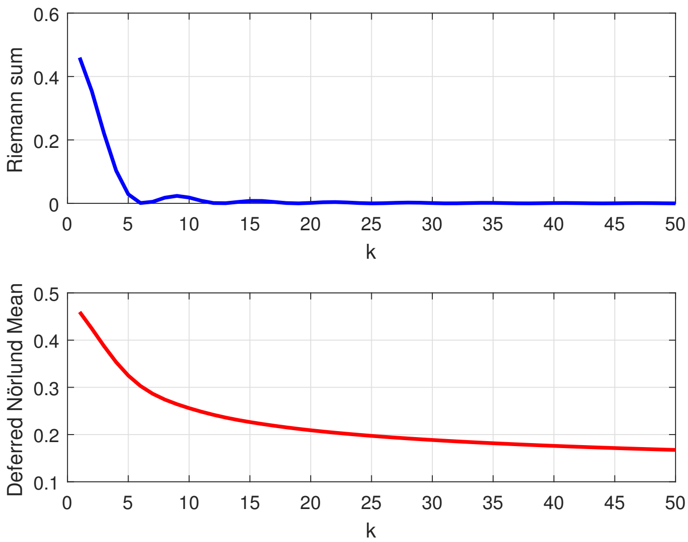

Figure 1: Deferred Nörlund Riemann sum

Figure 1 (under usual Riemann sum), confirms the expected behavior of the sequence of functions $\tilde{g}(x) = \frac{\sin(kx)}{k}$ *in terms of Riemann integrability. As k increases, the function values decrease, reducing their contribution to the integral. However, due to the inherent oscillatory nature, the sequence does not necessarily converge in the usual Riemann sense. In contrast, Figure 1 (under Deferred Nörlund mean) highlights the smoothing effect of sequence of functions via Deferred Nörlund summability. Unlike standard Riemann sums, this summability method ensures stabilization of the sequence, making it more effective for analyzing integral convergence. This suggests that while* $\tilde{g}(x)$ *may not be traditionally Riemann integrable, it is statistically Riemann integrable under Deferred Nörlund summability mean.*

Overall, the first plot of Figure 1 illustrates the oscillatory and decaying nature of Riemann sums, while the second plot of Figure 1 demonstrates how Deferred Nörlund summability regularizes the sum, leading to smoother convergence. This validates the use of summability techniques like Deferred Nörlund means to handle sequences that are not conventionally integrable but still exhibit meaningful statistical convergence.

Definition 2.3. *Let* (ϕ_k) *and* (φ_k) *be sequences in* $\mathbb{Z}^{0+}$*, and let* (p_k) *be a sequence of non-negative real numbers. A sequence* $(\tilde{g}_k)_{k\in\mathbb{N}}$ *of fuzzy number-valued functions is said to be statistically deferred Nörlund Riemann summable* $(stat_{\mathrm{DNFR}})$ *to a fuzzy number-valued function* $\tilde{g}$ *on* $[a,b]$ *if, for every* $\epsilon > 0$*, there exists* $\sigma_\epsilon > 0$ *such that for any tagged partition* $\dot{\mathcal{P}}$ *of* $[a,b]$ *with* $|\dot{\mathcal{P}}| < \sigma_\epsilon$*, the set*

$$\{\xi : \xi \leqq k \quad and \quad \mathcal{D}(\mathcal{N}(\delta(\tilde{g}_\xi;\dot{\mathcal{P}})),\tilde{g}) \geqq \epsilon\}$$

has natural density zero. This condition implies that for all $\epsilon > 0$*,*

$$\lim_{k\to\infty} \frac{|\{\xi : \xi \leqq k \quad and \quad \mathcal{D}(\mathcal{N}(\delta(\tilde{g}_\xi;\dot{\mathcal{P}})),\tilde{g}) \geqq \epsilon\}|}{k} = 0.$$

We denote this as

$$stat_{\mathrm{DNFR}} \lim_{k\to\infty} \mathcal{D}(\delta(\tilde{g}_k;\dot{\mathcal{P}}),\tilde{g}) = 0.$$

We now present an inclusion theorem that connects these two important concepts: every sequence of fuzzy number-valued functions that is deferred Nörlund statistically Riemann integrable is also statistically deferred Nörlund Riemann summable. However, the reverse implication does not necessarily hold.

Theorem 2.4. *Let* (ϕ_k) *and* (φ_k) *be sequences in* $\mathbb{Z}^{0+}$*, and let* (p_k) *be a sequence of non-negative real numbers. If a sequence* $(\tilde{g}_k)_{k\in\mathbb{N}}$ *of fuzzy number-valued functions is deferred Nörlund statistically Riemann integrable to a fuzzy number-valued function* $\tilde{g}$ *on* $[a,b]$*, then it is statistically deferred Nörlund Riemann summable to the same fuzzy number-valued function* $\tilde{g}$ *on* $[a,b]$*. However, the reverse implication does not necessarily hold.*

Proof. Suppose the sequence $(\tilde{g}_k)_{k\in\mathbb{N}}$ is deferred Nörlund statistically Riemann integrable to a fuzzy number valued function $\tilde{g}$ on $[a,b]$. Then, according to Definition 2.1, we obtain the following result:

$$\lim_{k\to\infty} \frac{|\{\xi : \xi \leqq P_k \quad \text{and} \quad p_\xi\mathcal{D}(\delta(\tilde{g}_\xi;\dot{\mathcal{P}}),\tilde{g}) \geqq \epsilon\}|}{P_k} = 0.$$

Now assuming two sets as follows:

$$\mathcal{L}_\epsilon = \{\xi : \xi \leqq P_k \quad \text{and} \quad p_\xi\mathcal{D}(\delta(\tilde{g}_\xi;\dot{\mathcal{P}}),\tilde{g}) \geqq \epsilon\}$$

and

$$\mathcal{L}_\epsilon^c = \{\xi : \xi \leqq P_k \quad \text{and} \quad p_\xi\mathcal{D}(\delta(\tilde{g}_\xi;\dot{\mathcal{P}}),\tilde{g}) < \epsilon\},$$

we have

$$\begin{aligned}
\mathcal{D}\Big(\mathcal{N}(\delta(\tilde{g}_k;\dot{\mathcal{P}})),\tilde{g}\Big) &= \frac{1}{P_k}\sum_{\lambda=\phi_k+1}^{\varphi_k} p_{\varphi_k-\lambda}\mathcal{D}(\delta(\tilde{g}_\lambda;\dot{\mathcal{P}}),\tilde{g}) \\
&\leqq \frac{1}{P_k}\sum_{\lambda=\phi_k+1}^{\varphi_k} p_{\varphi_k-\lambda}\mathcal{D}\Big(\delta(\tilde{g}_\lambda;\dot{\mathcal{P}}),\tilde{g}\Big) + \frac{1}{P_k}\sum_{\lambda=\phi_k+1}^{\varphi_k}\mathcal{D}(p_{\varphi_k-\lambda}\tilde{g},\tilde{g}) \\
&\leqq \frac{1}{P_k}\sum_{\substack{\lambda=\phi_k+1\\(\xi\in\mathcal{L}_\epsilon)}}^{\varphi_k} p_{\varphi_k-\lambda}\mathcal{D}\Big(\delta(\tilde{g}_\lambda;\dot{\mathcal{P}}),\tilde{g}\Big) + \frac{1}{P_k}\sum_{\substack{\lambda=\phi_k+1\\(\xi\in\mathcal{L}_\epsilon^c)}}^{\varphi_k} p_{\varphi_k-\lambda}\mathcal{D}\Big(\delta(\tilde{g}_\lambda;\dot{\mathcal{P}}),\tilde{g}\Big) \\
&\quad + |\tilde{g}|\left(\frac{1}{P_k}\sum_{\lambda=\phi_k+1}^{\varphi_k} p_{\varphi_k-\lambda} - 1\right) \\
&\leqq \frac{1}{P_k}|\mathcal{L}_\epsilon| + \frac{1}{P_k}|\mathcal{L}_\epsilon^c| = 0.
\end{aligned}$$

This implies that

$$\mathcal{D}(\mathcal{N}(\delta(\tilde{g}_k;\dot{\mathcal{P}})),\tilde{g}) < \epsilon.$$

Therefore, the sequence of fuzzy number-valued functions $(\tilde{g}_k)$ is statistically deferred Nörlund Riemann summable to a fuzzy number-valued function $\tilde{g}$ on the interval $[a,b]$.

Since the converse statement does not always hold, the following example demonstrates that a sequence of fuzzy number-valued functions that is statistically deferred Nörlund Riemann summable is not necessarily deferred Nörlund statistically Riemann integrable.

Example 2.5. *Let $\phi_k = 2k-1$, $\varphi_k = 4k-1$ and $p_k = 1$ and let $\tilde{g}_k : [0,1] \to \mathbb{R}$ be a sequence of functions of the form given by*

$$\tilde{g}_k(t) = \begin{cases} \frac{1}{k} & (t \in [0,1/2]) \\ \\ 1 & (t \in (1/2,1]). \end{cases} \tag{3}$$

The given sequence $(\tilde{g}_k)$ of functions clearly demonstrates that it is neither Riemann integrable nor deferred Nörlund statistically Riemann integrable. However, using the proposed mean (2), it is easy to see that

$$\begin{aligned}
\mathcal{N}(\delta(\tilde{g}_k;\dot{\mathcal{P}})) &= \frac{1}{\varphi_k-\phi_k}\sum_{\varrho=\phi_k+1}^{\varphi_k}\delta(\tilde{g}_\varrho;\dot{\mathcal{P}}) \\
&= \frac{1}{2k}\sum_{\varrho=2k+1}^{4k}\delta(\tilde{g}_\varrho;\dot{\mathcal{P}}) = \frac{1}{2}.
\end{aligned}$$

This example illustrates that while the sequence $(\tilde{g}_k)$ is statistically deferred Nörlund Riemann summable, it is not necessarily deferred Nörlund statistically Riemann integrable. This confirms that summability does not imply integrability in the deferred Nörlund statistical sense.

□

3. Fuzzy Korovkin-type Approximation Theorems

The fuzzy Korovkin-type theorem is an extension of the classical Korovkin approximation theorem to function spaces involving fuzzy numbers. In traditional Korovkin theory, a sequence of positive linear operators $\mathfrak{L}$ acting on a function space is said to approximate a function f if it converges to f for a specific set of test functions (typically 1, x and x^2 in classical cases). The fuzzy Korovkin-type theorem generalizes this idea by considering sequences of fuzzy-number-valued operators acting on fuzzy function spaces ensuring uniform convergence in the fuzzy setting. This theorem is particularly useful in fuzzy approximation theory, as it provides a criterion for approximating fuzzy continuous functions using linear operators. Such results have applications in numerical analysis, machine learning, and fuzzy control systems, where uncertainties and imprecisions in data representation play a crucial role. The theorem bridges classical approximation techniques with fuzzy set theory, enhancing computational methods in uncertain environments. In this context, we would like to refer interested readers to some recent works, including [8], [9], [12], [13], [22], [23], [24], [25], [26], [27], [28], [30], [33], [35], [36], [37], and [40].

Let $\tilde{g} : [a, b] \to \mathcal{E}$ be a fuzzy number-valued function. We define $\tilde{g}$ as continuous at a point $t_0 \in [a, b]$ if, for every $\epsilon > 0$, there exists $\delta > 0$ such that whenever $t_i \to t_0$ and $\mathcal{D}(t_i, t_0) < \delta$, it follows that $\mathcal{D}(\tilde{g}(t_i), \tilde{g}(t_0)) < \epsilon$. Furthermore, if $\tilde{g}$ satisfies this condition at every $t \in [a, b]$ then it is said to be fuzzy continuous over the entire interval $[a, b]$.

Let $C_{\mathcal{L}}[a, b]$ represent the set of all continuous fuzzy number-valued functions over the interval $[a, b]$.

Now, suppose $\mathfrak{L} : C_{\mathcal{L}}[a, b] \to C_{\mathcal{L}}[a, b]$ is a fuzzy linear operator. This means that for any scalars $\lambda_1, \lambda_2 \in \mathbb{R}$ and functions $\tilde{g}_1, \tilde{g}_2 \in C_{\mathcal{F}}[a, b]$, the operator satisfies the following linearity condition:

$$\mathfrak{L}(\lambda_1 \odot \tilde{g}_1 \oplus \lambda_2 \odot \tilde{g}_2; t) = \lambda_1 \odot \mathfrak{L}(\tilde{g}_1) \oplus \lambda_2 \odot \mathfrak{L}(\tilde{g}_2).$$

Furthermore, we say that $\mathfrak{L}$ is a positive fuzzy linear operator if it satisfies the condition

$$\tilde{g}_1(t) \preceq \tilde{g}_2(t)) \Longrightarrow \mathfrak{L}(\tilde{g}_1; t) \preceq \mathfrak{L}(\tilde{g}_2; t),$$

for all $\tilde{g}_1, \tilde{g}_2 \in C_{\mathcal{L}}[a, b]$ and $t \in [a, b]$, where $\preceq$ denotes the fuzzy ordering.

Theorem 3.1. *Let (ϕ_i) and $(\varphi_i) \in \mathbb{Z}^{0+}$, and let $\mathfrak{L}_i : C_{\mathcal{L}}[a, b] \to C_{\mathcal{L}}[a, b] \quad (i \in \mathbb{N})$ be a sequence of fuzzy number-valued positive linear operators. Additionally, let $\{\mathfrak{L}_i^*\}_{\in\mathbb{N}}$ represent the corresponding sequence of positive linear operators mapping from $C[a, b]$ into itself, with the relationship*

$$\{\mathfrak{L}_i(\tilde{g}; t)\}_{\pm}^{\Lambda} = \mathfrak{L}_i^*(\tilde{g}_{\pm}^{\Lambda}; t) \tag{4}$$

holding for all $t \in [a, b]$, $\Lambda \in [0, 1]$, $i \in \mathbb{N}$. Then, for $\tilde{g} \in C_{\mathcal{L}}[a, b]$, the following equivalence holds:

$$\mathrm{DNFR}_{\mathrm{stat}} \lim_{i\to\infty} \mathcal{D}^* \left(\mathfrak{L}_i(\tilde{g}; t), \tilde{g}(t)\right) = 0 \tag{5}$$

if and only if

$$\mathrm{DNFR}_{\mathrm{stat}} \lim_{i\to\infty} \mathcal{D}\left(\mathfrak{L}_i^*(1; t), 1\right) = 0, \tag{6}$$

$$\mathrm{DNFR}_{\mathrm{stat}} \lim_{i\to\infty} \mathcal{D}\left(\mathfrak{L}_i^*(t; t), t\right) = 0 \tag{7}$$

and

$$\mathrm{DNFR}_{\mathrm{stat}} \lim_{i\to\infty} \mathcal{D}\left(\mathfrak{L}_i^*(t^2; t), t^2\right) = 0. \tag{8}$$

Proof. Let $\tilde{g} \in C_{\mathcal{L}}[a,b]$, $t \in [a,b]$ and $\Lambda \in [0,1]$. Since $g_{\pm}^{\Lambda}(t) \in C[a,b]$, so for each $\varepsilon > 0$, there exists $\delta > 0$, such that

$$|\tilde{g}_{\pm}^{\Lambda}(r) - \tilde{g}_{\pm}^{\Lambda}(t)| < \varepsilon \quad \text{whenever} \quad |r-t| < \delta \quad (\forall\, t, r \in [a,b]). \tag{9}$$

Next, for $\tilde{g}$ is fuzzy bounded, $|\tilde{g}_{\pm}^{\Lambda}(t)| \le \mathcal{M}_{\pm}^{\Lambda}$ $(a < t < b)$. Clearly, we have

$$|\tilde{g}_{\pm}^{\Lambda}(r) - \tilde{\tilde{g}}_{\pm}^{\Lambda}(t)| \leqq 2\mathcal{M}_{\pm}^{\Lambda} \quad (a < t, r < b).$$

Let us choose $\theta(r,t) = (r-t)^2$. Then,

$$|\tilde{g}_{\pm}^{\Lambda}(r) - \tilde{g}_{\pm}^{\Lambda}(t)| < \varepsilon + \frac{2\mathcal{M}_{\pm}^{\Lambda}}{\delta^2}\theta(r,t)$$

which yields

$$-\varepsilon - \frac{2\mathcal{M}_{\pm}^{\Lambda}}{\delta^2}\theta(r,t) < \left(\tilde{g}_{\pm}^{\Lambda}(r) - \tilde{g}_{\pm}^{\Lambda}(t)\right) < \varepsilon + \frac{2\mathcal{M}_{\pm}^{\Lambda}}{\delta^2}\theta(r,t). \tag{10}$$

Now the operator $\mathfrak{L}_i^*$ is linear and monotone Thus, by applying the operator $\mathfrak{L}_i^*(1,t)$ in (10), we get

$$\begin{aligned}\mathfrak{L}_i^*(1,t)\left(-\varepsilon - \frac{2\mathcal{M}_{\pm}^{\Lambda}}{\delta^2}\theta(r,t)\right) &< \mathfrak{L}_i^*(1,t)\left(\tilde{g}_{\pm}^{\Lambda}(r) - \tilde{g}_{\pm}^{\Lambda}(t)\right)\\ &< \mathfrak{L}_i^*(1,t)\left(\varepsilon + \frac{2\mathcal{M}_{\pm}^{\Lambda}}{\delta^2}\theta(r,t)\right).\end{aligned} \tag{11}$$

We note that t is fixed and $\tilde{g}_{\pm}^{\Lambda}(t)$ is a constant number, we thus obtain

$$\begin{aligned}-\varepsilon\mathfrak{L}_i^*(1,t) - \frac{2\mathcal{M}_{\pm}^{\Lambda}}{\delta^2}\mathfrak{L}_i^*(\theta,t) &< \mathfrak{L}_i^*(\tilde{g}_{\pm}^{\Lambda},t) - \tilde{g}_{\pm}^{\Lambda}(t)\mathfrak{L}_i^*(1,t)\\ &< \varepsilon\mathfrak{L}_i^*(1,t) + \frac{2\mathcal{M}_{\pm}^{\Lambda}}{\delta^2}\mathfrak{L}_i^*(\theta,t).\end{aligned} \tag{12}$$

Also, we know that

$$\mathfrak{L}_i^*(\tilde{g}_{\pm}^{\Lambda},t) - \tilde{g}_{\pm}^{\Lambda}(t) = [\mathfrak{L}_i^*(\tilde{g}_{\pm}^{\Lambda},t) - \tilde{g}_{\pm}^{\Lambda}(t)\mathfrak{L}_i^*(1,t)] + \tilde{g}_{\pm}^{\Lambda}(t)[\mathfrak{L}_i^*(1,t) - 1]. \tag{13}$$

Using (12) and (13), we get

$$\mathfrak{L}_i^*(\tilde{g}_{\pm}^{\Lambda},t) - \tilde{g}_{\pm}^{\Lambda}(t) < \varepsilon\mathfrak{L}_i^*(1,t) + \frac{2\mathcal{M}_{\pm}^{\Lambda}}{\delta^2}\mathfrak{L}_i^*(\theta,t) + \tilde{g}_{\pm}^{\Lambda}(t)[\mathfrak{L}_i^{\Lambda}(1,t) - 1]. \tag{14}$$

Now, we compute $\mathfrak{L}_i^*(\theta,t)$ as follows:

$$\begin{aligned}\mathfrak{L}_i^*(\theta,t) &= \mathfrak{L}_i^*(r^2 - 2tr + t^2, t)\\ &= \mathfrak{L}_i^*(r^2,t) - 2t\mathfrak{L}_i^*(r,t) + t^2\mathfrak{L}_i^*(1,t)\\ &= [\mathfrak{L}_i^*(r^2,t) - t^2] - 2t[\mathfrak{L}_i^*(r,t) - t] + t^2[\mathfrak{L}_i^*(1,t) - 1].\end{aligned}$$

Using (14), we get

$$\begin{aligned}\mathfrak{L}_i^*(\tilde{g}_{\pm}^{\Lambda},t) - \tilde{g}_{\pm}^{\Lambda}(t) &< \varepsilon\mathfrak{L}_i^*(1,t) + \frac{2\mathcal{M}_{\pm}^{\Lambda}}{\delta^2}\{[\mathfrak{L}_i^*(r^2,t) - t^2] - 2t[\mathfrak{L}_i^*(r,t) - t]\\ &\quad + t^2[\mathfrak{L}_i^*(1,t) - 1]\} + \tilde{g}_{\pm}^{\Lambda}(t)[\mathfrak{L}_i^*(1,t) - 1]\\ &= \varepsilon[\mathfrak{L}_i^*(1,t) - 1] + \varepsilon + \frac{2\mathcal{M}_{\pm}^{\Lambda}}{\delta^2}\{[\mathfrak{L}_i^*(r^2,t) - t^2]\\ &\quad - 2t[\mathfrak{L}_i^{\Lambda}(r,t) - t] + t^2[\mathfrak{L}_i^*(1,t) - 1]\} + \tilde{g}_{\pm}^{\Lambda}(t)[\mathfrak{L}_i^*(1,t) - 1].\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we can write

$$|\mathfrak{L}_i^*(\tilde{g}_\pm^\Lambda, t) - \tilde{g}_\pm^\Lambda(t)| \leqq \varepsilon + \Big(\varepsilon + \frac{2\mathcal{M}_\pm^\Lambda h^2}{\delta^2} + \mathcal{M}_\pm^\Lambda\Big)|\mathfrak{L}_i^*(1,t) - 1| + \frac{4\mathcal{M}_\pm^\Lambda h}{\delta^2}|\mathfrak{L}_i^\Lambda(r,t) - t| + \frac{2\mathcal{M}_\pm^\Lambda}{\delta^2}|\mathfrak{L}_i^\Lambda(r^2,t) - t^2|,$$

where $h = \max\{|a|, |b|\}$.

Consequently, we get

$$|\mathfrak{L}_i^*(\tilde{g}_\pm^\Lambda, t) - \tilde{g}_\pm^\Lambda(t)| \leqq \epsilon + \mathcal{H}_\pm^\Lambda(\epsilon)\Big(|\mathfrak{L}_i^*(1,t) - 1| + |\mathfrak{L}_i^*(r,t) - t| + |\mathfrak{L}_i^*(r^2,t) - t^2|\Big), \tag{15}$$

where

$$\mathcal{H}_\pm^r(\epsilon) = \max\left(\epsilon + \frac{2\mathcal{M}_\pm^\Lambda h^2}{\delta^2} + \mathcal{M}_\pm^\Lambda,\ \frac{4\mathcal{M}_\pm^\Lambda h}{\delta^2},\ \frac{2\mathcal{M}_\pm^\Lambda}{\delta^2}\right).$$

Now it clearly follows from (4) that,

$$\begin{aligned}\mathcal{D}^*(\mathfrak{L}_i(\tilde{g}), \tilde{g}) &= \sup_{t\in[a,b]} \mathcal{D}\left(\mathfrak{L}_i^*(\tilde{g}; t), \tilde{g}\right)\\ &= \sup_{t\in[a,b]} \sup_{\Lambda\in[0,1]} \max\left\{\left|\mathfrak{L}_i^*(\tilde{g}_-^\Lambda; t) - \tilde{g}_-^\Lambda\right|, \left|\mathfrak{L}_i^*(\tilde{g}_+^\Lambda; t) - \tilde{g}_+^\Lambda(t)\right|\right\}.\end{aligned}$$

Considering (15) with the last equality, one can easily write

$$\mathcal{D}^*(\mathfrak{L}_i(\tilde{g}), \tilde{g}) \leqq \sup_{t\in[a,b]} \varepsilon + \mathcal{M}(\varepsilon)\left(\sup_{x\in[a,b]}\left|\mathfrak{L}_i^*(1,t) - 1\right| + \sup_{x\in[a,b]}\left|\mathfrak{L}_i^*(r,t) - t\right| + \sup_{x\in[a,b]}\left|\mathfrak{L}_i^*(r^2,t) - t^2\right|\right),$$

where

$$\mathcal{M}(\epsilon) = \sup_{\Lambda\in[0,1]} \max\left\{\mathcal{M}_-^\Lambda(\epsilon), \mathcal{M}_+^\Lambda(\epsilon)\right\}.$$

Therefore,

$$\begin{aligned}p_{\varphi_k-\lambda}\mathcal{D}^*(\mathfrak{L}_i(\tilde{g}), \tilde{g}) \leqq p_{\varphi_k-\lambda} \sup_{t\in[a,b]} \varepsilon + \mathcal{H}(\varepsilon)\Bigg(&p_{\varphi_k-\lambda} \sup_{t\in[a,b]}\left|\mathfrak{L}_i^*(1,t) - 1\right|\\ &+ p_{\varphi_k-\lambda} \sup_{t\in[a,b]}\left|\mathfrak{L}_i^*(r,t) - t\right| + p_{\varphi_k-\lambda} \sup_{t\in[a,b]}\left|\mathfrak{L}_i^*(r^2,t) - t^2\right|\Bigg).\end{aligned} \tag{16}$$

Next, for given $\kappa > 0$, choose $\varepsilon > 0$ such that $p_{\varphi_k-\lambda} \sup_{t\in[a,b]} \varepsilon < \kappa$.

Then, we can write

$$\Theta_i(t; \varepsilon) = \left|\left\{i : i \leqq P_i \quad \text{and} \quad p_{\varphi_k-\lambda}\mathcal{D}^*\left(\mathfrak{L}_m(\tilde{g}), \tilde{g}\right) \geqq \varepsilon'\right\}\right|$$

and

$$\Theta_{j,i}(t, \varepsilon) = \left|\left\{i : i \leqq P_i \quad \text{and} \quad p_{\varphi_k-\lambda}\mathcal{D}\left(\mathfrak{L}_i^*\tilde{g}_j(t), \tilde{g}_j(t)\right) \geqq \frac{\epsilon' - \epsilon}{3\mathcal{H}_\pm^\Lambda}\right\}\right|,$$

we easily obtain from (16) that

$$\Theta_i(t, \varepsilon) \leqq \sum_{j=0}^{2} \Theta_{j,i}(t, \varepsilon).$$

Thus, we fairly have

$$\frac{\|\Theta_i(t,\varepsilon)\|}{P_i} \leqq \sum_{j=0}^{2} \frac{\left\|\Theta_{j,i}(t,\varepsilon)\right\|}{P_i}. \tag{17}$$

Thus, based on Definition 2.1 and the assumptions outlined for the implications in equations (6) through (8), the right-hand side of equation (17) approaches zero as $i \to \infty$. Therefore, we obtain the following result

$$\lim_{i\to\infty} \frac{\|\Theta_i(t,\varepsilon)\|}{P_i} = 0 \quad (\varepsilon > 0).$$

As a result, the implication in equation (5) holds true. □

Theorem 3.2. *Let* (ϕ_i) *and* $(\varphi_i) \in \mathbb{Z}^{0+}$, *and let* $\mathfrak{L}_i : C_{\mathcal{L}}[a,b] \to C_{\mathcal{L}}[a,b] \quad (i \in \mathbb{N})$ *be the fuzzy number valued sequence of positive linear operators. Also, let* $\{\mathfrak{L}_i^*\}_{\in\mathbb{N}}$ *be the respective sequence of positive linear operators from* $C[a,b]$ *into itself with*

$$\{\mathfrak{L}_i(\tilde{g};t)\}_{\pm}^{\Lambda} = \mathfrak{L}_i^*(\tilde{g}_{\pm}^{\Lambda};t) \tag{18}$$

for all $t \in [a,b]$, $\Lambda \in [0,1]$, $i \in \mathbb{N}$. *Then, for* $\tilde{g} \in C_{\mathcal{L}}[a,b]$

$$\mathrm{stat}_{\mathrm{DNFR}} \lim_{i\to\infty} \mathcal{D}^* \left(\mathfrak{L}_i(\tilde{g};t), \tilde{g}(t)\right) = 0 \tag{19}$$

if and only if

$$\mathrm{stat}_{\mathrm{DNFR}} \lim_{i\to\infty} \mathcal{D}\left(\mathfrak{L}_i^*(1;t), 1\right) = 0, \tag{20}$$

$$\mathrm{stat}_{\mathrm{DNFR}} \lim_{i\to\infty} \mathcal{D}\left(\mathfrak{L}_i^*(t;t), t\right) = 0 \tag{21}$$

and

$$\mathrm{stat}_{\mathrm{DNFR}} \lim_{i\to\infty} \mathcal{D}\left(\mathfrak{L}_i^*(t^2;t), t^2\right) = 0. \tag{22}$$

Proof. In a similar manner to the proof of Theorem 3.1, Theorem 3.2 can be demonstrated. Therefore, we will omit the detailed steps of the proof. □

Based on Theorem 3.2, we provide an example of a sequence of positive linear operators that does not align with the statistical version of the deferred Nörlund Riemann integrable sequence of fuzzy number-valued functions (as outlined in Theorem 3.1), but instead works effectively with Theorem 3.2. From this, we conclude that Theorem 3.2 represents a significant extension of the statistical Nörlund Riemann integrable sequence of fuzzy number-valued functions presented in Theorem 3.1.

We now revisit the operator given by

$$\varpi(1+\varpi D) \qquad \left(D = \frac{d}{d\varpi}\right), \tag{23}$$

which was previously employed by Al-Salam [1] and more recently by Viskov and Srivastava [39].

Example 3.3. *Consider the Bernstein polynomial* $\mathfrak{B}_k(\tilde{g};\beta)$ *[4] on* $C[0,1]$ *given by*

$$\mathfrak{B}_\varrho(\tilde{g};\beta) = \sum_{\varrho=0}^{k} \tilde{g}\left(\frac{\varrho}{k}\right)\binom{k}{\varrho}\beta^{\varrho}(1-b)^{k-\varrho} \quad (\beta \in [0,1]; \varrho = 0,1,\cdots). \tag{24}$$

We now define the positive linear operators on $C[0,1]$ *through the composition of the Bernstein polynomial and the operators described in equation (23) as follows:*

$$\mathfrak{A}_\varrho(\tilde{g};\beta) = [1+\tilde{g}_\varrho]\beta(1+\beta D)\mathfrak{B}_\varrho(\tilde{g};\beta) \quad (\forall\ \tilde{g} \in C[0,1]), \tag{25}$$

where $(\tilde{g}_\varrho)$ *is the same as mentioned in Example 2.5.*

We now evaluate the values of each of the test functions 1, β *and* β^2 *by using the operators proposed in equation (25) as follows:*

$$\mathfrak{L}_\varrho(1;\beta) = [1+\tilde{g}_\varrho]\beta(1+\beta D)1 = [1+\tilde{g}_\varrho]\beta,$$

$$\mathfrak{L}_\varrho(t;\beta) = [1+\tilde{g}_\varrho]\beta(1+\beta D)\beta = [1+\tilde{g}_\varrho]\beta(1+\beta)$$

and

$$\begin{aligned}\mathfrak{L}_\varrho(t^2;\beta) &= [1+\tilde{g}_\varrho]\beta(1+\beta D)\left\{\beta^2 + \frac{\beta(1-\beta)}{\varrho}\right\}\\ &= [1+\tilde{g}_\varrho]\left\{\beta^2\left(2-\frac{3\beta}{\varrho}\right)\right\}.\end{aligned}$$

Consequently, we have

$$\mathrm{stat}_{\mathrm{DNFR}} \lim_{\varrho\to\infty} \mathcal{D}(\mathfrak{L}^*_\varrho(1;\beta),1) = 0, \tag{26}$$

$$\mathrm{stat}_{\mathrm{DNFR}} \lim_{\varrho\to\infty} \mathcal{D}(\mathfrak{L}^*_\varrho(\beta;\beta),\beta) = 0 \tag{27}$$

and

$$\mathrm{stat}_{\mathrm{DNFR}} \lim_{\varrho\to\infty} \mathcal{D}(\mathfrak{L}^*_\varrho(\beta^2;\beta),\beta^2) = 0, \tag{28}$$

that is, the sequence $\mathfrak{L}^*_\varrho(\tilde{g};\beta)$ *satisfies the conditions (20) to (22). Therefore, by Theorem 3.2, we have*

$$\mathrm{stat}_{\mathrm{DNFR}} \lim_{\varrho\to\infty} \mathcal{D}^*(\mathfrak{L}_\varrho(\tilde{g};\beta),g) = 0.$$

The sequence $(\tilde{g}_\varrho)$ *of fuzzy number-valued functions presented in Example 2.5 is statistically deferred Nörlund Riemann summable, but not deferred Nörlund statistically Riemann integrable. As a result, the operators defined by (25) fulfill the conditions of Theorem 3.2, but they do not meet the criteria for the statistical version of the deferred Nörlund Riemann integrable sequence of fuzzy number-valued functions as stated in Theorem 3.1.*

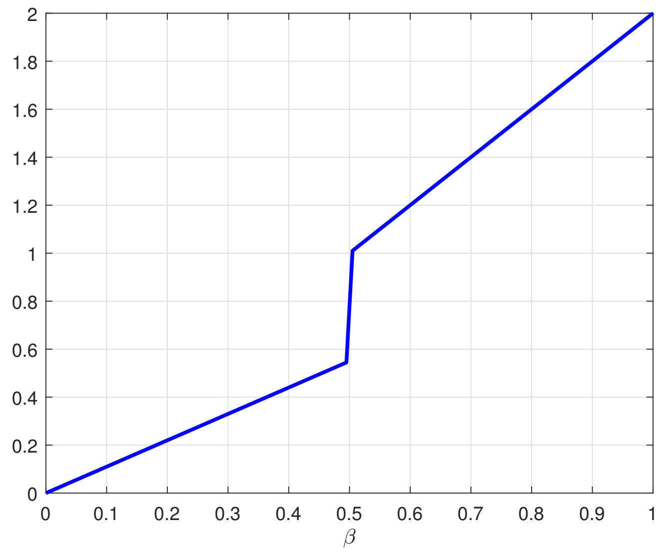

Figure 2:
(Convergence Behavior of the Operator $\mathfrak{L}_{\varrho}(1;\beta)$)

Figure 2 represents the convergence behavior of the operator $\mathfrak{L}_{\varrho}(1;\beta)$ *as a function of* β*. The graph depicts how the function evolves under the action of the operator involving the sequence* $(\tilde{g}_{\varrho})$*, which exhibits distinct behavior in the two regions:* $[0,1/2]$ *and* $(1/2,1]$*. In view of convergence condition given in equation* $\mathfrak{L}_{\varrho}(1;\beta) = [1+\tilde{g}_{\varrho}]\beta$*, the graph shows how the function evolves as* $\varrho \to \infty$*. In light of the convergence condition under equation (26), the statistical deferred Nörlund Riemann summability requires that as* ϱ *increases,* $\mathfrak{L}_{\varrho}(1;\beta)$ *should converge to* 1*. This is reflected in the graph, where the function stabilizes (or converges) to a constant value as* β *approaches its boundaries* 0 *and* 1*.*

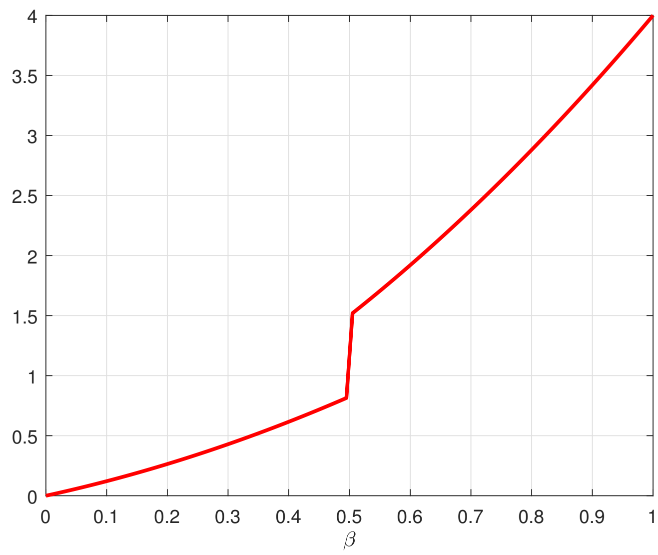

Figure 3:
(Convergence Behavior of the Operator $\mathfrak{L}_{\varrho}(\beta;\beta)$)

Figure 3 represents how the operator $\mathfrak{L}_\varrho(\beta;\beta)$ behaves for different values of β. In view of the operator defined as $\mathfrak{L}_\varrho(\beta;\beta) = [1+\tilde{g}_\varrho]\beta(1+\beta)$, the graph demonstrates how the function responds to variations in β within the interval $[0,1]$. In light of the convergence condition under equation (27), this operator should approach the function β as $\varrho \to \infty$.This expectation is reinforced by the convergence condition $\mathrm{stat_{DNFR}} \lim_{\varrho\to\infty} \mathcal{D}(\mathfrak{L}^*_\varrho(\beta;\beta),\beta) = 0$, which implies that $\mathfrak{L}^*_\varrho(\beta;\beta)$ increasingly align with β as ϱ grows. Consequently, the plot exhibits a trend where the operator approaches the identity function for β, reflecting the desired convergence behavior.

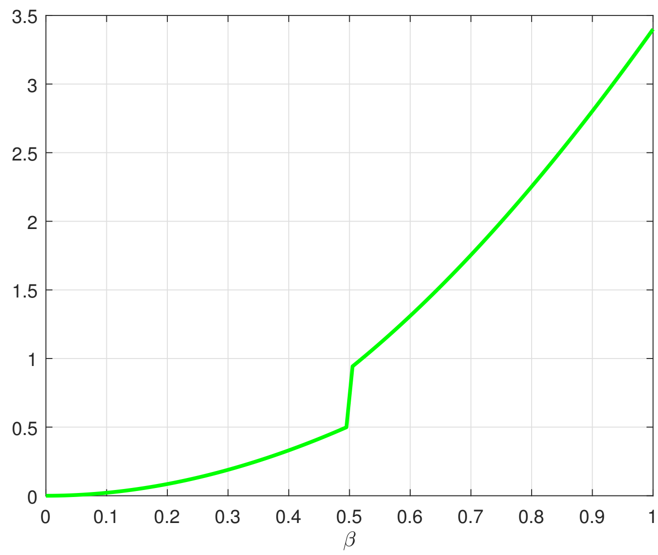

Figure 4:
(Convergence Behavior of the Operator $\mathfrak{L}_\varrho(\beta^2;\beta)$)

Figure 4 represents the behavior of the operator $\mathfrak{L}_\varrho(\beta^2;\beta)$, defined as $\mathfrak{L}_\varrho(\beta^2;\beta) = [1+\tilde{g}_\varrho]\left\{\beta^2\left(2-\frac{3\beta}{\varrho}\right)\right\}$. This plot demonstrates how the function responds to variations in β under the influence of this operator, highlighting the effect of the parameter ϱ on the quadratic term β^2. From the convergence condition given in equation (28), we expect that as $\varrho\to\infty$, the operator $\mathfrak{L}_\varrho(\beta^2;\beta)$ should approach β^2. The graph fairly shows that the operator converges to the quadratic function β^2 as ϱ increases, with the term $\frac{3\beta}{\varrho}$ diminishing over time. This trend confirms that the operator is correctly summable, satisfying the required convergence condition:

$$\mathrm{stat_{DNFR}} \lim_{\varrho\to\infty} \mathcal{D}(\mathfrak{L}^*_\varrho(\beta^2;\beta),\beta^2) = 0.$$

Each of the operators $\mathfrak{L}_\varrho(1;\beta)$, $\mathfrak{L}_\varrho(\beta;\beta)$ and $\mathfrak{L}_\varrho(\beta^2;\beta)$ demonstrates how the sequence of fuzzy number-valued functions behaves in the statistical sense under the action of the operator. The convergence of each operator to its respective function 1, β and β^2 as $\varrho\to\infty$ validates the conditions of deferred Nörlund Riemann summability in the statistical sense. The plots 2 to 4 highlight that Theorem 3.2 extends classical Korovkin-type approximation theorems, showing that even though the sequence is statistically deferred Nörlund Riemann summable, it may not necessarily be deferred Nörlund statistically Riemann integrable. The graphs visually support this extension by demonstrating convergence in terms of statistical summability, while also suggesting that classical Riemann integrability may not hold for the sequence.

4. Conclusion and Discussion

In this final section, we highlight the practical significance and theoretical advantages of Theorem 3.2 over Theorem 3.1, as well as its improvements upon classical fuzzy Korovkin-type approximation results.

While Theorem 3.1 establishes convergence criteria based on statistical deferred Nörlund Riemann integrability, Theorem 3.2 introduces statistical deferred Nörlund Riemann summability a broader and more flexible framework. This generalization allows for a wider class of fuzzy-number-valued functions to be approximated effectively, even in cases where classical summability methods may fail to guarantee convergence.

The fuzzy Korovkin-type theorem presented in Theorem 3.2 extends the classical Korovkin approximation theorem into the realm of fuzzy function spaces. It ensures uniform convergence of sequences of fuzzy positive linear operators by testing their behavior on the standard algebraic test functions 1, x and x^2. While this test set is classical, our use of it in conjunction with the newly proposed statistical deferred Nörlund summability method allows for stronger convergence results under uncertainty. This framework provides a robust mathematical tool for approximating fuzzy continuous functions in the presence of imprecision or incomplete information.

Our work further illustrates the applicability of Theorem 3.2 through an example involving fuzzy Bernstein-type operators. By analyzing their convergence behavior both analytically and graphically (using MATLAB visualizations), we demonstrate the enhanced approximation accuracy and stability offered by our approach. These results have promising implications for computational methods in fields such as fuzzy control systems, machine learning, and numerical analysis, where fuzzy approximations are essential. Thus, the contributions made in this paper not only bridge classical approximation theory with fuzzy set theory but also offer new techniques for handling uncertainty in real-world data.

Consider the sequence $(\tilde{g}_\varrho)_{\varrho\in\mathbb{N}}$ of functions from Example 2.5. Additionally, assume that $(\tilde{g}_\varrho)$ is statistically deferred Nörlund Riemann summable, so that we have the following limit

$$\mathrm{stat}_{\mathrm{DNFR}} \lim_{\varrho\to\infty} \mathcal{D}\left(\delta(\tilde{g}_\varrho;\dot{\mathcal{P}}), \frac{1}{2}\right) \text{ on } [0,1].$$

Then, we have

$$\mathrm{stat}_{\mathrm{DNFR}} \lim_{\varrho\to\infty} \mathcal{D}(\mathfrak{L}^*_\varrho(\tilde{g}_\nu;\varpi), \tilde{g}_\nu(\varpi)) = 0 \quad (\nu = 0,1,2). \tag{29}$$

Thus, by Theorem 3.2, we immediately get

$$\mathrm{stat}_{\mathrm{DNFR}} \lim_{\varrho\to\infty} \mathcal{D}^*(\mathfrak{L}_\varrho(\tilde{g};\varpi), \tilde{g}(\varpi)) = 0, \tag{30}$$

where

$$\tilde{g}_0(\varpi) = 1, \quad \tilde{g}_1(\varpi) = \varpi \quad \text{and} \quad \tilde{g}_2(\varpi) = \varpi^2.$$

The sequence $(\tilde{g}_\varrho)$ of fuzzy number-valued functions is statistically deferred Nörlund Riemann summable, but it is neither deferred Nörlund statistically Riemann integrable nor classically Riemann integrable. Consequently, our fuzzy Korovkin-type approximation result in Theorem 3.2 holds for the operators defined in equation (25), while both the classical and statistical versions of the deferred Nörlund Riemann integrable sequence of fuzzy number-valued functions do not apply to these operators. From this, we conclude that Theorem 3.2 serves as a significant extension of both Theorem 3.1 and the classical Korovkin-type approximation theorem [20].

Conflicts of Interest: The authors declare that they have no conflicts of interest.

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