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Different stabilities for integro–differential evolution equations with nonlocal conditions and application to epidemiology

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Abstract. Integro–differential evolution equations are becoming increasingly popular in many fields because of their ability to model and assess complicated procedures. In this paper, we study different kinds of stabilities for integro–differential evolution equations with nonlocal conditions. The concept of σ–semi–Ulam–Hyers stability, which lies somewhere between the Ulam–Hyers and Ulam–Hyers–Rassias stabilities, will be specifically discussed. To ensure Ulam–Hyers–Rassias stability, σ–semi–Ulam–Hyers stability, and Ulam–Hyers stability for integro–differential evolution equations with nonlocal conditions, this is considered within the framework of suitable metric spaces. We will examine the many situations in which the integrals are specified on both bounded and unbounded intervals. Techniques such as fixed–point arguments and generalizations of the Bielecki metric are utilized. To illustrate the main results, we also provide examples. The epidemiology application for modeling the transmission of infectious diseases served as a source of interest.

1. Introduction and preliminaries

Exact solutions are essential for practical research for any physical model. An exact answer of this type confirms the approximations obtained by analytical or numerical methods and offers the appropriate physical interpretation. A given (integral, functional, differential, difference, or fractional differential) equation is said to be stable (in the Ulam definition) if there exists an exact solution that is, in some sense, close to each approximation (in a particular sense) solution. S. M. Ulam posed an open issue that became the foundation for the theory of stability in a well–known 1940 speech at the University of Wisconsin (see, e.g., [8, 50] for further information). The stability theory originated from an intriguing open topic, which can be best described as follows. Let H, H^* stand for certain groups, where H^* is a metric group with a metric of $\mathfrak{N}$. Ulam questioned if there is $\delta > 0$, given $\epsilon > 0$: suppose $\varrho : H \to H^* : \forall(x_1, x_2 \in H)$

$$\mathfrak{N}(\varrho(x_1x_2), \varrho(x_1)\varrho(x_2)) < \delta,$$

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then there exists a homomorphism $\hbar : H \to H^*$:

$$\mathfrak{R}(\varrho(x_1), \hbar(x_1)) < \epsilon?$$

For further references, we urge the reader to [27], which covers most of the recent few decades study on the stability challenge in mathematical analysis.

For the Ulam investigation, there are a few responses accessible. For example, D. H. Hyers provided a positive response in 1941 when it came to Banach spaces (see [8]). This problem, concerning additive mappings, received attention from T. Aoki (see [49]). By examining the case of unbounded Cauchy differences, T. M. Rassias provided a generalized version of the same theorem in 1978 (see [51]). Ulam–Hyers–Rassias stability is the new name for the stability issue that Rassias (see [51]) invented.

Building upon these foundational and emerging studies, several recent works have focused specifically on the stability analysis of fractional and integral equations using various generalized concepts. For instance, Irshad et al. [26] examined the stability of time–fractional nonlinear Schrödinger equations, while Shah and Irshad [32] addressed Ulam–Hyers–Mittag–Leffler stability in nonlinear fractional reaction–diffusion equations with delays. The Ulam–Hyers–Rassias stability of nonlinear convolution integral equations has been explored in the work of Irshad, Shah, and Liaquat [25]. Related contributions include studies on the Ulam–Hyers stability of Bernoulli's differential equation [33] and the impulsive Fredholm integral equations on finite intervals [34]. Shah and Irshad further extended this investigation to oscillatory Volterra integral equations [35], hybrid differential equations involving Gronwall–type inequalities [36], and impulsive Hammerstein integral systems [37].

Additional advancements include the application of the Gronwall lemma to Ulam–type stability of integral equations by Shah et al. [38] and the use of fixed–point methods for stability analysis of delay fractional integro–differential equations with almost sectorial operators [39]. In the context of generalized integral systems, Shah and Tanveer [40] studied (k, ψ)–fractional order quadratic integral equations, while Shah and Abbasi [41] focused on impulsive Hammerstein integral equations. Further related efforts by Shah and collaborators address the Ulam–Hyers stability of weakly singular Volterra equations [42], nonlinear Volterra–Fredholm equations [43], and impulsive Volterra integral equations using fixed–point theory [44]. A foundational fixed–point analysis for Volterra–type systems with delay can also be found in [45].

In a parallel line of research integrating fixed–point theory with biological modeling, Turab and Sintunavarat contributed several influential works. The investigations include the analysis of traumatic avoidance learning models via the Banach fixed–point theorem [3], behavioral modeling of two–choice decision–making in fish using analytic techniques [4], and a novel study on nonlinear fractional boundary value problems defined over graph structures such as the ethane graph [5]. These cumulative efforts demonstrate the flexibility and depth of fixed–point techniques and fractional analysis in addressing a wide variety of nonlinear systems across mathematical and applied domains.

However, Byszewski [16] introduced the concept of a nonlocal condition by examining the existence and uniqueness of solutions to nonlocal Cauchy problems. As Byszewski [14, 17] and Deng [13] have noted, the motivation originates from physics, as nonlocal beginning conditions can characterize some physical processes more effectively and practically than classical ones. Day [54, 55] suggested a linear parabolic equation with a nonlocal boundary condition arising from static thermoelasticity. In order to examine the dynamics of gas in a transparent tube, Kerefov [2] and Vabishchevich [29] took into consideration a one-dimensional parabolic equation with nonlocal initial condition. The nonlocal function, for instance, could resemble this:

$$\mathfrak{I}(z(\zeta)) = \sum_{i=1}^{p} c_i z(\zeta_i)$$

where $c_i = 1, ..., p$, are given constants and $0 < \zeta_1 < ... < \zeta_p \leq T$. Afterwards, the issue has been covered by a number of authors [12, 15] for various kinds of integro–differential and nonlinear differential equations, such as functional differential equations in Banach spaces.

In this paper, we investigate the Ulam–Hyers stability, the Ulam–Hyers–Rassias stability, and a novel type of stability, the σ–semi–Ulam–Hyers stability, for the integro–differential evolution equation with nonlocal condition of the form:

$$\begin{cases} \mathfrak{R}'(\zeta) = \Psi\mathfrak{R}(\zeta) + \int_0^\zeta \psi(\zeta-\wp)\mathfrak{R}(\wp)\, d\wp + \aleph\nu(\zeta) + \varpi(\zeta, \mathfrak{R}(\zeta)), \quad \zeta \in [0, b], \\ \mathfrak{R}(0) = \mathfrak{R}_0 + \mathfrak{I}(\mathfrak{R}(\zeta)), \end{cases} \tag{1}$$

with $\zeta \in [0, b]$, where 0 and b are fixed real numbers, $\mathfrak{R} : [0, b] \to \mathbb{C}$ is the state function, $\nu(.) \in \mathbb{L}^2([0, b], \mathbb{C})$, $\Psi : D(\Psi) \subset \mathbb{C} \to \mathbb{C}$, and $\psi(\zeta) : D(\Psi) \subset \mathbb{C} \to \mathbb{C}$, $\zeta \geq 0$, $\psi \neq 0$, are closed operators, $\aleph : F \to \mathbb{C}$ is a bounded operator, F is a separable Hilbert space, $\varpi : [0, b] \times \mathbb{C} \to \mathbb{C}$ and $\mathfrak{I} : C([0, b], \mathbb{C}) \to \mathbb{C}$ are given functions to be specified later, and ϑ_0 is given element of $\mathbb{C}$.

A function $\mathfrak{R} : [0, b] \to \mathbb{C}$ is called a mild solution of Equation (1) if

$$\mathfrak{R}(\zeta) = \psi(\zeta)[\vartheta_0 + \mathfrak{I}(\mathfrak{R}(\zeta))] + \int_0^\zeta \psi(\zeta-\wp)[\aleph\nu(\wp) + \varpi(\wp, \mathfrak{R}(\wp))]\, d\wp. \tag{2}$$

Integro–differential evolution equations find use in numerous domains to simulate intricate dynamic systems (see, e.g., earlier studies [1, 10, 18, 28, 30, 31, 47, 53]), in which modifications rely on both past and present states. These equations simulate population dynamics in biology and account for delays brought on by gestation or maturation times. In thermodynamics, they capture the influence of previous temperature distributions to represent heat transfer in materials with thermal memory. By simulating the stress–strain relationship over time, material science uses these equations to understand viscoelastic materials, which have both viscous and elastic properties. Integro–differential equations are used in electromagnetic theory to examine electromagnetic fields in materials that possess memory or dispersive properties. They simulate how economic variables, such as the price of financial derivatives, change over time under the influence of historical data in economics. These equations are useful to neuroscience because they simulate the dynamics of neural networks, in which the future state of a neuron is dependent upon its past state as well as the states of other neurons. To design systems in which past states affect control actions, as in feedback loops with delays, control systems engineering uses integro–differential equations. These equations are used by epidemiologists to simulate the transmission of infectious diseases, taking into account the historical number of afflicted people and their patterns of interaction. Systems with memory effects or non–local interactions are described using integro–differential equations in quantum mechanics. Subsequently, they incorporate past states and variables like absorption and dispersion into the hydrodynamics model to simulate fluid flow in porous media. These varied applications demonstrate how important integro–differential evolution equations are to providing a thorough knowledge of dynamic, complex systems in a variety of academic fields.

We now want to quickly summarize this paper's originality and contributions: Several articles and books have been written about the study of Ulam–Hyers stability and Ulam–Hyers–Rassias stability of integro–differential equations, according to the relevant research. To the best of our knowledge, there aren't many publications on the Ulam–Hyers and Ulam–Hyers–Rassias stability of integro–differential equations with nonlocal conditions, according to the pertinent literature (see [52]). Regarding Ulam–Hyers and Ulam–Hyers–Rassias stability of integro–differential evolution equations with nonlocal conditions on bounded and unbounded intervals, we didn't find any previous work. This is the first study and contribution on the stability of integro–differential evolution Equation (1) with nonlocal conditions under the Ulam–Hyers and Ulam–Hyers–Rassias. Significantly, our findings also identify a σ–semi–Ulam–Hyers stability, which falls in between Ulam–Hyers and Ulam–Hyers–Rassias stability. This helps to provide an expanded view of the stability field. The results for integro–differential equations obtained earlier are refined and extended to integro–differential evolution equations with nonlocal conditions in this research. Additionally, in the context of Ulam [48], this is a novel addition to stability theory.

The structure of this paper is as follows: The Ulam–Hyers–Rassias stability of the integro–differential evolution Equation (1) on a bounded interval is demonstrated in Section 2. In Section 3, the σ–semi–Ulam–Hyers stability and the Ulam–Hyers stability of the integro–differential evolution Equation (1) on a bounded interval are established. The Ulam–Hyers–Rassias stability and the σ–semi–Ulam–Hyers stability of the integro–differential evolution Equation (1) are extended to an unbounded interval in Section 4. To demonstrate the findings, examples are presented in Section 5. In Section 6, an application of the integro–differential evolution equation in epidemiology is discussed. Finally, the conclusion of the paper is given in Section 7.

For the integro–differential evolution Equation (1), the stabilities previously described are now formally defined.

Definition 1.1. *Let ϑ be a continuous function on $[0, b]$ such that*

$$\left|\vartheta'(\zeta) - \Psi\vartheta(\zeta) - \int_0^\zeta \psi(\zeta-\wp)\vartheta(\wp)\,d\wp - \aleph\nu(\zeta) - \varpi(\zeta,\vartheta(\zeta))\right| \leq \sigma(\zeta),$$

where σ is a nonnegative function. If there is a solution $\mathfrak{R}$ of the integro–differential evolution equation and a constant $C > 0$, independent of ϑ and $\mathfrak{R}$, satisfying

$$\left|\vartheta(\zeta) - \mathfrak{R}(\zeta)\right| \leq C\sigma(\zeta),$$

for all $\zeta \in [0, b]$, then we say that the integro–differential evolution Equation (1) has the Ulam–Hyers–Rassias stability.

Definition 1.2. *Let ϑ be a continuous function on $[0, b]$ such that*

$$\left|\vartheta'(\zeta) - \Psi\vartheta(\zeta) - \int_0^\zeta \psi(\zeta-\wp)\vartheta(\wp)\,d\wp - \aleph\nu(\zeta) - \varpi(\zeta,\vartheta(\zeta))\right| \leq \theta,$$

where $\theta \geq 0$. If there is a solution $\mathfrak{R}$ of the integro–differential evolution equation and a constant $C > 0$, independent of ϑ and $\mathfrak{R}$, satisfying

$$\left|\vartheta(\zeta) - \mathfrak{R}(\zeta)\right| \leq C\theta,$$

for all $\zeta \in [0, b]$, then we say that the integro–differential evolution Equation (1) has the Ulam–Hyers stability.

A new kind of stability introduced in Castro and Simões [24] is now presented. This stability can be characterized as lying between the two previously mentioned stabilities of Ulam–Hyers–Rassias and Ulam–Hyers.

Definition 1.3. *Let σ a nondecreasing function defined on $[0, b]$. If for each continuous function ϑ satisfying*

$$\left|\vartheta'(\zeta) - \Psi\vartheta(\zeta) - \int_0^\zeta \psi(\zeta-\wp)\vartheta(\wp)\,d\wp - \aleph\nu(\zeta) - \varpi(\zeta,\vartheta(\zeta))\right| \leq \theta, \tag{3}$$

where $\theta \geq 0$, there is a solution $\mathfrak{R}$ of the integro–differential evolution equation and a constant $C > 0$, independent of ϑ and $\mathfrak{R}$, such that

$$\left|\vartheta(\zeta) - \mathfrak{R}(\zeta)\right| \leq C\sigma(\zeta),\ \zeta \in [0, b], \tag{4}$$

then we say that the integro–differential evolution Equation (1) has the σ–semi–Ulam–Hyers stability.

In some cases, fixed–point results are combined with a generalized metric to assess the stability of integral, differential, and integro–differential equations; previous research on this topic is available at [6, 11, 19–22].

Consequently, let us revisit what a generalized metric is and examine the significance of the Banach fixed–point theorem in achieving our goals.

Definition 1.4 ([46]). *Let Y be a nonempty set and $d : Y \times Y \to [0, +\infty]$ be a given mapping. We say that d is a generalized metric on Y if and only if d satisfies the following:*

(G_1) $d(\mathfrak{x}, \mathfrak{y}) = 0$ *if and only if* $\mathfrak{x} = \mathfrak{y}$*;*

(G_2) $d(\mathfrak{x},\mathfrak{y}) = d(\mathfrak{y},\mathfrak{x})$ *for all* $\mathfrak{x},\mathfrak{y} \in Y$;

(G_3) $d(\mathfrak{x},\mathfrak{z}) \leq d(\mathfrak{x},\mathfrak{y}) + d(\mathfrak{y},\mathfrak{z})$ *for all* $\mathfrak{x},\mathfrak{y},\mathfrak{z} \in Y$.

Theorem 1.5 ([9]). *Let* (Y, d) *be a generalized complete metric space and consider a mapping* $\Delta : Y \to Y$ *which is a strictly contractive operator, that is,*

$$d(\Delta\mathfrak{x}, \Delta\mathfrak{y}) \leq L\, d(\mathfrak{x},\mathfrak{y}), \quad \mathfrak{x},\mathfrak{y} \in Y,$$

for some Lipschitz constant $0 \leq L < 1$. *If there exists a nonnegative integer k such that* $d(\Delta^{k+1}\mathfrak{x}, \Delta^k\mathfrak{x}) < \infty$ *for some* $\mathfrak{x} \in Y$, *then the following three statements hold:*

(M_1) *The sequence* $(\Delta^n\mathfrak{x})_{n\in\mathbb{N}}$ *converges to a fixed–point* $\mathfrak{x}^*$ *of* Δ.

(M_2) $\mathfrak{x}^*$ *is the unique fixed–point of* Δ *in the set*

$$Y^* = \left\{\mathfrak{y} \in Y \,\middle|\, d(\Delta^k\mathfrak{x}, \mathfrak{y}) < \infty\right\}.$$

(M_3) *If* $\mathfrak{y} \in Y^*$, *then*

$$d(\mathfrak{y},\mathfrak{x}^*) \leq \frac{1}{1-L}\, d(\Delta\mathfrak{y}, \mathfrak{y}). \tag{5}$$

2. Ulam–Hyers–Rassias stability on a bounded interval

This section will provide adequate requirements for the Ulam–Hyers–Rassias stability of the integro–differential evolution Equation (1) for a given $b > 0$, where $\zeta \in [0, b]$.

We shall examine the space of continuous functions $C([0, b])$ on $[0, b]$ equipped with a metric [22] that is a generalization of the Bielecki one

$$d(\vartheta, \omega) = \sup_{\zeta\in[0,b]} \frac{|\vartheta(\zeta) - \omega(\zeta)|}{\sigma(\zeta)}, \tag{6}$$

where the nondecreasing continuous function $\sigma : [0, b] \to (0, \infty)$ is represented by σ. The well–known Bielecki metric can be found in (6) if we have $\sigma(\zeta) = e^{f\zeta}$ with $f > 0$. In this study, we decided to look at a broader type of metric to expand its scope.

Remember that the generalized metric d in this space $C([0, b])$ is a complete metric space (see, for example, earlier research [23], [7]).

Theorem 2.1. *Let us consider a closed operator* $\Psi : D(\Psi) \subset \mathbb{C} \to \mathbb{C}$. *Moreover, assume that* $\psi(\zeta) : D(\Psi) \subset \mathbb{C} \to \mathbb{C}$, $\zeta \geq 0$, $\psi \neq 0$ *is a closed operator such that there exists* $M > 0$ *so that*

$$\int_0^\zeta \psi(\zeta - \wp)\sigma(\wp)\, d\wp \leq M\sigma(\zeta), \tag{7}$$

for all $\zeta \in [0, b]$ *and* $\aleph : F \to \mathbb{C}$ *is a bounded operator. In addition, let* $\varpi : [0, b] \times \mathbb{C} \to \mathbb{C}$ *be a continuous function for which there exists a constant* $L_\varpi > 0$ *that satisfies the condition*

$$|\varpi(\zeta, \vartheta) - \varpi(\zeta, \omega)| \leq L_\varpi |\vartheta - \omega| \tag{8}$$

for all $\zeta \in [0, b]$, $\vartheta, \omega \in C([0, b])$ *and* $\mathfrak{I} : C([0, b], \mathbb{C}) \to \mathbb{C}$ *is a continuous function, and there exists a positive constant* $L_{\mathfrak{I}} > 0$ *that fulfills the following condition*

$$|\psi(\zeta)\mathfrak{I}(\vartheta) - \psi(\zeta)\mathfrak{I}(\omega)| \leq L_{\mathfrak{I}} |\vartheta - \omega| \tag{9}$$

for all $\zeta \in [0, b]$ *and* $\vartheta, \omega \in C([0, b])$.

If $\vartheta \in C([0, b])$ *is such that*

$$\left|\vartheta'(\zeta) - \Psi\vartheta(\zeta) - \int_0^{\zeta} \psi(\zeta - \wp)\vartheta(\wp)\, d\wp - \aleph\nu(\zeta) - \varpi(\zeta, \vartheta(\zeta))\right| \leq \sigma(\zeta), \tag{10}$$

for all $\zeta \in [0, b]$ *and* $L_{\mathfrak{I}} + ML_{\varpi} < 1$, *then there exists a unique function* $\mathfrak{R} \in C([0, b])$ *that serves as the solution to Equation* (1) *defined as*

$$\mathfrak{R}(\zeta) = \psi(\zeta)[\vartheta_0 + \mathfrak{I}(\mathfrak{R}(\zeta))] + \int_0^{\zeta} \psi(\zeta - \wp)[\aleph\nu(\wp) + \varpi(\wp, \mathfrak{R}(\wp))]\, d\wp \tag{11}$$

such that

$$|\vartheta(\zeta) - \mathfrak{R}(\zeta)| \leq \frac{M\sigma(\zeta)}{1 - (L_{\mathfrak{I}} + ML_{\varpi})} \tag{12}$$

for all $\zeta \in [0, b]$, *this implies that the integro–differential evolution Equation* (1) *is Ulam–Hyers–Rassias stable.*

Proof. We will examine the operator $\Delta : C([0, b]) \to C([0, b])$, defined by

$$(\Delta\vartheta)(\zeta) = \psi(\zeta)[\vartheta_0 + \mathfrak{I}(\vartheta(\zeta))] + \int_0^{\zeta} \psi(\zeta - \wp)[\aleph\nu(\wp) + \varpi(\wp, \vartheta(\wp))]\, d\wp, \tag{13}$$

for all $\zeta \in [0, b]$ and $\vartheta \in C([0, b])$.

Observe that if ϑ is a continuous function, then $\Delta\vartheta$ is also continuous. In fact,

$$\begin{aligned}
|(\Delta\vartheta)(\zeta) - (\Delta\vartheta)(\zeta_o)| &= \Big|\psi(\zeta)[\vartheta_0 + \mathfrak{I}(\vartheta(\zeta))] + \int_0^{\zeta} \psi(\zeta - \wp)[\aleph\nu(\wp) \\
&\quad + \varpi(\wp, \vartheta(\wp))]\, d\wp - \psi(\zeta_0)[\vartheta_0 + \mathfrak{I}(\vartheta(\zeta_0))] - \int_0^{\zeta_0} \psi(\zeta_0 - \wp)[\aleph\nu(\wp) + \varpi(\wp, \vartheta(\wp))]\, d\wp\Big| \\
&\leq \Big|\psi(\zeta)[\vartheta_0 + \mathfrak{I}(\vartheta(\zeta))] - \psi(\zeta_0)[\vartheta_0 + \mathfrak{I}(\vartheta(\zeta_0))]\Big| \\
&\quad + \Big|\int_0^{\zeta} \psi(\zeta - \wp)[\aleph\nu(\wp) + \varpi(\wp, \vartheta(\wp))]\, d\wp - \int_0^{\zeta_0} \psi(\zeta_0 - \wp)[\aleph\nu(\wp) + \varpi(\wp, \vartheta(\wp))]\, d\wp\Big| \\
&= \Big|\vartheta_0\psi(\zeta) + \mathfrak{I}(\vartheta(\zeta))\psi(\zeta) - \vartheta_0\psi(\zeta_0) - \mathfrak{I}(\vartheta(\zeta_0))\psi(\zeta_0)\Big| \\
&\quad + \Big|\int_0^{\zeta} \psi(\zeta - \wp)[\aleph\nu(\wp) + \varpi(\wp, \vartheta(\wp))]\, d\wp - \int_0^{\zeta} \psi(\zeta_0 - \wp)[\aleph\nu(\wp) + \varpi(\wp, \vartheta(\wp))]\, d\wp \\
&\quad + \int_0^{\zeta} \psi(\zeta_0 - \wp)[\aleph\nu(\wp) + \varpi(\wp, \vartheta(\wp))]\, d\wp - \int_0^{\zeta_0} \psi(\zeta_0 - \wp)[\aleph\nu(\wp) + \varpi(\wp, \vartheta(\wp))]\, d\wp\Big| \\
&\leq \Big|[\vartheta_0\psi(\zeta) - \vartheta_0\psi(\zeta_0)] + [\mathfrak{I}(\vartheta(\zeta))\psi(\zeta) - \mathfrak{I}(\vartheta(\zeta_0))\psi(\zeta_0)]\Big| \\
&\quad + \Big|\int_0^{\zeta} \psi(\zeta - \wp)[\aleph\nu(\wp) + \varpi(\wp, \vartheta(\wp))]\, d\wp - \int_0^{\zeta} \psi(\zeta_0 - \wp)[\aleph\nu(\wp) + \varpi(\wp, \vartheta(\wp))]\, d\wp \\
&\quad + \int_{\zeta_0}^{\zeta} \psi(\zeta_0 - \wp)[\aleph\nu(\wp) + \varpi(\wp, \vartheta(\wp))]\, d\wp\Big| \\
&\leq \Big|\vartheta_0[\psi(\zeta) - \psi(\zeta_0)]\Big| + \Big|\mathfrak{I}(\vartheta(\zeta))\psi(\zeta) - \mathfrak{I}(\vartheta(\zeta_0))\psi(\zeta_0)\Big| \\
&\quad + \int_0^{\zeta} \Big|\psi(\zeta - \wp) - \psi(\zeta_0 - \wp)\Big|\Big|\aleph\nu(\wp) + \varpi(\wp, \vartheta(\wp))\Big|\, d\wp
\end{aligned}$$

$$+\left|\int_{\zeta_0}^{\zeta}\psi(\zeta_0-\wp)[\aleph\nu(\wp)+\varpi(\wp,\vartheta(\wp))]\,d\wp\right|\to 0$$

when $\zeta\to\zeta_o$.

We will conclude that, with respect to the metric (6), the operator Δ is strictly contractive under the current conditions.

$$\begin{aligned}
d(\Delta\vartheta,\Delta\omega) &= \sup_{\zeta\in[0,b]}\frac{\left|(\Delta\vartheta)(\zeta)-(\Delta\omega)(\zeta)\right|}{\sigma(\zeta)}\\
&= \sup_{\zeta\in[0,b]}\frac{1}{\sigma(\zeta)}\Big|\psi(\zeta)[\vartheta_0+\Im(\vartheta(\zeta))]+\int_0^{\zeta}\psi(\zeta-\wp)[\aleph\nu(\wp)\\
&\quad+\varpi(\wp,\vartheta(\wp))]\,d\wp-\psi(\zeta)[\vartheta_0+\Im(\omega(\zeta))]\\
&\quad-\int_0^{\zeta}\psi(\zeta-\wp)[\aleph\nu(\wp)+\varpi(\wp,\omega(\wp))]\,d\wp\Big|\\
&= \sup_{\zeta\in[0,b]}\frac{1}{\sigma(\zeta)}\Big|\psi(\zeta)\Im(\vartheta(\zeta))-\psi(\zeta)\Im(\omega(\zeta))\\
&\quad+\int_0^{\zeta}\psi(\zeta-\wp)[\varpi(\wp,\vartheta(\wp))-\varpi(\wp,\omega(\wp))]\,d\wp\Big|\\
&\le \sup_{\zeta\in[0,b]}\frac{1}{\sigma(\zeta)}\Big|\psi(\zeta)\Im(\vartheta(\zeta))-\psi(\zeta)\Im(\omega(\zeta))\Big|\\
&\quad+\sup_{\zeta\in[0,b]}\frac{1}{\sigma(\zeta)}\Big|\int_0^{\zeta}\psi(\zeta-\wp)[\varpi(\wp,\vartheta(\wp))-\varpi(\wp,\omega(\wp))]\,d\wp\Big|\\
&\le L_{\Im}\sup_{\zeta\in[0,b]}\frac{|\vartheta(\zeta)-\omega(\zeta)|}{\sigma(\zeta)}+L_{\varpi}\sup_{\zeta\in[0,b]}\frac{\int_0^{\zeta}|\psi(\zeta-\wp)||\vartheta(\wp)-\omega(\wp)|\,d\wp}{\sigma(\zeta)}\\
&= L_{\Im}\sup_{\zeta\in[0,b]}\frac{|\vartheta(\zeta)-\omega(\zeta)|}{\sigma(\zeta)}+L_{\varpi}\sup_{\zeta\in[0,b]}\frac{\int_0^{\zeta}|\psi(\zeta-\wp)|\frac{|\vartheta(\wp)-\omega(\wp)|}{\sigma(\wp)}\sigma(\wp)\,d\wp}{\sigma(\zeta)}\\
&\le L_{\Im}\sup_{\zeta\in[0,b]}\frac{|\vartheta(\zeta)-\omega(\zeta)|}{\sigma(\zeta)}+L_{\varpi}\sup_{\wp\in[0,b]}\frac{|\vartheta(\wp)-\omega(\wp)|}{\sigma(\wp)}\\
&\quad\sup_{\zeta\in[0,b]}\frac{\int_0^{\zeta}|\psi(\zeta-\wp)|\sigma(\wp)\,d\wp}{\sigma(\zeta)}\\
&\le L_{\Im}d(\vartheta,\omega)+ML_{\varpi}d(\vartheta,\omega)\\
&= (L_{\Im}+ML_{\varpi})d(\vartheta,\omega).
\end{aligned}$$

Since $L_{\Im}+ML_{\varpi}<1$, it follows that Δ is strictly contractive. Therefore, to ensure the Ulam–Hyers–Rassias stability for the integro–differential evolution equation, we can apply the aforementioned Banach fixed–point theorem.

Additionally, (12) follows from (5) and (10). Indeed, from (10), we have

$$|\vartheta(\zeta)-(\Delta\vartheta)(\zeta)|\le\sigma(\zeta),\ \zeta\in[0,b]. \tag{14}$$

We are now able to reapply the Banach fixed–point theorem, resulting from (5) that

$$d(\vartheta,\Re)\le\frac{1}{1-(L_{\Im}+ML_{\varpi})}d(\Delta\vartheta,\vartheta). \tag{15}$$

Based on the definition of the metric d and Equation (14), it can be concluded that

$$\sup_{\zeta\in[0,b]} \frac{|\vartheta(\zeta)-\mathfrak{R}(\zeta)|}{\sigma(\zeta)} \leq \frac{M}{1-(L_{\mathfrak{J}}+ML_{\varpi})}, \tag{16}$$

and consequently, (12) is valid. □

3. σ–semi–Ulam–Hyers stability and Ulam–Hyers stability on a bounded interval

In this section, we will discuss the adequate conditions for the σ–semi–Ulam–Hyers stability and for the Ulam–Hyers stability of the integro–differential evolution Equation (1).

Theorem 3.1. *Let us consider a closed operator $\Psi : D(\Psi) \subset \mathbb{C} \to \mathbb{C}$. Moreover, assume that $\psi(\zeta) : D(\Psi) \subset \mathbb{C} \to \mathbb{C}$, $\zeta \geq 0$, $\psi \neq 0$ is a closed operator such that there exists $M > 0$ so that*

$$\int_0^\zeta \psi(\zeta-\wp)\sigma(\wp)\,d\wp \leq M\sigma(\zeta), \tag{17}$$

for all $\zeta \in [0,b]$ and $\aleph : F \to \mathbb{C}$ is a bounded operator. In addition, let $\varpi : [0,b]\times\mathbb{C} \to \mathbb{C}$ be a continuous function for which there exists a constant $L_{\varpi} > 0$ that satisfies the condition

$$|\varpi(\zeta,\vartheta)-\varpi(\zeta,\omega)| \leq L_{\varpi}|\vartheta-\omega| \tag{18}$$

for all $\zeta \in [0,b]$, $\vartheta,\omega \in C([0,b])$ and $\mathfrak{J} : C([0,b],\mathbb{C}) \to \mathbb{C}$ is a continuous function, and there exists a positive constant $L_{\mathfrak{J}} > 0$ that fulfills the following condition

$$|\psi(\zeta)\mathfrak{J}(\vartheta)-\psi(\zeta)\mathfrak{J}(\omega)| \leq L_{\mathfrak{J}}|\vartheta-\omega| \tag{19}$$

for all $\zeta \in [0,b]$ and $\vartheta,\omega \in C([0,b])$.

If $\vartheta \in C([0,b])$ is such that

$$\left|\vartheta'(\zeta)-\Psi\vartheta(\zeta)-\int_0^\zeta \psi(\zeta-\wp)\vartheta(\wp)\,d\wp-\aleph\nu(\zeta)-\varpi(\zeta,\vartheta(\zeta))\right| \leq \theta, \tag{20}$$

for all $\zeta \in [0,b]$, where $\theta \geq 0$ and $L_{\mathfrak{J}}+ML_{\varpi} < 1$, then there exists a unique function $\mathfrak{R} \in C([0,b])$ that serves as the solution to Equation (1) defined as

$$\mathfrak{R}(\zeta) = \psi(\zeta)[\vartheta_0+\mathfrak{J}(\mathfrak{R}(\zeta))]+\int_0^\zeta \psi(\zeta-\wp)[\aleph\nu(\wp)+\varpi(\wp,\mathfrak{R}(\wp))]\,d\wp \tag{21}$$

such that

$$|\vartheta(\zeta)-\mathfrak{R}(\zeta)| \leq \frac{b\theta}{[1-(L_{\mathfrak{J}}+ML_{\varpi})]\sigma(0)}\sigma(\zeta) \tag{22}$$

for all $\zeta \in [0,b]$, this implies that the integro–differential evolution Equation (1) is σ–semi–Ulam–Hyers stable.

Proof. Using the same process as previously described, we deduce that Δ is strictly contractive with respect to the metric (6), as $L_{\mathfrak{J}}+ML_{\varpi} < 1$. As a result, the Banach fixed–point theorem can be used to guarantee that the integro–differential evolution Equation (1) is σ–semi–Ulam–Hyers stable.

However, keeping in mind (20) and the definition of Δ, we have that

$$|\vartheta(\zeta)-(\Delta\vartheta)(\zeta)| \leq \theta,\ \zeta \in [0,b]. \tag{23}$$

From (5), the definition of the metric d and by (23) implies that

$$\sup_{\zeta\in[0,b]} \frac{|\vartheta(\zeta)-\mathfrak{R}(\zeta)|}{\sigma(\zeta)} \leq \frac{1}{1-(L_{\mathfrak{J}}+ML_{\varpi})}\sup_{\zeta\in[0,b]}\frac{b\theta}{\sigma(\zeta)}$$

$$\leq \frac{1}{1-(L_{\mathfrak{I}}+ML_{\varpi})}\frac{b\theta}{\sigma(0)}$$

and as a result, by the definition of σ, it follows that (22) is satisfied. □

Corollary 3.2. *Let us consider a closed operator $\Psi : D(\Psi) \subset \mathbb{C} \to \mathbb{C}$. Moreover, assume that $\psi(\zeta) : D(\Psi) \subset \mathbb{C} \to \mathbb{C}$, $\zeta \geq 0$, $\psi \neq 0$ is a closed operator such that there exists $M > 0$ so that*

$$\int_0^{\zeta} \psi(\zeta-\wp)\sigma(\wp)\,d\wp \leq M\sigma(\zeta), \tag{24}$$

for all $\zeta \in [0,b]$ and $\aleph : F \to \mathbb{C}$ is a bounded operator. In addition, let $\varpi : [0,b]\times\mathbb{C} \to \mathbb{C}$ be a continuous function for which there exists a constant $L_{\varpi} > 0$ that satisfies the condition

$$|\varpi(\zeta,\vartheta)-\varpi(\zeta,\omega)| \leq L_{\varpi}|\vartheta-\omega| \tag{25}$$

for all $\zeta \in [0,b]$, $\vartheta,\omega \in C([0,b])$ and $\mathfrak{I} : C([0,b],\mathbb{C}) \to \mathbb{C}$ is a continuous function, and there exists a positive constant $L_{\mathfrak{I}} > 0$ that fulfills the following condition

$$|\psi(\zeta)\mathfrak{I}(\vartheta)-\psi(\zeta)\mathfrak{I}(\omega)| \leq L_{\mathfrak{I}}|\vartheta-\omega| \tag{26}$$

for all $\zeta \in [0,b]$ and $\vartheta,\omega \in C([0,b])$.

If $\vartheta \in C([0,b])$ is such that

$$\left|\vartheta'(\zeta)-\Psi\vartheta(\zeta)-\int_0^{\zeta}\psi(\zeta-\wp)\vartheta(\wp)\,d\wp-\aleph\nu(\zeta)-\varpi(\zeta,\vartheta(\zeta))\right| \leq \theta, \tag{27}$$

for all $\zeta \in [0,b]$, where $\theta \geq 0$ and $L_{\mathfrak{I}}+ML_{\varpi} < 1$, then there exists a unique function $\mathfrak{R} \in C([0,b])$ that serves as the solution to Equation (1) defined as

$$\mathfrak{R}(\zeta) = \psi(\zeta)[\vartheta_0+\mathfrak{I}(\mathfrak{R}(\zeta))]+\int_0^{\zeta}\psi(\zeta-\wp)[\aleph\nu(\wp)+\varpi(\wp,\mathfrak{R}(\wp))]\,d\wp \tag{28}$$

such that

$$|\vartheta(\zeta)-\mathfrak{R}(\zeta)| \leq \frac{b\sigma(b)}{[1-(L_{\mathfrak{I}}+ML_{\varpi})]\sigma(0)}\theta \tag{29}$$

for all $\zeta \in [0,b]$, which implies that the integro–differential evolution Equation (1) is Ulam–Hyers stable.

4. Stabilities on an unbounded interval

We will now examine the Ulam–Hyers–Rassias and the σ–semi–Ulam–Hyers stabilities of the integro–differential evolution Equation (1) considering the unbounded interval $[0,\infty)$, for some fixed $0 \in \mathbb{R}$, rather than considering a finite interval $[0,b]$ (with $0, b \in \mathbb{R}$). The results for infinite intervals $(-\infty,0]$, with $0 \in \mathbb{R}$, and for $(-\infty,\infty)$ can be presented with the necessary adaptations. Therefore, let's focus on the integro–differential evolution equation,

$$\begin{cases} \mathfrak{R}'(\zeta) = \Psi\mathfrak{R}(\zeta)+\int_0^{\zeta}\psi(\zeta-\wp)\mathfrak{R}(\wp)\,d\wp+\aleph\nu(\zeta)+\varpi(\zeta,\mathfrak{R}(\zeta)), & \zeta\in[0,\infty), \\ \mathfrak{R}(0) = \mathfrak{R}_0+\mathfrak{I}(\mathfrak{R}(\zeta)), \end{cases} \tag{30}$$

with $\zeta \in [0,\infty)$, where 0 is a fixed real number, $\mathfrak{R} : [0,\infty) \to \mathbb{C}$ is the state function, $\nu(\cdot) \in \mathbb{L}^2([0,\infty),\mathbb{C})$ is a bounded function, $\Psi : D(\Psi) \subset \mathbb{C} \to \mathbb{C}$ is a closed bounded operator, and $\psi(\zeta) : D(\Psi) \subset \mathbb{C} \to \mathbb{C}$, with $\zeta \geq 0$,

$\psi \neq 0$, is a closed operator. Moreover, $\aleph : F \to \mathbb{C}$ is a bounded operator, where F is a separable Hilbert space, and $\varpi : [0, \infty) \times \mathbb{C} \to \mathbb{C}$ and $\Im : C([0, \infty), \mathbb{C}) \to \mathbb{C}$ are given functions to be specified later. Here, ϑ_0 is a given element of $\mathbb{C}$.

A function $\Re : [0, b] \to \mathbb{C}$ is called a mild solution of Equation (30) if

$$\Re(\zeta) = \psi(\zeta)[\vartheta_0 + \Im(\Re(\zeta))] + \int_0^{\zeta} \psi(\zeta - \wp)[\aleph\nu(\wp) + \varpi(\wp, \Re(\wp))]\, d\wp. \tag{31}$$

Here, our approach will use a recurrence procedure based on the results obtained for the corresponding finite interval case.

Let us consider a fixed non–decreasing continuous function $\sigma : [0, \infty) \to (\jmath, \xi)$, for some $\jmath, \xi > 0$ and the space $C_b([0, \infty))$ of bounded continuous functions equipped with the metric [22]

$$d_b(\vartheta, \omega) = \sup_{\zeta \in [0,\infty)} \frac{|\vartheta(\zeta) - \omega(\zeta)|}{\sigma(\zeta)}. \tag{32}$$

Theorem 4.1. *Let us consider a closed bounded operator $\Psi : D(\Psi) \subset \mathbb{C} \to \mathbb{C}$. Moreover, assume that $\psi(\zeta) : D(\Psi) \subset \mathbb{C} \to \mathbb{C}$, $\zeta \geq 0$, $\psi \neq 0$ is a closed operator such that there exists $M > 0$ so that*

$$\int_0^{\zeta} \psi(\zeta - \wp)\sigma(\wp)\, d\wp \leq M\sigma(\zeta), \tag{33}$$

for all $\zeta \in [0, \infty)$ and $\aleph : F \to \mathbb{C}$ is a bounded operator. In addition, let $\varpi : [0, \infty) \times \mathbb{C} \to \mathbb{C}$ be a bounded continuous function for which there exists a constant $L_{\varpi} > 0$ that satisfies the condition

$$|\varpi(\zeta, \vartheta) - \varpi(\zeta, \omega)| \leq L_{\varpi}|\vartheta - \omega| \tag{34}$$

for all $\zeta \in [0, \infty)$, $\vartheta, \omega \in C_b([0, \infty))$ and $\Im : C_b([0, \infty), \mathbb{C}) \to \mathbb{C}$ is a bounded continuous function, and there exists a positive constant $L_{\Im} > 0$ that fulfills the following condition

$$|\psi(\zeta)\Im(\vartheta) - \psi(\zeta)\Im(\omega)| \leq L_{\Im}|\vartheta - \omega| \tag{35}$$

for all $\zeta \in [0, \infty)$ and $\vartheta, \omega \in C_b([0, \infty))$.

In addition, suppose that

$$\int_0^{\zeta} \psi(\zeta - \wp)\Re(\wp)\, d\wp$$

is a bounded continuous function for any bounded continuous function $\Re$.

If $\vartheta \in C_b([0, \infty))$ is such that

$$\left|\vartheta'(\zeta) - \Psi\vartheta(\zeta) - \int_0^{\zeta} \psi(\zeta - \wp)\vartheta(\wp)\, d\wp - \aleph\nu(\zeta) - \varpi(\zeta, \vartheta(\zeta))\right| \leq \sigma(\zeta), \tag{36}$$

for all $\zeta \in [0, \infty)$ and $L_{\Im} + ML_{\varpi} < 1$, then there exists a unique function $\Re \in C_b([0, \infty))$ that serves as the solution to Equation (30) defined as

$$\Re(\zeta) = \psi(\zeta)[\vartheta_0 + \Im(\Re(\zeta))] + \int_0^{\zeta} \psi(\zeta - \wp)[\aleph\nu(\wp) + \varpi(\wp, \Re(\wp))]\, d\wp \tag{37}$$

such that

$$|\vartheta(\zeta) - \Re(\zeta)| \leq \frac{M\sigma(\zeta)}{1 - (L_{\Im} + ML_{\varpi})} \tag{38}$$

for all $\zeta \in [0, \infty)$, which means that the integro–differential evolution Equation (30) is Ulam–Hyers–Rassias stable.

Proof. For any $m \in \mathbb{N}$, we will define $I_m = [0, m]$. By Theorem 2.1, there exists a unique bounded continuous function $\mathfrak{R}_m \in C(I_m)$ such that

$$\mathfrak{R}_m(\zeta) = \psi(\zeta)[\vartheta_0 + \mathfrak{I}(\mathfrak{R}_m(\zeta)] + \int_0^\zeta \psi(\zeta - \wp)[\aleph\nu(\wp) + \varpi(\wp, \mathfrak{R}_m(\wp))]\, d\wp \tag{39}$$

and

$$|\vartheta(\zeta) - \mathfrak{R}_m(\zeta)| \leq \frac{M\sigma(\zeta)}{1 - (L_{\mathfrak{I}} + ML_\varpi)} \tag{40}$$

for all $\zeta \in I_m$. The uniqueness of $\mathfrak{R}_m$ implies that if $\zeta \in I_m$, then

$$\mathfrak{R}_m(\zeta) = \mathfrak{R}_{m+1}(\zeta) = \mathfrak{R}_{m+2}(\zeta) = \cdots . \tag{41}$$

For any $\zeta \in [0, \infty)$, let us define $m(\zeta) \in \mathbb{N}$ as

$$m(\zeta) = \min\{m \in \mathbb{N} \mid \zeta \in I_m\}.$$

We also define a function $\mathfrak{R} : [0, \infty) \to \mathbb{C}$ by

$$\mathfrak{R}(\zeta) = \mathfrak{R}_{m(\zeta)}(\zeta). \tag{42}$$

For any $\zeta_1 \in [0, \infty)$, let $m_1 = m(\zeta_1)$. Then $\zeta_1 \in IntI_{m_1+1}$ and there exists an $\jmath > 0$ such that $\mathfrak{R}(\zeta) = \mathfrak{R}_{m_1+1}(\zeta)$ for all $\zeta \in (\zeta_1 - \jmath, \zeta_1 + \jmath)$, (where $IntI_{m_1+1}$ represents the interior of the set I_{m_1+1}). By Theorem 2.1, $\mathfrak{R}_{m_1+1}$ is continuous at ζ_1, and so it is $\mathfrak{R}$.

Now, we will prove that $\mathfrak{R}$ satisfies

$$\mathfrak{R}(\zeta) = \psi(\zeta)[\vartheta_0 + \mathfrak{I}(\mathfrak{R}(\zeta))] + \int_0^\zeta \psi(\zeta - \wp)[\aleph\nu(\wp) + \varpi(\wp, \mathfrak{R}(\wp))]\, d\wp \tag{43}$$

and

$$|\vartheta(\zeta) - \mathfrak{R}(\zeta)| \leq \frac{M\sigma(\zeta)}{1 - (L_{\mathfrak{I}} + ML_\varpi)}. \tag{44}$$

For an arbitrary $\zeta \in [0, \infty)$, we choose $m(\zeta)$ such that $\zeta \in I_{m(\zeta)}$. By (39) and (42), we have

$$\begin{aligned}\mathfrak{R}(\zeta) &= \mathfrak{R}_{m(\zeta)}(\zeta) \\ &= \psi(\zeta)[\vartheta_0 + \mathfrak{I}(\mathfrak{R}_{m(\zeta)}(\zeta))] + \int_0^\zeta \psi(\zeta - \wp)[\aleph\nu(\wp) + \varpi(\wp, \mathfrak{R}_{m(\zeta)}(\wp))]\, d\wp \\ &= \psi(\zeta)[\vartheta_0 + \mathfrak{I}(\mathfrak{R}(\zeta))] + \int_0^\zeta \psi(\zeta - \wp)[\aleph\nu(\wp) + \varpi(\wp, \mathfrak{R}(\wp))]\, d\wp.\end{aligned} \tag{45}$$

Note that $m(\tau) \leq m(\zeta)$, for any $\tau \in I_{m(\zeta)}$, and it follows from (41) that $\mathfrak{R}(\tau) = \mathfrak{R}_{m(\tau)}(\tau) = \mathfrak{R}_{m(\zeta)}(\tau)$, so the last equality in (45) holds.

To prove (38), by (42) and (40), we have that for all $\zeta \in [0, \infty)$,

$$|\vartheta(\zeta) - \mathfrak{R}(\zeta)| = |\vartheta(\zeta) - \mathfrak{R}_{m(\zeta)}(\zeta)| \leq \frac{M\sigma(\zeta)}{1 - (L_{\mathfrak{I}} + ML_\varpi)}. \tag{46}$$

Finally, we will prove the uniqueness of $\mathfrak{R}$. Let us consider another bounded continuous function $\mathfrak{R}_1$ which satisfies (37) and (38), for all $\zeta \in [0, \infty)$. By the uniqueness of the solution on $I_{m(\zeta)}$ for any $m(\zeta) \in \mathbb{N}$, we have that $\mathfrak{R}_{|I_{m(\zeta)}} = \mathfrak{R}_{m(\zeta)}$ and $\mathfrak{R}_{1|I_{m(\zeta)}}$ satisfies (37) and (38) for all $\zeta \in I_{m(\zeta)}$, so

$$\mathfrak{R}(\zeta) = \mathfrak{R} \mid_{I_{m(\zeta)}} (\zeta) = \mathfrak{R}_1 \mid_{I_{m(\zeta)}} (\zeta) = \mathfrak{R}_1(\zeta).$$

□

Now, we will present adequate conditions for the σ–semi–Ulam–Hyers stability of the integro–differential evolution Equation (30).

Theorem 4.2. *Let us consider a closed bounded operator $\Psi : D(\Psi) \subset \mathbb{C} \to \mathbb{C}$. Moreover, assume that $\psi(\zeta) : D(\Psi) \subset \mathbb{C} \to \mathbb{C}$, $\zeta \geq 0$, $\psi \neq 0$ is a closed operator such that there exists $M > 0$ so that*

$$\int_0^\zeta \psi(\zeta - \wp)\sigma(\wp)\, d\wp \leq M\sigma(\zeta), \tag{47}$$

for all $\zeta \in [0, \infty)$ and $\aleph : F \to \mathbb{C}$ is a bounded operator. In addition, let $\varpi : [0, \infty) \times \mathbb{C} \to \mathbb{C}$ be a bounded continuous function for which there exists a constant $L_\varpi > 0$ that satisfies the condition

$$|\varpi(\zeta, \vartheta) - \varpi(\zeta, \omega)| \leq L_\varpi |\vartheta - \omega| \tag{48}$$

for all $\zeta \in [0, \infty)$, $\vartheta, \omega \in C_b([0, \infty))$ and $\mathfrak{I} : C_b([0, \infty), \mathbb{C}) \to \mathbb{C}$ is a bounded continuous function, and there exists a positive constant $L_\mathfrak{I} > 0$ that fulfills the following condition

$$|\psi(\zeta)\mathfrak{I}(\vartheta) - \psi(\zeta)\mathfrak{I}(\omega)| \leq L_\mathfrak{I} |\vartheta - \omega| \tag{49}$$

for all $\zeta \in [0, \infty)$ and $\vartheta, \omega \in C_b([0, \infty))$.

In addition, suppose that

$$\int_0^\zeta \psi(\zeta - \wp)\mathfrak{R}(\wp)\, d\wp$$

is a bounded continuous function for any bounded continuous function $\mathfrak{R}$.

If $\vartheta \in C_b([0, \infty))$ is such that

$$\left| \vartheta'(\zeta) - \Psi\vartheta(\zeta) - \int_0^\zeta \psi(\zeta - \wp)\vartheta(\wp)\, d\wp - \aleph\nu(\zeta) - \varpi(\zeta, \vartheta(\zeta)) \right| \leq \theta, \tag{50}$$

for all $\zeta \in [0, \infty)$, where $\theta \geq 0$ and $L_\mathfrak{I} + ML_\varpi < 1$, then there exists a unique function $\mathfrak{R} \in C_b([0, \infty))$ that serves as the solution to Equation (30) *defined as*

$$\mathfrak{R}(\zeta) = \psi(\zeta)[\vartheta_0 + \mathfrak{I}(\mathfrak{R}(\zeta))] + \int_0^\zeta \psi(\zeta - \wp)[\aleph\nu(\wp) + \varpi(\wp, \mathfrak{R}(\wp))]\, d\wp \tag{51}$$

such that

$$|\vartheta(\zeta) - \mathfrak{R}(\zeta)| \leq \frac{b\theta}{[1 - (L_\mathfrak{I} + ML_\varpi)]\sigma(0)}\sigma(\zeta) \tag{52}$$

for all $\zeta \in [0, \infty)$, which means that the integro–differential evolution Equation (30) *is σ–semi– Ulam–Hyers stable.*

Proof. The proof can be established by using the same method as previously used in Theorem 3.1. For that reason, we leave it out here. □

5. Examples

We now go over examples demonstrating the applicability and reliability of established findings.

Example 5.1. *Let us consider the integro–differential evolution equation with nonlocal condition of the form*

$$\begin{cases} \mathfrak{R}'(\zeta) = \dfrac{9}{2}\mathfrak{R}(\zeta) + \dfrac{1}{4}\displaystyle\int_0^\zeta \sin(\zeta - \wp)\mathfrak{R}(\wp)\, d\wp + \dfrac{1}{7}\nu(\zeta) + \dfrac{1}{30}\sin(\mathfrak{R}(\zeta)) - \dfrac{1}{20}\cos(\mathfrak{R}(\zeta)), \quad \zeta \in [0, 1], \\ \mathfrak{R}(0) = \dfrac{\mathfrak{R}(\zeta)}{90 + \mathfrak{R}(\zeta)}. \end{cases} \tag{53}$$

We know that the mild solution of this equation is

$$\mathfrak{R}(\zeta) = \sin(\zeta)\left[\frac{\mathfrak{R}(\zeta)}{90+\mathfrak{R}(\zeta)}\right] + \frac{1}{4}\int_0^{\zeta}\sin(\zeta-\wp)\left[\frac{1}{7}\nu(\wp) + \frac{1}{30}\sin(\mathfrak{R}(\wp)) - \frac{1}{20}\cos(\mathfrak{R}(\wp))\right] d\wp. \tag{54}$$

We observe that all the conditions of Theorem 2.1 are fulfilled in this case. In fact, $\Psi : D(\Psi) \subset \mathbb{C} \to \mathbb{C}$ *such that* $\Psi\mathfrak{R}(\zeta) = \frac{9}{2}\mathfrak{R}(\zeta)$ *is a closed operator;* $\psi(\zeta) : D(\Psi) \subset \mathbb{C} \to \mathbb{C}$ *defined by* $\psi(\zeta-\wp) = \frac{1}{4}\sin(\zeta-\wp)$ *is also a closed operator; there exists* $M > 0$ *such that*

$$\begin{aligned}
\int_0^{\zeta}\psi(\zeta-\wp)\sigma(\wp)\,d\wp &= \frac{1}{4}\int_0^{\zeta}\sin(\zeta-\wp)\wp d\wp \\
&= \frac{1}{4}(\zeta\cos(\zeta)-\sin(\zeta)) \\
&\leq \frac{1}{4}\zeta \\
&= M\sigma(\zeta),
\end{aligned}$$

where $\sigma : [0,1] \to (0,\infty)$ *is the non–decreasing continuous function given by* $\sigma(\zeta) = \zeta$ *and* $\aleph : F \to \mathbb{C}$ *such that* $\aleph\nu(\zeta) = \frac{1}{7}\nu(\zeta)$ *is a bounded operator. Moreover,* $\varpi : [0,1]\times\mathbb{C} \to \mathbb{C}$ *such that* $\varpi(\zeta,\mathfrak{R}(\zeta)) = \frac{1}{30}\sin(\mathfrak{R}(\zeta)) - \frac{1}{20}\cos(\mathfrak{R}(\zeta))$ *is a continuous function that satisfies the condition*

$$\begin{aligned}
|\varpi(\zeta,\vartheta)-\varpi(\zeta,\omega)| &= \left|\left(\frac{1}{30}\sin(\vartheta)-\frac{1}{20}\cos(\vartheta)\right)-\left(\frac{1}{30}\sin(\omega)-\frac{1}{20}\cos(\omega)\right)\right| \\
&\leq \frac{1}{30}|\sin(\vartheta)-\sin(\omega)| + \frac{1}{20}|\cos(\vartheta)-\cos(\omega)| \\
&\leq \frac{1}{30}|\vartheta-\omega| + \frac{1}{20}|\vartheta-\omega| \\
&= \frac{1}{12}|\vartheta-\omega| \\
&= L_{\varpi}|\vartheta-\omega|, \quad \vartheta,\omega\in\mathbb{C},\ \zeta\in[0,1].
\end{aligned}$$

Further, $\mathfrak{I} : C([0,1],\mathbb{C}) \to \mathbb{C}$ *such that* $\mathfrak{I}(\mathfrak{R}(\zeta)) = \frac{\mathfrak{R}(\zeta)}{90+\mathfrak{R}(\zeta)}$ *is a continuous function that satisfies the condition*

$$\begin{aligned}
|\psi(\zeta)\mathfrak{I}(\vartheta)-\psi(\zeta)\mathfrak{I}(\omega)| &= \left|\frac{\vartheta\sin(\zeta)}{90+\vartheta}-\frac{\omega\sin(\zeta)}{90+\omega}\right| \\
&= \left|\frac{[\vartheta(90+\omega)-\omega(90+\vartheta)]\sin(\zeta)}{(90+\vartheta)(90+\omega)}\right| \\
&\leq \frac{1}{90}|\vartheta-\omega| \\
&= L_{\mathfrak{I}}|\vartheta-\omega|, \quad \vartheta,\omega\in\mathbb{C},\ \zeta\in[0,1].
\end{aligned}$$

Let $\vartheta \in C([0,1])$ *is such that*

$$\left|\vartheta'(\zeta)-\frac{9}{2}\vartheta(\zeta)-\frac{1}{4}\int_0^{\zeta}\sin(\zeta-\wp)\vartheta(\wp)\,d\wp-\frac{1}{7}\nu(\zeta)-\frac{1}{30}\sin(\vartheta(\zeta))+\frac{1}{20}\cos(\vartheta(\zeta))\right| \leq \sigma(\zeta)=\zeta, \quad \zeta\in[0,1].$$

This demonstrates the Ulam–Hyers–Rassias stability of the integro–differential evolution Equation (53). *Furthermore, taking into account the mild solution* (54) *and* $L_{\mathfrak{I}} + ML_{\varpi} = 0.03194444444 < 1$, *we have*

$$|\vartheta(\zeta)-\mathfrak{R}(\zeta)| \leq \frac{\zeta}{4\left[1-\left(\frac{1}{90}+\frac{1}{4}\times\frac{1}{12}\right)\right]}, \quad \zeta\in[0,1].$$

Example 5.2. *Consider the following integro–differential evolution equation with nonlocal condition of the form*

$$\begin{cases} \mathfrak{R}'(\zeta) = \dfrac{1}{25} + \dfrac{1}{15}\sin(\mathfrak{R}(\zeta)) + \dfrac{1}{12}\displaystyle\int_0^\zeta (\zeta-\wp)^3 \mathfrak{R}(\wp)\,d\wp + \dfrac{\nu(\zeta)}{9}\ln\left(\dfrac{\nu(\zeta)+1}{5}\right) + \dfrac{\cos(\mathfrak{R}(\zeta))}{10(\zeta+2)}, \quad \zeta \in [0,1], \\ \mathfrak{R}(0) = \mathfrak{R}_0 + \dfrac{1}{7}\mathfrak{R}(\zeta). \end{cases} \tag{55}$$

We know that the mild solution of this equation is

$$\mathfrak{R}(\zeta) = \frac{1}{7}\zeta^3\mathfrak{R}(\zeta) + \int_0^\zeta (\zeta-\wp)^3\left[\frac{\nu(\wp)}{9}\ln\Big(\frac{\nu(\wp)+1}{5}\Big) + \frac{\cos(\mathfrak{R}(\wp))}{10(\wp+2)}\right] d\wp. \tag{56}$$

It is obvious that each condition stated in the Theorem 3.1 is satisfied. In fact, $\Psi : D(\Psi) \subset \mathbb{C} \to \mathbb{C}$ *such that*

$$\Psi\mathfrak{R}(\zeta) = \frac{1}{25} + \frac{1}{15}\sin(\mathfrak{R}(\zeta))$$

is a closed operator; $\psi(\zeta) : D(\Psi) \subset \mathbb{C} \to \mathbb{C}$ *defined by* $\psi(\zeta-\wp) = \frac{1}{12}(\zeta-\wp)^3$ *is also a closed operator; there exists* $M > 0$ *such that*

$$\int_0^\zeta \psi(\zeta-\wp)\sigma(\wp)\,d\wp = \frac{1}{12}\int_0^\zeta (\zeta-\wp)^3 e^{\wp} d\wp \le \frac{1}{2}e^\zeta = M\sigma(\zeta),$$

where $\sigma : [0,1] \to (0,\infty)$ *is the non–decreasing continuous function given by* $\sigma(\zeta) = e^\zeta$ *and* $\aleph : F \to \mathbb{C}$ *such that* $\aleph\nu(\zeta) = \frac{\nu(\zeta)}{9}\ln\left(\frac{\nu(\zeta)+1}{5}\right)$ *is a bounded operator. Moreover,* $\varpi : [0,1]\times\mathbb{C} \to \mathbb{C}$ *such that* $\varpi(\zeta,\mathfrak{R}(\zeta)) = \frac{\cos(\mathfrak{R}(\zeta))}{10(\zeta+2)}$ *is a continuous function that satisfies the condition*

$$\begin{aligned} |\varpi(\zeta,\vartheta) - \varpi(\zeta,\omega)| &= \left|\frac{\cos(\vartheta)}{10(\zeta+2)} - \frac{\cos(\omega)}{10(\zeta+2)}\right| \\ &\le \frac{1}{10}|\vartheta-\omega| \\ &= L_\varpi|\vartheta-\omega|, \quad \vartheta,\omega \in \mathbb{C},\ \zeta \in [0,1]. \end{aligned}$$

Further, $\mathfrak{I} : C([0,1],\mathbb{C}) \to \mathbb{C}$ *such that* $\mathfrak{I}(\mathfrak{R}(\zeta)) = \frac{1}{7}\mathfrak{R}(\zeta)$ *is a continuous function that satisfies the condition*

$$\begin{aligned} |\psi(\zeta)\mathfrak{I}(\vartheta) - \psi(\zeta)\mathfrak{I}(\omega)| &= \left|\frac{\zeta^3}{7}\vartheta - \frac{\zeta^3}{7}\omega\right| \\ &\le \frac{1}{7}|\vartheta-\omega| \\ &= L_{\mathfrak{I}}|\vartheta-\omega|, \quad \vartheta,\omega \in \mathbb{C},\ \zeta \in [0,1]. \end{aligned}$$

Let $\vartheta \in C([0,1])$ *be such that*

$$\left|\vartheta'(\zeta) - \frac{1}{25} - \frac{1}{15}\sin(\vartheta)(\zeta) - \frac{1}{12}\int_0^\zeta (\zeta-\wp)^3\vartheta(\wp)\,d\wp - \frac{\nu(\zeta)}{9}\ln\left(\frac{\nu(\zeta)+1}{5}\right) - \frac{\cos(\vartheta)(\zeta)}{10(\zeta+2)}\right| \le \theta, \quad \zeta \in [0,1].$$

Therefore, this demonstrates the σ*–semi–Ulam–Hyers stability of the integro–differential evolution Equation* (55)*. Furthermore, considering the mild solution* (56) *and* $L_{\mathfrak{I}} + ML_\varpi = 0.1928571429 < 1$*, we have*

$$|\vartheta(\zeta) - \mathfrak{R}(\zeta)| \le \frac{\theta}{1 - \left[\frac{1}{7} + \frac{1}{2}\times\frac{1}{10}\right]}e^\zeta, \quad \zeta \in [0,1].$$

Example 5.3. *Consider the integro–differential evolution equation with nonlocal condition of the form*

$$\begin{cases} \Re'(\zeta) = \tan(\zeta)(\sin(\zeta) - \zeta + \sqrt{5}) + \displaystyle\int_0^\zeta \cos(\zeta - \wp)\Re(\wp)\, d\wp + \frac{\zeta}{5}\nu(\zeta) + \frac{\Re(\zeta)}{3\cos^2(\zeta)}, \quad \zeta \in [0, 10], \\ \Re(0) = \Re_0 + \sin(\Re(\zeta)). \end{cases} \tag{57}$$

We know that the mild solution of this equation is

$$\Re(\zeta) = \cos(\zeta)\sin(\Re(\zeta)) + \int_0^\zeta \cos(\zeta - \wp)\left[\frac{1}{5}\nu(\wp) + \frac{1}{3}\frac{\Re(\wp)}{\cos^2(\wp)}\right] d\wp. \tag{58}$$

We observe that this instance satisfies each condition of Corollary 3.2. In fact, $\Psi : D(\Psi) \subset \mathbb{C} \to \mathbb{C}$ *such that* $\Psi\Re(\zeta) = \tan(\zeta)(\sin(\zeta) - \zeta + \sqrt{5})$ *is a closed operator;* $\psi(\zeta) : D(\Psi) \subset \mathbb{C} \to \mathbb{C}$ *defined by* $\psi(\zeta - \wp) = \cos(\zeta - \wp)$ *is also a closed operator; there exists* $M > 0$ *such that*

$$\int_0^\zeta \psi(\zeta - \wp)\sigma(\wp)\, d\wp = \int_0^\zeta \cos(\zeta - \wp)e^{2\wp} d\wp \leq \frac{2}{3}e^{2\zeta} = M\sigma(\zeta),$$

where $\sigma : [0, 10] \to (0, \infty)$ *is the non–decreasing continuous function given by* $\sigma(\zeta) = e^{2\zeta}$ *and* $\aleph : F \to \mathbb{C}$ *such that* $\aleph\nu(\zeta) = \frac{\zeta}{5}\nu(\zeta)$ *is a bounded operator. Moreover,* $\varpi : [0, 10] \times \mathbb{C} \to \mathbb{C}$ *such that* $\varpi(\zeta, \Re(\zeta)) = \frac{\Re(\zeta)}{3\cos^2(\zeta)}$ *is a continuous function that satisfies the condition*

$$\begin{aligned} |\varpi(\zeta, \vartheta) - \varpi(\zeta, \omega)| &= \left|\frac{\vartheta}{3\cos^2(\zeta)} - \frac{\omega}{3\cos^2(\zeta)}\right| \\ &\leq \frac{1}{3}|\vartheta - \omega| \\ &= L_\varpi|\vartheta - \omega|, \quad \vartheta, \omega \in \mathbb{C}, \ \zeta \in [0, 10]. \end{aligned}$$

In addition, $\Im : C([0, 10], \mathbb{C}) \to \mathbb{C}$ *such that* $\Im(\Re(\zeta)) = \sin(\Re(\zeta))$ *is a continuous function that satisfies the condition*

$$\begin{aligned} |\psi(\zeta)\Im(\vartheta) - \psi(\zeta)\Im(\omega)| &= \left|\cos(\zeta)\sin(\vartheta) - \cos(\zeta)\sin(\omega)\right| \\ &\leq |\sin(\vartheta) - \sin(\omega)| \\ &\leq |\vartheta - \omega| \\ &\leq \frac{1}{2}|\vartheta - \omega| \\ &= L_\Im|\vartheta - \omega|, \quad \vartheta, \omega \in \mathbb{C}, \ \zeta \in [0, 10]. \end{aligned}$$

Let $\vartheta \in C([0, 10])$ *is such that*

$$\left|\vartheta'(\zeta) - \tan(\zeta)(\sin(\zeta) - \zeta + \sqrt{5}) - \int_0^\zeta \cos(\zeta - \wp)\vartheta(\wp)\, d\wp - \frac{\zeta}{5}\nu(\zeta) - \frac{1}{3}\frac{\vartheta(\zeta)}{\cos^2(\zeta)}\right| \leq \theta, \quad \zeta \in [0, 10].$$

This demonstrates the Ulam–Hyers stability of the integro–differential evolution Equation (57)*. Furthermore, taking into account the mild solution* (58) *and* $L_\Im + ML_\varpi = 0.7222222222 < 1$*, we have*

$$|\vartheta(\zeta) - \Im(\zeta)| \leq \frac{10e^{20}}{1 - \left[\frac{1}{2} + \frac{2}{3} \times \frac{1}{3}\right]}\theta, \quad \zeta \in [0, 10].$$

Example 5.4. *We consider the following integro–differential evolution equation with nonlocal condition of the form*

$$\begin{cases} \Re'(\zeta) = \dfrac{3}{3 + e^{\Re(\zeta)}} + \displaystyle\int_0^\zeta (\zeta - \wp)\Re(\wp)\, d\wp + \sec(\nu(\zeta)) + \frac{\Re(\zeta)}{(\zeta + 9)^2}, \quad \zeta \in [0, \infty), \\ \Re(0) = \Re_0 + \dfrac{1}{7}\ln \Re(\zeta). \end{cases} \tag{59}$$

We know that the mild solution of this equation is

$$\mathfrak{R}(\zeta) = \frac{\zeta}{7}\ln\mathfrak{R}(\zeta) + \int_0^{\zeta}(\zeta-\wp)\left[\sec(\nu(\zeta)) + \frac{\mathfrak{R}(\wp)}{(\wp+9)^2}\right]d\wp. \tag{60}$$

We observe that all the conditions of Theorem 4.1 are here satisfied. In fact, $\Psi : D(\Psi) \subset \mathbb{C} \to \mathbb{C}$ *such that* $\Psi\mathfrak{R}(\zeta) = \frac{3}{3+e^{\mathfrak{R}(\zeta)}}$ *is a closed bounded operator;* $\psi(\zeta) : D(\Psi) \subset \mathbb{C} \to \mathbb{C}$ *defined by* $\psi(\zeta-\wp) = \zeta-\wp$ *is a closed operator; there exists* $M > 0$ *such that*

$$\int_0^{\zeta}\psi(\zeta-\wp)\sigma(\wp)\,d\wp = \int_0^{\zeta}(\zeta-\wp)e^{3\wp}d\wp \leq \frac{1}{9}e^{3\zeta} = M\sigma(\zeta),$$

where $\sigma : [0,\infty) \to (0,\infty)$ *is the non–decreasing continuous function given by* $\sigma(\zeta) = e^{3\zeta}$ *and* $\aleph : F \to \mathbb{C}$ *such that* $\aleph\nu(\zeta) = \sec(\nu(\zeta))$ *is a bounded operator. Moreover,* $\varpi : [0,\infty)\times\mathbb{C} \to \mathbb{C}$ *such that* $\varpi(\zeta,\mathfrak{R}(\zeta)) = \frac{\mathfrak{R}(\zeta)}{(\zeta+9)^2}$ *is a bounded continuous function that satisfies the condition*

$$\begin{aligned}
|\varpi(\zeta,\vartheta)-\varpi(\zeta,\omega)| &= \left|\frac{\vartheta}{(\zeta+9)^2} - \frac{\omega}{(\zeta+9)^2}\right| \\
&\leq \frac{1}{9}|\vartheta-\omega| \\
&= L_{\varpi}|\vartheta-\omega|, \quad \vartheta,\omega\in\mathbb{C},\ \zeta\in[0,\infty).
\end{aligned}$$

Further, $\mathfrak{I} : C_b([0,\infty),\mathbb{C}) \to \mathbb{C}$ *such that* $\mathfrak{I}(\mathfrak{R}(\zeta)) = \frac{1}{7}\ln\mathfrak{R}(\zeta)$ *is a bounded continuous function that satisfies the condition*

$$\begin{aligned}
|\psi(\zeta)\mathfrak{I}(\vartheta)-\psi(\zeta)\mathfrak{I}(\omega)| &= \left|\frac{\zeta}{7}\ln\vartheta - \frac{\zeta}{7}\ln\omega\right| \\
&\leq \frac{1}{7}|\vartheta-\omega| \\
&= L_{\mathfrak{I}}|\vartheta-\omega|, \quad \vartheta,\omega\in\mathbb{C},\ \zeta\in[0,\infty).
\end{aligned}$$

Let $\vartheta \in C_b([0,\infty))$ *is such that*

$$\left|\vartheta'(\zeta) - \frac{3}{3+e^{\vartheta(\zeta)}} - \int_0^{\zeta}(\zeta-\wp)\vartheta(\wp)\,d\wp - \sec(\nu(\zeta)) - \frac{\vartheta(\zeta)}{(\zeta+9)^2}\right| \leq \sigma(\zeta) = e^{3\zeta}, \quad \zeta\in[0,\infty).$$

This demonstrates the Ulam–Hyers–Rassias stability of the integro–differential evolution Equation (59)*. Furthermore, considering the mild solution* (60) *and* $L_{\mathfrak{I}} + ML_{\varpi} = 0.1552028219 < 1$*, we have*

$$|\vartheta(\zeta)-\mathfrak{R}(\zeta)| \leq \frac{e^{3\zeta}}{9\left[1-\left(\frac{1}{7}+\frac{1}{9}\times\frac{1}{9}\right)\right]}, \quad \zeta\in[0,\infty).$$

Example 5.5. *Finally, we will consider the integro–differential evolution equation with nonlocal condition of the form*

$$\begin{cases}\mathfrak{R}'(\zeta) = \dfrac{\arccos(\mathfrak{R}(\zeta))}{\zeta^2+1} + \dfrac{1}{29}\displaystyle\int_0^{\zeta}e^{\zeta-\wp}\mathfrak{R}(\wp)\,d\wp + e^{\frac{1}{13}\nu(\zeta)+2} + \dfrac{\arctan(\mathfrak{R}(\zeta))}{36}, \quad \zeta\in[0,\infty),\\ \mathfrak{R}(0) = \mathfrak{R}_0 + e^{\frac{1}{14}\mathfrak{R}(\zeta)}.\end{cases} \tag{61}$$

We know that the mild solution of this equation is

$$\mathfrak{R}(\zeta) = e^{\frac{1}{14}\mathfrak{R}(\zeta)+\zeta} + \int_0^{\zeta}e^{\zeta-\wp}\left[e^{\frac{1}{13}\nu(\wp)+2} + \frac{\arctan(\mathfrak{R}(\wp))}{36}\right]d\wp. \tag{62}$$

We observe that all the conditions of Theorem 4.2 are here satisfied. In fact, $\Psi : D(\Psi) \subset \mathbb{C} \to \mathbb{C}$ *such that* $\Psi\Re(\zeta) = \frac{\arccos(\Re(\zeta))}{\zeta^2+1}$ *is a closed bounded operator;* $\psi(\zeta) : D(\Psi) \subset \mathbb{C} \to \mathbb{C}$ *defined by* $\psi(\zeta - \wp) = \frac{1}{29}e^{\zeta-\wp}$ *is also a closed bounded operator; there exists* $M > 0$ *such that*

$$\int_0^\zeta \psi(\zeta-\wp)\sigma(\wp)\,d\wp = \frac{1}{29}\int_0^\zeta e^{\zeta-\wp}(0.2+e^{0.2\wp})\,d\wp \le \frac{1}{20}e^\zeta = M\sigma(\zeta),$$

where $\sigma : [0,\infty) \to (0,\infty)$ *is the non–decreasing continuous function given by* $\sigma(\zeta) = 0.2 + e^{0.2\zeta}$ *and* $\aleph : F \to \mathbb{C}$ *such that* $\aleph\nu(\zeta) = e^{\frac{1}{13}\nu(\zeta)+2}$ *is a bounded operator. Moreover,* $\varpi : [0,\infty)\times\mathbb{C} \to \mathbb{C}$ *such that* $\varpi(\zeta,\Re(\zeta)) = \frac{\arctan(\Re(\zeta))}{36}$ *is a bounded continuous function that satisfies the condition*

$$\begin{aligned} |\varpi(\zeta,\vartheta)-\varpi(\zeta,\omega)| &= \left|\frac{\arctan(\vartheta)}{36} - \frac{\arctan(\omega)}{36}\right| \\ &\le \frac{1}{36}|\vartheta-\omega| \\ &= L_\varpi|\vartheta-\omega|, \quad \vartheta,\omega\in\mathbb{C},\ \zeta\in[0,\infty). \end{aligned}$$

Furthermore, $\Im : C_b([0,\infty),\mathbb{C}) \to \mathbb{C}$ *such that* $\Im(\Re(\zeta)) = e^{\frac{1}{14}\Re(\zeta)}$ *is a bounded continuous function that satisfies the condition*

$$\begin{aligned} |\psi(\zeta)\Im(\vartheta)-\psi(\zeta)\Im(\omega)| &= \left|e^\zeta e^{\frac{\vartheta}{14}} - e^\zeta e^{\frac{\omega}{14}}\right| \\ &\le \frac{1}{14}|\vartheta-\omega| \\ &= L_\Im|\vartheta-\omega|, \quad \vartheta,\omega\in\mathbb{C},\ \zeta\in[0,\infty). \end{aligned}$$

Let $\vartheta \in C_b([0,\infty))$ *is such that*

$$\left|\vartheta'(\zeta) - \frac{\arccos(\vartheta(\zeta))}{\zeta^2+1} - \frac{1}{29}\int_0^\zeta e^{\zeta-\wp}\vartheta(\wp)\,d\wp - e^{\frac{1}{13}\nu(\zeta)+2} - \frac{\arctan(\vartheta(\zeta))}{36}\right| \le \theta, \quad \zeta\in[0,\infty).$$

This demonstrates the σ*–semi–Ulam–Hyers stability of the integro–differential evolution Equation* (61)*. Additionally, considering the mild solution* (62) *and* $L_\Im + ML_\varpi = 0.07281746032 < 1$*, we have*

$$|\vartheta(\zeta) - \Re(\zeta)| \le \frac{20\,\theta(0.2+e^{0.2\zeta})}{\frac{6}{5}\left[1-\left(\frac{1}{14}+\frac{1}{20}\times\frac{1}{36}\right)\right]}, \quad \zeta\in[0,\infty).$$

6. Application

In this section, we describe the application that we used to assess the validity of our theoretical claims: the integro–differential representation for the spread of infectious diseases in epidemiology that have an effect on cognition.

6.1. Simulating the transmission of infectious diseases with memory effects

Direct or indirect contact between individuals can result in the transmission of infectious diseases, which are mostly brought on by pathogenic microorganisms such as bacteria, viruses, parasites, or fungi. For public health planning and response, understanding how these diseases spread and change over time is essential. The susceptible–infected–Recovered (SIR) model is one of the classic models of infectious disease transmission that offers a fundamental framework for comprehending the dynamics of disease propagation. These models describe variations in the proportion of susceptible, infected, and recovered people within a population using equations. These models, however, frequently make the unreal assumption that the rates of transmission and recovery remain constant across time.

Indeed, memory affects the way that past conditions and experiences affect present and future transmission rates, which can have an impact on the dynamics of infectious disease transmission. For instance, past health exposure might impact an individual's immunity, and previous epidemics can alter population behavior like vaccination uptake or social distancing. Advanced models include memory effects in the equations regulating the transmission of disease in order to represent these complicated procedures. By incorporating previous states into the present dynamics of disease transmission, these models can incorporate these memory effects through the use of integro–differential, differential, and integral equations.

Furthermore, infectious diseases have a variety of effects on cognition. Infections can cause fever, exhaustion, or neurological problems, which can be directly harmful to cognitive processes including memory, concentration, and decision–making. Furthermore, the anxiety and stress that come with disease outbreaks might affect behavioral responses and increase cognitive problems. The realism of disease propagation simulations is improved by our technique, which incorporates cognitive variables into epidemiological models, including the impact of disease–induced cognitive changes on decision–making and adherence to preventive measures.

By taking into account variables like changing infectiousness over time and the delayed impacts of interventions, integro–differential evolution equations provide a more sophisticated simulation of epidemiological processes. By using advanced computational techniques and numerical approaches to solve these improved models, researchers are able to simulate different situations and create efficient plans for the control and prevention of disease. We can strengthen the efficacy of public health interventions, better address the cognitive effects of infectious diseases on individuals and populations, and better understand the dynamics of disease by employing integro–differential evolution equations.

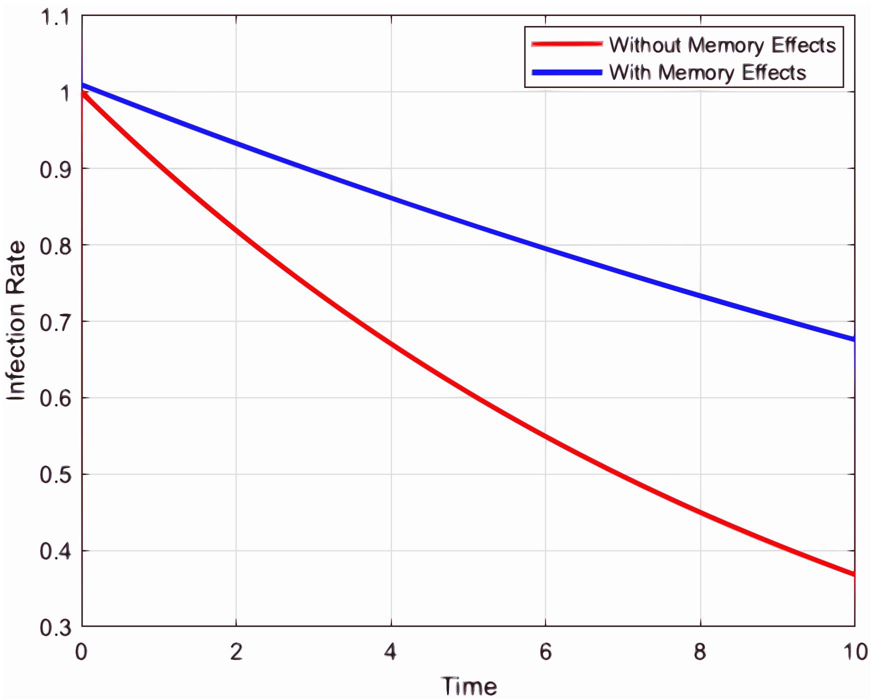


Figure 1: 2D plot of infection rates over time, showing differences between models without memory effects and with memory effects.

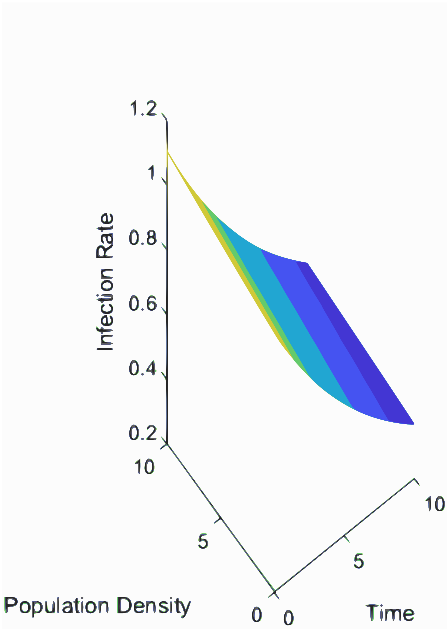
In order to demonstrate infection rates over time, we can make a plot that illustrates the relationship between models with and without memory effects. This graph shows how the dynamics of infection rates over time are influenced by memory effects in disease transmission models and how these dynamics may be modified by various parameters. The graph is made up of visual data that shows how memory effects affect the dynamics of infection rates over time in models of disease transmission. We can visually examine the relationship or connection that these points display over the specified time period by graphing them and connecting them. With MATLAB and MATHEMATICA, the obtained answers were simulated using 2D and 3D graphs. As seen in Figures 1 and 2, the resulting graph can have the following features:

Rapid Deterioration Over Time: A slow decrease in the infection rate is one important aspect. In the absence of memory effects, the change happens quickly, indicating that infection rates will decrease rapidly over time. The decrease is slower in memory effects, indicating a more gradual decrease in infection rates over time.

Population Density and Exponential Growth: The infection rate grows exponentially with population density, which is another characteristic. This feature is present in both models, suggesting that increased infection rates are a function of population densities.

Variations: Periodic variations in the infection rate are a specific characteristic of the memory effects hypothesis. These variances show how memory effects affect infection rates over time. Without memory effects, the model is free of this characteristic and declines smoothly and gradually.

External attributes: In the absence of memory effects, the 3D surface is smooth and predictable, exhibiting a distinct pattern of steady development with increasing population density and rapid degeneration over time. When memory effects are present, the surface becomes more convoluted and unpredictable as a result of the combined impacts of oscillations and slower decay.



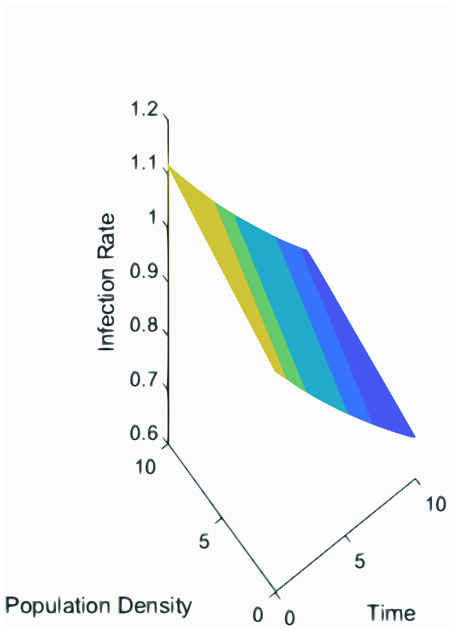


Figure 2: 3D plot of infection rates with and without memory effects.

It's crucial to understand that temporal dynamics are drastically changed when memory effects are included in infection rate models. Instead of fast decay in the absence of memory, there is delayed decay and oscillations in the presence of memory. These changes impact not only the prediction of infection rates but also the understanding of seasonal variations and disease persistence in the study of epidemiology.

The integro–differential evolution equation for the transmission of infectious diseases with memory effects is

$$\begin{cases} \mathfrak{R}'(\zeta) = \frac{3}{10}\mathfrak{R}(\zeta) + \frac{1}{10}\int_0^{\zeta} e^{-\frac{1}{2}(3\zeta-2\wp)}\mathfrak{R}(\wp)\,d\wp + \frac{3}{10}\tan(\zeta) - \frac{1}{2}\sin^2(\mathfrak{R}(\zeta)), \quad \zeta \in [0,10], \\ \mathfrak{R}(0) = \frac{9}{10}\mathfrak{R}(\zeta), \end{cases} \tag{63}$$

in which $\mathfrak{R}'(\zeta)$ is the rate of change of the infected population over time ζ. It explains the changes in the number of infections throughout time. We know that the mild solution of this equation is

$$\mathfrak{R}(\zeta) = \frac{1}{10}e^{-\frac{3}{2}\zeta}\left[\frac{9}{10}\mathfrak{R}(\zeta)\right] + \frac{1}{10}\int_0^{\zeta} e^{-\frac{1}{2}(3\zeta-2\wp)}\left[\frac{3}{10}\tan(\wp) - \frac{1}{2}\sin^2(\mathfrak{R}(\wp))\right] d\wp. \tag{64}$$

We realize that all the conditions of Corollary 3.2 are here satisfied.

In order to establish a connection between the system (63) and our provided integro–differential evolution Equation (1), we can identify the following:

- $\Psi\mathfrak{R}(\zeta) = \frac{3}{10}\mathfrak{R}(\zeta)$ represents the rate of new infections, and the constant rate at which new infections occur is denoted by $\Psi = \frac{3}{10}$. It implies that the number of new infections is proportional to the total number of infections at any particular time.
- $\psi(\zeta-\wp) = \frac{1}{10}e^{-\frac{1}{2}(3\zeta-2\wp)}$ represents that the influence of previous infections reduces exponentially with time, with the influence of more recent infections being stronger on the current state, which fulfills the condition

$$\int_0^{\zeta} \psi(\zeta-\wp)\sigma(\wp)\,d\wp = \frac{1}{10}\int_0^{\zeta} e^{-\frac{1}{2}(3\zeta-2\wp)}e^{19\wp}d\wp \leq \frac{1}{200}e^{19\zeta} = M\sigma(\zeta),$$

  where $\sigma : [0,10] \to (0,\infty)$ is the non–decreasing continuous function $\sigma(\zeta) = e^{19\zeta}$.
- $\aleph\nu(\zeta) = \frac{3}{10}\tan(\zeta)$ represents external interventions or control measures, and $\aleph = \frac{3}{10}$ sets the impact of these interventions on lowering infection rates. In this case, periodic interventions that could influence the spread of infectious diseases are modeled by $\tan(\zeta)$.
- $\varpi(\zeta,\mathfrak{R}(\zeta)) = -\frac{1}{2}\sin^2(\mathfrak{R}(\zeta))$ represents the infection–dynamic saturation effect. It simulates how the number of affected people could rise while the rate of new infections might fall. Herd immunity and restricted transmission possibilities are two examples of factors that may affect this non–linear behavior. It fulfils the condition

$$\begin{aligned}|\varpi(\zeta,\vartheta)-\varpi(\zeta,\omega)| &= \left|\frac{1}{2}\sin^2(\vartheta) - \frac{1}{2}\sin^2(\omega)\right| \\ &\leq \frac{1}{2}|\vartheta-\omega| \\ &= L_{\varpi}|\vartheta-\omega|, \quad \vartheta,\omega\in\mathbb{C},\ \zeta\in[0,10].\end{aligned}$$

- $\mathfrak{I}(\mathfrak{R}(\zeta)) = \frac{9}{10}\mathfrak{R}(\zeta)$ represents a feedback mechanism reflecting the memory and dynamics of the system, where the current state depends on a future state. It fulfils the condition

$$\begin{aligned}|\psi(\zeta)\mathfrak{I}(\vartheta)-\psi(\zeta)\mathfrak{I}(\omega)| &= \left|\frac{1}{10}e^{-\frac{3}{2}\zeta}\frac{9}{10}\vartheta - \frac{1}{10}e^{-\frac{3}{2}\zeta}\frac{9}{10}\omega\right| \\ &= \left|\frac{1}{10}e^{-\frac{3}{2}\zeta}\left(\frac{9}{10}\vartheta - \frac{9}{10}\omega\right)\right| \\ &\leq \frac{9}{10}|\vartheta-\omega| \\ &= L_{\mathfrak{I}}|\vartheta-\omega|, \quad \vartheta,\omega\in\mathbb{C},\ \zeta\in[0,10].\end{aligned}$$

Let $\vartheta \in C([0,10])$ is such that

$$\left|\vartheta'(\zeta) - \frac{3}{10}\vartheta(\zeta) - \frac{1}{10}\int_0^{\zeta} e^{-\frac{1}{2}(3\zeta-2\wp)}\vartheta(\wp)\,d\wp - \frac{3}{10}\tan(\zeta) + \frac{1}{2}\sin^2(\vartheta)(\zeta)\right| \leq \theta, \quad \zeta\in[0,10].$$

This demonstrates the Ulam–Hyers stability of the integro–differential evolution Equation (63). In addition, considering the mild solution (64) and $L_{\mathfrak{I}} + ML_{\varpi} = 0.9025 < 1$, we have

$$|\vartheta(\zeta) - \mathfrak{I}(\zeta)| \leq \frac{10e^{190}\theta}{1 - \left[\frac{9}{10} + \frac{1}{200} \times \frac{1}{2}\right]}, \quad \zeta \in [0, 10].$$

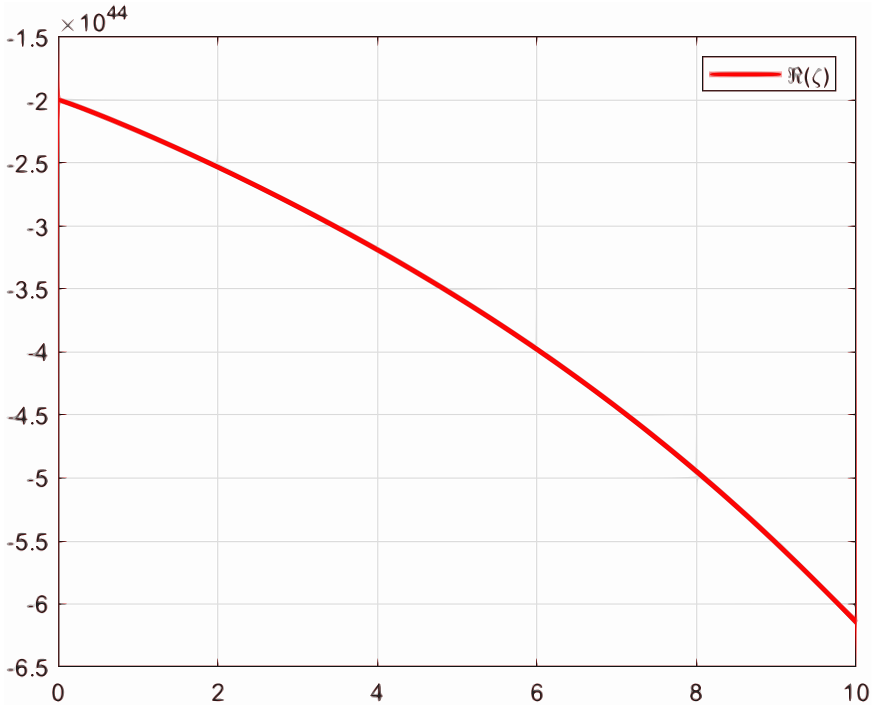


Figure 3: A simulation showing the dynamics of infection over time.

The graph 3 illustrates the evolution of infection dynamics impacted by memory effects, infection rate, external interventions, and nonlinear saturation effects. It shows the number of infected individuals over time ζ.

7. Conclusion

Nowadays, studies on the stability of differential equations have made a significant contribution to the literature. Specifically, the literature has discussed the σ–semi–Ulam–Hyers stability of differential equations, where a variety of conditions have been applied and the Banach fixed–point theorem has been used to obtain most results. In this paper, we have studied the Ulam–Hyers, the Ulam–Hyers–Rassias, and the σ–semi–Ulam–Hyers stability of the integro–differential evolution equations with nonlocal conditions through the Banach fixed–point theorem. This study is crucial for the mathematical community, especially for researchers who deal with differential equation problems.

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