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New perspective on Jensen type inequalities pertaining to local fractional derivatives

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Abstract. We examined and analyzed the characteristics of generalized convex functions defined on fractal sets. We then conducted a comprehensive analysis of the properties associated with these generalized convex functions, and these characteristics were utilized in proving two significant inequalities: the extended Jensen's inequality and the generalized Hermite-Hadamard inequality. Through these inequalities, we derived valuable insights into the behavior of these functions and their relationships with other mathematical concepts. Additionally, practical applications that showcase the significance and applicability of these generalized inequalities in various fields are discussed.

1. Introduction

Fractional calculus has advanced significantly recently, especially in developing local fractional derivatives. These mathematical tools are now widely used to model complex systems with memory effects and nonlocal behaviors in physics, engineering, and biology. This hybrid operator combines features of both Riemann-Liouville and Caputo fractional derivatives. It has proven effective for solving nonlinear fractional differential equations with nonlocal initial conditions by establishing connections to cotangent Volterra integral equations. These new operators maintain beneficial mathematical properties while extending classical fractional calculus concepts. Numerical methods for fractional systems have also improved. Wang [1] used the local fractional derivative for the first time in his research to define a novel fractional low-pass electrical transmission lines model (LPETLM). Two special functions are used in conjunction with Yang's non-differentiable transformation to find the non-differentiable (ND) exact solutions (ESs) by defining the Mittag-Leffler function (MLF) on the Cantor set (CS). Saleh et al. [2] recently worked on integral inequalities

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utilizing a new identity involving fractal–fractional integrals. This identity enabled them to use generalized convexity to derive new Bullen-type inequalities. By giving a variety of conclusions for fractional integrals and fractal calculus and a revision of the well-known Bullen-type inequality, this work represents a substantial advancement in the field of fractal–fractional inequalities. Lakhdari et al. [3] addressed fractal sets and corrected Simpson-type inequalities. Based on an introduced identity, they used the generalized s-convexity and s-concavity of the local fractional derivative to derive certain error bounds for the formula under consideration. Alqhtani et al. [4] created the Mittag-Leffler-Caputo-Fabrizio derivative, which uses a special kernel based on Mittag-Leffler functions. Their numerical approach shows good convergence properties as computational steps become smaller. Researchers have also created specialized fractional tools for specific applications. For quantum physics problems, Sadek et al. [5] combined fractional calculus with trigonometric functions to develop new analysis methods. The field has also expanded to include fractal mathematics. Further, Sadek et al. [6] also generalized fractal derivatives by incorporating arbitrary functions, developing new transform methods and chain rules for solving differential equations in fractal spaces. These advances demonstrate how fractional calculus continues to grow as researchers develop both theoretical foundations and practical applications across multiple scientific disciplines.

A function $\check{H} : I \subseteq \mathbb{R} \to \mathbb{R}$ is considered convex on the interval I when, for any $\mathfrak{s}_1, \mathfrak{s}_2 \in I$ and $\mu \in [0, 1]$, the subsequent inequality is satisfied:

$$\check{H}\left(\mu\mathfrak{s}_1 + (1-\mu)\mathfrak{s}_2\right) \leq \mu\check{H}\left(\mathfrak{s}_1\right) + (1-\mu)\check{H}\left(\mathfrak{s}_2\right).$$

In other words, a function $\check{H}$ is convex on I if the value of the function at the midpoint of any interval in I is less than or equal to the weighted average of the function values at the endpoints of the interval, where the weights are determined by the parameter μ.

Definition 1.1. *[7] Suppose $\varrho : J \to \mathbb{R}$ is a function that is both non-negative and non-zero. We define an ϱ-convex function as $\check{H} : I \to \mathbb{R}$ that satisfies the following conditions: (1) $\check{H}$ is non-negative, and (2) for any $\mathfrak{s}_1, \mathfrak{s}_2 \in I$ and $\mu \in [0, 1]$.*

$$\check{H}((1-\mu)\mathfrak{s}_1 + \mu\mathfrak{s}_2) \leq \varrho(1-\mu)\check{H}(\mathfrak{s}_1) + \varrho(\mu)\check{H}(\mathfrak{s}_2). \tag{1}$$

If this inequality is reversed, then $\check{H}$ is said to be ϱ-concave.

Convex functions hold great importance in various fields such as biology, economics, optimization, and more [8, 9]. They are associated with several significant inequalities that have been established in the literature. Among these, the well-known Jensen's inequality and Hermite-Hadamard's inequality are particularly noteworthy.

According to Jensen's inequality [10], if $\check{H}$ is a convex function defined on the interval $[a, b]$, then for any $\mathfrak{s}_i \in [a, b]$ and $\mu_i \in [0, 1]$ $(i = 1, 2, \ldots, n)$ with the condition that $\sum_{i=1}^{n} \mu_i = 1$, the following inequality holds:

$$\check{H}\left(\sum_{i=1}^{n} \mu_i \mathfrak{s}_i\right) \leq \sum_{i=1}^{n} \mu_i \check{H}\left(\mathfrak{s}_i\right).$$

Hermite-Hadamard's inequality [11] can be stated in simpler language as follows:

If we have a function $\check{H}$ that is convex on the interval $[u, v]$, where $u < v$ and if the function is also defined for all values between u and v, then the average value of $\check{H}$ over this interval is always greater than or equal to the integral of $\check{H}$ over the same interval i.e.

$$\check{H}\left(\frac{u+v}{2}\right) \leq \frac{1}{v-u}\int_u^v \check{H}(\mathfrak{s})d\mathfrak{s} \leq \frac{\check{H}(u) + \check{H}(v)}{2}.$$

In recent times, there has been a noticeable upsurge of interest among scientists and engineers in the realm of fractals. Fractal sets, as defined by Mandelbrot, exhibit a Hausdorff dimension that surpasses their topological dimension [12–16]. A multitude of researchers have dedicated their efforts to exploring the properties of functions defined on fractal spaces, employing diverse approaches to fractional calculus within this domain (see [17–21]). Notably, Yang's comprehensive analysis of local fractional functions on fractal spaces in [20] encompasses topics such as local fractional calculus and the monotonicity of functions.

Building upon comprehensive research in the field, our main goal is to introduce the concept of generalized convex functions on fractal sets and present associated results. This encompasses the development of the generalized Jensen's inequality and the generalized Hermite-Hadamard inequality. Our particular emphasis centers on convexity, as we note a profound interconnection between concave functions and their convex equivalents, where a function $\check{H}$ is concave if and only if $-\check{H}$ is convex. Consequently, any results concerning convex functions can be readily adapted for concave functions. Some of recent works related to fractional calculus and inequalities can be found in the following articles and the references therein [22–28].

The article follows this structure:

Section 2 dives into how we work with numbers on fractals, explaining things like local fractional derivatives and integrals. In Section 3, we define a new kind of function on fractals and explore what makes them tick. Then, in Section 4, we introduce two special rules for fractals: the generalized Jensen's inequality and the generalized Hermite-Hadamard's inequality. We show why they matter. Finally, in Section 5, we see how these rules play out in real-life situations on fractals.

2. Preliminaries

The definitions of the local fractional derivative and local fractional integral on the real number line set $\mathbb{R}^r$ can be described using the concept proposed by Gao-Yang-Kang. A new theory called Yang's fractional sets, introduced by Yang [20], provides valuable insights into the following aspects.

For values of $0 < r \leq 1$, a set consisting of various element sets can be defined, referred to as the r-type set.

$\mathbb{Z}^r$: The set of integers with an r-type is defined as follows: $\{0^r, \pm 1^r, \pm 2^r, \cdots, \pm n^r, \cdots\}$.

$\mathbb{Q}^r$: The set of rational numbers with an r-type is defined as follows: $\{m^r = (p/q)^r : \quad p \in \mathbb{Z}, q \neq 0\}$.

$\mathbb{J}^r$: The set of irrational numbers with an r-type is defined as follows: $\{m^r \neq (p/q)^r : \quad p \in \mathbb{Z}, q \neq 0\}$.

$\mathbb{R}^r$: The set of real numbers with an r-type is defined as follows: $\mathbb{R}^r = \mathbb{Q}^r \cup \mathbb{J}^r$.

If a^r, b^r and c^r belong to the set $\mathbb{R}^r$ of real line numbers, then

(1) $a^r + b^r$ and $a^r b^r$ belong to the set $\mathbb{R}^r$;
(2) $a^r + b^r = b^r + a^r = (a+b)^r = (b+a)^r$;
(3) $a^r + (b^r + c^r) = (a+b)^r + c^r$;
(4) $a^r b^r = b^a a^r = (ab)^r = (ba)^r$;
(5) $a^r (b^r c^r) = (a^r b^r) c^r$
(6) $a^r (b^r + c^r) = a^r b^r + a^r c^r$;
(7) $a^r + 0^r = 0^r + a^r = a^r$ and $a^r 1^r = 1^r a^r = a^r$.

Now, let's introduce some definitions regarding local fractional calculus on the real number line set $\mathbb{R}^r$.

Definition 2.1. *[20] If we have a function $\check{H} : \mathbb{R} \to \mathbb{R}^r, \check{H} \to \check{H}(\mathfrak{s})$ we say that $\check{H}$ is locally fractionally continuous at $\check{H}_0$ if, for any positive number 'ε', there is a positive number δ such that*

$$\left|\check{H}(\mathfrak{s}) - \check{H}(\mathfrak{s}_0)\right| < \varepsilon^r$$

holds for $|\mathfrak{s} - \mathfrak{s}_0| < \delta$, where $\varepsilon, \delta \in \mathbb{R}$. If $\check{H}(\mathfrak{s})$ is local fractional continuous on the interval (a,b), we denote $\check{H}(\mathfrak{s}) \in C_r(a,b)$.

Definition 2.2. *[20] The local fractional derivative of $\check{H}(\mathfrak{s})$ with an order of r at $\mathfrak{s} = \mathfrak{s}_0$ is determined as*

$$\check{H}^{(r)}(\mathfrak{s}_0) = \left.\frac{d^r \check{H}(\mathfrak{s})}{d\mathfrak{s}^r}\right|_{\mathfrak{s}=\mathfrak{s}_0} = \lim_{\mathfrak{s} \to \mathfrak{s}_0} \frac{\Delta^r\left(\check{H}(\mathfrak{s}) - \check{H}(\mathfrak{s}_0)\right)}{\left(\mathfrak{s} - \check{H}_0\right)^r}$$

where $\Delta^r\left(\check{H}(\mathfrak{s}) - \check{H}(\mathfrak{s}_0)\right) = \Gamma(1+r)\left(\check{H}(\mathfrak{s}) - \check{H}(\mathfrak{s}_0)\right)$.

Definition 2.3. *[20] The local fractional integral of the function $\check{H}(\mathfrak{s})$ of order r is defined by*

$$\begin{aligned} {}_aI_b^{(r)}\check{H}(\mathfrak{s}) &= \frac{1}{\Gamma(1+r)}\int_a^b \check{H}(\mathfrak{s})(d\mathfrak{s})^r \\ &= \frac{1}{\Gamma(1+r)}\lim_{\Delta\mathfrak{s}\to 0}\sum_{j=0}^{N-1}\check{H}\left(\mathfrak{s}_j\right)\left(\Delta\mathfrak{s}_j\right)^r, \end{aligned}$$

with $\Delta\mathfrak{s}_j = \mathfrak{s}_{j+1}-\mathfrak{s}_j$ and $\Delta\mathfrak{s} = \max\left\{\Delta\mathfrak{s}_j \mid j=0,1,2,\ldots,N-1\right\}$, where $\left[\mathfrak{s}_j,\mathfrak{s}_{j+1}\right]$, $j=0,\ldots,N-1$ and $\mathfrak{s}_0 = a < \mathfrak{s}_1 < \cdots < \mathfrak{s}_i < \cdots < \mathfrak{s}_{N-1} < \mathfrak{s}_N = b$ is a partition of the interval $[a,b]$.

Here, it follows that ${}_aI_a^{(r)}\check{H}(\mathfrak{s}) = 0$ if $a=b$ and ${}_aI_b^{(r)}\check{H}(\mathfrak{s}) = -{}_bI_a^{(r)}\check{H}(\mathfrak{s})$ if $a<b$. If for any $\mathfrak{s}\in[a,b]$, there exists $I_{\mathfrak{s}}^{(r)}\check{H}(\mathfrak{s})$, then it is denoted by $\check{H}(\mathfrak{s})\in I_{\mathfrak{s}}^{(r)}[a,b]$.

Main Results:

3. A local fractional derivative with respect to another function and their integral

Let $C^1(I) = \{\varrho : I \to \mathbb{R} \mid \varrho$ is differentiable and ϱ' is continuous on $I\}$.

Definition 3.1. *Let $0 < r \leq 1$, $\varrho \in C^1(I)$ and function $\check{H}$. Let ϱ be a strictly increasing function with continuous derivative ϱ' on the interval $]a,b[$. The local fractional derivative with respect to another function ϱ of $\check{H}(\mathfrak{s})$ of order r at $\mathfrak{s}=\mathfrak{s}_0$ is defined by*

$$\check{H}^{(r,\varrho)}\left(\mathfrak{s}_0\right) = \left.\frac{d^{r,\varrho}\check{H}(\mathfrak{s})}{d\mathfrak{s}^{r,\varrho}}\right|_{\mathfrak{s}=\mathfrak{s}_0} = \lim_{\mathfrak{s}\to\mathfrak{s}_0}\frac{\Delta^r\left(\check{H}(\mathfrak{s})-\check{H}\left(\mathfrak{s}_0\right)\right)}{\left(\varrho(\mathfrak{s})-\varrho(\mathfrak{s}_0)\right)^r}.$$

If $\check{H}^{(r,\varrho)}\left(\mathfrak{s}_0\right)$ exists for all $\mathfrak{s}_0$, so $\check{H}$ is r-differentiable.

Remark 3.2. *If $\varrho(\mathfrak{s}) = \mathfrak{s}$, then $\check{H}^{(r,\varrho)}\left(\mathfrak{s}_0\right) = \check{H}^{(r)}\left(\mathfrak{s}_0\right)$.*

Lemma 3.3. *We have*

- $(\mu)^{(r,\varrho)} = 0$.
- $(\frac{(\varrho(\mathfrak{s}))^r}{\Gamma(r+1)})^{(r,\varrho)} = 1$.

Proof. 1. From the definition, we have $(\mu)^{(r,\varrho)} = \lim_{\mathfrak{s}\to\mathfrak{s}_0}\frac{\Delta^r(\mu-\mu)}{\left(\varrho(\mathfrak{s})-\varrho(\mathfrak{s}_0)\right)^r} = 0$.

2. Using the definition, we have $(\frac{(\varrho(\mathfrak{s}))^r}{\Gamma(r+1)})^{(r,\varrho)} = \lim_{\Delta\mathfrak{s}\to 0}\frac{1}{\Gamma(1+r)}\frac{\Gamma(1+r)\left[(\varrho(\mathfrak{s})+\Delta\mathfrak{s})^r-(\varrho(\mathfrak{s}))^r\right]}{(\Delta\mathfrak{s})^r} = 1$.
□

This proves the desired equality.

Lemma 3.4. *We have*

$$(E_r(\mu(\varrho(\mathfrak{s}))^r))^{(r,\varrho)} = \mu E_r(\mu(\varrho(\mathfrak{s}))^r).$$

Proof. Using the definition of the fractional operator, we have

$$
\begin{aligned}
\left(\frac{(\varrho(\mathfrak{s}))^{kr}}{\Gamma(1+kr)}\right)^{(r,\varrho)} &= \lim_{\Delta\mathfrak{s}\to 0}\left\{\frac{\Gamma(r+1)}{\Gamma(kr+1)}\frac{\left[(\varrho(\mathfrak{s})+\Delta\mathfrak{s})^{kr}-(\varrho(\mathfrak{s}))^{kr}\right]}{(\Delta\mathfrak{s})^{r}}\right\}\\
&= \lim_{\Delta\mathfrak{s}\to 0}\left\{\frac{\Gamma(r+1)}{\Gamma(kr+1)}\frac{\left[(\varrho(\mathfrak{s}))^{kr}+\frac{\Gamma(1+kr)}{\Gamma(1+r)\Gamma(1+(k-1)r)}(\varrho(\mathfrak{s}))^{(k-1)r}(\Delta\mathfrak{s})^{r}+\cdots-(\varrho(\mathfrak{s}))^{kr}\right]}{(\Delta\mathfrak{s})^{r}}\right\}\\
&= \lim_{\Delta\mathfrak{s}\to 0}\left\{\frac{\Gamma(r+1)}{\Gamma(1+kr)}\frac{\left[\frac{\Gamma(1+kr)}{\Gamma(1+r)\Gamma(1+(k-1)r)}(\varrho(\mathfrak{s}))^{(k-1)r}(\Delta\mathfrak{s})^{r}\right]}{(\Delta\mathfrak{s})^{r}}\right\}\\
&= \frac{(\varrho(\mathfrak{s}))^{(k-1)r}}{\Gamma((k-1)r+1)}.
\end{aligned}
\tag{2}
$$

In this case, from (2) we have

$$
(E_r(\mu(\varrho(\mathfrak{s}))^r))^{(r,\varrho)} = \left(\sum_{k=0}^{\infty}\frac{\mu^k(\varrho(\mathfrak{s}))^{kr}}{\Gamma(1+kr)}\right)^{(r,\varrho)} = \sum_{k=1}^{\infty}\frac{\mu^k(\varrho(\mathfrak{s}))^{(k-1)r}}{\Gamma((k-1)r+1)},
$$

Therefore, we conclude that

$$
(E_r(\mu(\varrho(\mathfrak{s}))^r))^{(r,\varrho)} = \mu E_r(\mu(\varrho(\mathfrak{s}))^r).
$$

□

Theorem 3.5. *Let $I\subset\mathbb{R}$, $a,b,\mu\in I$ and $\check{H}:I\longrightarrow\mathbb{R}^r$ and $g:I\longrightarrow\mathbb{R}^r$ are r-differentiable. We have:*

1. $(a\check{H}+bg)^{(r,\varrho)} = a\check{H}^{(r,\varrho)}+bg^{(r,\varrho)}$.
2. $(\check{H}g)^{(r,\varrho)} = \check{H}^{(r,\varrho)}g+\check{H}g^{(r,\varrho)}$.
3. $\left(\frac{\check{H}}{g}\right)^{(r,\varrho)} = \frac{\check{H}^{(r,\varrho)}g-g^{(r,\varrho)}\check{H}}{g^2}$.

Proof. Parts (1) using the Definition 3.1. For (2): Now, for fixed $\check{H}_0$,

$$
\begin{aligned}
(\check{H}g)^{(r,\varrho)}(\mathfrak{s}_0) &= \lim_{\mathfrak{s}\to\mathfrak{s}_0}\frac{\Delta^r(\check{H}(\mathfrak{s})g(\mathfrak{s})-\check{H}(\mathfrak{s}_0)g(\mathfrak{s}_0))}{(\varrho(\mathfrak{s})-\varrho(\mathfrak{s}_0))^r}\\
&= \lim_{\mathfrak{s}\to\mathfrak{s}_0}\frac{\Delta^r(\check{H}(\mathfrak{s})g(\mathfrak{s})-\check{H}(\mathfrak{s}_0)g(\mathfrak{s})+\check{H}(\mathfrak{s}_0)g(\mathfrak{s})-\check{H}(\mathfrak{s}_0)g(\mathfrak{s}_0))}{(\varrho(\mathfrak{s})-\varrho(\mathfrak{s}_0))^r}\\
&= \lim_{\mathfrak{s}\to\mathfrak{s}_0}\frac{\Delta^r(\check{H}(\mathfrak{s}_0)(g(\mathfrak{s})-g(\mathfrak{s}_0))+\check{H}(\mathfrak{s})(g(\mathfrak{s})-\check{H}(\mathfrak{s}_0)))}{(\varrho(\mathfrak{s})-\varrho(\mathfrak{s}_0))^r}\\
&= \lim_{\mathfrak{s}\to\mathfrak{s}_0}\frac{\Delta^r(g(\mathfrak{s})(\check{H}(\mathfrak{s})-\check{H}(\mathfrak{s}_0))}{(\varrho(\mathfrak{s})-\varrho(\mathfrak{s}_0))^r}+\frac{\Delta^r(\check{H}(\mathfrak{s}_0)(g(\mathfrak{s})-g(\mathfrak{s}_0)))}{(\varrho(\mathfrak{s})-\varrho(\mathfrak{s}_0))^r}\\
&= \lim_{\mathfrak{s}\to\mathfrak{s}_0}g(\mathfrak{s})\frac{\Delta^r(\check{H}(\mathfrak{s})-\check{H}(\mathfrak{s}_0)}{(\varrho(\mathfrak{s})-\varrho(\mathfrak{s}_0))^r}+\check{H}(\mathfrak{s}_0)\frac{\Delta^r(g(\mathfrak{s})-g(\mathfrak{s}_0))}{(\varrho(\mathfrak{s})-\varrho(\mathfrak{s}_0))^r}\\
&= \check{H}^{(r,\varrho)}(\mathfrak{s})g(\mathfrak{s})+g^{(r,\varrho)}(\mathfrak{s})\check{H}(\mathfrak{s}).
\end{aligned}
$$

For (3): Now, for fixed $\mathfrak{s}_0$,

$$\begin{aligned}
(\frac{\check{H}}{g})^{(r,\varrho)}(\mathfrak{s}_0) &= \lim_{\mathfrak{s}\to\mathfrak{s}_0} \frac{\Delta^r(\frac{\check{H}(\mathfrak{s})}{g(\mathfrak{s})} - \frac{\check{H}(\mathfrak{s}_0)}{g(\mathfrak{s}_0)})}{(\varrho(\mathfrak{s}) - \varrho(\mathfrak{s}_0))^r} \\
&= \lim_{\mathfrak{s}\to\mathfrak{s}_0} \frac{\frac{\check{H}(\mathfrak{s}_0)g(\mathfrak{s}) - g(\mathfrak{s}_0)\check{H}(\mathfrak{s})}{g(\mathfrak{s}_0)g(\mathfrak{s})}}{(\varrho(\mathfrak{s}) - \varrho(\mathfrak{s}_0))^r} \\
&= \lim_{\mathfrak{s}\to\mathfrak{s}_0} \frac{\check{H}(\mathfrak{s})g(\mathfrak{s}) - \check{H}(\mathfrak{s})g(\mathfrak{s}) + \check{H}(\mathfrak{s})g(\mathfrak{s}) - g(\mathfrak{s}_0)\check{H}(\mathfrak{s})}{g(\mathfrak{s}_0)g(\mathfrak{s})((\varrho(\mathfrak{s}) - \varrho(\mathfrak{s}_0))^r)} \\
&= \lim_{\mathfrak{s}\to\mathfrak{s}_0} \frac{\check{H}(\mathfrak{s}_0) - \check{H}(\mathfrak{s})}{(\varrho(\mathfrak{s}) - \varrho(\mathfrak{s}_0))^r} \frac{g(\mathfrak{s})}{g(\mathfrak{s}_0)g(\mathfrak{s})} - \lim_{\check{H}\to\check{H}_0} \frac{\check{H}(\mathfrak{s})}{g(\mathfrak{s}_0)g(\mathfrak{s})} \frac{g(\mathfrak{s}_0) - g(\mathfrak{s})}{(\varrho(\mathfrak{s}) - \varrho(\mathfrak{s}_0))^r}.
\end{aligned}$$

Since, $\check{H}$ and g are r-differentiable, so $\left(\frac{\check{H}}{g}\right)^{(r,\varrho)} = \frac{\check{H}^{(r,\varrho)}g - g^{(r,\varrho)}\check{H}}{g^2}$. □

Definition 3.6. *Let $I \subset \mathbb{R}$ and the continuous function $\check{H} : I \longrightarrow \mathbb{R}^r$. Then the local fractional integral I_r of $\check{H}$ of order $r \in (0,1]$ is defined by:*

$$ {}_aI_b^{r,\varrho}\check{H}(\mathfrak{s}) = \frac{1}{\Gamma(r+1)} \int_a^b \check{H}(\mathfrak{s})(\varrho'(\mathfrak{s})d\mathfrak{s})^r. $$

The following results are easy to obtain.

Proposition 3.7. *If $\check{H}(\mathfrak{s}) \in C_r[a,b]$, then it can be considered as a local fractional integral on the interval $[a,b]$. To make things easier, we can summarize the following rules:*

(a) ${}_aI_b^{(r,\varrho)}\check{H}(\mathfrak{s}) = 0$ *if* $a = b$.

(b) ${}_aI_b^{(r,\varrho)}\check{H}(\mathfrak{s}) = -{}_bI_a^{(r,\varrho)}\check{H}(\mathfrak{s})$ *if* $a < b$.

(c) ${}_aI_b^{(r,\varrho)}\check{H}(\mathfrak{s}) = \check{H}(\mathfrak{s})$ *if* $r = 0$.

Proposition 3.8. *Let's say we have three functions: $\check{H}(\mathfrak{s})$, $\check{H}_1(\mathfrak{s})$, and $\check{H}_2(\mathfrak{s}) \in C_r[a,b]$. Now, we'll discuss the rules for performing local fractional integration on these functions, even if they are not differentiable and are defined on fractal sets. Here are the rules you need to know:*

(a) ${}_aI_b^{(r,\varrho)}\left[\check{H}_1(\mathfrak{s}) + \check{H}_2(\mathfrak{s})\right] = {}_aI_b^{(r,\varrho)}\check{H}_1(\mathfrak{s}) + {}_aI_b^{(r,\varrho)}\check{H}_2(\mathfrak{s})$.

(b) ${}_aI_b^{(r,\varrho)}[C\check{H}(\mathfrak{s})] = C{}_aI_b^{(r,\varrho)}\check{H}(\mathfrak{s})$, *provided a constant C;*

(c) ${}_aI_b^{(r,\varrho)}1 = (\varrho(b) - \varrho(a))^r/\Gamma(1+r)$;

(d) ${}_aI_b^{(r,\varrho)}\check{H}(\mathfrak{s}) \geq 0$, *provided* $\check{H}(\mathfrak{s}) \geq 0$;

(e) $\left|{}_aI_b^{(r,\varrho)}\check{H}(\mathfrak{s})\right| \leq {}_aI_b^{(r,\varrho)}|\check{H}(\mathfrak{s})|$;

(f) ${}_aI_b^{(r,\varrho)}\check{H}(\mathfrak{s}) = {}_aI_c^{(r,\varrho)}\check{H}(\mathfrak{s}) + {}_cI_b^{(r,\varrho)}\check{H}(\mathfrak{s})$, *provided* $a < c < b$;

(g) ${}_aI_b^{(r,\varrho)}\check{H}(\mathfrak{s}) \in \left[\frac{T(\varrho(b)-\varrho(a))^r}{\Gamma(1+r)}, \frac{\Pi(\varrho(b)-\varrho(a))^r}{\Gamma(1+r)}\right]$, *if we know that the highest value of $\check{H}(\mathfrak{s})$ is Π and the lowest value is T.*

• Fractal Mean Value Theorem: Insights into Local Fractional Integrals

Theorem 3.9. *If we have a function $\check{H}(\mathfrak{s}) \in C_r[a,b]$, then there is a point ξ within the interval (a,b) such that the following condition holds:*

$$ {}_aI_b^{(r,\varrho)}\check{H}(\mathfrak{s}) = \check{H}(\xi)\frac{(\varrho(b) - \varrho(a))^r}{\Gamma(1+r)}. $$

Proof. In view of $\check{H}(\mathfrak{s}) \in C_r[a,b]$, we have

$$ {}_aI_b^{(r,\varrho)}\check{H}(\mathfrak{s}) \in \left[\frac{T(\varrho(b) - \varrho(a))^r}{\Gamma(1+r)}, \frac{\Pi(\varrho(b) - \varrho(a))^r}{\Gamma(1+r)}\right], $$

which leads us to

$$\frac{{}_aI_b^{(r,\varrho)}\check{H}(\mathfrak{s})}{\frac{(\varrho(b)-\varrho(a))^r}{\Gamma(1+r)}} \in [\mathrm{T},\Pi].$$

Therefore, for $\xi \in (a,b)$, we have

$$\frac{{}_aI_b^{(r,\varrho)}\check{H}(\mathfrak{s})}{\frac{(\varrho(b)-\varrho(a))^r}{\Gamma(1+r)}} = \check{H}(\xi),$$

which yields the required proof. □

• Local Fractional Integrals: A Newton-Leibniz Perspective

Theorem 3.10. *Suppose that*

$$\Phi^{(r,\varrho)}(\mathfrak{s}) = \check{H}(\mathfrak{s}) \in C_r[a,b].$$

Then,

$${}_aI_b^{(r,\varrho)}\check{H}(\mathfrak{s}) = \Phi(b) - \Phi(a).$$

Proof. Now, we will define the function $\Phi_0(\mathfrak{s}) = {}_aI_{\check{H}}^{(r,\varrho)}\check{H}(\mathfrak{s})$. Thus, we have

$$\frac{\partial^{r,\varrho}}{\partial \check{H}^{r,\varrho}}\left(\Phi_0(\mathfrak{s}) - \Phi(\mathfrak{s})\right) = \frac{\partial^{r,\varrho}}{\partial \check{H}^{r,\varrho}}\Phi_0(\mathfrak{s}) - \frac{\partial^{r,\varrho}}{\partial \check{H}^{r,\varrho}}\Phi(\mathfrak{s}) = \check{H}(\mathfrak{s}) - \check{H}(\mathfrak{s}) = 0. \tag{3}$$

This leads to

$$\Phi_0(\mathfrak{s}) - \Phi(\mathfrak{s}) = C,$$

with C representing a constant, we can use equation (3) to establish the following identity:

$${}_aI_b^{(r,\varrho)}\check{H}(\mathfrak{s}) = \Phi_0(b) - \Phi_0(a) = \Phi(b) - \Phi(a).$$

Thus, the desired result is obtained. □

Theorem 3.11. *Suppose that $D^{(kr,\varrho)}\check{H}(\mathfrak{s}), D^{((k+1)r,\varrho)}\check{H}(\mathfrak{s}) \in C_r(a,b)$. For values of $0 < r < 1$, there exists a point $\mathfrak{s}_0 \in (a,b)$ such that*

$${}_{\mathfrak{s}_0}I_{\mathfrak{s}}^{(kr,\varrho)}\left[D^{(kr,\varrho)}\check{H}(\mathfrak{s})\right] - {}_{\mathfrak{s}_0}I_{\mathfrak{s}}^{((k+1)r,\varrho)}\left[D^{((k+1)r,\varrho)}\check{H}(\mathfrak{s})\right] = D^{(kr,\varrho)}\check{H}(\mathfrak{s}_0)\frac{(\varrho(\mathfrak{s})-\varrho(\mathfrak{s}_0))^{kr}}{\Gamma(1+kr)},$$

where ${}_{\mathfrak{s}_0}I_{\mathfrak{s}}^{(kr,\varrho)}\check{H}(\mathfrak{s}) = \overbrace{{}_{\mathfrak{s}_0}I_{\mathfrak{s}}^{(r,\varrho)}\cdots{}_{\mathfrak{s}_0}I_{\mathfrak{s}}^{(r,\varrho)}}^{k\text{-times}}\check{H}(\mathfrak{s})$ *and* $D^{(kr,\varrho)}\check{H}(\mathfrak{s}) = \overbrace{D^{(r,\varrho)}\cdots D^{(r,\varrho)}}^{k\text{-times}}\check{H}(\mathfrak{s})$.

Proof. We present the formula

$$\begin{aligned}
{}_{\mathfrak{s}_0}I_{\mathfrak{s}}^{((k+1)r,\varrho)}\left[D^{((k+1)r,\varrho)}\check{H}(\mathfrak{s})\right] &= {}_{\mathfrak{s}_0}I_{\mathfrak{s}}^{(kr,\varrho)}\left\{{}_{\mathfrak{s}_0}I_{\mathfrak{s}}^{(r,\varrho)}\left[D^{((k+1)r,\varrho)}\check{H}(\mathfrak{s})\right]\right\}\\
&= {}_{\mathfrak{s}_0}I_{\mathfrak{s}}^{(kr,\varrho)}\left\{D^{(kr,\varrho)}\check{H}(\mathfrak{s}) - D^{(kr,\varrho)}\check{H}(\mathfrak{s}_0)\right\}\\
&= {}_{\mathfrak{s}_0}I_{\mathfrak{s}}^{(kr,\varrho)}\left[D^{(kr,\varrho)}\check{H}(\mathfrak{s})\right] - {}_{\mathfrak{s}_0}I_{\mathfrak{s}}^{(kr,\varrho)}\left[D^{(kr,\varrho)}\check{H}(\mathfrak{s}_0)\right].
\end{aligned}$$

Adopting the formula

$$\begin{aligned}
{}_{\mathfrak{s}_0}I_{\mathfrak{s}}^{(kr,\varrho)}\left[D^{(kr,\varrho)}\check{H}(\mathfrak{s}_0)\right] &= D^{(kr,\varrho)}\check{H}(\mathfrak{s}_0)\,{}_{\mathfrak{s}_0}I_{\mathfrak{s}}^{(kr,\varrho)}1\\
&= D^{(kr,\varrho)}\check{H}(\mathfrak{s}_0)\,{}_{\mathfrak{s}_0}I_{\mathfrak{s}}^{((k-1)r,\varrho)}\frac{(\varrho(\mathfrak{s})-\varrho(\mathfrak{s}_0))^{r}}{\Gamma(1+r)}\\
&= D^{(kr,\varrho)}\check{H}(\mathfrak{s}_0)\frac{(\varrho(\mathfrak{s})-\varrho(\mathfrak{s}_0))^{kr}}{\Gamma(1+kr)},
\end{aligned}$$

there is

$$ {}_{\mathfrak{s}_0}I_{\mathfrak{s}}^{(kr,\varrho)}\left[D^{(kr,\varrho)}\check{H}(\mathfrak{s})\right] - {}_{\mathfrak{s}_0}I_{\mathfrak{s}}^{((k+1)r,\varrho)}\left[D^{((k+1)r,\varrho)}\check{H}(\mathfrak{s})\right] = D^{(kr,\varrho)}\check{H}\left(\mathfrak{s}_0\right)\frac{\left(\varrho(\mathfrak{s})-\varrho(\mathfrak{s}_0)\right)^{kr}}{\Gamma(1+kr)}. $$

□

• Generalized local fractional Taylor theorem

Lemma 3.12. *Let's say we have a function $\check{H}^{(k+1)r,\varrho}(\mathfrak{s}) \in C_r(I)$, where I is an interval in the real numbers. Here, k takes values from 0 to n, and $0 < r \leq 1$. Suppose $\mathfrak{s}_0 \in [a,b]$. Now, for any value of t within the interval I, we can guarantee the existence of at least one point ξ. This point ξ lies between $\mathfrak{s}$ and $\mathfrak{s}_0$, and satisfies the following condition:*

$$ \check{H}(\mathfrak{s}) = \sum_{k=0}^{n}\frac{\check{H}^{(kr,\varrho)}\left(\mathfrak{s}_0\right)}{\Gamma(1+kr)}\left(\varrho(\mathfrak{s})-\varrho(\mathfrak{s}_0)\right)^{kr} + \frac{\check{H}^{((n+1)r,\varrho)}(\xi)}{\Gamma(1+(n+1)a)}\left(\varrho(\mathfrak{s})-\varrho(\mathfrak{s}_0)\right)^{(n+1)r}. $$

Proof. By making use of

$$ {}_{\mathfrak{s}_0}I_{\mathfrak{s}}^{(kr,\varrho)}\left[D^{(kr,\varrho)}\check{H}(\mathfrak{s})\right] - {}_{\mathfrak{s}_0}I_{\mathfrak{s}}^{((k+1)r,\varrho)}\left[D^{((k+1)r,\varrho)}\check{H}(\mathfrak{s})\right] = D^{(kr,\varrho)}\check{H}\left(\mathfrak{s}_0\right)\frac{\left(\varrho(\mathfrak{s})-\varrho(\mathfrak{s}_0)\right)^{kr}}{\Gamma(1+kr)}, $$

we conclude that

$$ \begin{aligned} &\sum_{k=0}^{n}\left\{{}_{\mathfrak{s}_0}I_{\mathfrak{s}}^{(kr,\varrho)}\left[D^{(kr,\varrho)}\check{H}(\mathfrak{s})\right] - {}_{\mathfrak{s}_0}I_{\mathfrak{s}}^{((k+1)r,\varrho)}\left[D^{((k+1)r,\varrho)}\check{H}(\mathfrak{s})\right]\right\} \\ &= \check{H}(\mathfrak{s}) - {\mathfrak{s}_0}I_{\mathfrak{s}}^{((n+1)r,\varrho)}\left[D^{((n+1)r,\varrho)}\check{H}(\mathfrak{s})\right] \\ &= \sum_{k=0}^{n}\left\{D^{(kr,\varrho)}\check{H}\left(\mathfrak{s}_0\right)\frac{\left(\varrho(\mathfrak{s})-\varrho(\mathfrak{s}_0)\right)^{kr}}{\Gamma(1+kr)}\right\}. \end{aligned} $$

Thus, we show that

$$ \begin{aligned} {\mathfrak{s}_0}I_{\mathfrak{s}}^{((n+1)r,\varrho)}\left[D^{((n+1)r,\varrho)}\check{H}(\mathfrak{s})\right] &= {\mathfrak{s}_0}I_{\mathfrak{s}}^{(r,\varrho)}\left\{{\mathfrak{s}_0}I_{\mathfrak{s}}^{(nr,\varrho)}\left[D^{((n+1)r,\varrho)}\check{H}(\mathfrak{s})\right]\right\} \\ &= D^{((n+1)r,\varrho)}\check{H}(\xi)_{\mathfrak{s}_0}I_{\mathfrak{s}}^{((n+1)r,\varrho)}1 \\ &= D^{((n+1)r,\varrho)}\check{H}(\xi)\frac{\left(\varrho(\mathfrak{s})-\varrho(\mathfrak{s}_0)\right)^{(n+1)r}}{\Gamma(1+(n+1)r)}, \end{aligned} $$

where $\mathfrak{s}_0 < \xi < t, \forall t \in (a,b)$. Therefore, we have proved the result. □

4. Generalized convex functions

Looking at it analytically, we can define it as follows.

Definition 4.1. *Consider a function $\check{H} : I \subseteq \mathbb{R} \to \mathbb{R}^r$. Now, for any two points $\mathfrak{s}_1$ and $\mathfrak{s}_2 \in I$, and for any value of $\mu \in [0,1]$, if the following inequality holds:*

$$ \check{H}\left(\mu\mathfrak{s}_1 + (1-\mu)\mathfrak{s}_2\right) \leq \mu^r\check{H}\left(\mathfrak{s}_1\right) + (1-\mu)^r\check{H}\left(\mathfrak{s}_2\right). $$

In that case, we say that $\check{H}$ is a generalized convex function on the interval I.

Definition 4.2. *Consider a function $\check{H} : I \to \mathbb{R}^r$. Now, for any two distinct points $\mathfrak{s}_1$ and $\mathfrak{s}_2 \in I$, and for any value of $\mu \in [0,1]$, if the following inequality holds:*

$$ \check{H}\left(\mu\mathfrak{s}_1 + (1-\mu)\mathfrak{s}_2\right) < \mu^r\check{H}\left(\mathfrak{s}_1\right) + (1-\mu)^r\check{H}\left(\mathfrak{s}_2\right). $$

Then we refer to $\check{H}$ as a generalized strictly convex function on the interval $I \subseteq \mathbb{R}$.

Definition 4.3. *A non-negative and non-zero function* $\varrho : J \to \mathbb{R}$ *defines the conditions for considering* $\check{H} : I \to \mathbb{R}$ *as a generalized* ϱ*-convex function or a member of the class* $SX(I)$*. For* $\check{H}$ *to qualify, it must satisfy two requirements:* $\check{H}$ *is non-negative and for all* $\mathfrak{s}, z \in I$ *and* $\mu \in]0,1[$,

$$\check{H}((1-\mu)\mathfrak{s} + \mu z) \leq \varrho(1-\mu)^r \check{H}(\mathfrak{s}) + \varrho(\mu)^r \check{H}(z). \tag{4}$$

If this inequality is reversed, then $\check{H}$ *is said to be generalized* ϱ*-concave.*

Definition 4.4. *Suppose* $\varrho : J \to \mathbb{R}$ *be a function that is both non-negative and non-zero. We define a function* $\check{H} : I \to \mathbb{R}^r$ *to be a generalized* (p,ϱ)*-convex function or a member of the class* $ghx(\varrho, p, I)$ *if it fulfills the properties of being non-negative and*

$$\check{H}\left(\left[(1-\mu)\mathfrak{s}^p + \mu z^p\right]^{\frac{1}{p}}\right) \leq \varrho(1-\mu)^r \check{H}(\mathfrak{s}) + \varrho(\mu)^r \check{H}(z) \tag{5}$$

for all $\mathfrak{s}, z \in I$ *and consider a value of* $\mu \in (0,1)$*. In a similar manner, if we reverse the inequality sign in equation* (5)*, we refer to the function* $\check{H}$ *as a generalized* (p,ϱ)*-concave function or as a member of the class* $ghv(\varrho, p, I)$*.*

If $r = 1$ we obtain in [29].

Definition 4.5. *[30] Let* $\varrho : J \to \mathbb{R}$*. If*

$$\varrho(\mathfrak{s})\varrho(z) \leq \varrho(\mathfrak{s}z), \tag{6}$$

for all $\mathfrak{s}, z \in J$*, then* ϱ *is said to be a super-multiplicative function. When the inequality in equation (6) is reversed, we classify the function* ϱ *as a sub-multiplicative function. On the other hand, if equality is achieved in equation (6), we refer to* ϱ *as a multiplicative function.*

Definition 4.6. *[30] Consider a function* $\varrho : J \to \mathbb{R}$*. If for every* $\mathfrak{s}, z \in J$*, the following condition holds true:*

$$\varrho(\mathfrak{s}) + \varrho(z) \leq \varrho(\mathfrak{s} + z), \tag{7}$$

then we classify it as a super-additive function. Conversely, if the inequality (7) is reversed, we refer to ϱ *as a sub-additive function. Furthermore, if equality is satisfied in (7), we describe* ϱ *as an additive function.*

Example 4.7. *Let* $\varrho : I \to (0,\infty)$ *be defined by* $\varrho(\mathfrak{s}) = \mathfrak{s}^k, \check{H} > 0$*. Then* ϱ *is*

1. *additive if* $k = 1$,
2. *sub-additive if* $k \in (-\infty, -1] \cup [0, 1)$,
3. *super-additive if* $k \in (-1, 0) \cup (1, \infty)$.

Let $\varrho : [1, +\infty) \mapsto \mathbb{R}^+$ *be defined by* $\varrho(\mathfrak{s}) = \mathfrak{s}^3 - \mathfrak{s}^2 + \mathfrak{s}$*. We have*

1. $\varrho(\mathfrak{s}z) - \varrho(\mathfrak{s})\varrho(z) = \check{H}(\mathfrak{s} + z)(1-\mathfrak{s})(1-z) \geq 0$
2. $\varrho(\mathfrak{s} + z) - \varrho(\mathfrak{s}) - \varrho(z) = \mathfrak{s}z(\mathfrak{s} + z + (\mathfrak{s}-1) + (z-1)) \geq 0.$

Then ϱ *is a super-multiplicative and super-additive function.*

The definitions provided indicate that a generalized strictly convex function falls under the category of generalized convex functions. However, it is important to note that the opposite is not necessarily true. If we reverse these two inequalities, we categorize the function as a generalized concave function or a generalized strictly concave function, respectively.

Let's consider two simple examples of generalized strictly convex functions:

(1) $\check{H}(\mathfrak{s}) = \mathfrak{s}^{rp}, \mathfrak{s} \geq 0, p > 1$

(2) $E_r(\mathfrak{s}^r) = \check{H}(\mathfrak{s}), \mathfrak{s} \in R$, where $E_r(\mathfrak{s}^r) = \sum_{k=0}^{\infty} \frac{\mathfrak{s}^{rk}}{\Gamma(1+kr)}$ is the Mittag-Leffer function.

It's worth noting that the linear function $\check{H}(\mathfrak{s}) = a^r\mathfrak{s}^r + b^r$, where $\mathfrak{s} \in \mathbb{R}$, is an example of a function that is both generalized convex and generalized concave.

We will primarily focus on convexity, as it is important to note that a function $\check{H}$ is concave if and only if the function $-\check{H}$ is convex. As a result, any findings or conclusions regarding convex functions can be easily translated into statements about concave functions.

Next, we will examine the characteristics and properties of generalized convex functions.

Theorem 4.8. *Let $\check{H} : I \to \mathbb{R}^r$ and let ϱ super-additive function and super-multiplicative function. A function $\check{H}$ is considered a generalized convex function if and only if the inequality*

$$\frac{\check{H}(\mathfrak{s}_1) - \check{H}(\mathfrak{s}_2)}{(\varrho(\mathfrak{s}_1) - \varrho(\mathfrak{s}_2))^r} \le \frac{\check{H}(\mathfrak{s}_3) - \check{H}(\mathfrak{s}_2)}{(\varrho(\mathfrak{s}_3) - \varrho(\mathfrak{s}_2))^r},$$

holds, for any $\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3 \in I$ with $\mathfrak{s}_1 < \mathfrak{s}_2 < \mathfrak{s}_3$.

Proof. In fact, take $\mu = \frac{\mathfrak{s}_3 - \mathfrak{s}_2}{\mathfrak{s}_3 - \mathfrak{s}_1}$, then $\mathfrak{s}_2 = \mu\mathfrak{s}_1 + (1-\mu)\mathfrak{s}_3$. And by the generalized convexity of $\check{H}$, we get

$$\begin{aligned}\check{H}(\mathfrak{s}_2) &= \check{H}(\mu\mathfrak{s}_1 + (1-\mu)\mathfrak{s}_3) \\ &\le \varrho(\mu)^r \check{H}(\mathfrak{s}_1) + \varrho((1-\mu))^r \check{H}(\mathfrak{s}_3) \\ &= \left(\varrho\left(\frac{\mathfrak{s}_3 - \mathfrak{s}_2}{\mathfrak{s}_3 - \mathfrak{s}_1}\right)\right)^r \check{H}(\mathfrak{s}_1) + \left(\varrho\left(\frac{\mathfrak{s}_2 - \mathfrak{s}_1}{\mathfrak{s}_3 - \mathfrak{s}_1}\right)\right)^r \check{H}(\mathfrak{s}_3).\end{aligned}$$

Since ϱ super-additive and super-multiplicative function we get

$$\check{H}(\mathfrak{s}_2) \le \frac{(\varrho(\mathfrak{s}_3) - \varrho(\mathfrak{s}_2))^r}{(\varrho(\mathfrak{s}_3) - \varrho(\mathfrak{s}_1))^r}\check{H}(\mathfrak{s}_1) + \frac{(\varrho(\mathfrak{s}_2) - \varrho(\mathfrak{s}_1))^r}{(\varrho(\mathfrak{s}_3) - \varrho(\mathfrak{s}_1))^r}\check{H}(\mathfrak{s}_3).$$

From the above formula, it is easy to see that

$$\frac{\check{H}(\mathfrak{s}_1) - \check{H}(\mathfrak{s}_2)}{(\varrho(\mathfrak{s}_1) - \varrho(\mathfrak{s}_2))^r} \le \frac{\check{H}(\mathfrak{s}_3) - \check{H}(\mathfrak{s}_2)}{(\varrho(\mathfrak{s}_3) - \varrho(\mathfrak{s}_2))^r}.$$

Reversely, for any two points $\mathfrak{s}_1, \mathfrak{s}_3\, (\mathfrak{s}_1 < \mathfrak{s}_3)$ on $I \subseteq R$, we take $\mathfrak{s}_2 = \mu\mathfrak{s}_1 + (1-\mu)\mathfrak{s}_3$ for $\mu \in (0,1)$. Then $\mathfrak{s}_1 < \mathfrak{s}_2 < \mathfrak{s}_3$ and $\mu = \frac{\mathfrak{s}_3 - \mathfrak{s}_2}{\mathfrak{s}_3 - \mathfrak{s}_1}$. Using the above inverse process, we have

$$\check{H}(\mu\mathfrak{s}_1 + (1-\mu)\mathfrak{s}_3) \le \varrho(\mu)^r \check{H}(\mathfrak{s}_1) + (\varrho(1-\mu))^r \check{H}(\mathfrak{s}_3).$$

Thus, if $\check{H}$ satisfies this inequality, it is a convex function on $I \subseteq \mathbb{R}$. Similarly, it can be demonstrated that $\check{H}$ is a generalized convex function on $I \subseteq \mathbb{R}$ if and only if

$$\frac{\check{H}(\mathfrak{s}_2) - \check{H}(\mathfrak{s}_1)}{(\varrho(\mathfrak{s}_2) - \varrho(\mathfrak{s}_1))^r} \le \frac{\check{H}(\mathfrak{s}_3) - \check{H}(\mathfrak{s}_1)}{(\varrho(\mathfrak{s}_3) - \varrho(\mathfrak{s}_1))^r} \le \frac{\check{H}(\mathfrak{s}_3) - \check{H}(\mathfrak{s}_2)}{(\varrho(\mathfrak{s}_3) - \varrho(\mathfrak{s}_2))^r},$$

for any $\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3 \in I$ with $\mathfrak{s}_1 < \mathfrak{s}_2 < \mathfrak{s}_3$. □

Theorem 4.9. *Let $\check{H} \in D_r(I)$ and ϱ super-additive, super-multiplicative function, the following conditions hold the same meaning:*

(1) $\check{H}$ exhibits generalized ϱ-convexity over the interval I.

(2) $\check{H}^{(r,\varrho)}$ displays increasing behavior across the interval I.

(3) For any $\mathfrak{s}_1, \mathfrak{s}_2 \in I$,

$$\check{H}(\mathfrak{s}_2) \ge \check{H}(\mathfrak{s}_1) + \frac{\check{H}^{(r,\varrho)}(\mathfrak{s}_1)}{\Gamma(1+r)}(\mathfrak{s}_2 - \mathfrak{s}_1)^r.$$

Proof. $(1 \to 2)$ Consider $\mathfrak{s}_1$ and $\mathfrak{s}_2$ as elements of I with $\mathfrak{s}_1 < \mathfrak{s}_2$. Let $h > 0$ be a positive value small enough such that $\mathfrak{s}_1 - h$ and $\mathfrak{s}_2 + h$ also belong to I. Since $\mathfrak{s}_1 - h < \mathfrak{s}_1 < \mathfrak{s}_2 < \mathfrak{s}_2 + h$, we can apply Theorem 4.8 to obtain

$$\Gamma(1+r)\frac{\check{H}(\mathfrak{s}_1) - \check{H}(\mathfrak{s}_1 - h)}{(\varrho(\mathfrak{s}_1) - \varrho(\mathfrak{s}_1 - h))^r} \le \Gamma(1+r)\frac{\check{H}(\mathfrak{s}_2) - \check{H}(\mathfrak{s}_1)}{(\varrho(\mathfrak{s}_2) - \varrho(\mathfrak{s}_1))^r} \le \Gamma(1+r)\frac{\check{H}(\mathfrak{s}_2 + h) - \check{H}(\mathfrak{s}_2)}{(\varrho(\mathfrak{s}_2 + h) - \varrho(\mathfrak{s}_2))^r}.$$

Since ϱ super-additive, super-multiplicative function, we have

$$\Gamma(1+r)\frac{\check{H}(\mathfrak{s}_1) - \check{H}(\mathfrak{s}_1 - h)}{\varrho(h)^r} \le \Gamma(1+r)\frac{\check{H}(\mathfrak{s}_2) - \check{H}(\mathfrak{s}_1)}{(\varrho(\mathfrak{s}_2) - \varrho(\mathfrak{s}_1))^r} \le \Gamma(1+r)\frac{\check{H}(\mathfrak{s}_2 + h) - \check{H}(\mathfrak{s}_2)}{\varrho(h)^r}.$$

As, $f \in D_a(I)$, we can consider $\varrho(h) \to 0^+$. Consequently, we can deduce that

$$\check{H}^{(r,\varrho)}(\mathfrak{s}_1) \leq \Gamma(1+r)\frac{\check{H}(\mathfrak{s}_2) - \check{H}(\mathfrak{s}_1)}{(\varrho(\mathfrak{s}_2) - \varrho(\mathfrak{s}_1))^r} \leq \check{H}^{(r,\varrho)}(\mathfrak{s}_2).$$

So, $\check{H}^{(r,\varrho)}$ is increasing in I.

($2 \to 3$) Let $\mathfrak{s}_1$ and $\mathfrak{s}_2$ be elements of I, with $\mathfrak{s}_1 < \mathfrak{s}_2$. Since $\check{H}^{(r,\varrho)}$ is an increasing function in the interval I, we can utilize the generalized local fractional Taylor theorem to obtain the following expression:

$$\check{H}(\mathfrak{s}_2) - \check{H}(\mathfrak{s}_1) = \frac{\check{H}^{(r,\varrho)}(\xi)}{\Gamma(1+r)}(\varrho(\mathfrak{s}_2) - \varrho(\mathfrak{s}_1))^r \geq \frac{\check{H}^{(r,\varrho)}(\mathfrak{s}_1)}{\Gamma(1+r)}(\varrho(\mathfrak{s}_2) - \varrho(\mathfrak{s}_1))^r,$$

where $\xi \in (\mathfrak{s}_1, \mathfrak{s}_2)$. That is to say

$$\check{H}(\mathfrak{s}_2) \geq \check{H}(\mathfrak{s}_1) + \frac{\check{H}^{(r,\varrho)}(\mathfrak{s}_1)}{\Gamma(1+r)}(\varrho(\mathfrak{s}_2) - \varrho(\mathfrak{s}_1))^r.$$

($3 \to 1$) For any $\mathfrak{s}_1, \mathfrak{s}_2 \in I$, we let $\mathfrak{s}_3 = \mu\mathfrak{s}_1 + (1-\mu)\mathfrak{s}_2$, where $0 < \mu < 1$. It is easy to see that $\mathfrak{s}_1 - \mathfrak{s}_3 = (1-\mu)(\mathfrak{s}_1 - \mathfrak{s}_2)$ and $\mathfrak{s}_2 - \mathfrak{s}_3 = \mu(\mathfrak{s}_2 - \mathfrak{s}_1)$. Then from the third condition, we have

$$\check{H}(\mathfrak{s}_1) \geq \check{H}(\mathfrak{s}_3) + \frac{\check{H}^{(r,\varrho)}(\mathfrak{s}_3)}{\Gamma(1+r)}(\varrho(\mathfrak{s}_1) - \varrho(\mathfrak{s}_3))^r \geq \check{H}(\mathfrak{s}_3) + (1-\mu)^r\frac{\check{H}^{(r,\varrho)}(\mathfrak{s}_3)}{\Gamma(1+r)}(\varrho(\mathfrak{s}_1) - \varrho(\mathfrak{s}_2))^r,$$

and

$$\check{H}(\mathfrak{s}_2) \geq \check{H}(\mathfrak{s}_3) + \frac{\check{H}^{(r,\varrho)}(\mathfrak{s}_3)}{\Gamma(1+r)}(\varrho(\mathfrak{s}_2) - \varrho(\mathfrak{s}_3))^r \geq \check{H}(\mathfrak{s}_3) + \mu^r\frac{\check{H}^{(r,\varrho)}(\mathfrak{s}_3)}{\Gamma(1+r)}(\varrho(\mathfrak{s}_2) - \varrho(\mathfrak{s}_1))^r.$$

At the above two formulas, multiply $\varrho(\mu)^r$ and $\varrho((1-\mu))^r$, respectively, then we obtain

$$\varrho(\mu)^r\check{H}(\mathfrak{s}_1) + (\varrho(1-\mu))^r\check{H}(\mathfrak{s}_2) \geq \check{H}(\mathfrak{s}_3) = \check{H}(\mu\mathfrak{s}_1 + (1-\mu)\mathfrak{s}_2).$$

Therefore, we can establish that $\check{H}$ is a generalized ϱ-convex function on I. □

Corollary 4.10. *Consider a function $\check{H}$ in $D_{2r}(a,b)$. If $\check{H}$ is a generalized ϱ-convex function (or a generalized ϱ-concave function), then we have the following equivalence:*

$$\check{H}^{(2r,\varrho)}(\mathfrak{s}) \geq 0 \left(\text{or } \check{H}^{(2r,\varrho)}(\mathfrak{s}) \leq 0 \right),$$

for any $\check{H} \in (a,b)$.

• Generalized Jensen's inequality

Theorem 4.11. *Let's consider $\check{H}$ as a generalized ϱ-convex function defined on the interval $[a,b]$. Then, for any $\mathfrak{s}_i$ belonging to $[a,b]$ and μ_i within the range $[0,1]$ $(i = 1,2,\ldots,n)$ with $\sum_{i=1}^n \mu_i = 1$, the following relation applies:*

$$\check{H}\left(\sum_{i=1}^n \mu_i\mathfrak{s}_i\right) \leq \sum_{i=1}^n \varrho(\mu_i)^r\check{H}(\mathfrak{s}_i). \tag{8}$$

Proof. For $n = 2$, the inequality is evidently valid. Let's assume that the inequality holds for $n = k$. Then, for any $\mathfrak{s}_1, \mathfrak{s}_2, \ldots, \mathfrak{s}_k \in [a,b]$ and $\gamma_i > 0$ $(i = 1,2,\ldots,k)$ with $\sum_{i=1}^k \gamma_i = 1$, we can state that:

$$\check{H}\left(\sum_{i=1}^k \gamma_i\mathfrak{s}_i\right) \leq \sum_{i=1}^k \varrho(\gamma_i)^r\check{H}(\mathfrak{s}_i).$$

If $\mathfrak{s}_1, \mathfrak{s}_2, \ldots, \mathfrak{s}_k, \mathfrak{s}_{(k+1)}$ lie within the interval $[a, b]$, and $\mu_i > 0$ for $i = 1, 2, \ldots, k+1$ with $\sum_{i=1}^{k+1} \mu_i = 1$, then one can define $\gamma_i = \frac{\mu_i}{1-\mu_{k+1}}, i = 1, 2, \ldots, k$. It's evident that $\sum_{i=1}^{k} \gamma_i = 1$ is straightforward to observe. Thus,

$$\begin{aligned}
&\check{H}\left(\mu_1\mathfrak{s}_1 + \mu_2\mathfrak{s}_2 + \ldots + \mu_k\check{H}_k + \mu_{k+1}\check{H}_{k+1}\right) \\
&= \check{H}\left((1-\mu_{k+1})\frac{\mu_1\mathfrak{s}_1 + \mu_2\mathfrak{s}_2 + \ldots + \mu_k\check{H}_k}{1-\mu_{k+1}} + \mu_{k+1}\check{H}_{k+1}\right) \\
&\leq (\varrho(1-\mu_{k+1}))^r \check{H}\left(\gamma_1\mathfrak{s}_1 + \gamma_2\mathfrak{s}_2 + \ldots + \gamma_k\check{H}_k\right) + \varrho(\mu_{k+1})^r \check{H}(\mathfrak{s}_{k+1}) \\
&\leq (\varrho(1-\mu_{k+1}))^r \left[\varrho(\gamma_1)^r\check{H}(\mathfrak{s}_1) + \varrho(\gamma_2)^r\check{H}(\mathfrak{s}_2) + \ldots + \varrho(\gamma_k)^r\check{H}(\mathfrak{s}_k)\right] + \varrho(\mu_{k+1})^r\check{H}(\mathfrak{s}_{k+1}) \\
&= (\varrho(1-\mu_{k+1}))^r \left[\left(\varrho(\frac{\mu_1}{1-\mu_{k+1}})\right)^r \check{H}(\mathfrak{s}_1) + \left(\varrho(\frac{\mu_2}{1-\mu_{k+1}})\right)^r \check{H}(\mathfrak{s}_2) + \ldots \right. \\
&\left. + \left(\varrho(\frac{\mu_k}{1-\mu_{k+1}})\right)^r \check{H}(\mathfrak{s}_k)\right] + \varrho(\mu_{k+1})^r\check{H}(\mathfrak{s}_{k+1}) \\
&= \sum_{i=1}^{k} \varrho(\mu_i)^r \check{H}(\mathfrak{s}_i).
\end{aligned}$$

Therefore, the proof of Theorem 4.11 is established through mathematical induction. □

Remark 4.12. *For $\varrho(\mathfrak{s}) = \mathfrak{s}$, inequality (8) becomes the classical Jensen's inequality.*

Corollary 4.13. *Let $\check{H} \in D_{2r,\varrho}[a, b]$ and $\check{H}^{(2r,\varrho)}(\mathfrak{s}) \geq 0$ for any $\check{H} \in [a, b]$. Then for any $\mathfrak{s}_i \in [a, b]$ and $\mu_i \in [0, 1]$($i = 1, 2, \ldots, n$) with $\sum_{i=1}^{n} \mu_i = 1$, we have*

$$\check{H}\left(\sum_{i=1}^{n} \mu_i\mathfrak{s}_i\right) \leq \sum_{i=1}^{n} \varrho(\mu_i)^r \check{H}(\mathfrak{s}_i).$$

Utilizing the generalized Jensen's inequality alongside the convexity of functions enables the derivation of various integral inequalities.

In the paper by Yang [20], the generalized Cauchy-Schwarz's inequality was established by estimating $a^{\frac{r}{p}} b^{\frac{r}{q}} \leq \frac{a^r}{p}\frac{b^r}{q}$, where $a^r, b^r > 0, p, q \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Employing the generalized Jensen's inequality offers a clearer path to demonstrating the generalized Cauchy-Schwarz's inequality.

- **The extended version of the Cauchy-Schwarz inequality**

Corollary 4.14. *Let $|a_k| > 0, |b_k| > 0, k = 1, 2, \ldots, n$, and let ϱ super-multiplicative, super-additive function. Then we have*

$$\sum_{k=1}^{n} \varrho(|a_k|)^r \varrho(|b_k|)^r \leq \left(\sum_{k=1}^{n} \varrho(|a_k|)^{2r}\right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} \varrho(|b_k|)^{2r}\right)^{\frac{1}{2}}.$$

Proof. Take $\check{H}(\mathfrak{s}) = \varrho(\mathfrak{s})^{2r}$. It is easy to see that $\check{H}^{(2r,\varrho)}(\mathfrak{s}) \geq 0$ for any $\check{H} \in (a, b)$. Take

$$\mu_k = \frac{|b_k|^2}{\sum_{k=1}^{n} |b_k|^2}, \mathfrak{s}_k = \frac{|a_k|}{|b_k|}. \text{ Then } 0 \leq \mu_k \leq 1 \quad (k = 1, 2, \ldots, n) \text{ with } \sum_{k=1}^{n} \mu_k = 1.$$

Therefore, according to Jensen's inequality $\check{H}(\sum_{k=1}^{n} \mu_k\mathfrak{s}_k) \leq \sum_{k=1}^{n} \varrho(\mu_k)^{(r)}\check{H}(\mathfrak{s}_k)$, we have

$$\left[\varrho(\sum_{k=1}^{n} \frac{|b_k|^2}{\sum_{k=1}^{n} |b_k|^2}\frac{|a_k|}{|b_k|})\right]^{2r} \leq \sum_{k=1}^{n} \left[\varrho(\frac{|b_k|^2}{\sum_{k=1}^{n} |b_k|^2})\right]^r \left[\varrho(\frac{|a_k|}{|b_k|})\right]^{2r}.$$

Since ϱ super-multiplicative function, we have

$$\left[\varrho(\sum_{k=1}^{n}\frac{|b_k|\,|a_k|}{\sum_{j=1}^{n}\left|b_j\right|^2})\right]^{2r}\leq\sum_{k=1}^{n}\left[\varrho(\frac{|b_k|^2}{\sum_{j=1}^{n}\left|b_j\right|^2}\frac{|a_k|^2}{|b_k|^2})\right]^{r}.$$

So

$$\left[\varrho(\sum_{k=1}^{n}\frac{|b_k|\,|a_k|}{\sum_{j=1}^{n}\left|b_j\right|^2})\right]^{2r}\leq\sum_{k=1}^{n}\left[\varrho(\frac{|a_k|^2}{\sum_{j=1}^{n}\left|b_j\right|^2}\right]^{r}.$$

Since, ϱ super-additive function, we have

$$\left[\sum_{k=1}^{n}\varrho(\frac{|b_k|\,|a_k|}{\sum_{j=1}^{n}\left|b_j\right|^2})\right]^{2r}\leq\sum_{k=1}^{n}\left[\varrho(\frac{|a_k|^2}{\sum_{j=1}^{n}\left|b_j\right|^2}\right]^{r}.$$

The formula above simplifies to

$$\left[\sum_{k=1}^{n}\varrho(|b_k|\,|a_k|)\varrho(\frac{1}{\sum_{j=1}^{n}\left|b_j\right|^2})\right]^{2r}\leq\sum_{k=1}^{n}\left[\frac{\varrho(|a_k|^2)}{\varrho(\sum_{j=1}^{n}\left|b_j\right|^2)}\right]^{r}.$$

This indicates that

$$\left[\sum_{k=1}^{n}\varrho(|b_k||a_k|)\right]^{2r}\leq\sum_{k=1}^{n}\varrho(|a_k|)^{2r}\sum_{k=1}^{n}\varrho(|b_k|)^{2r}.$$

Thus, we have

$$\sum_{k=1}^{n}\varrho(|a_k|)^{r}\varrho(|b_k|)^{r}\leq\left(\sum_{k=1}^{n}\varrho(|a_k|)^{2r}\right)^{\frac{1}{2}}\left(\sum_{k=1}^{n}\varrho(|b_k|)^{2r}\right)^{\frac{1}{2}}.$$

□

Theorem 4.15. *(Generalized Hermite-Hadmard's inequality) Consider $\check{H}(\mathfrak{s})\in I_{\mathfrak{s}}^{(r)}[a,b]$, a generalized ϱ-convex function on the interval $[a,b]$, where $a<b$. Assume ϱ is both super-multiplicative and super-additive. Then*

$$\check{H}\left(\frac{b+a}{2}\right)\leq\frac{\Gamma(1+r)}{(\varrho(b-a))^{r}}\,{}_{a}I_{b}^{(r,\varrho)}\check{H}(\mathfrak{s})\leq\varrho(\frac{1}{2})^{r}(\check{H}(b)+\check{H}(a)).$$

Proof. Let $\mathfrak{s}=a+b-y$. Then

$$\int_{a}^{\frac{b+a}{2}}\check{H}(\mathfrak{s})(\varrho'(\mathfrak{s})d\mathfrak{s})^{r}=\int_{\frac{b+a}{2}}^{b}\check{H}(b+a-y)(\varrho'(y)dy)^{r}.$$

Furthermore, when $\mathfrak{s}\in\left[\frac{b+a}{2},b\right]$, $a+b-\mathfrak{s}\in\left[a,\frac{b+a}{2}\right]$. By the hypotheses of convexity of $\check{H}$, we have

$$\check{H}(b+a-\mathfrak{s})+\check{H}(\mathfrak{s})\geq\frac{1}{\varrho(\frac{1}{2})^{r}}\check{H}\left(\frac{b+a}{2}\right).$$

Thus,

$$\begin{aligned}
&\int_a^b \check{H}(\mathfrak{s})(\varrho'(\mathfrak{s})d\mathfrak{s})^r\\
&= \int_a^{\frac{b+a}{2}} \check{H}(\mathfrak{s})(\varrho'(\mathfrak{s})d\mathfrak{s})^r + \int_{\frac{b+a}{2}}^b \check{H}(\mathfrak{s})(\varrho'(\mathfrak{s})d\mathfrak{s})^r\\
&= \int_{\frac{b+a}{2}}^b [\check{H}(b+a-\mathfrak{s}) + \check{H}(\mathfrak{s})](\varrho'(\mathfrak{s})d\mathfrak{s})^r\\
&\geq \int_{\frac{b+a}{2}}^b \frac{1}{\varrho(\frac{1}{2})^r}\check{H}\left(\frac{b+a}{2}\right)(\varrho'(\mathfrak{s})d\mathfrak{s})^r.
\end{aligned}$$

Since ϱ super-multiplicative, super-additive function, we have

$$\begin{aligned}
&\int_a^b \check{H}(\mathfrak{s})(\varrho'(\mathfrak{s})d\mathfrak{s})^r\\
&\geq \int_{\frac{b+a}{2}}^b \frac{1}{\varrho(\frac{1}{2})^r}\check{H}\left(\frac{b+a}{2}\right)(\varrho'(\mathfrak{s})d\mathfrak{s})^r\\
&= (\varrho(b) - \varrho(\frac{a+b}{2}))^r \frac{1}{\varrho(\frac{1}{2})^r}\check{H}\left(\frac{b+a}{2}\right)\\
&\geq (\varrho(\frac{b-a}{2})^r \frac{1}{\varrho(\frac{1}{2})^r}\check{H}\left(\frac{b+a}{2}\right)\\
&\geq \varrho(b-a)^r \varrho(\frac{1}{2})^r \frac{1}{\varrho(\frac{1}{2})^r}\check{H}\left(\frac{b+a}{2}\right)\\
&= \varrho(b-a)^r\check{H}\left(\frac{b+a}{2}\right).
\end{aligned} \tag{9}$$

In another aspect, it is important to highlight that when considering a generalized ϱ-convex function $\check{H}$, it follows that for $\mathfrak{s} \in [0,1]$, the following relationship holds:

$$\check{H}((1-\mathfrak{s})b + \mathfrak{s}a) \leq \varrho(\mathfrak{s})^r\check{H}(a) + (\varrho(1-\mathfrak{s}))^r\check{H}(b),$$

and

$$\check{H}(\mathfrak{s}b + (1-\mathfrak{s})a) \leq (\varrho(1-\mathfrak{s}))^r\check{H}(a) + (\varrho(\mathfrak{s})^r\check{H}(b).$$

By summing up these inequalities, we arrive at the following expression:

$$\begin{aligned}
&\check{H}(\mathfrak{s}a + (1-\mathfrak{s})b) + \check{H}((1-\mathfrak{s})a + \mathfrak{s}b)\\
&\leq \varrho(\mathfrak{s})^r\check{H}(a) + (\varrho(1-\mathfrak{s}))^r\check{H}(b) + (\varrho(1-\mathfrak{s}))^r\check{H}(a) + \varrho(\mathfrak{s})^r\check{H}(b)\\
&\leq \varrho(1)^r(\check{H}(a) + \check{H}(b)).
\end{aligned}$$

Next, by integrating the resulting inequality with respect to $\mathfrak{s}$ over the interval $[0,1]$, we derive the following expression:

$$\begin{aligned}
&\frac{1}{\Gamma(1+r)}\int_0^1 [\check{H}((1-\mathfrak{s})a + \mathfrak{s}b) + \check{H}(\mathfrak{s}a + (1-\mathfrak{s})b)](\varrho'(\mathfrak{s})d\mathfrak{s})^r\\
&\leq \frac{1}{\Gamma(1+r)}\int_0^1 \varrho(1)^r(\check{H}(a) + \check{H}(b))(\varrho'(\mathfrak{s})d\mathfrak{s})^r.
\end{aligned}$$

It is obvious that

$$\frac{1}{\Gamma(1+r)}\int_0^1 [\check{H}((1-\mathfrak{s})a + \mathfrak{s}b) + \check{H}(\mathfrak{s}a + (1-\mathfrak{s})b)](\varrho'(\mathfrak{s})d\mathfrak{s})^r = \frac{2^r}{(\varrho(b-a))^r}{}_aI_b^{(r,\varrho)}\check{H}(\mathfrak{s}),$$

and

$$\frac{1}{\Gamma(1+r)}\int_0^1 \varrho(1)^r(\check{H}(a)+\check{H}(b))(\varrho'(\mathfrak{s})d\mathfrak{s})^r = \varrho(1)^r(\varrho(1)^r-\varrho(0)^r)\frac{\check{H}(a)+\check{H}(b)}{\Gamma(1+r)}$$
$$\leq \varrho(1)^{2r}\frac{\check{H}(a)+\check{H}(b)}{\Gamma(1+r)}.$$

So,

$$\frac{\Gamma(1+r)}{(\varrho(b-a))^r}{}_aI_b^{(r,\varrho)}\check{H}(\mathfrak{s}) \leq \varrho(1)^{2r}\frac{\check{H}(a)+\check{H}(b)}{2^r}. \tag{10}$$

By merging the inequalities (9) and (10), we obtain the following expression:

$$\check{H}\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(1+r)}{(\varrho(b-a))^r}{}_aI_b^{(r)}\check{H}(\mathfrak{s}) \leq \varrho(1)^{2r}\frac{\check{H}(a)+\check{H}(b)}{2^r}.$$

It's worth noting that if $r=1$ and $\varrho(\mathfrak{s})=\mathfrak{s}$, the inequality simplifies to the Hermite-Hadamard inequality. □

5. Versatility of Generalized Jensen's Inequality: Diverse Applications

Example 1

Let $a_1>0, a_2>0$ and $a_2^{3r}+a_1^{3r} \leq \frac{1^r}{\varrho(\frac{1}{2})^r}$. Then $a_1+a_2 \leq 2$.

Proof. Let $\check{H}(\mathfrak{s})=\mathfrak{s}^{3r}, \mathfrak{s}\in(0,+\infty)$. It is evident that the function $\check{H}$ is a generalized ϱ-convex function. So,

$$\check{H}\left(\frac{a_2+a_1}{2}\right) \leq \varrho(\frac{1}{2})^r(\check{H}(a_2)+\check{H}(a_1)).$$

That is

$$\frac{(a_2+a_1)^{3r}}{8^r} \leq \varrho(\frac{1}{2})^r(a_2^{3r}+a_1^{3r}) \leq 1^r.$$

Therefore, it can be concluded that the inequality $a_2+a_1 \leq 2$ holds. □

Example 2

Consider $\mathfrak{s}$ and y as elements of the real numbers. Then

$$E_r\left(\left(\frac{\varrho(\mathfrak{s})+\varrho(y)}{2}\right)^r\right) \leq \varrho(\frac{1}{2})^r\left(E_r\left(\varrho(\mathfrak{s})^r\right)+E_r\left(\varrho(y)^r\right)\right),$$

where $E_r(\mathfrak{s}^r)=\sum_{k=0}^{\infty}\frac{\mathfrak{s}^{rk}}{\Gamma(1+kr)}$ is the Mittag-Leffer function.

Proof. Take $\check{H}(\mathfrak{s})=E_r\left(\varrho(\mathfrak{s})^r\right)$. It is easy to see $(E_r(\mathfrak{s}^r))^{(2r,\varrho)}=E_r\left(\varrho(\mathfrak{s})^r\right)>0$. Consequently, the application of the generalized Jensen's inequality yields

$$E_r\left(\left(\frac{\varrho(\mathfrak{s})+\varrho(y)}{2}\right)^r\right) \leq \varrho(\frac{1}{2})^r\left(E_r\left(\varrho(\mathfrak{s})^r\right)+E_r\left(\varrho(y)^r\right)\right).$$

□

Example 3

Let's say we have positive numbers $a_1, a_2, \ldots, a_n > 0$. We also have two cases: either s is between 0 and t, or t is between 0 and s.

$$S_r = \left(\varrho(\frac{1}{n})^r(\varrho(a_1^r)^r + \varrho(a_2^r)^r + \cdots + \varrho(a_n^r)^r)\right)^{1/r}, r \in R,$$

$$S_r = \left(\frac{a_1^{rr} + a_2^{rr} + \cdots + a_n^{rr}}{n^r}\right)^{1/r}, r \in R.$$

So, we conclude that S_s is less than or equal to S_t. Furthermore, S_s equals S_t if and only if $a_1 = a_2 = \cdots = a_n$.

The inequality holds: $S_s \leq S_t$. Moreover, equality between S_s and S_t occurs solely when all values of $a_1, a_2, \ldots, a_n$ are identical.

Proof. Scenario I: $0 < s < t$. Take $\check{H}(\mathfrak{s}) = [\varrho(\mathfrak{s}^{(\frac{t}{s})})]^r, t > 0$. Then

$$\check{H}^{(2r,\varrho)}(\mathfrak{s}) = \frac{\Gamma\left(1 + \frac{tr}{s}\right)}{\Gamma\left(1 + \left(\frac{t}{s} - 1\right)r\right)}\varrho(\mathfrak{s})^{(t/s-2)r} > 0.$$

Through the application of the generalized Jensen's inequality, we obtain the following expression:

$$\check{H}\left(\frac{a_1^s + a_2^s + \cdots + a_n^s}{n}\right) \leq \varrho(\frac{1}{n})^r(\check{H}\left(a_1^s\right) + \check{H}\left(a_2^s\right) + \cdots + \check{H}\left(a_n^s\right)).$$

That is,

$$[\varrho(\left(\frac{a_1^s + a_2^s + \cdots + a_n^s}{n}\right)^{(t/s)})]^r \leq \varrho(\frac{1}{n})^r(\varrho(\left(a_1^s\right)^{(t/s)})^r + \varrho(\left(a_2^s\right)^{(t/s)})^r + \cdots + \varrho((a_n^s)^{(t/s)})^r).$$

So

$$[\varrho(\left(\frac{a_1^s + a_2^s + \cdots + a_n^s}{n}\right)^{(t/s)})]^r \leq \varrho(\frac{1}{n})^r(\varrho(\left(a_1^t\right))^r + \varrho(\left(a_2^t\right))^r + \cdots + \varrho(\left(a_n^t\right))^r).$$

Since ϱ super-multiplicative then

$$[\varrho(\left(\frac{a_1^s + a_2^s + \cdots + a_n^s}{n}\right)^{(1/s)})]^r \leq [\varrho(\frac{1}{n})^r(\varrho(\left(a_1^t\right))^r + \varrho(\left(a_2^t\right))^r + \cdots + \varrho(\left(a_n^t\right))^r)]^{\frac{1}{t}}.$$

So

$$[\varrho(\left(\frac{a_1^s + a_2^s + \cdots + a_n^s}{n}\right))^r]^{(1/s)} \leq [\varrho(\frac{1}{n})^r(\varrho(\left(a_1^t\right))^r + \varrho(\left(a_2^t\right))^r + \cdots + \varrho(\left(a_n^t\right))^r)]^{\frac{1}{t}}.$$

Since ϱ super-multiplicative and super-additive then

$$[\varrho(\frac{1}{n})^r\left(\varrho(a_1^s)^r + \varrho(a_2^s)^r + \cdots + \varrho(a_n^s)^r\right))]^{(1/s)}$$
$$\leq [\varrho(\frac{1}{n})^r(\varrho(\left(a_1^t\right))^r + \varrho(\left(a_2^t\right))^r + \cdots + \varrho(\left(a_n^t\right))^r)]^{\frac{1}{t}}.$$

Thus, we establish $S_t \geq S_s$.

Scenario II: When $s < t < 0$.

If we consider the case where $0 < -t < -s$ and define b_i as the reciprocal of a_i, we can reach a conclusion. □

Example 4

If we have positive numbers a_1, a_2, a_3, and their sum is 1 (i.e., $a_1 + a_2 + a_3 = 1$), we need to find the minimum value of the expression

$$\left(\frac{1}{a_1} + a_1\right)^{10r} + \left(\frac{1}{a_2} + a_2\right)^{10r} + \left(\frac{1}{a_3} + a_3\right)^{10r}.$$

Consider the stipulation: $0 < a_1, a_2, a_3 < 1$. Let $\check{H}(\mathfrak{s}) = \left(\varrho(\mathfrak{s}) + \frac{1}{\varrho(\mathfrak{s})}\right)^{10r}, \mathfrak{s} \in (0,1)$. Then, via the formula

$$\frac{d^{r,\varrho}\varrho(\mathfrak{s})^{kr}}{dt^{r,\varrho}} = \frac{\Gamma(kr+1)}{\Gamma(1+(k-1)r)}\varrho(\mathfrak{s})^{(k-1)r}.$$

We have

$$\check{H}^{(2r,\varrho)}(\mathfrak{s}) = \frac{\Gamma(1+10r)}{\Gamma(1+8r)}\left(\varrho(\mathfrak{s}) + \frac{1}{\varrho(\mathfrak{s})}\right)^{8r}\left(1 - \frac{1}{\varrho(\mathfrak{s})^2}\right)^{2r}$$
$$+\frac{\Gamma(1+r)\Gamma(1+10r)}{\Gamma(9r+1)}\left(\varrho(\mathfrak{s}) + \frac{1}{\varrho(\mathfrak{s})}\right)^{9r}\left(\frac{2}{\varrho(\mathfrak{s})^3}\right)^{r} > 0.$$

By the generalized Jensen's inequality,

$$\begin{aligned}
&\left(\varrho(1/3) + \frac{1}{\varrho(1/3)}\right)^{10r} = \check{H}\left(\frac{a_1+a_2+a_3}{3}\right)\\
&\le \varrho(\frac{1}{3})^r[\check{H}(a_1) + \check{H}(a_2) + \check{H}(a_3)]\\
&= \varrho(\frac{1}{3})^r\left[\left(\varrho(a_1) + \frac{1}{\varrho(a_1)}\right)^{10r} + \left(\varrho(a_2) + \frac{1}{\varrho(a_2)}\right)^{10r} + \left(\varrho(a_3) + \frac{1}{\varrho(a_3)}\right)^{10r}\right].
\end{aligned}$$

The minimum value occurs at $a_1 = a_2 = a_3 = \frac{1}{3}$ and equals $\frac{\left(\varrho(1/3) + \frac{1}{\varrho(1/3)}\right)^{10r}}{\varrho(\frac{1}{3})^r}$.

When we have $\varrho(\mathfrak{s}) = \mathfrak{s}$, the minimum value is obtained as $\frac{10^{10r}}{3^{9r}}$ when $a_1 = a_2 = a_3 = \frac{1}{3}$.

6. Conclusions

This paper's major goal is to present generalized convex functions and give them a precise definition with respect to fractal sets. Before unveiling two significant inequalities, we thoroughly scrutinized the properties linked with these generalized convex functions. Additionally, we provide real-world examples that highlight the usefulness of these inequalities in the context of fractal collections. In the future, the concepts of fractal sets can be further investigated using coordinated convexity and interval-valued functions.

The field of fractal sets using inequalities is still a new area. We can use our new definition about generalized local fractional derivative with respect to another function to derive Markov inequality, Yang inequality, Hölder inequality, Power-mean inequality, and so on. These will be our future research and recommendations for interested researchers.

References

[1] K. J. Wang, *An effective computational approach to the local fractional low-pass electrical transmission lines model*, Alex. Eng. J. **110** (2025), 629–635.

[2] W. Saleh, H. Boulares, A. Moumen, H. Albala, B. Meftah, *New approaches to fractal–fractional Bullen's inequalities through generalized convexity*, Fractal Fract. **9** (2025), 25.

[3] A. Lakhdari, B. Meftah, W. Saleh, *On corrected Simpson-type inequalities via local fractional integrals*, Georgian Math. J. **32** (2025), 111–121.

[4] M. Alqhtani, L. Sadek, K. M. Saad, *The Mittag-Leffler–Caputo–Fabrizio fractional derivative and its numerical approach*, Symmetry **17** (2025), 800.

[5] L. Sadek, A. Algefary, *On quantum trigonometric fractional calculus*, Alex. Eng. J. **120** (2025), 371–377.

[6] L. Sadek, A. S. Bataineh, E. M. Sadek, I. Hashim, *A general definition of the fractal derivative: Theory and applications*, AIMS Math. **10** (2025), 15390–15409.

[7] S. Varanec, *h-Convexity*, J. Math. Anal. Appl. **326** (2007), 303–311.

[8] J. J. Ruel, M. P. Ayres, *Jensen's inequality predicts effects of environmental variation*, Trends Ecol. Evol. **14** (1999), 361–366.

[9] M. Grinblatt, J. T. Linnainmaa, *Jensen's inequality, parameter uncertainty, and multiperiod investment*, Rev. Asset Pricing Stud. **1** (2011), 1–34.

[10] P. R. Beesack, J. E. Pečarić, *On Jensen's inequality for convex functions*, J. Math. Anal. Appl. **110** (1985), 536–552.

[11] D. S. Mitrinović, I. B. Lacković, *Hermite and convexity*, Aequat. Math. **28** (1985), 229–232.

[12] K. M. Kolwankar, A. D. Gangal, *Local fractional calculus: a calculus for fractal space-time*, in: *Fractals: Theory and Applications in Engineering*, Springer, London, 1999, pp. 171–181.

[13] A. K. Golmankhaneh, D. Baleanu, *On a new measure on fractals*, J. Inequal. Appl. **2013** (2013), 256.

[14] B. B. Mandelbrot, *The Fractal Geometry of Nature*, Macmillan, New York, 1983.

[15] K. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, John Wiley & Sons, Chichester, 2004.

[16] G. A. Edgar, *Integral Probability and Fractal Measures*, Springer, Berlin, 1998.

[17] A. Carpinteri, B. Chiaia, P. Cornetti, *Static-kinematic duality and the principle of virtual work in the mechanics of fractal media*, Comput. Methods Appl. Mech. Eng. **191** (2001), 3–19.

[18] W. P. Zhong, F. Gao, X. M. Shen, *Applications of Yang-Fourier transform to local fractional equations with local fractional derivative and local fractional integral*, Adv. Mater. Res. **461** (2012), 306–310.

[19] A. Babakhani, V. Daftardar-Gejji, *On calculus of local fractional derivatives*, J. Math. Anal. Appl. **270** (2002), 66–79.

[20] X. J. Yang, *Advanced Local Fractional Calculus and Its Applications*, World Science Publisher, 2012.

[21] Y. Zhao, D. F. Cheng, X. J. Yang, *Approximation solutions for local fractional Schrödinger equation in the one-dimensional Cantorian system*, Adv. Math. Phys. **2013** (2013), 291386.

[22] W. Liu, W. Wen, J. Park, *Hermite-Hadamard type inequalities for MT-convex functions via classical integrals and fractional integrals*, J. Nonlinear Sci. Appl. **9** (2016), 766–777.

[23] M. V. Mihai, *Some Hermite-Hadamard type inequalities via Riemann-Liouville fractional calculus*, Tamkang J. Math. **44** (2013), 411–416.

[24] W. Afzal, M. Abbas, S. M. Eldin, Z. A. Khan, *Some well-known inequalities for* (h_1, h_2)*-convex stochastic process via interval set inclusion relation*, AIMS Math. **8** (2023), 19913–19932.

[25] W. Afzal, S. M. Eldin, W. Nazeer, A. M. Galal, *Some integral inequalities for harmonical cr-h-Godunova-Levin stochastic processes*, AIMS Math. **8** (2023), 13473–13491.

[26] S. K. Sahoo, F. Jarad, B. Kodamasingh, A. Kashuri, *Hermite-Hadamard type inclusions via generalized Atangana-Baleanu fractional operator with application*, AIMS Math. **7** (2022), 12303–12321.

[27] M. Tariq, H. Ahmad, S. K. Sahoo, A. Kashuri, T. A. Nofal, C. H. Hsu, *Inequalities of Simpson-Mercer-type including Atangana-Baleanu fractional operators and their applications*, AIMS Math. **7** (2022), 15159–15181.

[28] A. Kashuri, S. K. Sahoo, P. O. Mohammed, E. A. Sarairah, Y. S. Hamed, *Some new Hermite-Hadamard type inequalities pertaining to fractional integrals with an exponential kernel for subadditive functions*, Symmetry **15** (2023), 748.

[29] Z. B. Fang, R. Shi, *On the* (p, h)*-convex function and some integral inequalities*, J. Inequal. Appl. **2014** (2014), 45.

[30] X. Jin, B. Jin, J. Ruan, X. Ma, *Some characterization of h-convex functions*, J. Math. Inequal. **16** (2022), 751–764.