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Revisiting Chen's α-Bernstein operators: A new representation, generalization and novel insights

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Abstract. The α-Bernstein operators were initially introduced in the paper by Chen, X., Tan, J., Liu, Z., Xie, J. (2017) titled "Approximation of Functions by a New Family of Generalized Bernstein Operators" (Journal of Mathematical Analysis and Applications, 450(1), 244-261). Since their introduction, these operators have served as a source of inspiration for numerous research endeavors. In this study, we propose a novel technique founded on a recursive relation for constructing Bernstein-like bases. A special case of this new representation yields a novel portrayal of Chen's operators. This innovative representation enables the discovery of additional properties of α-Bernstein operators and facilitates alternative and more straightforward proofs of certain theorems.

1. Introduction

The Bernstein polynomials, rooted in the fertile ground of approximation theory, stand as an enduring foundation of mathematical research and practice. These polynomials, introduced by S.N. Bernstein in 1912, not only define a fundamental class of algebraic polynomials but also wield a profound influence on mathematical analysis. It was through the lens of Bernstein polynomials that S.N. Bernstein provided the first constructive proof of Weierstrass' approximation theorem, a seminal contribution that continues to reverberate through the annals of mathematical history [11].

Defined as:

$$B_{n,k}(z) = \binom{n}{k} z^k (1-z)^{n-k}, \quad k = 0, ..., n \;\; \& \;\; n \in \mathbb{N},$$

these polynomials are instrumental in approximating functions over the interval $[0, 1]$. The core of this approximation is encapsulated in Bernstein operators with the formula:

$$\mathcal{B}_n(f; z) = \sum_{k=0}^{n} B_{n,k}(z) f(\frac{k}{n}), \quad z \in [0, 1],$$

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where a weighted sum of function values at equidistant points on the interval offers a versatile tool for approximation.

In 2017, Chen et al. [8] proposed an innovative modification of Bernstein operators, centered around the concept of α-Bernstein polynomials as follows:

Definition 1.1. *[8] The α-Bernstein polynomials of degree n, $B^{(\alpha)}_{n,k}(z)$, are defined by $B^{(\alpha)}_{1,0}(z) = 1 - z$, $B^{(\alpha)}_{1,1}(z) = z$ and*

$$B^{(\alpha)}_{n,k}(z) = \left[\binom{n-2}{k}(1-\alpha)z + \binom{n-2}{k-2}(1-\alpha)(1-z) + \binom{n}{k}\alpha z(1-z)\right] z^{k-1}(1-z)^{n-k-1} \tag{1}$$

where $n \geq 2$, $z \in [0, 1]$ and $\binom{m}{l}$ denotes the binomial coefficients.

Then, the corresponding α-Bernstein operators are given by

$$\mathscr{B}^{\alpha}_n(f; z) = \sum_{k=0}^{n} B^{\alpha}_{n,k}(z) f\left(\frac{k}{n}\right), \quad z \in [0, 1],$$

for any $f \in C[0, 1]$, where $C[0, 1]$ is the space of continuous functions on $[0, 1]$.

In recent years, the field of approximation theory has experienced noteworthy advancements, propelled in part by the pioneering work of Chen and his colleagues. Their contributions to the theory of Bernstein-like operators have not only broadened our comprehension of the theory, but also catalyzed further exploration and refinement. This seminal work has sparked significant interest and inspired further investigations into various families of Bernstein-like operators [13, 20, 22, 27].

The α-Bernstein operators are commonly identified as generalized Bernstein operators in the literature, and researchers have further developed the underlying concept of this modification to generalize other prominent operators in the field. These efforts can be broadly categorized into two main directions. The first group of works aims to connect Chen's operator with classical families of positive linear operators, such as those based on Schurer polynomials, Kantorovich polynomials, Stancu polynomials, q-Bernstein polynomials, and Durrmeyer polynomials, as well as other operator types like Favard–Szász–Mirakjan and Baskakov. These extensions and related studies can be found in [3, 9, 15, 16, 20, 21, 25–27, 29, 30]. The second group of contributions are devoted to further generalizing the idea introduced by Chen, with particular emphasis on constructing new operator families and conducting comparative analyses. These comparisons often focus on approximation properties, convergence behavior, and other performance metrics. These aspects are explored in [1, 5, 7, 14].

Building upon the initial ideas introduced in Chen's work, this paper proposes a novel framework for generating Bernstein-like basis functions. While Chen's formulation [8] served as a motivational starting point, the approach developed here gradually evolved into a broader generalization. This innovative structure can be viewed as an extension of Bernstein polynomials, aligning with these classical polynomials and their α-Bernstein modifications [8] in specific scenarios. The proposed framework allows for a recursive representation of the α-Bernstein operators, representing an advancement that deepens our understanding of their theoretical foundations and practical applications. By revisiting the operator's structure through a new lens, we aim to offer a distinct perspective on its mathematical properties and computational implications.

Our primary focus lies in presenting this new representation and demonstrating its utility through alternative proofs for selected theorems associated with α-Bernstein operators. Moreover, this representation provides us with the opportunity to uncover some additional properties of α-Bernstein operators.

The structure of the paper is as follows: Section 2 delineates the novel representation and validates its equivalence with Chen's formula. Section 3 examines the properties of this new presentation within the context of the Bernstein-like bases. In Section 4, two novel properties of the α-Bernstein operators are derived, and subsequently, we present alternative proofs for key theorems, highlighting the practical implications of our proposed representation. Finally, Section 5 presents concluding remarks and proposes potential avenues for future research.

2. A general structure for constructing Bernstein-like bases

The idea of recursively constructing Bernstein-like basis functions has already been employed by researchers; Yan and Liang [32] utilized this approach to construct a Bézier-like model and Li [19] extended the idea to define a parameter-based Bernstein-like basis, suitable for constructing Bézier-type curves. Additionally, Bibi et al. [4] introduced the so-called *generalized hybrid trigonometric Bernstein basis*, and Ameer and his colleagues [2] developed generalized Bernstein-like basis functions. Both of these studies further contribute to the recursive construction of Bernstein-like bases.

The idea is straightforward: identify an appropriate set of initial bases, such as $\{F_{2,0}, F_{2,1}, F_{2,2}\}$, and generate higher-order bases through the established recursive relation of Bernstein polynomials [10]:

$$F_{n,i}(z) = (1-z)F_{n-1,i}(z) + zF_{n-1,i-1}(z) \quad \text{for } z \in [0,1], \quad i = 0,1,2,\ldots,n. \tag{2}$$

The number of initial bases, hereafter referred to as the *starting basis*, can be any natural number. Yan and Liang [32] and Ameer et al. [2] both employed three polynomials of degree 4; Li [19] selected four polynomials of degree 3 as the starting basis; and Bibi et al. [4] utilized three hybrid trigonometric functions. It is essential that the starting basis functions exhibit as many fundamental properties of Bernstein bases as possible.

We follow this technique and propose a general structure to define a family of Bernstein-like bases.

Definition 2.1. *The starting Bernstein-like basis functions of order two are defined for $z \in [0,1]$ as*

$$\begin{aligned} g_{2,0}(z) &= az + b + \varphi(z), \\ g_{2,1}(z) &= 1 - 2b - a - 2\varphi(z), \\ g_{2,2}(z) &= -az + b + a + \varphi(z), \end{aligned} \tag{3}$$

where $a, b \in \mathbb{R}$ and the real-valued function φ must satisfy some conditions (specified below). The higher order Bernstein-like bases are generated by recursive relation (2).

Lemma 2.2. *The Bernstein-like basis functions of order $n \geq 2$, as defined in Definition 2.1, satisfy the partition of unity property, namely $\sum_{i=0}^{n} g_{n,i}(z) = 1$. Additionally, if the function φ is constrained to satisfy $\varphi(1-z) = \varphi(z)$, then a symmetry property emerges: $g_{n,i}(z) = g_{n,n-i}(1-z)$ for $i = 0,1,2,\ldots,n$.*

Further restrictions may be imposed on the function φ and the real values a and b in order to satisfy more fundamental properties of classical Bernstein polynomials. In fact, the non-negativity and end-point coincidence with Bernstein bases are two essential features.

Example 2.3. *When we set $a = -1$, $b = 0$ and $\varphi(z) = z^2 - z + 1$, the polynomials in equation (3) reduce to the Bernstein polynomials of degree 2, thus Definition 2.1 constructs the classical Bernstein polynomials.*

Example 2.4. *Authors in [23] presented a special case of (3) where they used the function $\varphi(z) = \sqrt{(1-\nu)(z^2 - z) + \frac{1}{4}}$, along with $a = -1$ and $b = \frac{1}{2}$. It is verified that the so-called sq-basis functions also satisfy the non-negativity and end-point interpolation properties. Furthermore, the corresponding operators exhibit the most common features of the classical Bernstein operators [24].*

Example 2.5. *By choosing the the trigonometric function $\varphi = \cos(\eta\pi z(1-z))$, where $\eta \in [0, 1.2]$, along with $a = -1$ and $b = 0$, results in a completely new family of basis functions, which are examined in a recent study [28].*

Example 2.6. *Employing the function*

$$\varphi(z) = \sqrt{(z - \frac{1}{2})^2 + \gamma z^2(1-z)^2} + \frac{1}{2},$$

along with $a = -1$ and $b = 0$ results in a new family of basis function. The three terms of order 2 are the same as the p-basis functions introduced by Stivan and Varady in [18], However, the higher order basis elements are different.

3. New representation of the α-Bernstein polynomials

Setting $a = -1$, $b = 1$ and $\varphi(z) = \alpha(z^2 - z)$, in equation (3), leads us to a set of starting basis functions as follows:

$$\begin{aligned} F_{2,0}(z) &= 1 - z + \alpha\left(z^2 - z\right), \\ F_{2,1}(z) &= -2\alpha\left(z^2 - z\right), \\ F_{2,2}(z) &= z + \alpha\left(z^2 - z\right). \end{aligned} \tag{4}$$

Now, employing the recurrence relation (2), we construct a set of basis functions that can be readily verified to be equivalent to the α-Bernstein polynomials of Chen [8]. Thus, we have:

Theorem 3.1. *The procedure presented in Definition 2.1, along with starting basis functions (4), generates the α-Bernstein polynomials.*

The recursive definition of the α-Bernstein polynomials provides an opportunity to derive practical computational formulas, one of which is articulated in the following lemma.

Lemma 3.2. *The α-Bernstein polynomials, defined by equations (2) and (4), can be derived through the following relation:*

$$F_{n,i}(z) = \sum_{j=0}^{n-2} B_{n-2,j}(z)F_{2,i-j}(z), \quad i = 0, 1, \cdots, n, \tag{5}$$

where $B_{n-2,j}$ are the classical Bernstein polynomials of order $n - 2$.

Proof. We employ mathematical induction to establish the validity of the result, beginning with the base case $n = 3$:

$$\begin{aligned} F_{n,0}(z) &= \sum_{j=0}^{1} B_{1,j}(z)F_{2,-j}(z) = B_{1,0}(z)F_{2,0}(z) = (1-z)F_{2,0}(z), \\ F_{n,1}(z) &= \sum_{j=0}^{1} B_{1,j}(z)F_{2,1-j}(z) = B_{1,0}(z)F_{2,1}(z) + B_{1,1}(z)F_{2,0}(z) = (1-z)F_{2,1}(z) + zF_{2,0}(z), \\ F_{n,2}(z) &= \sum_{j=0}^{1} B_{1,j}(z)F_{2,2-j}(z) = B_{1,0}(z)F_{2,2}(z) + B_{1,1}(z)F_{2,1}(z) = (1-z)F_{2,2}(z) + zF_{2,1}(z), \\ F_{n,3}(z) &= \sum_{j=0}^{1} B_{1,j}(z)F_{2,3-j}(z) = B_{1,1}(z)F_{2,2}(z) = zF_{2,2}(z). \end{aligned}$$

Thus, the base case holds, and the desired outcome is obtained.

Now, assuming the validity of the relation for $n - 1$, we obtain:

$$F_{n-1,i}(z) = \sum_{j=0}^{n-3} B_{n-3,j}(z)F_{2,i-j}(z), \quad i = 0, 1, \cdots, n-1.$$

In the next step, we aim to establish the validity of the relation for n. Utilizing the recursive relation (2), for $i = 0, 1, \cdots, n$, we have:

$$\begin{aligned} F_{n,i}(z) &= (1-z)F_{n-1,i}(z) + zF_{n-1,i-1}(z) \\ &= (1-z)\sum_{j=0}^{n-3} B_{n-3,j}(z)F_{2,i-j}(z) + z\sum_{j=0}^{n-3} B_{n-3,j}(z)F_{2,i-1-j}(z) \end{aligned}$$

$$
\begin{aligned}
&= (1-z)B_{n-3,0}(z)F_{2,i-0}(z) + \sum_{j=1}^{n-3}(1-z)B_{n-3,j}(z)F_{2,i-j}(z) \\
&\quad + \sum_{j=1}^{n-3} zB_{n-3,j-1}(z)F_{2,i-j}(z) + zB_{n-3,n-3}(z)F_{2,i-(n-2)}(z) \\
&= (1-z)B_{n-3,0}(z)F_{2,i-0}(z) + \sum_{j=1}^{n-3}[(1-z)B_{n-3,j}(z) + zB_{n-3,j-1}(z)]F_{2,i-j}(z) \\
&\quad + zB_{n-3,n-3}(z)F_{2,i-(n-2)}(z) \\
&= B_{n-2,0}(z)F_{2,i-0}(z) + \sum_{j=1}^{n-3} B_{n-2,j}(z)F_{2,i-j}(z) + B_{n-2,n-2}(z)F_{2,i-(n-2)}(z) \\
&= \sum_{j=0}^{n-2} B_{n-2,j}(z)F_{2,i-j}(z).
\end{aligned}
$$

□

Remark 3.3. *For any given value of n, the right-hand side of (5) has at most three non-zero terms, providing a computational advantage. Moreover, relation (5) can be represented in a matrix form as follows:*

$$
\begin{bmatrix} F_{n,0}(z) \\ F_{n,1}(z) \\ \vdots \\ F_{n,n}(z) \end{bmatrix}
=
\begin{bmatrix}
B_{n-2,0}(z) & 0 & 0 \\
B_{n-2,1}(z) & B_{n-2,0}(z) & 0 \\
B_{n-2,2}(z) & B_{n-2,1}(z) & B_{n-2,0}(z) \\
B_{n-2,3}(z) & B_{n-2,2}(z) & B_{n-2,1}(z) \\
\vdots & \vdots & \vdots \\
B_{n-2,n-3}(z) & B_{n-2,n-4}(z) & B_{n-2,n-5}(z) \\
B_{n-2,n-2}(z) & B_{n-2,n-3}(z) & B_{n-2,n-3}(z) \\
0 & B_{n-2,n-2}(z) & B_{n-2,n-3}(z) \\
0 & 0 & B_{n-2,n-2}(z)
\end{bmatrix}
\begin{bmatrix} F_{2,0}(z) \\ F_{2,1}(z) \\ F_{2,2}(z) \end{bmatrix}.
$$

From the proof of Lemma 3.2, the following fact can be derived directly:

Lemma 3.4. *For any set of basis functions $\{F_{n,i}\}_{i=0}^{n}$, which are constructed using the recursive relation (2), any basis of order n could be represented in terms of the basis elements of order $m < n$, i.e.*

$$
F_{n,i}(z) = \sum_{j=0}^{n-m} B_{n-m,j}(z)F_{m,i-j}(z), \quad i = 0, \cdots, n. \tag{6}
$$

Remark 3.5. *Lemma 3.4 provides a general formula for expressing the basis functions in terms of the initial ones, regardless of how many starting basis functions are used.*

The next result demonstrates that the α-Bernstein polynomials possess many of the properties of classical Bernstein polynomials, making them well-suited for function approximation.

Theorem 3.6. *The α-Bernstein polynomials derived from equations (2) and (4) exhibit the following properties:*

(a) *Non-negativity: $F_{n,i}(z) \geq 0$ for $i = 0, 1, 2, \ldots, n$.*

(b) *Partition of unity: $\sum_{i=0}^{n} F_{n,i}(z) = 1$.*

(c) *Symmetry: $F_{n,i}(z) = F_{n,n-i}(1-z)$ for $i = 0, 1, 2, \ldots, n$.*

(d) *End-point values:*

$$F_{n,i}(0)=\begin{cases}1, & i=0,\\ 0, & i\neq 0,\end{cases}\qquad F_{n,i}(1)=\begin{cases}1, & i=n,\\ 0, & i\neq n.\end{cases} \tag{7}$$

Proof. The proof of this theorem relies on the application of the mathematical induction.

(a) Non-negativity can be deduced from (2) and (4).

(b) It is already stated in Lemma 2.2.

(c) The starting basis functions exhibit symmetry. Suppose that the α-Bernstein polynomials of order k hold symmetry. Now, by considering this inductive hypothesis along with the recurrence relation (2), we can conclude that

$$\begin{aligned} F_{k+1,i}(1-z) &= (1-(1-z))F_{k,i}(1-z)+(1-z)F_{k,i-1}(1-z)\\ &= zF_{k,k-i}(z)+(1-z)F_{k,k-i+1}(z)=F_{k+1,k-i+1}(z).\end{aligned}$$

(d) Based on a straightforward deduction from (4), it can be concluded that the results in (7) apply to the case when $n=2$. Assume that the properties at the endpoints are satisfied by the α-Bernstein polynomials of order k. Consequently, by utilizing the inductive hypothesis and equation (2), it follows that:

$$\begin{aligned} F_{k+1,i}(0) &= F_{k,i}(0)=\begin{cases}1 & i=0,\\ 0, & i\neq 0,\end{cases}\\ F_{k+1,i}(1) &= F_{k,i-1}(1)=\begin{cases}1 & i=k+1,\\ 0, & i\neq k+1.\end{cases}\end{aligned}$$

□

In Figure 1, an illustration of the α-Bernstein polynomials is presented. These functions are generated for varying values of α (including $0, 0.2, 0.4, 0.6, 0.8$ and 1), with $n=2,3,4$.

4. New results and alternative proofs

This section is dedicated to showcasing the benefits of the new representation of α-Bernstein operators. We introduce a new formula for computing the moments and present an analogue of Vornovskaja's theorem. Subsequently, we verify the shape-preserving properties of these operators using entirely different strategies from those employed in the existing literature.

Theorem 4.1. *For all $j\in\mathbb{N}\cup\{0\}$, $n\in\mathbb{N}$, $\alpha\in[0,1]$ and $z\in[0,1]$, we have the recurrence formula:*

$$\mathcal{B}_n^{\alpha}\left(z^{j+1};z\right)=\left(1-\frac{1}{n}\right)^{j+1}\left[(1-z)\,\mathcal{B}_{n-1}^{\alpha}\left(z^{j+1};z\right)+z\sum_{k=0}^{j+1}\binom{j+1}{k}\frac{1}{(n-1)^k}\mathcal{B}_{n-1}^{\alpha}\left(z^{j+1-k};z\right)\right].$$

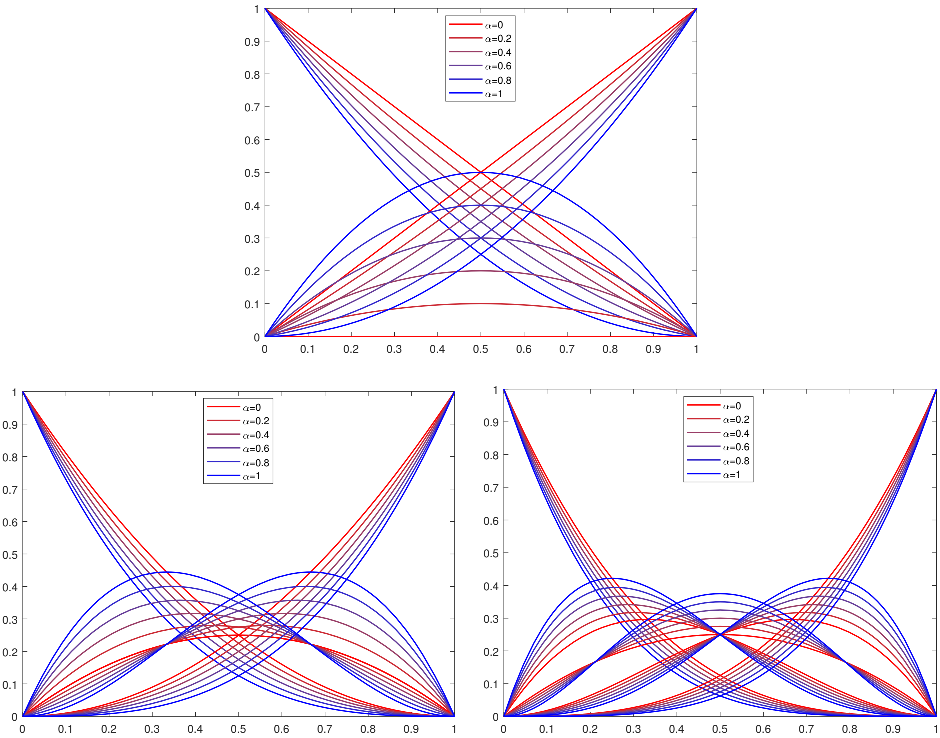


Figure 1: The α-Bernstein polynomials by adjusting α from $0, 0.2, \ldots, 1$.

Proof. Employing equation (2) we derive:

$$\begin{aligned}
\mathcal{B}_n^{\alpha}\left(z^{j+1};z\right) &= \sum_{i=0}^{n}\left(\frac{i}{n}\right)^{j+1} F_{n,i}(z)\\
&= \sum_{i=0}^{n}\left(\frac{i}{n}\right)^{j+1}\left[(1-z)\,F_{n-1,i}(z) + zF_{n-1,i-1}(z)\right]\\
&= \frac{(n-1)^{j+1}}{n^{j+1}}\left[(1-z)\sum_{i=0}^{n}\left(\frac{i}{n-1}\right)^{j+1} F_{n-1,i}(z) + z\sum_{i=0}^{n}\left(\frac{i}{n-1}\right)^{j+1} F_{n-1,i-1}(z)\right]\\
&= \left(1-\frac{1}{n}\right)^{j+1}\left[(1-z)\,\mathcal{B}_{n-1}^{\alpha}\left(z^{j+1};z\right) + z\sum_{i=0}^{n}\left(\frac{i-1+1}{n-1}\right)^{j+1} F_{n-1,i-1}(z)\right]\\
&= \left(1-\frac{1}{n}\right)^{j+1}\left[(1-z)\,\mathcal{B}_{n-1}^{\alpha}\left(z^{j+1};z\right) + z\sum_{k=0}^{j+1}\binom{j+1}{k}\frac{1}{(n-1)^k}\sum_{i=0}^{n-1}\left(\frac{i}{n-1}\right)^{j+1-k} F_{n-1,i}(z)\right]\\
&= \left(1-\frac{1}{n}\right)^{j+1}\left[(1-z)\,\mathcal{B}_{n-1}^{\alpha}\left(z^{j+1};z\right) + z\sum_{k=0}^{j+1}\binom{j+1}{k}\frac{1}{(n-1)^k}\mathcal{B}_{n-1}^{\alpha}\left(z^{j+1-k};z\right)\right].
\end{aligned}$$

We utilized the outcome of the following relation in the above calculations.

$$\sum_{i=0}^{n}\left(\frac{i-1+1}{n-1}\right)^{j+1} F_{n-1,i-1}(z) = \sum_{i=0}^{n}\frac{1}{(n-1)^{j+1}}\left[\sum_{k=0}^{j+1}\binom{j+1}{k}(i-1)^{j+1-k}\right] F_{n-1,i-1}(z)$$

$$= \sum_{i=0}^{n}\left[\sum_{k=0}^{j+1}\binom{j+1}{k}\frac{1}{(n-1)^k}\left(\frac{i-1}{n-1}\right)^{j+1-k}\right]F_{n-1,i-1}(z)$$
$$= \sum_{k=0}^{j+1}\binom{j+1}{k}\frac{1}{(n-1)^k}\sum_{i=0}^{n-1}\left(\frac{i}{n-1}\right)^{j+1-k}F_{n-1,i}(z).$$

□

Subsequently, we present a Grüss-Voronovskaja-type theorem for the α-Bernstein operators, utilizing the methodology outlined in [12].

Theorem 4.2. *Let $f, h \in C^2[0,1]$, then for any $z \in [0,1]$, we have*

$$\lim_{n\to\infty} n\left[\mathcal{B}_n^\alpha(fh;z) - \mathcal{B}_n^\alpha(f;z)\mathcal{B}_n^\alpha(h;z)\right] = z(1-z)f'(z)h'(z), \tag{8}$$

where $\alpha \in [0,1]$.

Proof. It is straightforward to express

$$\begin{aligned}\mathcal{B}_n^\alpha(fh;z) - \mathcal{B}_n^\alpha(f;z)\mathcal{B}_n^\alpha(h;z) &\\ &= \mathcal{B}_n^\alpha(fh;z) - f(z)h(z) + f(z)h(z) - \mathcal{B}_n^\alpha(f;z)\mathcal{B}_n^\alpha(h;z) + \mathcal{B}_n^\alpha(f;z)h(z) - \mathcal{B}_n^\alpha(f;z)h(z)\\ &= \left[\mathcal{B}_n^\alpha(fh;z) - f(z)h(z)\right] + \left[h(z)\left(f(z) - \mathcal{B}_n^\alpha(f;z)\right)\right] + \left[\mathcal{B}_n^\alpha(f;z)\left(h(z) - \mathcal{B}_n^\alpha(h;z)\right)\right].\end{aligned}$$

Considering the corresponding Korovkin [17] and Vornovskaja [31] theorems from the original paper [8], we derive the following result

$$\begin{aligned}\lim_{n\to\infty} n\left[\mathcal{B}_n^\alpha(fh;z) - \mathcal{B}_n^\alpha(f;z)\mathcal{B}_n^\alpha(h;z)\right] &= \lim_{n\to\infty} n\left[\mathcal{B}_n^\alpha(fh;z) - f(z)h(z)\right] + \left[h(z)\left(\lim_{n\to\infty} n[f(z) - \mathcal{B}_n^\alpha(f;z)]\right)\right]\\ &\quad + \left[\lim_{n\to\infty}\mathcal{B}_n^\alpha(f;z)\left(\lim_{n\to\infty} n[h(z) - \mathcal{B}_n^\alpha(h;z)]\right)\right]\\ &= \frac{1}{2}z(1-z)(fh)''(z) - \frac{1}{2}z(1-z)f''(z)h(z) - \frac{1}{2}z(1-z)h''(z)f(z)\\ &= \frac{1}{2}z(1-z)\left[(fh)''(z) - f''(z)h(z) - h''(z)f(z)\right]\\ &= z(1-z)f'(z)h'(z).\end{aligned}$$

□

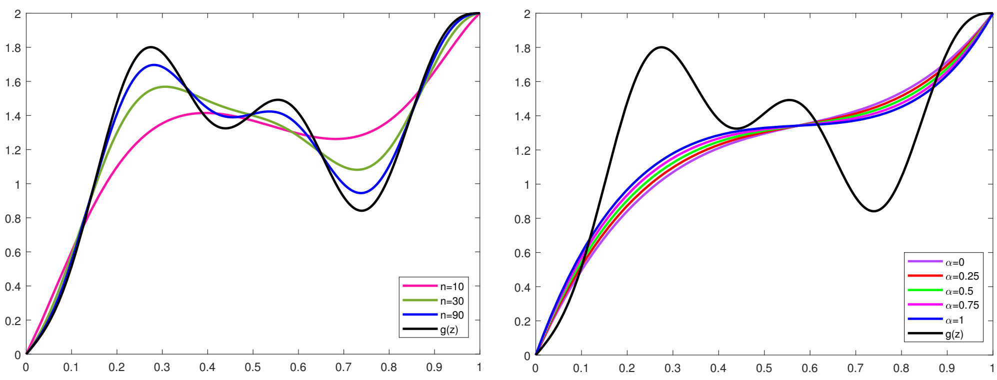
Example 4.3. *To analyze the approximation behavior of our α-Bernstein operators, we examine their performance for the function $g(z) = 2\sin\left(\frac{\pi z}{2}\right) + \sin^3(2\pi z)$. Figures 3(a) and 3(b) demonstrate the convergence properties from two perspectives. Figure 3(a) shows how increasing the polynomial degree n from 10 to 90 improves the approximation quality with fixed $\alpha = 0.8$. Figure 3(b) illustrates how varying the parameter α from 0 to 1 affects the approximation accuracy for fixed $n = 5$. This analysis reveals how both the polynomial degree and the shape parameter influence the performance of α-Bernstein operators.*

4.1. Monotonicity preservation

Definition 4.4. *A system of functions $\{h_0, \cdots, h_n\}$ is monotonicity preserving if for any $\nu_0 \le \nu_1 \le \cdots \le \nu_n$ in $\mathbb{R}$, the function $\sum_{i=0}^{n}\nu_i h_i$ is increasing.*

Proposition 2.3 of [6] outlines the characterization of systems that preserve monotonicity, as presented in the following result.

Proposition 4.5. *Let $\{h_0, \cdots, h_n\}$ be a system of functions defined on an interval $[a,b]$. Let $k_i := \sum_{j=i}^{n} h_j$ for $i \in \{0, 1, \cdots, n\}$. Then $\{h_0, \cdots, h_n\}$ is monotonicity preserving if and only if k_0 is a constant function and the functions k_i are increasing for $i = 1, \cdots, n$.*



(a) $n = 10, 30, 90, \alpha = 0.8$ (b) $n = 5, \alpha = 0, 0.25, \ldots, 1$

Figure 2: Approximation of $g(z)$ using $R_{n,\alpha}(\cdot; z)$ operators.

The subsequent result is derived from the preceding proposition.

Proposition 4.6. *The α-Bernstein polynomials $\{F_{n,0}, F_{n,1}, \cdots, F_{n,n}\}$, defined by equations (4) and (2), are monotonicity preserving.*

Proof. We use mathematical induction to verify the result. Consider the α-Bernstein polynomials defined by equation (2) for $n = 2$. According to Proposition 4.5, the set $\{F_{2,0}, F_{2,1}, F_{2,2}\}$ preserves monotonicity under the conditions that k_0 is a constant function, and both k_1 and k_2 are increasing functions.

Utilizing the partition of unity of α-Bernstein polynomials, we obtain $k_0 = \sum_{j=0}^{2} F_{2,j}(z) = 1$. On the other hand

$$\sum_{j=1}^{2} F_{2,j}(z) = z - \alpha(z^2 - z) \quad \Longrightarrow \quad \frac{d}{dz}\Big(\sum_{j=1}^{2} F_{2,j}(z)\Big) = 1 - \alpha(2z - 1) \geq 1 - \alpha,$$

where $1 - \alpha$ takes a non-negative value, therefore $\sum_{j=1}^{2} F_{2,j}(z)$ is an increasing function. Similarly, we show the monotonicity of the last function by taking derivative

$$\sum_{j=2}^{2} F_{2,j}(z) = z + \alpha(z^2 - z) \quad \Longrightarrow \quad \frac{d}{dz}\Big(F_{2,2}(z)\Big) = 1 + \alpha(2z - 1),$$

where obviously $1 + \alpha(2z - 1) \geq 0$, so we have the desired result.

Assume that the α-Bernstein polynomials $\{F_{n-1,0}, F_{n-1,1}, \cdots, F_{n-1,n-1}\}$ are monotonicity preserving, and it remains to conclude the same result for any α-Bernstein polynomials of order n, $\{F_{n,0}, F_{n,1}, \cdots, F_{n,n}\}$.

To do so, we first observe that $k_0 = \sum_{j=0}^{n} F_{n,j}(z) = 1$, i.e., it is a constant function due to partition of unity. For $k_i(z) = \sum_{j=i}^{n} F_{n,j}(z)$, we show the monotonicity by verifying the non-negativity of its derivative, $\frac{d}{dz}k_i(z) = \sum_{j=i}^{n} \frac{d}{dz}F_{n,j}(z)$.

Employing relation (2), we have

$$\begin{aligned}
\frac{d}{dz}k_i(z) &= -\sum_{j=i}^{n} F_{n-1,j}(z) + (1-z)\sum_{j=i}^{n} \frac{d}{dz}F_{n-1,j}(z) + \sum_{j=i}^{n} F_{n-1,j-1}(z) + z\sum_{j=i}^{n} \frac{d}{dz}F_{n-1,j-1}(z) \\
&= F_{n-1,n}(z) + \left(-\sum_{j=i}^{n-1} F_{n-1,j}(z) + \sum_{j=i}^{n-1} F_{n-1,j}(z)\right) + F_{n-1,i-1}(z)
\end{aligned}$$

$$
\begin{aligned}
& +(1-z)\sum_{j=i}^{n}\frac{d}{dz}F_{n-1,j}(z)+z\sum_{j=i}^{n}\frac{d}{dz}F_{n-1,j-1}(z) \\
= \; & F_{n-1,i-1}(z)+(1-z)\sum_{j=i}^{n-1}\frac{d}{dz}F_{n-1,j}(z)+z\sum_{j=i-1}^{n-1}\frac{d}{dz}F_{n-1,j}(z). \qquad (9)
\end{aligned}
$$

Now, we use the induction hypothesis to see that

$$
\sum_{j=i}^{n-1}\frac{d}{dz}F_{n-1,j}(z)\geq 0, \qquad \sum_{j=i-1}^{n-1}\frac{d}{dz}F_{n-1,j}(z)\geq 0. \qquad (10)
$$

Therefore, it becomes evident that $\frac{d}{dz}k_i(z)\geq 0$. □

The monotonicity preservation of the α-Bernstein operators follows directly from the preceding proposition.

Theorem 4.7. *For a function q that is continuous and monotonically increasing on the interval $[0,1]$, the corresponding α-Bernstein operators are also increasing.*

Proof. Given the monotonically increasing function q, we observe the inequality $q\left(\frac{0}{n}\right)\leq q\left(\frac{1}{n}\right)\leq\cdots\leq q\left(\frac{n}{n}\right)$. Consequently, the result is directly derived from Proposition 4.6. □

4.2. Convexity preservation

The preservation of convexity by the α-Bernstein operators is verified by the subsequent theorem.

Theorem 4.8. *If $q\in C[0,1]$ is a convex function, then its α-Bernstein operators are all convex.*

Proof. We employ the method of induction to establish the validity of the result, starting with the base case $n=2$. Consider a set of real values $\lambda_0,\lambda_1,\lambda_2$ forming a convex set of data, ensuring $\lambda_2-2\lambda_1+\lambda_0\geq 0$. Our objective is to demonstrate the convexity of the function $G(z,\alpha)=\sum_{i=0}^{2}\lambda_iF_{2,i}(z)$ over the interval $[0,1]$. Evaluating the second derivative, we get

$$
\frac{d^2}{dz^2}G(z,\alpha)=\sum_{i=0}^{2}\lambda_i\frac{d^2}{dz^2}F_{2,i}(z)=2\alpha\left(\lambda_2-2\lambda_1+\lambda_0\right).
$$

Given this expression and $\lambda_2-2\lambda_1+\lambda_0\geq 0$, we can affirm that $\frac{d^2}{dz^2}G(z,\alpha)\geq 0$.

Presuming the convexity of a set of real values $\{\beta_i\}_{i=0}^{n}$, we take as our induction hypothesis that the expression $\sum_{i=0}^{n}\beta_iF_{n,i}(z)$ is convex.

Next, we introduce a new set of convex values $\{\eta_i\}_{i=0}^{n+1}$. The goal is to establish the convexity of the function $\sum_{i=0}^{n+1}\eta_iF_{n+1,i}(z)$. To accomplish this, we consider

$$
G(z,\alpha)=\sum_{i=0}^{n+1}\eta_iF_{n+1,i}(z)=\sum_{i=0}^{n+1}\eta_i\left[(1-z)F_{n,i}(z)+zF_{n,i-1}(z)\right],
$$

and establish that the function $\frac{d}{dz}G(z,\alpha)$ is increasing.

$$
\frac{d}{dz}G(z,\alpha) = \sum_{i=0}^{n+1}\eta_i\frac{d}{dz}\left[(1-z)F_{n,i}(z)+zF_{n,i-1}(z)\right]
$$

$$= \sum_{i=0}^{n+1} \eta_i \left[-F_{n,i}(z) + (1-z)\frac{d}{dz}F_{n,i}(z) + F_{n,i-1}(z) + z\frac{d}{dz}F_{n,i-1}(z) \right]$$

$$= \sum_{i=0}^{n} (\eta_{i+1} - \eta_i)F_{n,i}(z) + (1-z)\sum_{i=0}^{n} \eta_i \frac{d}{dz}F_{n,i}(z) + z\sum_{i=0}^{n} \eta_{i+1}\frac{d}{dz}F_{n,i}(z).$$

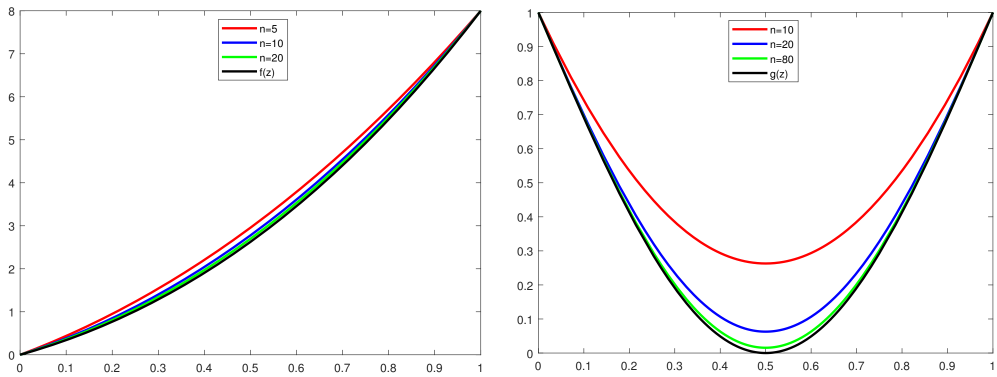
Given that $\{\eta_i\}_{i=0}^{n+1}$ represents convex data, we observe that $\eta_{i+1} - 2\eta_i + \eta_{i-1} \geq 0, i = 1, \cdots, n-1$. This inequality implies $\eta_{i+1} - \eta_i \geq \eta_i - \eta_{i-1}, i = 1, \cdots, n-1$. In simpler terms, the differences $\eta_i - \eta_{i-1}, i = 1, \cdots, n-1$ form an increasing sequence. By applying Proposition 4.6, it concludes that the function $\sum_{i=0}^{n}(\eta_{i+1} - \eta_i)F_{n,i}(z)$ is an increasing function.

Assuming the induction hypothesis and given that $\{\eta_i\}_{i=0}^{n+1}$ represents convex data, it follows that both $\sum_{i=0}^{n} \eta_i F_{n,i}(z)$ and $\sum_{i=0}^{n} \eta_{i+1} F_{n,i}(z)$ are convex functions. This implies that their respective derivatives, $\sum_{i=0}^{n} \eta_i \frac{d}{dz}F_{n,i}(z)$ and $\sum_{i=0}^{n} \eta_{i+1} \frac{d}{dz}F_{n,i}(z)$, are increasing functions.

By the derived results and considering $z \in [0,1]$, it follows that $\frac{d}{dz}G(z,\alpha)$ is an increasing function. Consequently, this implies $\frac{d^2}{dz^2}G(z,\alpha) \geq 0$, signifying that $G(z,\alpha)$ is a convex function.

We observe that if q is a convex function, then the values $q\left(\frac{0}{n}\right), q\left(\frac{1}{n}\right), \ldots, q\left(\frac{n}{n}\right)$ form a set of convex data. Consequently, this directly implies that $R_{n,\alpha}(q;z)$ is also a convex function. □

Example 4.9. *In this example, we present graphics illustrating the approximation of $R_{n,\alpha}(.;z)$ to a function g, as well as its shape-preserving properties. We provide examples for a monotone function, $f(z) = z^3 + 4z^2 + 3z$, and a convex function, $g(z) = 1 - \sin(\pi z)$, to demonstrate these characteristics.*



(a) $n = 5, 10, 20, \alpha = 0.5$ (b) $n = 10, 20, 80, \alpha = 0.5$

Figure 3: Monotonicity and Convexity Preservation by $R_{n,\alpha}(.;z)$.

5. Conclusion

The present study makes two significant contributions to the field of approximation theory, particularly in relation to Bernstein-like operators.

First, it introduces a generalized formula for generating Bernstein-like functions. The paper illustrates this approach through three specific families and opens pathways for constructing new and innovative ones.

Second, it demonstrates that the proposed structure yields a recursive formula for generating the α-Bernstein polynomials. This recursive framework for Chen's basis allows many of Chen's theorems to be proved in a substantially simpler manner than in existing literature. Furthermore, this perspective enables the derivation of new theoretical results and provides alternative proofs for known properties that are less transparent in the original closed-form representation. The recursive approach thus offers both computational benefits and deeper theoretical insight into the underlying mathematical structure.

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