



Some results on additivity of Moore-Penrose inverse in matrix settings

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Abstract. We present the necessary and sufficient conditions for the Moore-Penrose inverse of the sum of two matrices to be equal to the sum of Moore-Penrose inverses of those matrices when these matrices are invertible, or one matrix is invertible and the other satisfies certain range inclusions. Similar results are presented for the group inverse.

1. Introduction

In this paper, we will investigate the additivity of Moore-Penrose and group inverse in the case of two matrices. $\mathbb{C}^{m \times n}$ will denote the set of $m \times n$ complex matrices while for given $A \in \mathbb{C}^{m \times n}$, A^* , $r(A)$, $\mathcal{R}(A)$ and $\mathcal{N}(A)$ will denote the conjugate transpose, rank, the column space (range) and the null space of A , respectively. If matrix $A \in \mathbb{C}^{n \times n}$ satisfies $A^2 = A$, we say that A is idempotent (projection). The symbol $\cdot X$ ($X \cdot$) written after an equality means that in the next step we will multiply that equality from the left (right) side by the matrix X .

In [5] the authors investigate the equation $\frac{1}{a+b} = \frac{1}{a} + \frac{1}{b}$ for real and complex numbers and, further, its generalization for matrices with real and complex entries. Here we will restate some of the existing results concerning complex matrices since the generalization of these results will be the subject of our study. For the sake of completeness, we give the proof of the following

Lemma 1.1. [5] *Let $A, B \in \mathbb{C}^{n \times n}$ be invertible matrices. Then $A + B$ is an invertible matrix and*

$$(A + B)^{-1} = A^{-1} + B^{-1} \quad (1.1)$$

if and only if AB^{-1} is a diagonalizable matrix whose eigenvalues belong to the set $X = \{\frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}\}$, i.e. $A = PDP^{-1}B$, for some invertible matrix P and diagonal matrix D whose entries on the main diagonal belong to the set X .

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Proof. Notice that $(A+B)^{-1} = A^{-1} + B^{-1}$ if and only if $A^{-1} + B^{-1}$ is right inverse of $A+B$. By direct computation, we get

$$\begin{aligned} I_n = (A+B)(A^{-1} + B^{-1}) &\Leftrightarrow AB^{-1} + BA^{-1} + I_n = 0 \quad \setminus \cdot AB^{-1} (\text{invertible matrix}) \\ &\Leftrightarrow (AB^{-1})^2 + AB^{-1} + I_n = 0. \end{aligned}$$

Hence, condition (1.1) holds if and only if the matrix $M = AB^{-1}$ satisfies

$$M^2 + M + I_n = 0. \quad (1.2)$$

If $M = PJP^{-1}$, where J is Jordan normal form of M , (1.2) holds if and only if J satisfies the same equation, i.e. for all Jordan blocks J_i of M holds $J_i^2 + J_i + I_{k_i} = 0$. If J_i is of dimension greater than one,

$$\begin{pmatrix} \lambda_i^2 + \lambda_i + 1 & 2\lambda_i + 1 & & \\ 0 & \lambda_i^2 + \lambda_i + 1 & & \\ & & \ddots & \\ & & & \ddots \end{pmatrix} = 0$$

must hold, which is impossible, so all Jordan blocks are of dimension one, and their entries satisfy $\lambda_i^2 + \lambda_i + 1 = 0$, i.e. $\lambda_i \in X = \{\frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}\}$. Hence (1.2) is true if and only if M is diagonalizable matrix whose eigenvalues belong to the set X . (That M is a diagonalizable matrix, we could conclude from the fact that the polynomial $x^2 + x + 1 = 0$ does not have multiple roots.) $\square$

The concept of regular inverse is generalized by Penrose [17] who showed that for any complex matrix $A \in \mathbb{C}^{m \times n}$ there exists unique matrix $X \in \mathbb{C}^{n \times m}$ which satisfies the four Penrose equations

$$(1) AXA = A, \quad (2) XAX = X, \quad (3) (AX)^* = AX, \quad (4) (XA)^* = XA.$$

This inverse is known as the Moore-Penrose inverse of A and is denoted by $A^\dagger$. Actually, Penrose rediscovered this inverse, which was presented first by Moore [15]. Properties of this inverse can be found in [2, 7]. If matrix X satisfies the Penrose equation (i) for each $i \in K \subseteq \{1, 2, 3, 4\}$ we say that X is a K -inverse of A . In particular, we call the $\{1\}$ -inverse of A the inner inverse of A , while the $\{2\}$ -inverse of A is called the outer inverse of A . The question naturally arises when the equality analogous to (1.1) holds for the Moore-Penrose inverse, i.e.

$$(A+B)^\dagger = A^\dagger + B^\dagger. \quad (1.3)$$

This problem is not solved in general, while first results on it can already be found in [17].

Lemma 1.2. [17] If $A_i \in \mathbb{C}^{m \times n}$, $i = \overline{1, n}$, $A = \sum_{i=1}^n A_i$ and $(\forall i, j = \overline{1, n})(i \neq j \Rightarrow A_i A_j^* = A_i^* A_j = 0)$ then $A^\dagger = \sum_{i=1}^n A_i^\dagger$.

Baksalary et al. [1] gave other sufficient conditions for (1.3) independent of those in Lemma 1.2.

Proposition 1.1. [1] Let $A, B \in \mathbb{C}^{m \times n}$ be such that the identities

$$(A+B)B^* = B^*(A+B) = 0$$

are satisfied. Then the condition (1.3) is satisfied as well.

Notice that Proposition 1.1 can be derived from Lemma 1.2 if we take $n = 2$, $A_1 = A+B$ and $A_2 = -B$. Necessary and sufficient conditions for (1.3) when A and B are certain functions of orthogonal projectors or A and B are rank additive are also presented in [1]. In [16] it is shown that if $A, B \in \mathbb{C}^{n \times n}$ and $\mathcal{R}(B^*) = \mathcal{N}(A)$ and $\mathcal{R}(B) \subseteq \mathcal{N}(A^*)$, then $A+B$ is invertible and, of course, $(A+B)^{-1} = A^\dagger + B^\dagger$. An example given in [16], which relies on singular value decomposition, shows that conditions from Lemma 1.2 and Proposition 1.1 are not necessary for the obedience of equality (1.3). Various formulas for the Moore-Penrose inverse of a sum of two matrices can be found in [4, 9, 11].

The second essential inverse for each square matrix (in terms of spectral properties) is the Drazin inverse. Recall that for a square matrix $A \in \mathbb{C}^{n \times n}$ the smallest nonnegative integer k such that $r(A^k) = r(A^{k+1})$ is called the index of A and is denoted by $\text{ind}(A)$. If matrix A is of index k , there exists the unique matrix X which satisfies equations

$$(1k) A^k XA = A^k, \quad (2) XAX = X, \quad (5) AX = XA.$$

This matrix is called the Drazin inverse of A and is denoted by A^D . First results on the additivity of the Drazin inverse were published by Drazin himself in the paper [8].

Lemma 1.3. [8] *If $x_1, \dots, x_j$ are given Drazin-invertible elements (of some ring) with $x_s x_t = 0$ ($s, t = 1, \dots, j$; $s \neq t$), then $x_1 + \dots + x_j$ is also Drazin-invertible, with $(x_1 + \dots + x_j)^D = x_1^D + \dots + x_j^D$.*

Representations of the Drazin inverse of the sum of two matrices satisfying different conditions can be found in the papers [6, 10, 14, 19].

In the case when $\text{ind}(A) \leq 1$ we say that $A \in \mathbb{C}^{n \times n}$ is group invertible and its Drazin inverse is called the group inverse. In other words, the matrix X which satisfies

$$(1) AXA = A, \quad (2) XAX = X, \quad (5) AX = XA$$

is the group inverse of A and it is denoted by A^g . In [12] some formulas for the group inverse of the sum of two group invertible matrices that satisfy certain conditions are presented, while in [3, 13] the group invertibility of the linear combination of two k -potent matrices and the group invertibility of some expressions of two idempotent matrices are investigated, respectively. Among the results on the topic of additivity of the group inverse, we came across the following result.

Proposition 1.2. [18] *Let $A, B \in \mathbb{C}^{n \times n}$. Suppose that A^g and B^g exist and that one has the following relationships between A and B :*

$$(A + B)B^g = B^g(A + B) = 0. \quad (1.4)$$

Then $A + B$ is group invertible and

$$(A + B)^g = A^g + B^g. \quad (1.5)$$

Let us mention that condition (1.4) is equivalent to

$$(A + B)B = B(A + B) = 0, \quad (1.6)$$

when B^g exists. Actually, in [18], it is stated that condition (1.4) and group invertibility of B imply the group invertibility of A and $A + B$. This is not true and the following example illustrates that. If we suppose that A and B are group invertible (or B and $A + B$) as in the Proposition 1.2 the statement is valid.

Example 1.1. Let $A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then matrix B is group invertible and $B^g = B$, $(A + B)B^g = 0 = B^g(A + B)$, while $\text{ind}(A) = \text{ind}(A + B) = 2$.

In the same manner as in the case of the Moore-Penrose inverse, i.e. by using Lemma 1.3 for $j = 2$, $x_1 = A + B$ and $x_2 = -B$, we can conclude that condition (1.6) implies equality $(A + B)^D = A^D + B^D$, for arbitrary matrices $A, B \in \mathbb{C}^{n \times n}$.

Now, let $B \in \mathbb{C}^{n \times n}$ be an invertible matrix. The only matrix $A \in \mathbb{C}^{n \times n}$ which satisfies the conditions from Lemma 1.2 or Lemma 1.3, for $n = 2$ and matrices A and B is $A = 0$, while the only matrix $A \in \mathbb{C}^{n \times n}$ that satisfies the conditions from Proposition 1.1 or Proposition 1.2 is $A = -B$. Therefore, if B is invertible, in the literature, there is only a narrow set of matrices A such that the equality (1.3) or equality (1.5) holds. This fact has motivated our research.

2. Additivity of Moore-Penrose inverse

First, we will examine when (1.3) holds if A and B are invertible matrices. Of course, the conditions for that will be weaker than those from Lemma 1.1.

Theorem 2.1. *Let $A, B \in \mathbb{C}^{n \times n}$ be invertible matrices. Then*

$$(A + B)^+ = A^{-1} + B^{-1} = A^\dagger + B^\dagger$$

if and only if AB^{-1} is a diagonalizable matrix whose eigenvalues belong to the set $Y = \{-1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}\}$, its eigenspace associated with eigenvalue -1 , E_{-1} , is orthogonal to its eigenspaces associated with eigenvalues from the set $X = \{\frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}\}$ and E_{-1} is invariant under the transformation BB^ .*

Proof. The matrix $A^{-1} + B^{-1}$ is $\{1\}$ -inverse of the matrix $A + B$ if and only if

$$\begin{aligned} (A + B)(A^{-1} + B^{-1})(A + B) &= A + B \Leftrightarrow (2I_n + AB^{-1} + BA^{-1})(A + B) = A + B \\ &\Leftrightarrow 2(A + B) + AB^{-1}A + BA^{-1}B = 0 \quad \setminus \cdot B^{-1}AB^{-1}(\text{invertible matrix}) \\ &\Leftrightarrow (AB^{-1})^3 + 2(AB^{-1})^2 + 2AB^{-1} + I_n = 0, \end{aligned}$$

and $A^{-1} + B^{-1}$ is $\{2\}$ -inverse of the matrix $A + B$ if and only if

$$\begin{aligned} (A^{-1} + B^{-1})(A + B)(A^{-1} + B^{-1}) &= A^{-1} + B^{-1} \Leftrightarrow (A^{-1} + B^{-1})(2I_n + AB^{-1} + BA^{-1}) = A^{-1} + B^{-1} \\ &\Leftrightarrow 2(A^{-1} + B^{-1}) + B^{-1}AB^{-1} + A^{-1}BA^{-1} = 0, \quad A \cdot / \quad \setminus \cdot AB^{-1} \\ &\Leftrightarrow (AB^{-1})^3 + 2(AB^{-1})^2 + 2AB^{-1} + I_n = 0. \end{aligned}$$

So, $A^{-1} + B^{-1}$ is inner inverse of the matrix $A + B$ if and only if it is the outer inverse of this matrix, and that happens exactly when matrix $M = AB^{-1}$ satisfies

$$M^3 + 2M^2 + 2M + I_n = 0. \quad (2.1)$$

Similarly to the proof of Lemma 1.1, if $M = PJP^{-1}$, where J is Jordan normal form of M , we can conclude that (2.1) holds if and only if all Jordan blocks J_i of M are of dimension one, and their entries satisfy $\lambda_i^3 + 2\lambda_i^2 + 2\lambda_i + 1 = 0$, i.e. $\lambda_i \in Y = \{-1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}\}$. Hence, (2.1) is true if and only if M is diagonalizable matrix whose eigenvalues belong to the set Y .

Further, the matrix $A^{-1} + B^{-1}$ is $\{3, 4\}$ -inverse of the matrix $A + B$ if and only if the matrices

$$(A + B)(A^{-1} + B^{-1}) = 2I_n + AB^{-1} + BA^{-1} \quad \text{and} \quad (A^{-1} + B^{-1})(A + B) = 2I_n + B^{-1}A + A^{-1}B$$

are hermitian i.e. if and only if matrices $M + M^{-1}$ and $B^{-1}(M + M^{-1})B$ are hermitian. Let matrix P be $P = \begin{bmatrix} P_1 & P_2 \end{bmatrix}$, where the columns of matrix $P_1 \in \mathbb{C}^{n \times s}$ span E_{-1} , the eigenspace of M associated with eigenvalue -1 , while the columns of matrix $P_2 \in \mathbb{C}^{n \times (n-s)}$ are the eigenvectors of M corresponding to eigenvalues $\frac{-1+i\sqrt{3}}{2}$ and $\frac{-1-i\sqrt{3}}{2}$ and $P^{-1} = \begin{bmatrix} U \\ V \end{bmatrix}$, where $U \in \mathbb{C}^{s \times n}$ and $V \in \mathbb{C}^{(n-s) \times n}$. Then, the matrix

$$M + M^{-1} = P(J + J^{-1})P^{-1} = \begin{bmatrix} P_1 & P_2 \end{bmatrix} \begin{bmatrix} -2I_s & 0 \\ 0 & -I_{n-s} \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix} = -I_n - P_1U \quad (2.2)$$

is hermitian if and only if the matrix P_1U is hermitian. From the identity

$$I_n = P^{-1}P = \begin{bmatrix} U \\ V \end{bmatrix} \begin{bmatrix} P_1 & P_2 \end{bmatrix} = \begin{bmatrix} UP_1 & UP_2 \\ VP_1 & VP_2 \end{bmatrix} \quad (2.3)$$

we get that $UP_1 = I_s$ and $VP_2 = I_{n-s}$ which implies that U and V are $\{1, 2, 4\}$ -inverses of P_1 and P_2 , respectively. So P_1U is hermitian if and only if $U = P_1^\dagger$.

If $U = P_1^\dagger$, we have that

$$I_n = PP^{-1} = \begin{bmatrix} P_1 & P_2 \end{bmatrix} \begin{bmatrix} P_1^\dagger \\ V \end{bmatrix} = P_1 P_1^\dagger + P_2 V,$$

which implies that $V = P_2^\dagger$ and $I_n - P_1 P_1^\dagger = P_2 P_2^\dagger$. So, $\mathcal{R}(P_1) \oplus^\perp \mathcal{R}(P_2) = \mathbb{C}^n$. Suppose now that $\mathcal{R}(P_1) \oplus^\perp \mathcal{R}(P_2) = \mathbb{C}^n$. We have that

$$\begin{bmatrix} P_1 & P_2 \end{bmatrix} \begin{bmatrix} P_1^\dagger \\ P_2^\dagger \end{bmatrix} = P_1 P_1^\dagger + P_2 P_2^\dagger = P_1 P_1^\dagger + I_n - P_1 P_1^\dagger = I_n,$$

which implies that $P^{-1} = \begin{bmatrix} P_1^\dagger \\ P_2^\dagger \end{bmatrix}$, and $U = P_1^\dagger$.

Now, let us discuss when the matrix $B^{-1}(M + M^{-1})B = B^{-1}(-I_n - P_1 P_1^\dagger)B = -I_n - B^{-1}P_1 P_1^\dagger B$ is hermitian. This happens exactly when the matrix $B^{-1}P_1 P_1^\dagger B$ is hermitian i.e. $P_1 P_1^\dagger = (BB^*)^{-1}P_1 P_1^\dagger BB^*$. This is true if and only if the subspace $\mathcal{R}(P_1)$ is invariant under the transformation BB^* . Let us prove it. Suppose that $P_1 P_1^\dagger = (BB^*)^{-1}P_1 P_1^\dagger BB^*$ and let $x \in \mathcal{R}(P_1)$ be arbitrary. Then, $x = P_1 P_1^\dagger x = (BB^*)^{-1}P_1 P_1^\dagger BB^* x$ implies $BB^* x = P_1 P_1^\dagger BB^* x$, which means that $BB^* x \in \mathcal{R}(P_1)$. So, we proved that $BB^*(\mathcal{R}(P_1)) \subseteq \mathcal{R}(P_1)$. Now let us suppose that $BB^*(\mathcal{R}(P_1)) \subseteq \mathcal{R}(P_1)$ and let $v \in \mathcal{R}(P_1)^\perp = \mathcal{R}(P_2)$ be arbitrary. Then

$$((\forall u \in \mathcal{R}(P_1)) \quad 0 = \langle BB^* u, v \rangle = \langle u, BB^* v \rangle) \Rightarrow BB^* v \in \mathcal{R}(P_1)^\perp,$$

and consequently $BB^*(\mathcal{R}(P_1)^\perp) \subseteq \mathcal{R}(P_1)^\perp$. If $x \in \mathbb{C}^n$ is arbitrary there are unique $u \in \mathcal{R}(P_1)$ and $v \in \mathcal{R}(P_1)^\perp$ such that $x = u + v$. We have that $(BB^*)^{-1}P_1 P_1^\dagger BB^* x = (BB^*)^{-1}P_1 P_1^\dagger BB^* u + (BB^*)^{-1}P_1 P_1^\dagger BB^* v = (BB^*)^{-1}BB^* u = u = P_1 P_1^\dagger x$, which implies $P_1 P_1^\dagger = (BB^*)^{-1}P_1 P_1^\dagger BB^*$. $\square$

In previous theorem, the condition E_{-1} is invariant under the transformation BB^* can be replaced with the condition sum of eigenspaces of matrix AB^{-1} associated with eigenvalues from the set $X = \{\frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}\}$, $E_{\frac{-1+i\sqrt{3}}{2}} \oplus E_{\frac{-1-i\sqrt{3}}{2}}$, is invariant under the transformation BB^* .

Example 2.1. Let $A = \begin{pmatrix} -\frac{7}{26} + \frac{9i\sqrt{3}}{13} & \frac{i(64\sqrt{3}+87i)}{52} & \frac{i(48\sqrt{3}+49i)}{52} \\ \frac{11i(2\sqrt{3}+i)}{26} & \frac{i(84\sqrt{3}+55i)}{52} & \frac{i(76\sqrt{3}+25i)}{52} \\ \frac{i(19\sqrt{3}+34i)}{26} & \frac{i(69\sqrt{3}+40i)}{52} & \frac{i(55\sqrt{3}+56i)}{52} \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 3 & 2 \\ 1 & 2 & 1 \\ 2 & 2 & 2 \end{pmatrix}$. Then $AB^{-1} = P \cdot D \cdot P^{-1}$

where $P = \begin{pmatrix} -1 & 1 & 3 \\ 7 & 1 & 1 \\ 1 & 1 & -4 \end{pmatrix}$ and $D = \begin{pmatrix} \frac{-1-i\sqrt{3}}{2} & 0 & 0 \\ 0 & \frac{-1+i\sqrt{3}}{2} & 0 \\ 0 & 0 & -1 \end{pmatrix}$. It is evident that the eigenspace associated with

the eigenvalue -1 is orthogonal to the eigenspaces associated with eigenvalues $\{\frac{-1-i\sqrt{3}}{2}, \frac{-1+i\sqrt{3}}{2}\}$. Since $BB^* \begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix} =$

$\begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix}$, eigenspace E_{-1} is invariant under the transformation BB^* . Matrices A and B are invertible, $A + B$ is singular matrix and

$$(A + B)^\dagger = \begin{pmatrix} -\frac{7}{26} - \frac{57i\sqrt{3}}{104} & \frac{15}{26} + \frac{59i\sqrt{3}}{104} & -\frac{3}{52} - \frac{7i\sqrt{3}}{26} \\ -\frac{9}{52} - \frac{47i\sqrt{3}}{104} & \frac{23}{52} + \frac{45i\sqrt{3}}{104} & -\frac{1}{52} - \frac{3i\sqrt{3}}{13} \\ \frac{29}{52} + \frac{87i\sqrt{3}}{104} & -\frac{51}{52} - \frac{101i\sqrt{3}}{104} & \frac{9}{52} + \frac{5i\sqrt{3}}{13} \end{pmatrix} = A^{-1} + B^{-1}.$$

Example 2.2. Let $A = \begin{pmatrix} -\frac{5}{7} - \frac{2i\sqrt{3}}{7} & \frac{-31-11i\sqrt{3}}{14} & \frac{-19-9i\sqrt{3}}{14} \\ \frac{i(5\sqrt{3}+23i)}{28} & -\frac{i(\sqrt{3}-29i)}{14} & -\frac{i(\sqrt{3}-29i)}{28} \\ -\frac{i(\sqrt{3}-25i)}{28} & \frac{-9-25i\sqrt{3}}{14} & \frac{-23-39i\sqrt{3}}{28} \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 3 & 2 \\ 1 & 2 & 1 \\ 2 & 2 & 2 \end{pmatrix}$. Then $AB^{-1} = P \cdot D \cdot P^{-1}$

where $P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & \frac{3}{2} \\ 2 & 3 & -\frac{1}{2} \end{pmatrix}$ and $D = \begin{pmatrix} \frac{-1-i\sqrt{3}}{2} & 0 & 0 \\ 0 & \frac{-1+i\sqrt{3}}{2} & 0 \\ 0 & 0 & -1 \end{pmatrix}$. It is evident that the eigenspace associated with the

eigenvalue -1 is orthogonal to the eigenspaces associated with eigenvalues $\{\frac{-1-i\sqrt{3}}{2}, \frac{-1+i\sqrt{3}}{2}\}$. E_{-1} is not invariant under the transformation BB^* since $BB^* \begin{pmatrix} 1 \\ \frac{3}{2} \\ -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{43}{2} \\ 14 \\ 18 \end{pmatrix} \notin E_{-1}$. In this case

$$(A+B)^\dagger = \begin{pmatrix} \frac{i(1097\sqrt{3}+847i)}{1834} & \frac{15(49-59i\sqrt{3})}{1834} & \frac{511-461i\sqrt{3}}{1834} \\ \frac{5(49-5i\sqrt{3})}{917} & \frac{3(-119+31i\sqrt{3})}{1834} & \frac{i(179\sqrt{3}+91i)}{1834} \\ \frac{287+83i\sqrt{3}}{1834} & \frac{-98-13i\sqrt{3}}{917} & \frac{i(44\sqrt{3}+7i)}{917} \end{pmatrix}$$

and

$$A^{-1} + B^{-1} = \begin{pmatrix} \frac{15i(\sqrt{3}+i)}{28} & \frac{25(1-i\sqrt{3})}{56} & \frac{15(1-i\sqrt{3})}{56} \\ -\frac{1}{4} - \frac{13i\sqrt{3}}{28} & \frac{7+17i\sqrt{3}}{56} & -\frac{i(\sqrt{3}-7i)}{56} \\ \frac{23+17i\sqrt{3}}{28} & \frac{-29-19i\sqrt{3}}{56} & \frac{5+11i\sqrt{3}}{56} \end{pmatrix},$$

so we have

$$(A+B)^\dagger \neq A^{-1} + B^{-1}.$$

Let $B \in \mathbb{C}^{n \times n}$ be an invertible matrix and $A \in \mathbb{C}^{n \times n}$ an arbitrary matrix. As a continuation of Theorem 2.1, the question naturally arises when is

$$(A+B)^\dagger = A^\dagger + B^\dagger = A^\dagger + B^{-1}. \quad (2.4)$$

Let us see when the matrix $A^\dagger + B^{-1}$ satisfies the four Penrose equations for the matrix $A+B$:

$$\begin{aligned} (P1) \quad (A+B)(A^\dagger + B^{-1})(A+B) &= A+B \Leftrightarrow (AA^\dagger + AB^{-1} + BA^\dagger + I_n)(A+B) = A+B \\ &\Leftrightarrow 2A + AA^\dagger B + AB^{-1}A + BA^\dagger A + BA^\dagger B = 0 \\ &\stackrel{\setminus B^{-1}}{\Leftrightarrow} 2AB^{-1} + AA^\dagger + AB^{-1}AB^{-1} + BA^\dagger AB^{-1} + BA^\dagger = 0, \\ (P2) \quad (A^\dagger + B^{-1})(A+B)(A^\dagger + B^{-1}) &= A^\dagger + B^{-1} \Leftrightarrow (A^\dagger + B^{-1})(AA^\dagger + AB^{-1} + BA^\dagger + I_n) = A^\dagger + B^{-1} \\ &\Leftrightarrow 2A^\dagger + B^{-1}AA^\dagger + A^\dagger AB^{-1} + B^{-1}AB^{-1} + A^\dagger BA^\dagger = 0 \\ &\stackrel{B^{-1}}{\Leftrightarrow} 2BA^\dagger + AA^\dagger + BA^\dagger AB^{-1} + AB^{-1} + BA^\dagger BA^\dagger = 0, \\ (P3) \quad (A+B)(A^\dagger + B^{-1}) &\text{ is hermitian } \Leftrightarrow AA^\dagger + AB^{-1} + BA^\dagger + I_n \text{ is hermitian} \\ &\Leftrightarrow AB^{-1} + BA^\dagger \text{ is hermitian,} \\ (P4) \quad (A^\dagger + B^{-1})(A+B) &\text{ is hermitian } \Leftrightarrow A^\dagger A + A^\dagger B + B^{-1}A + I_n \text{ is hermitian} \\ &\Leftrightarrow A^\dagger B + B^{-1}A \text{ is hermitian.} \end{aligned}$$

Notice that matrix $BA^\dagger$ is $\{1, 2, 3\}$ -inverse of matrix $M = AB^{-1}$, so the last equality of (P1) can be written as

$$2M + MM^{(1,2,3)} + MM + M^{(1,2,3)}M + M^{(1,2,3)} = 0, \quad (2.5)$$

for $M^{(1,2,3)} = BA^\dagger$.

Lemma 2.1. Let $B \in \mathbb{C}^{n \times n}$ be an invertible matrix and $A \in \mathbb{C}^{n \times n}$ be an arbitrary matrix. If matrix $A^\dagger + B^{-1} = A^\dagger + B^\dagger$ is inner inverse of matrix $A+B$ then AB^{-1} is diagonalizable matrix whose eigenvalues belong to the set $Z = \{0, -1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}\}$.

Proof. Since matrix $A^\dagger + B^{-1}$ is inner inverse of matrix $A+B$, matrix $M = AB^{-1}$ must satisfies (2.5). If we multiply (2.5) the left-hand side and the right-hand side by M we get

$$M^4 + 2M^3 + 2M^2 + M = 0. \quad (2.6)$$

Since polynomial $x^4 + 2x^3 + 2x^2 + x = 0$ does not have multiple roots, matrix M must be diagonalizable and its eigenvalues must satisfy $\lambda_i^4 + 2\lambda_i^3 + 2\lambda_i^2 + \lambda_i = 0$, i.e. $\lambda_i \in Z = \{0, -1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}\}$. $\square$

Notice that, by previous lemma, if (2.4) holds matrix $M = AB^{-1}$ is group invertible. In the next theorem, we derive necessary and sufficient condition for (2.4) to hold if $BA^\dagger$ is M^g . The resulting conditions will be analogous to those from Theorem 2.1. We have that

$$\begin{aligned} BA^\dagger = M^g &\Leftrightarrow BA^\dagger AB^{-1} = AA^\dagger \Leftrightarrow \mathcal{R}(BA^\dagger) = \mathcal{R}(A) \wedge \mathcal{N}(AB^{-1}) = \mathcal{N}(A^*) \\ &\Leftrightarrow B(\mathcal{R}(A^*)) = \mathcal{R}(A) \wedge \mathcal{R}((B^*)^{-1}A^*) = \mathcal{R}(A) \\ &\Leftrightarrow B(\mathcal{R}(A^*)) \subseteq \mathcal{R}(A) \wedge (B^*)^{-1}(\mathcal{R}(A^*)) = \mathcal{R}(A) \\ &\Leftrightarrow B(\mathcal{R}(A^*)) \subseteq \mathcal{R}(A) \wedge B^*(\mathcal{R}(A)) = \mathcal{R}(A^*) \Leftrightarrow B(\mathcal{R}(A^*)) \subseteq \mathcal{R}(A) \wedge B^*(\mathcal{R}(A)) \subseteq \mathcal{R}(A^*) \end{aligned} \quad (2.7)$$

Theorem 2.2. Let $B \in \mathbb{C}^{n \times n}$ be invertible matrix and $A \in \mathbb{C}^{n \times n}$ arbitrary matrix such that $B^*(\mathcal{R}(A)) \subseteq \mathcal{R}(A^*)$ and $B(\mathcal{R}(A^*)) \subseteq \mathcal{R}(A)$. Then

$$(A + B)^\dagger = A^\dagger + B^{-1} = A^\dagger + B^\dagger$$

if and only if AB^{-1} is a diagonalizable matrix whose eigenvalues belong to the set $Z = \{0, -1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}\}$, its eigenspace associated with eigenvalue -1 , E_{-1} , is orthogonal to its eigenspaces associated with eigenvalues from the set $X = \{\frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}\}$ and E_{-1} is invariant under the transformation BB^* .

Proof. If the equality (P1) holds, by Lemma 2.1, $M = AB^{-1}$ is a diagonalizable matrix whose eigenvalues belong to the set $Z = \{0, -1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}\}$. If $M = PJP^{-1}$, where J is Jordan normal form of M , let the matrix P be $P = \begin{bmatrix} P_1 & P_2 \end{bmatrix}$, where the columns of matrix $P_1 \in \mathbb{C}^{n \times s}$ span $\mathcal{R}(M)$, while the columns of matrix $P_2 \in \mathbb{C}^{n \times (n-s)}$ span $\mathcal{N}(M)$. Because $B^*(\mathcal{R}(A)) \subseteq \mathcal{R}(A^*)$ and $B(\mathcal{R}(A^*)) \subseteq \mathcal{R}(A)$ by (2.7) we have that $BA^\dagger = M^g = M^\dagger$, so the matrix M is rang-Hermitian and we have that $\mathcal{R}(P_1) \oplus^\perp \mathcal{R}(P_2) = \mathbb{C}^n$. As in the proof of Theorem 2.1 can be proven that $P^{-1} = \begin{bmatrix} P_1^\dagger \\ P_2^\dagger \end{bmatrix}$. Further, the matrix $BA^\dagger = M^g = PJ^\dagger P^{-1}$. From the other

side, if matrix M is a diagonalizable matrix whose eigenvalues belong to the set $Z = \{0, -1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}\}$, the identity (2.6) holds, and since M is group invertible, if we multiply this identity by $(M^g)^2$ and by $(M^g)^3$, respectively, we get that for all specified matrices M , the equations (P1) and (P2) are satisfied.

Let us see when the equations (P3) and (P4) are satisfied. Represent the matrix P_1 as $P_1 = \begin{bmatrix} P_{1,1} & P_{1,2} \end{bmatrix}$, where the columns of matrix $P_{1,1} \in \mathbb{C}^{n \times k}$ span E_{-1} , the eigenspace of M associated with eigenvalue -1 , while the columns of matrix $P_{1,2} \in \mathbb{C}^{n \times (s-k)}$ are the eigenvectors of M corresponding to eigenvalues $\frac{-1+i\sqrt{3}}{2}$ and $\frac{-1-i\sqrt{3}}{2}$ and $P_1^\dagger = \begin{bmatrix} U \\ V \end{bmatrix}$, where $U \in \mathbb{C}^{k \times n}$ and $V \in \mathbb{C}^{(s-k) \times n}$. Then, the matrix

$$M + M^g = P(J + J^\dagger)P^{-1} = \begin{bmatrix} P_{1,1} & P_{1,2} & P_2 \end{bmatrix} \begin{bmatrix} -2I_k & 0 & 0 \\ 0 & -I_{(s-k)} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} U \\ V \\ P_2^\dagger \end{bmatrix} = -P_{1,1}U - P_1P_1^\dagger$$

is hermitian if and only the matrix $P_{1,1}U$ is hermitian. As in the proof of Theorem 2.1 can be concluded that $P_{1,1}U$ is hermitian if and only if $U = P_{1,1}^\dagger$.

If $U = P_{1,1}^\dagger$, we have that $\mathcal{R}(P_{1,2}) \subseteq \mathcal{N}(U) = \mathcal{N}(P_{1,1}^*)$ which implies $\mathcal{R}(P_{1,1}) \perp \mathcal{R}(P_{1,2})$. From the other side if $\mathcal{R}(P_{1,1}) \perp \mathcal{R}(P_{1,2})$ we have that $\mathcal{R}(P_{1,1}) \oplus^\perp \mathcal{R}(P_{1,2}) = \mathcal{R}(P_1)$ and since $\mathcal{R}(P_1) \oplus^\perp \mathcal{R}(P_2) = \mathbb{C}^n$ we have that

$$P_{1,1}P_{1,1}^\dagger + P_{1,2}P_{1,2}^\dagger = P_1P_1^\dagger \text{ which implies that } P^{-1} = \begin{bmatrix} P_{1,1}^\dagger \\ P_{1,2}^\dagger \\ P_2^\dagger \end{bmatrix}, \text{ and } U = P_{1,1}^\dagger. \text{ So, the equation (P}_3\text{) is satisfied if}$$

and only if $\mathcal{R}(P_{1,1}) \perp \mathcal{R}(P_{1,2})$.

Now, let us discuss when the matrix

$$B^{-1}(M + M^g)B = B^{-1}(-P_{1,1}P_{1,1}^\dagger - P_1P_1^\dagger)B = -B^{-1}P_{1,1}P_{1,1}^\dagger B - B^{-1}P_1P_1^\dagger B = -B^{-1}P_{1,1}P_{1,1}^\dagger B - B^{-1}AA^\dagger B$$

is hermitian. Since the condition (2.7) is satisfied we have that $BA^\dagger AB^{-1} = AA^\dagger$ which implies that $A^\dagger A = B^{-1}AA^\dagger B$ so the equation (P4) is satisfied if and only if $B^{-1}P_{1,1}P_{1,1}^\dagger B$ is hermitian matrix i.e. $P_{1,1}P_{1,1}^\dagger = (BB^*)^{-1}P_{1,1}P_{1,1}^\dagger BB^*$. Similarly as in the proof of Theorem 2.1 can be shown that this is true if and only if the subspace $\mathcal{R}(P_{1,1}) = E_{-1}$ is invariant under the transformation BB^* . $\square$

Example 2.3. Let $A = \begin{pmatrix} -\frac{4}{5} & 0 & \frac{8}{5} & 0 \\ 0 & -2 + 2i\sqrt{3} & 0 & 0 \\ \frac{8}{5} & 0 & -\frac{16}{5} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 0 & 0 & -6 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & -3 \\ -2 & 0 & -1 & 6 \end{pmatrix}$. Then $AB^{-1} = P \cdot D \cdot P^{-1}$

where $P = \begin{pmatrix} 0 & -1 & 6 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 1 \\ 0 & 0 & -5 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} \frac{-1+i\sqrt{3}}{2} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Since $A^\dagger AB^*A = B^*A$ and $AA^\dagger BA^* = BA^*$, the

preconditions $B^*(\mathcal{R}(A)) \subseteq \mathcal{R}(A^*)$ and $B(\mathcal{R}(A^*)) \subseteq \mathcal{R}(A)$ are satisfied. Further, the eigenspace associated with the eigenvalue -1 is orthogonal to the eigenspaces associated with eigenvalues $\{\frac{-1-i\sqrt{3}}{2}, \frac{-1+i\sqrt{3}}{2}\}$. Since $BB^* \begin{pmatrix} -1 \\ 0 \\ 2 \\ 0 \end{pmatrix} =$

$16 \begin{pmatrix} -1 \\ 0 \\ 2 \\ 0 \end{pmatrix}$, eigenspace E_{-1} is invariant under the transformation BB^* . In this case

$$(A + B)^\dagger = \begin{pmatrix} \frac{8}{15} & 0 & \frac{4}{15} & \frac{2}{3} \\ 0 & \frac{1}{8} - \frac{i\sqrt{3}}{8} & 0 & 0 \\ \frac{4}{15} & 0 & \frac{2}{15} & \frac{1}{3} \\ \frac{2}{9} & 0 & \frac{1}{9} & \frac{4}{9} \end{pmatrix} = A^\dagger + B^{-1}.$$

Example 2.4. Let $A = \begin{pmatrix} -\frac{59}{20} + \frac{i\sqrt{3}}{4} & 0 & -\frac{59}{40} + \frac{i\sqrt{3}}{8} & \frac{3(17+i\sqrt{3})}{8} \\ 0 & \frac{i(\sqrt{3}+i)}{2} & 0 & 0 \\ -\frac{59}{40} + \frac{i\sqrt{3}}{8} & 0 & \frac{i(5\sqrt{3}+59i)}{80} & \frac{3(17+i\sqrt{3})}{16} \\ \frac{17+i\sqrt{3}}{8} & 0 & \frac{17+i\sqrt{3}}{16} & \frac{3i(\sqrt{3}+31i)}{16} \end{pmatrix}$ and

$B = \begin{pmatrix} 4 & 0 & 0 & -6 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & -3 \\ -2 & 0 & -1 & 6 \end{pmatrix}$. Then $AB^{-1} = P \cdot D \cdot P^{-1}$ where $P = \begin{pmatrix} 0 & 2 & 6 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 1 & -5 & 0 \end{pmatrix}$ and

$D = \begin{pmatrix} \frac{-1+i\sqrt{3}}{2} & 0 & 0 & 0 \\ 0 & \frac{-1+i\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Since $A^\dagger AB^*A = B^*A$ and $AA^\dagger BA^* = BA^*$, the preconditions $B^*(\mathcal{R}(A)) \subseteq \mathcal{R}(A^*)$

and $B(\mathcal{R}(A^*)) \subseteq \mathcal{R}(A)$ are satisfied. Further, the eigenspace associated with the eigenvalue -1 is not orthogonal to the eigenspace associated with the eigenvalues $\{\frac{-1-i\sqrt{3}}{2}, \frac{-1+i\sqrt{3}}{2}\}$, neither is the eigenspace E_{-1} invariant under the

transformation BB^* since $BB^* \begin{pmatrix} 6 \\ 0 \\ 3 \\ -5 \end{pmatrix} = \begin{pmatrix} 586 \\ 0 \\ 293 \\ -535 \end{pmatrix}$. In this case

$$(A+B)^{\dagger} = \begin{pmatrix} \frac{101-80i\sqrt{3}}{420} & 0 & -\frac{1}{210} - \frac{2i}{7\sqrt{3}} & \frac{2(1-i\sqrt{3})}{21} \\ 0 & \frac{1-i\sqrt{3}}{2} & 0 & 0 \\ -\frac{1}{210} - \frac{2i}{7\sqrt{3}} & 0 & \frac{26-5i\sqrt{3}}{105} & \frac{1-i\sqrt{3}}{21} \\ \frac{2(1-i\sqrt{3})}{7} & 0 & \frac{1-i\sqrt{3}}{7} & \frac{1-i\sqrt{3}}{7} \end{pmatrix}$$

and

$$A^{\dagger} + B^{-1} = \begin{pmatrix} \frac{339\sqrt{3}+318i}{80\sqrt{3}+1260i} & 0 & \frac{319\sqrt{3}+3i}{160\sqrt{3}+2520i} & \frac{3(67\sqrt{3}+51i)}{8(4\sqrt{3}+63i)} \\ 0 & \frac{1-i\sqrt{3}}{2} & 0 & 0 \\ \frac{319\sqrt{3}+3i}{160\sqrt{3}+2520i} & 0 & \frac{3(133\sqrt{3}+421i)}{80(4\sqrt{3}+63i)} & \frac{3(67\sqrt{3}+51i)}{16(4\sqrt{3}+63i)} \\ \frac{1-i\sqrt{3}}{8} & 0 & \frac{1-i\sqrt{3}}{16} & \frac{3(1-i\sqrt{3})}{16} \end{pmatrix},$$

so it follows

$$(A+B)^{\dagger} \neq A^{\dagger} + B^{-1}.$$

We complete this section with a generalization of Theorem 2.2, whose consequence will be a complete description of the matrices $A \in \mathbb{C}^{n \times n}$ for which equality (2.4) holds in the case when matrix B is unitary. Recall that the matrix $U \in \mathbb{C}^{n \times n}$ is unitary if $UU^* = U^*U = I_n$.

Theorem 2.3. Let $B \in \mathbb{C}^{n \times n}$ be invertible matrix and $A \in \mathbb{C}^{n \times n}$ arbitrary matrix such that $B^*B(\mathcal{R}(A^*)) \subseteq \mathcal{R}(A^*)$. Then (2.4) holds if and only if $B(\mathcal{R}(A^*)) \subseteq \mathcal{R}(A)$, AB^{-1} is diagonalizable matrix whose eigenvalues belong to the set $Z = \{0, -1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}\}$, its eigenspace associated with eigenvalue -1 , E_{-1} , is orthogonal to its eigenspaces associated with eigenvalues from the set $X = \{\frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}\}$ and E_{-1} is invariant under the transformation BB^* .

Proof. Since $B^*B(\mathcal{R}(A^*)) = \mathcal{R}(A^*) \Rightarrow B(\mathcal{R}(A^*)) = (B^*)^{-1}(\mathcal{R}(A^*)) \Rightarrow \mathcal{R}(BA^{\dagger}) = \mathcal{R}((AB^{-1})^*)$, we have that $BA^{\dagger} = (AB^{-1})^{\dagger}$.

Suppose that (2.4) holds and let $x \in \mathcal{N}(M)$ be arbitrary, where $M = AB^{-1}$. From Penrose equations (P1) and (P2) we get $MM^{\dagger}x + M^{\dagger}x = 0$ and $2M^{\dagger}x + MM^{\dagger}x + M^{\dagger}M^{\dagger}x = 0$ that imply $M^{\dagger}x = -MM^{\dagger}x \in \mathcal{R}(M)$ and $M^{\dagger}x + M^{\dagger}M^{\dagger}x = 0 \Rightarrow (I_n + M^{\dagger})x \in \mathcal{N}(M^{\dagger}) = \mathcal{N}(M^*) = \mathcal{R}(M)^{\perp}$. Further $0 = \langle M^{\dagger}x, (I_n + M^{\dagger})x \rangle = \|M^{\dagger}x\|^2 \Rightarrow M^{\dagger}x = 0$ that as a consequence has $\mathcal{N}(M) = \mathcal{N}(M^{\dagger})$. Now, $\mathcal{R}(M^{\dagger}) = \mathcal{N}(M)^{\perp} = \mathcal{N}(M^*)^{\perp} = \mathcal{R}(M)$ that gives $M^{\dagger} = M^*$ and by (2.7) we have that $B^*(\mathcal{R}(A)) \subseteq \mathcal{R}(A^*)$ and $B(\mathcal{R}(A^*)) \subseteq \mathcal{R}(A)$. The rest of the proof follows from Theorem 2.2.

Let now $A \in \mathbb{C}^{n \times n}$ be such that $B^*B(\mathcal{R}(A^*)) \subseteq \mathcal{R}(A^*)$, $B(\mathcal{R}(A^*)) \subseteq \mathcal{R}(A)$, AB^{-1} is a diagonalizable matrix whose eigenvalues belong to the set $Z = \{0, -1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}\}$, its eigenspace associated with eigenvalue -1 , E_{-1} , is orthogonal to its eigenspaces associated with eigenvalues from the set $X = \{\frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}\}$ and E_{-1} is invariant under the transformation BB^* . We have that $B^*(\mathcal{R}(A)) = B^*B(\mathcal{R}(A^*)) = \mathcal{R}(A^*)$, so by Theorem 2.2 the equality (2.4) holds. $\square$

Corollary 2.1. Let $B \in \mathbb{C}^{n \times n}$ be unitary matrix and $A \in \mathbb{C}^{n \times n}$ arbitrary matrix. Then (2.4) holds if and only if $B(\mathcal{R}(A^*)) \subseteq \mathcal{R}(A)$, AB^* is a diagonalizable matrix whose eigenvalues belong to the set $Z = \{0, -1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}\}$ and its eigenspace associated with eigenvalue -1 , E_{-1} , is orthogonal to its eigenspaces associated with eigenvalues from the set $X = \{\frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}\}$.

3. Additivity of group inverse

In this section, we will present results analogous to those from Section 2. Instead of the Moore-Penrose inverse, we will investigate the group inverse. As both Moore-Penrose and group inverse satisfy Penrose equations (1) and (2), we will carry over certain results from the previous section. We will start by examining the additivity of the group inverse of two invertible matrices.

Theorem 3.1. *Let $A, B \in \mathbb{C}^{n \times n}$ be invertible matrices. Then $A + B$ is group invertible and*

$$(A + B)^g = A^{-1} + B^{-1} = A^g + B^g$$

if and only if AB^{-1} is diagonalizable matrix whose eigenvalues belong to the set $Y = \{-1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}\}$, its eigenspace associated with eigenvalue -1 , E_{-1} , is invariant for the transformation B and sum of its eigenspaces associated with eigenvalues from the set $X = \{\frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}\}$, $E_{\frac{-1+i\sqrt{3}}{2}} \oplus E_{\frac{-1-i\sqrt{3}}{2}}$, is invariant under the transformation B .

Proof. From the proof of Theorem 2.1 we get that $A^{-1} + B^{-1}$ is $\{1, 2\}$ -inverse of the matrix $A + B$ if and only if matrix $M = AB^{-1}$ is a diagonalizable matrix whose eigenvalues belong to the set $Y = \{-1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}\}$. Further, $A^{-1} + B^{-1}$ is $\{5\}$ -inverse of the matrix $A + B$ if and only if

$$\begin{aligned} (A + B)(A^{-1} + B^{-1}) &= (A^{-1} + B^{-1})(A + B) \Leftrightarrow AB^{-1} + BA^{-1} = B^{-1}A + A^{-1}B \\ &\Leftrightarrow M + M^{-1} = B^{-1}(M + M^{-1})B. \end{aligned} \quad (3.1)$$

If $M = PJP^{-1}$, where J is Jordan normal form of M , matrix P be $P = \begin{bmatrix} P_1 & P_2 \end{bmatrix}$, where the columns of matrix $P_1 \in \mathbb{C}^{n \times s}$ span E_{-1} , the eigenspace of M associated with eigenvalue -1 , while the columns of matrix $P_2 \in \mathbb{C}^{n \times (n-s)}$ are the eigenvectors of M corresponding to eigenvalues $\frac{-1+i\sqrt{3}}{2}$ and $\frac{-1-i\sqrt{3}}{2}$ and then span $E_{\frac{-1+i\sqrt{3}}{2}} \oplus E_{\frac{-1-i\sqrt{3}}{2}}$, the sum of eigenspaces of M associated with eigenvalues $\frac{-1 \pm i\sqrt{3}}{2}$. Let P^{-1} be $P^{-1} = \begin{bmatrix} U \\ V \end{bmatrix}$, where $U \in \mathbb{C}^{s \times n}$ and $V \in \mathbb{C}^{(n-s) \times n}$. From (2.2), we get that (3.1) is satisfied if and only if

$$P_1 U = B^{-1} P_1 U B. \quad (3.2)$$

Since $U \in P_1\{1, 2, 4\}$ and $UB \in (B^{-1}P_1)\{1, 2, 4\}$, matrices $P_1 U$ and $B^{-1}P_1 U B$ are idempotents and equality (3.2) holds if and only if these idempotents have the same range and null space, i.e.

$$\begin{aligned} \mathcal{R}(P_1) = B^{-1}(\mathcal{R}(P_1)) \wedge \mathcal{N}(U) = \mathcal{N}(UB) &\Leftrightarrow \mathcal{R}(P_1) = B(\mathcal{R}(P_1)) \wedge \mathcal{N}(U) = B^{-1}(\mathcal{N}(U)) \\ &\Leftrightarrow B(\mathcal{R}(P_1)) \subseteq \mathcal{R}(P_1) \wedge B(\mathcal{N}(U)) = \mathcal{N}(U) \\ &\stackrel{\mathcal{N}(U)=\mathcal{R}(P_2)}{\Leftrightarrow} B(\mathcal{R}(P_1)) \subseteq \mathcal{R}(P_1) \wedge B(\mathcal{R}(P_2)) \subseteq \mathcal{R}(P_2). \end{aligned}$$

□

Example 3.1. Let $A = \begin{pmatrix} -26 - 6i\sqrt{3} & 12 + 3i\sqrt{3} & 15 + 4i\sqrt{3} \\ -\frac{41}{2} - \frac{11i\sqrt{3}}{2} & 9 + 3i\sqrt{3} & \frac{25}{2} + \frac{7i\sqrt{3}}{2} \\ -\frac{51}{2} - \frac{13i\sqrt{3}}{2} & 12 + 3i\sqrt{3} & \frac{29}{2} + \frac{9i\sqrt{3}}{2} \end{pmatrix}$ and $B = \begin{pmatrix} 34 & -15 & -21 \\ 29 & -12 & -19 \\ 33 & -15 & -20 \end{pmatrix}$. Then $AB^{-1} = P \cdot D \cdot P^{-1}$ where $P = \begin{pmatrix} 2 & 1 & 3 \\ 3 & 1 & 2 \\ 1 & 1 & 3 \end{pmatrix}$ and $D = \begin{pmatrix} \frac{-1+i\sqrt{3}}{2} & 0 & 0 \\ 0 & \frac{-1-i\sqrt{3}}{2} & 0 \\ 0 & 0 & -1 \end{pmatrix}$. Since $B \begin{pmatrix} 2 & 1 \\ 3 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 3 & -2 \\ 1 & -2 \end{pmatrix}$ we conclude that $E_{\frac{-1+i\sqrt{3}}{2}} \oplus E_{\frac{-1-i\sqrt{3}}{2}}$ is invariant under the transformation B . Since $B \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix} = 3 \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}$, E_{-1} is also invariant under

transformation B . Conditions of Theorem 1.3 are satisfied and

$$(A + B)^g = \begin{pmatrix} \frac{11+3i\sqrt{3}}{4} & -\frac{3(2+2i\sqrt{3})}{8} & -\frac{i(\sqrt{3}-9i)}{4} \\ \frac{13+i\sqrt{3}}{4} & -\frac{3(2+2i\sqrt{3})}{8} & \frac{i(\sqrt{3}+11i)}{4} \\ \frac{9+5i\sqrt{3}}{4} & -\frac{3(2+2i\sqrt{3})}{8} & -\frac{7}{4} - \frac{3i\sqrt{3}}{4} \end{pmatrix} = A^{-1} + B^{-1}.$$

Example 3.2. Let $A = \begin{pmatrix} -20 - 20i\sqrt{3} & 9 + 10i\sqrt{3} & 12 + 11i\sqrt{3} \\ -\frac{33}{2} - \frac{43i\sqrt{3}}{2} & 7 + 11i\sqrt{3} & \frac{21+23i\sqrt{3}}{2} \\ -\frac{35}{2} - \frac{37i\sqrt{3}}{2} & 8 + 9i\sqrt{3} & \frac{21(1+i\sqrt{3})}{2} \end{pmatrix}$ and $B = \begin{pmatrix} 34 & -15 & -21 \\ 29 & -12 & -19 \\ 33 & -15 & -20 \end{pmatrix}$. Then $AB^{-1} = P \cdot D \cdot P^{-1}$ where $P = \begin{pmatrix} 2 & 1 & 3 \\ 3 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} \frac{-1+i\sqrt{3}}{2} & 0 & 0 \\ 0 & \frac{-1-i\sqrt{3}}{2} & 0 \\ 0 & 0 & -1 \end{pmatrix}$. As in Example 3.1 we conclude that $E_{\frac{-1+i\sqrt{3}}{2}} \oplus E_{\frac{-1-i\sqrt{3}}{2}}$ is invariant under the transformation B , but $B \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 51 \\ 44 \\ 49 \end{pmatrix} \notin E_{-1}$. In this case

$$(A + B)^g = \begin{pmatrix} \frac{5}{4} + \frac{29i\sqrt{3}}{4} & -4i\sqrt{3} & -\frac{3}{2} - \frac{7i\sqrt{3}}{2} \\ -\frac{1}{4} + \frac{35i\sqrt{3}}{4} & 1 - 5i\sqrt{3} & -1 - 4i\sqrt{3} \\ \frac{11}{4} + \frac{23i\sqrt{3}}{4} & -1 - 3i\sqrt{3} & -2 - 3i\sqrt{3} \end{pmatrix}$$

but

$$A^{-1} + B^{-1} = \begin{pmatrix} -\frac{1}{4} + \frac{5i}{4\sqrt{3}} & \frac{9-7i\sqrt{3}}{12} & -\frac{i(\sqrt{3}-9i)}{12} \\ \frac{i(7\sqrt{3}+5i)}{12} & \frac{13-11i\sqrt{3}}{12} & \frac{i(\sqrt{3}+11i)}{12} \\ -\frac{1}{12} + \frac{i\sqrt{3}}{4} & \frac{5-3i\sqrt{3}}{12} & -\frac{7}{12} - \frac{i\sqrt{3}}{4} \end{pmatrix},$$

so

$$(A + B)^g \neq A^{-1} + B^{-1}.$$

Now we will consider what happens if the assumption of invertibility of the matrix A is omitted, but it is assumed that the matrix A is group invertible and satisfies certain conditions analogous to those in Section 2.

Lemma 3.1. Let $B \in \mathbb{C}^{n \times n}$ be invertible matrix and $A \in \mathbb{C}^{n \times n}$ be matrix such that $\text{ind}(A) \leq 1$. If matrix $A^g + B^{-1} = A^g + B^g$ is inner inverse of matrix $A + B$ then AB^{-1} is diagonalizable matrix whose eigenvalues belong to the set $Z = \{0, -1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}\}$.

Proof. As in the case of Moore-Penrose inverse, matrix $A^g + B^{-1}$ satisfies the first Penrose equation for the matrix $A + B$ if and only if $2AB^{-1} + AB^{-1}BA^g + AB^{-1}AB^{-1} + BA^gAB^{-1} + BA^g = 0$. Since $BA^g \in (AB^{-1})\{1, 2\}$, the rest of the proof is the same as in Lemma 2.1. $\square$

Theorem 3.2. Let $B \in \mathbb{C}^{n \times n}$ be invertible matrix and $A \in \mathbb{C}^{n \times n}$ be matrix such that $\text{ind}(A) \leq 1$, $B(\mathcal{R}(A)) \subseteq \mathcal{R}(A)$ and $B(\mathcal{N}(A)) \subseteq \mathcal{N}(A)$. Then matrix $A + B$ is group invertible and

$$(A + B)^g = A^g + B^{-1} = A^g + B^g \quad (3.3)$$

if and only if AB^{-1} is a diagonalizable matrix whose eigenvalues belong to the set $Z = \{0, -1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}\}$, its eigenspace associated with eigenvalue -1 , E_{-1} , is invariant for the transformation B and the sum of its eigenspaces associated with eigenvalues from the set $X = \{\frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}\}$, $E_{\frac{-1+i\sqrt{3}}{2}} \oplus E_{\frac{-1-i\sqrt{3}}{2}}$, is invariant under the transformation B .

Proof. Because of the following equivalences

$$\begin{aligned} BA^g = M^g &\Leftrightarrow BA^g AB^{-1} = AA^g \Leftrightarrow \mathcal{R}(BA^g) = \mathcal{R}(A) \wedge \mathcal{N}(AB^{-1}) = \mathcal{N}(A) \\ &\Leftrightarrow B(\mathcal{R}(A)) = \mathcal{R}(A) \wedge B(\mathcal{N}(A)) = \mathcal{N}(A) \Leftrightarrow B(\mathcal{R}(A)) \subseteq \mathcal{R}(A) \wedge B(\mathcal{N}(A)) \subseteq \mathcal{N}(A) \end{aligned}$$

we have that $BA^g = M^g$.

If matrix $A^g + B^{-1}$ satisfies first and second Penrose equation for matrix $A + B$ by Lemma 3.1 matrix $M = AB^{-1}$ is diagonalizable and its eigenvalues belong to the set $Z = \{0, -1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}\}$. Conversely, for all matrices A such that matrix M is diagonalizable matrix whose eigenvalues belong to the set Z , in the same manner as in Theorem 2.2, since $BA^g = M^g$, it can be concluded that $A^g + B^{-1} \in (A + B)\{1, 2\}$.

Further, $A^g + B^{-1}$ is $\{5\}$ -inverse of the matrix $A + B$ if and only if $M + M^g = B^{-1}(M + M^g)B$. If $M = PJP^{-1}$, where J is Jordan normal form of M , matrix $P = \begin{bmatrix} P_{1,1} & P_{1,2} & P_2 \end{bmatrix}$, where the columns of matrix $P_{1,1} \in \mathbb{C}^{n \times k}$ span E_{-1} , the eigenspace of M associated with eigenvalue -1 , the columns of matrix $P_{1,2} \in \mathbb{C}^{n \times (s-k)}$ are the eigenvectors of M corresponding to eigenvalues $\frac{-1+i\sqrt{3}}{2}$ and $\frac{-1-i\sqrt{3}}{2}$ and then span $E_{\frac{-1+i\sqrt{3}}{2}} \oplus E_{\frac{-1-i\sqrt{3}}{2}}$, the sum of eigenspaces of M associated with eigenvalues $\frac{-1+i\sqrt{3}}{2}$, and the columns of matrix $P_2 \in \mathbb{C}^{n \times (n-s)}$ span $\mathcal{N}(M)$.

Let $P^{-1} = \begin{bmatrix} U \\ V \\ W \end{bmatrix}$, where $U \in \mathbb{C}^{k \times n}$, $V \in \mathbb{C}^{(s-k) \times n}$ and $W \in \mathbb{C}^{(n-s) \times n}$. By direct computation we get that $P_2 W$ is projection with range $\mathcal{N}(M) = \mathcal{N}(AB^{-1}) = B\mathcal{N}(A) = \mathcal{N}(A)$ and null space $\mathcal{R}(A)$, so $P_{1,1}U + P_{1,2}V = AA^g$. Furthermore, $M^g = P J^\dagger P^{-1}$ and we obtain

$$M + M^g = P(J + J^\dagger)P^{-1} = \begin{bmatrix} P_{1,1} & P_{1,2} & P_2 \end{bmatrix} \begin{bmatrix} -2I_k & 0 & 0 \\ 0 & -I_{(s-k)} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} U \\ V \\ W \end{bmatrix} = -P_{1,1}U - AA^g.$$

Since $AA^g = B^{-1}AA^gB$, we have that $A^g + B^{-1}$ is $\{5\}$ -inverse of the matrix $A + B$ if and only if $P_{1,1}U = B^{-1}P_{1,1}UB$, i.e. if and only if $B(\mathcal{R}(P_{1,1})) \subseteq \mathcal{R}(P_{1,1})$ and $B(\mathcal{N}(U)) \subseteq \mathcal{N}(U)$, i.e. if and only if

$$B(E_{-1}) \subseteq E_{-1} \quad \text{and} \quad B\left(E_{\frac{-1+i\sqrt{3}}{2}} \oplus E_{\frac{-1-i\sqrt{3}}{2}} \oplus \mathcal{N}(A)\right) \subseteq E_{\frac{-1+i\sqrt{3}}{2}} \oplus E_{\frac{-1-i\sqrt{3}}{2}} \oplus \mathcal{N}(A). \quad (3.4)$$

Since $B(\mathcal{R}(A)) \subseteq \mathcal{R}(A)$, $E_{\frac{-1+i\sqrt{3}}{2}} \oplus E_{\frac{-1-i\sqrt{3}}{2}} \subseteq \mathcal{R}(A)$ and $B(\mathcal{N}(A)) \subseteq \mathcal{N}(A)$ conditions from (3.4) are equivalent with

$$B(E_{-1}) \subseteq E_{-1} \quad \text{and} \quad B\left(E_{\frac{-1+i\sqrt{3}}{2}} \oplus E_{\frac{-1-i\sqrt{3}}{2}}\right) \subseteq E_{\frac{-1+i\sqrt{3}}{2}} \oplus E_{\frac{-1-i\sqrt{3}}{2}}.$$

□

Let us mention that our Theorem 3.2 is in accordance with Corollary 2.5. from [12].

Corollary 3.1. [12] Let $P, Q \in \mathbb{C}^{n \times n}$ be two group invertible matrices and let a, b be two nonzero complex numbers. If $PQQ^g = QPP^g$ and $QQ^gP = PP^gQ$, then $aP + bQ$ and PQ are group invertible. If $a + b \neq 0$, then

$$(aP + bQ)^g = \frac{1}{a+b}P^gQQ^g + \frac{1}{a}(I_n - QQ^g)P^g + \frac{1}{b}(I_n - PP^g)Q^g. \quad (3.5)$$

Moreover,

$$(P - Q)^g = P^g - Q^g \quad \text{and} \quad (PQ)^g = (P^gQQ^g)^2 = (QP)^g. \quad (3.6)$$

Suppose that Q is invertible matrix in previous corollary. Then matrix P and Q satisfy $P = QPP^g = PP^gQ$. Set $A = aP$ and $B = bQ$. Then A is group invertible matrix, B is invertible matrix, $B(\mathcal{R}(A)) \subseteq \mathcal{R}(A)$ and $B(\mathcal{N}(A)) \subseteq \mathcal{N}(A)$, so Theorem 3.2 can be applied. Notice that $AB^{-1} = \frac{a}{b}PQ^{-1} = \frac{a}{b}PP^g$, and since PP^g is an

idempotent, by Theorem 3.2 the equality (3.3) is satisfied if and only if $P = 0$ or $\frac{a}{b} \in Y = \{-1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}\}$. (The subspaces which need to be invariant under transformation B are trivial or $\mathcal{R}(A)$.) Let us see what Corollary 3.1 says. If P is trivial in equation (3.5), the equality (3.3) is obviously satisfied. If $\frac{a}{b} = -1$ by (3.6), the equality (3.3) is satisfied. Since Q is invertible matrix, the equality (3.5) has a form

$$(aP + bQ)^g = \frac{1}{a+b}P^g + \frac{1}{b}Q^{-1} - \frac{1}{b}PP^gQ^{-1} = -\frac{a}{b(a+b)}P^g + \frac{1}{b}Q^{-1},$$

i.e. $(A+B)^g = -\frac{a^2}{b(a+b)}A^g + B^g$, so (3.3) holds if and only if $a^2 + ab + b^2 = 1$, i.e. $\frac{a}{b} \in X = \{\frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}\}$. However, notice that $B(\mathcal{R}(A)) \subseteq \mathcal{R}(A)$ and $B(\mathcal{N}(A)) \subseteq \mathcal{N}(A)$ if and only if $QPP^gQ^{-1} = PP^g$ which is weaker condition than $P = QPP^g = PP^gQ$.

Example 3.3. Let $A = \begin{pmatrix} -\frac{1}{2} & 0 & -\frac{1}{4} & -\frac{3}{4} \\ 0 & \frac{1}{2} - \frac{i\sqrt{3}}{2} & 0 & 0 \\ -\frac{1}{4} & 0 & -\frac{1}{8} & -\frac{3}{8} \\ -\frac{1}{4} & 0 & -\frac{1}{8} & -\frac{3}{8} \end{pmatrix}$ and $B = \begin{pmatrix} \frac{13}{10} & 0 & \frac{3}{20} & -\frac{3}{4} \\ 0 & -1 & 0 & 0 \\ \frac{3}{20} & 0 & \frac{43}{40} & -\frac{3}{8} \\ -\frac{1}{4} & 0 & -\frac{1}{8} & \frac{13}{8} \end{pmatrix}$. Then $AB^{-1} = P \cdot D \cdot P^{-1}$

where $P = \begin{pmatrix} 0 & 2 & 6 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 1 & -5 & 0 \end{pmatrix}$ and $D = \begin{pmatrix} \frac{-1+i\sqrt{3}}{2} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Since $AA^\dagger BA = BA$ and $AB = ABA^\dagger A$, the

preconditions $B(\mathcal{R}(A)) \subseteq \mathcal{R}(A)$ and $B(\mathcal{N}(A)) \subseteq \mathcal{N}(A)$ are satisfied. It can easily be checked that the eigenspaces $E_{\frac{-1+i\sqrt{3}}{2}} \oplus E_{\frac{-1-i\sqrt{3}}{2}}$ and E_{-1} are invariant under transformation B . In this case

$$(A+B)^g = \begin{pmatrix} \frac{7}{20} & 0 & -\frac{13}{40} & -\frac{3}{8} \\ 0 & \frac{i(\sqrt{3}+i)}{2} & 0 & 0 \\ -\frac{13}{40} & 0 & \frac{67}{80} & -\frac{3}{16} \\ -\frac{1}{8} & 0 & -\frac{1}{16} & \frac{5}{16} \end{pmatrix} = A^g + B^{-1}.$$

Example 3.4. Let $A = \begin{pmatrix} -\frac{5i}{\sqrt{3}} & -\frac{1}{2} + \frac{3i\sqrt{3}}{2} & \frac{3-23i\sqrt{3}}{6} & \frac{-3-11i\sqrt{3}}{6} \\ 1+2i\sqrt{3} & -\frac{3i(\sqrt{3}-i)}{2} & \frac{7+5i\sqrt{3}}{2} & \frac{1+5i\sqrt{3}}{2} \\ \frac{i(11\sqrt{3}+3i)}{6} & \frac{1-3i\sqrt{3}}{2} & \frac{i(19\sqrt{3}+3i)}{6} & \frac{i(13\sqrt{3}+3i)}{6} \\ 1 & -1 & 2 & 1 \end{pmatrix}$ and

$B = \begin{pmatrix} -3 & 4 & -7 & -8 \\ -2 & 3 & -7 & -4 \\ 0 & 0 & -1 & 0 \\ 2 & -2 & 4 & 5 \end{pmatrix}$. Then $AB^{-1} = P \cdot D \cdot P^{-1}$ where $P = \begin{pmatrix} 1 & 2 & 3 & 2 \\ 3 & 0 & 3 & 1 \\ 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$ and

$D = \begin{pmatrix} \frac{-1+i\sqrt{3}}{2} & 0 & 0 & 0 \\ 0 & \frac{-1-i\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Since $AA^\dagger BA = BA$ and $AB = ABA^\dagger A$, the preconditions $B(\mathcal{R}(A)) \subseteq \mathcal{R}(A)$ and

$B(\mathcal{N}(A)) \subseteq \mathcal{N}(A)$ are satisfied. Since $B \begin{pmatrix} 1 & 2 \\ 3 & 0 \\ 1 & -1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 3 \\ -1 & 1 \\ 0 & 0 \end{pmatrix}$ we conclude that $E_{\frac{-1+i\sqrt{3}}{2}} \oplus E_{\frac{-1-i\sqrt{3}}{2}}$ is invariant under

the transformation B . Since $B \begin{pmatrix} 3 \\ 3 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ 1 \\ 1 \end{pmatrix}$, E_{-1} is not invariant under transformation B . In this case

$$(A+B)^g \neq A^g + B^{-1}.$$

4. Conclusion

In this paper we presented the necessary and sufficient conditions for the Moore-Penrose inverse or the group inverse of the sum of two invertible matrices to be equal to the sum of (regular) inverses of those matrices.

We also stated the equivalent conditions to additivity of Moore-Penrose inverse for matrix A such that $B^*(\mathcal{R}(A)) \subseteq \mathcal{R}(A^*)$ and $B(\mathcal{R}(A^*)) \subseteq \mathcal{R}(A)$ and invertible matrix B . In the case when matrix B is unitary the set of matrices A for which additivity of Moore-Penrose inverse holds is fully described. The equivalent conditions to additivity of group inverse are given in the case when A is group invertible matrix such that $B(\mathcal{R}(A)) \subseteq \mathcal{R}(A)$ and $B(\mathcal{N}(A)) \subseteq \mathcal{N}(A)$ and B is invertible matrix.

The introduced results are illustrated with appropriate examples and compared with already existing results in the literature.

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