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On novel developments of Mean-type fractional integral inequalities and trapezoid type error bounds

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Abstract. In this article, we establish fractional integral inequalities based on fundamental integral identities involving the second-order derivatives of a given function. We utilize specific functions to explore these inequalities and illustrate their 2D and 3D graphs along with corresponding table values providing evidence for the validity of the obtained results. As applications, we present trapezoid bounds as error estimates for the obtained results. Many limiting outcomes are presented as remarks by drawing connections with the existing findings in the literature.

1. Introduction

The study of differentiation and integration extended to non-integer orders is identified as fractional calculus. This branch of mathematics deals with complicated real life problems for which traditional calculus is inadequate. It plays a significant role to increase forecast accuracy[12], build up control volume [30] and to improve problem-solving techniques across a wide range of industries[22, 30].

Moreover, the classes of convex functions are the key ideas for effective optimization, analytical methods and other various disciplines of research [8, 27]. Convex optimization is an effective tool for problems of scientific and engineering fields [2]. This theory is applicable in different ways in a variety of fields such as machine learning [29], game theory [11], economics [20] and other areas of study. The field of mathematical inequalities and theory of convex functions are correlated and have played significant role in developments of science [9]. Numerous mathematical problems are expressed in terms of inequalities. It finds multiple applications in different mathematical sciences [23]. The scientists have developed and extended the inequalities such as the Hardy type inequalities [13], Gagliardo-Nirenberg inequality [18], Ostrowski type inequalities [28], Grüss type inequalities [5], Milne type inequalities [6], weighted integral inequalities involving Chebyshev functionals [7], and Olsen inequality [19]. One of the majorally used inequality which is recognized in literature as Hermite-Hadamard's inequality was first presented in 1893 [14]. After this the mathematicians explored many refinements and extensions of this classical inequality

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[4, 15, 21, 25] via different convexities. These inequalities are well-known in the areas of convex functions. They have been studied and refined for different convexities under specific conditions and parameters. It is an important inequality of analysis among many classical inequalities due to its geometrical meaning and range of usefulness. This inequality has been applied to solve many kinds of fractional calculus problems [3]. In this work, we utilize the definition of the $(h-m)$-convex functions via fractional calculus to establish certain new results for the fractional integrals of Riemann-type, as a continuation of the research of the Hermite-Hadamard inequality.

Definition 1.1. *[25] Consider a non-negative function $h : I \subseteq \Re \to \Re$, then the non-negative function $¥ : [\nu_1, \nu_2] \to \Re$ is referred to as (h, m)-convex, for all $\theta_1, \theta_2 \in [\nu_1, \nu_2]$, $m \in [0, 1]$ and $\varphi \in (0, 1)$, if the following condition is satisfied:*

$$¥(\varphi\theta_1 + m(1-\varphi)\theta_2) \leq h(\varphi)¥(\theta_1) + mh(1-\varphi)¥(\theta_2). \tag{1}$$

Davis stated the definition of the gamma function in [24] which is as follows.

Definition 1.2. *For $Re(\theta) > 0$, the gamma function is expressed as*

$$\Gamma(\theta) = \int_0^\infty \varphi^{\theta-1} e^{-\varphi} d\varphi. \tag{2}$$

The k-gamma function describe in [26] is stated by the following definition.

Definition 1.3. *For $Re(\theta) > 0$, the k-gamma function is given by*

$$\Gamma_k(\theta) = \lim_{m\to\infty} \frac{m!k^m (mk)^{\frac{\theta}{k}-1}}{(\theta)_{m,k}},$$

where $(\theta)_{m,k}$ indicates the Pochhammer k-symbol for $k > 0$ and includes the factorial function. The integral form can be expressed by the following

$$\Gamma_k(\theta) = \int_0^\infty \varphi^{\theta-1} e^{(-\frac{\varphi^k}{k})} d\varphi.$$

Obviously, $\Gamma(\theta) = \lim_{k\to 1} \Gamma_k(\theta)$ and k-gamma and classical gamma have the relationship $\Gamma_k(\theta) = k^{\frac{\theta}{k}-1}\Gamma(\frac{\theta}{k})$.

Definition 1.4. *[17] The beta function is described by the equation*

$$\beta(\nu_1, \nu_2) = \int_0^1 \varphi^{\nu_1-1}(1-\varphi)^{\nu_2-1} d\varphi,$$

where $Re(\nu_1) > 0$, $Re(\nu_2) > 0$.

Definition 1.5. *[10] Let $¥ \in L[\nu_1, \nu_2]$, then the Riemann-Liouville fractional integrals having order ϑ are determined by*

$$(\beth^{\vartheta}_{\nu_1^+} ¥)(\theta) = \frac{1}{\Gamma(\vartheta)} \int_{\nu_1}^{\theta} (\theta-\varphi)^{\vartheta-1} ¥(\varphi) d\varphi, \quad (0 \leq \nu_1 < \theta),$$

and

$$(\beth^{\vartheta}_{\nu_2^-} ¥)(\theta) = \frac{1}{\Gamma(\vartheta)} \int_{\theta}^{\nu_2} (\varphi-\theta)^{\vartheta-1} ¥(\varphi) d\varphi, \quad (0 \leq \theta < \nu_2).$$

These expressions are identified as the left- and right-sided Riemann-Liouville fractional integrals, respectively.

Definition 1.6. *[26] Let ¥ ∈ $L[\nu_1,\nu_2]$, then the fractional integrals of order ϑ are determined by*

$$(\beth^{\vartheta}_{\nu_1^+,k}¥)(\theta)=\frac{1}{k\Gamma_k(\vartheta)}\int_{\nu_1}^{\theta}(\theta-\varphi)^{\frac{\vartheta}{k}-1}¥(\varphi)d\varphi,\ \ (0\le\nu_1<\theta),$$

and

$$(\beth^{\vartheta}_{\nu_2^-,k}¥)(\theta)=\frac{1}{k\Gamma_k(\vartheta)}\int_{\theta}^{\nu_2}(\varphi-\theta)^{\frac{\vartheta}{k}-1}¥(\varphi)d\varphi,\ \ (0\le\theta<\nu_2)$$

are identified as the left and right sided k-Riemann-Liouville fractional integrals respectively.

Definition 1.7. *(Hölder's inequality [1]) Let $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ and χ, ¥ : $[\nu_1,\nu_2]\to\mathfrak{R}$ be functions such that $|\chi|^p$, $|¥|^q\in L[\nu_1,\nu_2]$. Then the following inequality*

$$\int_{\nu_1}^{\nu_2}|\chi(\varphi)¥(\varphi)|d\varphi\le\left(\int_{\nu_1}^{\nu_2}|\chi(\varphi)|^p d\varphi\right)^{\frac{1}{p}}\left(\int_{\nu_1}^{\nu_2}|¥(\varphi)|^q d\varphi\right)^{\frac{1}{q}},$$

holds. The equality is valid if and only if $|\chi|^p$ is multiple of $|¥|^q$ almost everywhere.

Definition 1.8. *(Power mean inequality [1]) Suppose that $q\ge1$, $\frac{1}{p}+\frac{1}{q}=1$ and χ, ¥ : $[\nu_1,\nu_2]\to\mathfrak{R}$ are functions such that $|\chi|^p$, $|¥|^q\in L[\nu_1,\nu_2]$. Then the following relation*

$$\int_{\nu_1}^{\nu_2}|\chi(\varphi)¥(\varphi)|d\varphi\le\left(\int_{\nu_1}^{\nu_2}|\chi(\varphi)|d\varphi\right)^{1-\frac{1}{q}}\left(\int_{\nu_1}^{\nu_2}|\chi(\varphi)||¥(\varphi)|^q d\varphi\right)^{\frac{1}{q}}$$

holds.

Lemma 1.9. *[21] Let ¥ : $[\nu_1,\nu_2]\to\mathfrak{R}$ be a function such that ¥$''$ exists on (ν_1,ν_2). If ¥$''\in L[\nu_1,\nu_2]$, then, we have the following identity involving fractional integrals*

$$\frac{¥(\nu_1)+¥(\nu_2)}{2}-\frac{\Gamma_k(\vartheta+k)}{2(\nu_2-\nu_1)^{\frac{\vartheta}{k}}}\left[\beth^{\vartheta}_{\nu_1^+,k}¥(\nu_2)+\beth^{\vartheta}_{\nu_2^-,k}¥(\nu_1)\right]$$
$$=\frac{(\nu_2-\nu_1)^2}{2}\int_0^1\frac{1-(1-\varphi)^{\frac{\vartheta}{k}+1}-\varphi^{\frac{\vartheta}{k}+1}}{\frac{\vartheta}{k}+1}¥''(\varphi\nu_1+(1-\varphi)\nu_2)d\varphi.$$

2. Main Result

This section is dedicated to explore new mean inequalities via (h,m)-convexity. To establish these inequalities, we first need the following identity that help to derive the targeted inequalities.

Lemma 2.1. *Suppose that ¥ : $[\nu_1,\nu_2]\to\mathfrak{R}$ is a twice differentiable function on (ν_1,ν_2) with $\nu_1<m\nu_2\le\nu_2$. If ¥$''\in L[\nu_1,\nu_2]$, then the equation for Riemann-type fractional integrals*

$$\frac{¥(\nu_1)+¥(m\nu_2)}{2}-\frac{\Gamma_k(\vartheta+k)}{2(m\nu_2-\nu_1)^{\frac{\vartheta}{k}}}\left[\beth^{\vartheta}_{\nu_1^+,k}¥(m\nu_2)+\beth^{\vartheta}_{m\nu_2^-,k}¥(\nu_1)\right]$$
$$=\frac{k(m\nu_2-\nu_1)^2}{2(\vartheta+k)}\int_0^1\left(1-(1-\varphi)^{\frac{\vartheta}{k}+1}-\varphi^{\frac{\vartheta}{k}+1}\right)¥''(\varphi\nu_1+m(1-\varphi)\nu_2)d\varphi$$

holds.

Proof. Consider

$$I = \int_0^1 \left(\frac{1-(1-\varphi)^{\frac{\vartheta}{k}+1} - \varphi^{\frac{\vartheta}{k}+1}}{\frac{\vartheta}{k}+1} \right) \left(¥^{''}(\varphi\nu_1 + m(1-\varphi)\nu_2) \right) d\varphi.$$

By using integration by parts, we get

$$\begin{aligned}
&= \frac{1}{(m\nu_2-\nu_1)} \int_0^1 ¥'(\varphi\nu_1 + m(1-\varphi)\nu_2)(1-\varphi)^{\frac{\vartheta}{k}} d\varphi - \frac{1}{(m\nu_2-\nu_1)} \int_0^1 ¥^{'}(\varphi\nu_1 + m(1-\varphi)\nu_2)\varphi^{\frac{\vartheta}{k}} d\varphi \\
&= \frac{1}{(m\nu_2-\nu_1)} \left[\int_0^1 ¥^{'}(\varphi\nu_1 + m(1-\varphi)\nu_2)(1-\varphi)^{\frac{\vartheta}{k}} - \int_0^1 ¥^{'}(\varphi\nu_1 + m(1-\varphi)\nu_2)\varphi^{\frac{\vartheta}{k}} \right] d\varphi \qquad (3) \\
&= \frac{1}{(m\nu_2-\nu_1)} (I_1 + I_2) \qquad (4)
\end{aligned}$$

Let

$$\begin{aligned}
I_1 &= \int_0^1 ¥^{'}(\varphi\nu_1 + m(1-\varphi)\nu_2)(1-\varphi)^{\frac{\vartheta}{k}} d\varphi, \\
I_2 &= \int_0^1 ¥^{'}(\varphi\nu_1 + m(1-\varphi)\nu_2)\varphi^{\frac{\vartheta}{k}} d\varphi.
\end{aligned}$$

First, we solve I_1

$$I_1 = \int_0^1 ¥^{'}(\varphi\nu_1 + m(1-\varphi)\nu_2)(1-\varphi)^{\frac{\vartheta}{k}} d\varphi.$$

Integrating by parts, we get

$$= \frac{¥(m\nu_2)}{m\nu_2-\nu_1} - \frac{\frac{\vartheta}{k}}{m\nu_2-\nu_1} \int_0^1 (1-\varphi)^{\frac{\vartheta}{k}-1} ¥(\varphi\nu_1 + m(1-\varphi)\nu_2) d\varphi.$$

After changing the variable $\varphi\nu_1 + m(1-\varphi)\nu_2 = z$, we get

$$\begin{aligned}
&= \frac{¥(m\nu_2)}{(m\nu_2-\nu_1)} - \frac{\frac{\vartheta}{k}}{m\nu_2-\nu_1} \int_{m\nu_2}^{\nu_1} \left(\frac{z-\nu_1}{m\nu_2-\nu_1} \right)^{\frac{\vartheta}{k}-1} ¥(z) \frac{dz}{\nu_1 - m\nu_2} \\
&= \frac{¥(m\nu_2)}{(m\nu_2-\nu_1)} - \frac{\frac{\vartheta}{k}}{(m\nu_2-\nu_1)^2} \int_{\nu_1}^{m\nu_2} \left(\frac{z-\nu_1}{m\nu_2-\nu_1} \right)^{\frac{\vartheta}{k}-1} ¥(z) dz \\
&= \frac{¥(m\nu_2)}{(m\nu_2-\nu_1)} - \frac{\Gamma_k(\vartheta+k)}{(m\nu_2-\nu_1)^{\frac{\vartheta}{k}+1}} \beth^{\vartheta}_{m\nu_2^-,k} ¥(\nu_1).
\end{aligned}$$

By using the same procedure on I_2, we derive

$$I_2 = -\frac{¥(\nu_1)}{(m\nu_2-\nu_1)} + \frac{\Gamma_k(\vartheta+k)}{(m\nu_2-\nu_1)^{\frac{\vartheta}{k}+1}} \beth^{\vartheta}_{\nu_1^+,k} ¥(m\nu_2).$$

Now using the value of integrals I_1 and I_2 in (3), we get

$$\begin{aligned}
&= \frac{1}{(m\nu_2-\nu_1)} \left[\frac{¥(m\nu_2)}{(m\nu_2-\nu_1)} - \frac{\Gamma_k(\vartheta+k)}{(m\nu_2-\nu_1)^{\frac{\vartheta}{k}+1}} \beth^{\vartheta}_{m\nu_2^-,k} ¥(\nu_1) \right. \\
&\quad \left. + \frac{¥(\nu_1)}{(m\nu_2-\nu_1)} - \frac{\Gamma_k(\vartheta+k)}{(m\nu_2-\nu_1)^{\frac{\vartheta}{k}+1}} \beth^{\vartheta}_{\nu_1^+,k} ¥(m\nu_2) \right]
\end{aligned}$$

$$= \frac{¥(\nu_1) + ¥(m\nu_2)}{(m\nu_2 - \nu_1)^2} - \frac{\Gamma_k(\vartheta + k)}{(m\nu_2 - \nu_1)^{\frac{\vartheta}{k}+2}} \left[\beth^{\vartheta}_{\nu_1^+,k} ¥(m\nu_2) + \beth^{\vartheta}_{m\nu_2^-,k} ¥(\nu_1) \right].$$

Multiplying both sides by $\frac{(m\nu_2-\nu_1)^2}{2}$

$$\frac{¥(\nu_1) + ¥(m\nu_2)}{2} - \frac{\Gamma_k(\vartheta + k)}{2(m\nu_2 - \nu_1)^{\frac{\vartheta}{k}}} \left[\beth^{\vartheta}_{\nu_1^+,k} ¥(m\nu_2) + \beth^{\vartheta}_{m\nu_2^-,k} ¥(\nu_1) \right]$$
$$= \frac{k(m\nu_2 - \nu_1)^2}{2(\vartheta + k)} \int_0^1 \left(1 - (1-\varphi)^{\frac{\vartheta}{k}+1} - \varphi^{\frac{\vartheta}{k}+1} \right) ¥''(\varphi \nu_1 + m(1-\varphi)\nu_2) d\varphi$$

Hence, we obtained the required result. □

Theorem 2.2. *Suppose that $¥ : [\nu_1, \nu_2] \to \Re$ is a twice differentiable function. If the function $|¥''|^q$ is integrable and (h, m)-convex on $[\nu_1, \nu_2]$, then for a non-negative function $h : I \subseteq \Re \to \Re$, $q \geq 1$ and $m \in [0, 1]$, the fractional integrals inequality*

$$\left| \frac{¥(\nu_1) + ¥(m\nu_2)}{2} - \frac{\Gamma_k(\vartheta + k)}{2(m\nu_2 - \nu_1)^{\frac{\vartheta}{k}}} \left[\beth^{\vartheta}_{\nu_1^+,k} ¥(m\nu_2) + \beth^{\vartheta}_{m\nu_2^-,k} ¥(\nu_1) \right] \right|$$
$$\leq \frac{kM\vartheta(m\nu_2 - \nu_1)^2}{2(\vartheta + k)(\vartheta + 2k)} \left[\left| ¥''(\nu_1) \right|^q + m \left| ¥''(\nu_2) \right|^q \right]^{\frac{1}{q}}$$

is satisfied with $h(x) \leq M$.

Proof. First, we prove the result for the case $q = 1$. By using Lemma 2.1, we obtain

$$\left| \frac{¥(\nu_1) + ¥(m\nu_2)}{2} - \frac{\Gamma_k(\vartheta + k)}{2(m\nu_2 - \nu_1)^{\frac{\vartheta}{k}}} \left[\beth^{\vartheta}_{\nu_1^+,k} ¥(m\nu_2) + \beth^{\vartheta}_{m\nu_2^-,k} ¥(\nu_1) \right] \right|$$
$$\leq \frac{(m\nu_2 - \nu_1)^2}{2} \int_0^1 \left(\frac{1 - (1-\varphi)^{\frac{\vartheta}{k}+1} - \varphi^{\frac{\vartheta}{k}+1}}{\frac{\vartheta}{k} + 1} \right) \left| \left(¥''(\varphi \nu_1 + m(1-\varphi)\nu_2) \right) \right| d\varphi. \tag{5}$$

Since $|¥''|$ is (h, m)-convex, therefore we can write

$$\leq \frac{(m\nu_2 - \nu_1)^2}{2} \int_0^1 \left(\frac{1 - (1-\varphi)^{\frac{\vartheta}{k}+1} - \varphi^{\frac{\vartheta}{k}+1}}{\frac{\vartheta}{k} + 1} \right)$$
$$\times \left(h(\varphi) \left| ¥''(\nu_1) \right| + mh(1-\varphi) \left| ¥''(\nu_2) \right| \right) d\varphi$$
$$\leq \frac{k(m\nu_2 - \nu_1)^2}{2(\vartheta + k)} \left(\frac{\vartheta}{\vartheta + 2k} \right) \left(M \left| ¥''(\nu_1) \right| + mM \left| ¥''(\nu_2) \right| \right)$$
$$= \frac{k\vartheta M(m\nu_2 - \nu_1)^2}{2(\vartheta + k)(\vartheta + 2k)} \left(\left| ¥''(\nu_1) \right| + m \left| ¥''(\nu_2) \right| \right).$$

This proved the result for the case $q = 1$.
Now, we prove the result for the case $q > 1$. By applying Lemma 2.1 and power mean inequality, we achieve

$$\left| \frac{¥(\nu_1) + ¥(m\nu_2)}{2} - \frac{\Gamma_k(\vartheta + k)}{2(m\nu_2 - \nu_1)^{\frac{\vartheta}{k}}} \left[\beth^{\vartheta}_{\nu_1^+,k} ¥(m\nu_2) + \beth^{\vartheta}_{m\nu_2^-,k} ¥(\nu_1) \right] \right|$$
$$\leq \frac{k(m\nu_2 - \nu_1)^2}{2(\vartheta + k)} \int_0^1 \left(1 - (1-\varphi)^{\frac{\vartheta}{k}+1} - \varphi^{\frac{\vartheta}{k}+1} \right) \left| \left(¥''(\varphi \nu_1 + m(1-\varphi)\nu_2) \right) \right| d\varphi$$
$$\leq \left(\int_0^1 \left(1 - (1-\varphi)^{\frac{\vartheta}{k}+1} - \varphi^{\frac{\vartheta}{k}+1} \right) \right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 \left(1-(1-\varphi)^{\frac{\vartheta}{k}+1}-\varphi^{\frac{\vartheta}{k}+1}\right)\left|\left(¥''(\varphi\nu_1+m(1-\varphi)\nu_2)\right)\right|^q d\varphi\right)^{\frac{1}{q}}. \tag{6}$$

Since $|¥''|^q$ is (h,m)-convex, therefore we can have

$$\begin{aligned}
&\le \frac{k(m\nu_2-\nu_1)^2}{2(\vartheta+k)}\left(\int_0^1 \left(1-(1-\varphi)^{\frac{\vartheta}{k}+1}-\varphi^{\frac{\vartheta}{k}+1}\right)\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1 \left(1-(1-\varphi)^{\frac{\vartheta}{k}+1}-\varphi^{\frac{\vartheta}{k}+1}\right)\left(h(\varphi)\left|¥''(\nu_1)\right|^q+mh(1-\varphi)\left|¥''(\nu_2)\right|^q\right)d\varphi\right)^{\frac{1}{q}}\\
&\le \frac{Mk(m\nu_2-\nu_1)^2}{2(\vartheta+k)}\left(\frac{\vartheta}{\vartheta+2k}\right)^{1-\frac{1}{q}}\left(\frac{\vartheta}{\vartheta+2k}\right)^{\frac{1}{q}}\left[\left|¥''(\nu_1)\right|^q+m\left|¥''(\nu_2)\right|^q\right]^{\frac{1}{q}}\\
&= \frac{kM\vartheta(m\nu_2-\nu_1)^2}{2(\vartheta+k)(\vartheta+2k)}\left[\left|¥''(\nu_1)\right|^q+m\left|¥''(\nu_2)\right|^q\right]^{\frac{1}{q}}.
\end{aligned}$$

The proof is done. □

Remark 2.3. *By choosing $h(\varphi)=\varphi^s$ and $k=1$ in Theorem 2.2, we get*

$$\begin{aligned}
&\left|\frac{¥(\nu_1)+¥(m\nu_2)}{2}-\frac{\Gamma(\vartheta+1)}{2(m\nu_2-\nu_1)^{\vartheta}}\left[\beth^{\vartheta}_{\nu_1^+}¥(m\nu_2)+\beth^{\vartheta}_{m\nu_2^-}¥(\nu_1)\right]\right|\\
&\le \frac{(m\nu_2-\nu_1)^2}{2(\vartheta+1)}\left(\frac{\vartheta}{\vartheta+2}\right)^{1-\frac{1}{q}}\left(\frac{1}{s+1}-\beta(s+1,\vartheta+2)-\frac{1}{\vartheta+s+2}\right)^{\frac{1}{q}}\\
&\times\left(|¥''(\nu_1)|^q+m|¥''(\nu_2)|^q\right)^{\frac{1}{q}}.
\end{aligned}$$

Theorem 2.4. *Suppose that $¥:[\nu_1,\nu_2]\to\Re$ is a twice differentiable function. If the function $|¥''|^q$ is integrable and (h,m)-convex on $[\nu_1,\nu_2]$ for a non-negative function $h: I\subseteq\Re\to\Re$, $q\ge 1$ and $m\in[0,1]$, then for Riemann-type fractional integrals the following inequality*

$$\begin{aligned}
&\left|\frac{¥(\nu_1)+¥(m\nu_2)}{2}-\frac{\Gamma_k(\vartheta+k)}{2(m\nu_2-\nu_1)^{\frac{\vartheta}{k}}}\left[\beth^{\vartheta}_{\nu_1^+,k}¥(m\nu_2)+\beth^{\vartheta}_{m\nu_2^-,k}¥(\nu_1)\right]\right|\\
&\le \frac{kM(m\nu_2-\nu_1)^2}{2(\vartheta+k)}\left(1-\frac{2}{p\left(\frac{\vartheta}{k}+1\right)+1}\right)^{\frac{1}{p}}\left(\left|¥''(\nu_1)\right|^q+m\left|¥''(\nu_2)\right|^q\right)^{\frac{1}{q}}
\end{aligned}$$

is satisfied with $h(x)\le M$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Utilizing Lemma 2.1 and Hölder's inequality, we obtain

$$\begin{aligned}
&\left|\frac{¥(\nu_1)+¥(m\nu_2)}{2}-\frac{\Gamma_k(\vartheta+k)}{2(m\nu_2-\nu_1)^{\frac{\vartheta}{k}}}\left[\beth^{\vartheta}_{\nu_1^+,k}¥(m\nu_2)+\beth^{\vartheta}_{m\nu_2^-,k}¥(\nu_1)\right]\right|\\
&\le \frac{k(m\nu_2-\nu_1)^2}{2(\vartheta+k)}\left(\int_0^1 \left(1-(1-\varphi)^{\frac{\vartheta}{k}+1}-\varphi^{\frac{\vartheta}{k}+1}\right)^p d\varphi\right)^{\frac{1}{p}}\\
&\left(\int_0^1 \left|\left(¥''(\varphi\nu_1+m(1-\varphi)\nu_2)\right)\right|^q d\varphi\right)^{\frac{1}{q}}.
\end{aligned}$$

By using the relation

$$\left(1-(1-\varphi)^{\frac{\vartheta}{k}+1}-\varphi^{\frac{\vartheta}{k}+1}\right)^p\le 1-(1-\varphi)^{p\left(\frac{\vartheta}{k}+1\right)}-\varphi^{p\left(\frac{\vartheta}{k}+1\right)},$$

for any $\varphi \in (0,1)$ derived from

$$(E-F)^{P} \leq E^{p} - F^{p},$$

for any $E > F \geq 0$, $p \geq 1$. Also using the Definition of (h,m)-convexity, we get

$$\begin{aligned}
&\leq \frac{k(mv_2 - v_1)^2}{2(\vartheta + k)} \left(\int_0^1 \left(1 - (1-\varphi)^{p\left(\frac{\vartheta}{k}+1\right)} - \varphi^{p\left(\frac{\vartheta}{k}+1\right)} \right) dt \right)^{\frac{1}{p}} \\
&\times \left(\left| ¥''(v_1) \right|^q \int_0^1 h(\varphi) d\varphi + m \left| ¥''(v_2) \right|^q \int_0^1 h(1-\varphi) d\varphi \right)^{\frac{1}{q}} \\
&\leq \frac{Mk(mv_2 - v_1)^2}{2(\vartheta + k)} \left(1 - \frac{2}{p\left(\frac{\vartheta}{k}+1\right)+1} \right)^{\frac{1}{p}} \left(\left| ¥''(v_1) \right|^q + m \left| ¥''(v_2) \right|^q \right)^{\frac{1}{q}} \\
&= \frac{kM(mv_2 - v_1)^2}{2(\vartheta + k)} \left(1 - \frac{2}{p\left(\frac{\vartheta}{k}+1\right)+1} \right)^{\frac{1}{p}} \left(\left| ¥''(v_1) \right|^q + m \left| ¥''(v_2) \right|^q \right)^{\frac{1}{q}}.
\end{aligned}$$

Hence, we obtained the required result. □

Example 2.5. *Here, we confirm the validly of Theorem 2.4 via graphical representations. For this purpose, we substitute* $¥(\varphi) = e^{2\varphi}$ *to get the following integral values*

$$\beth^{\vartheta}_{v_1^+,k} e^{2mv_2} = \frac{1}{k\Gamma_k(\vartheta)} \int_{v_1}^{mv_2} (mv_2 - \varphi)^{\frac{\vartheta}{k}-1} e^{2\varphi} d\varphi, \tag{7}$$

and

$$\beth^{\vartheta}_{mv_2^-,k} e^{2v_1} = \frac{1}{k\Gamma_k(\vartheta)} \int_{v_1}^{mv_2} (\varphi - v_1)^{\frac{\vartheta}{k}-1} e^{2\varphi} d\varphi. \tag{8}$$

By using the expressions (7) *and* (8) *in Theorem 2.4, we get*

$$\begin{aligned}
&\frac{e^{2v_1} + e^{2mv_2}}{2} - \frac{kM(mv_2 - v_1)^2}{2(\vartheta + k)} \left(1 - \frac{2}{p(\frac{\vartheta}{k})+1} \right)^{\frac{1}{p}} \left(\left(4e^{2v_1}\right)^q + m(4e^{2(\frac{v_2}{m})})^q \right)^{\frac{1}{q}} \\
&\leq \frac{\Gamma_k(\vartheta + k)}{2k\Gamma_k(\vartheta)(mv_2 - v_1)^{\frac{\vartheta}{k}}} \int_{v_1}^{mv_2} \left[(mv_2 - \varphi)^{\frac{\vartheta}{k}-1} + (\varphi - v_1)^{\frac{\vartheta}{k}-1} \right] e^{2\varphi} d\varphi \\
&\leq \frac{e^{2v_1} + e^{2mv_2}}{2} + \frac{kM(mv_2 - v_1)^2}{2(\vartheta + k)} \left(1 - \frac{2}{p(\frac{\vartheta}{k})+1} \right)^{\frac{1}{p}} \left(\left(4e^{2v_1}\right)^q + m(4e^{2(\frac{v_2}{m})})^q \right)^{\frac{1}{q}}.
\end{aligned} \tag{9}$$

By setting the parameters $M = 1$, $k = 1$, $v_1 = 0$, $v_2 = 1$, $p = 2$, $q = 2$, $m = 1$ *in the double inequality* (9), *we obtain the following functions*

$$\begin{aligned}
p_o(\vartheta) &= 4.1945 - \frac{14.9128}{(\vartheta + 1)} \left(\frac{2\vartheta + 1}{2\vartheta + 3} \right)^{\frac{1}{2}}, \\
p_1(\vartheta) &= \frac{\vartheta}{2} \int_0^1 \left[(1-\varphi)^{\vartheta - 1} + \varphi^{\vartheta - 1} \right] e^{2\varphi} d\varphi, \\
p_2(\vartheta) &= 4.1945 + \frac{14.9128}{(\vartheta + 1)} \left(\frac{2\vartheta + 1}{2\vartheta + 3} \right)^{\frac{1}{2}}.
\end{aligned}$$

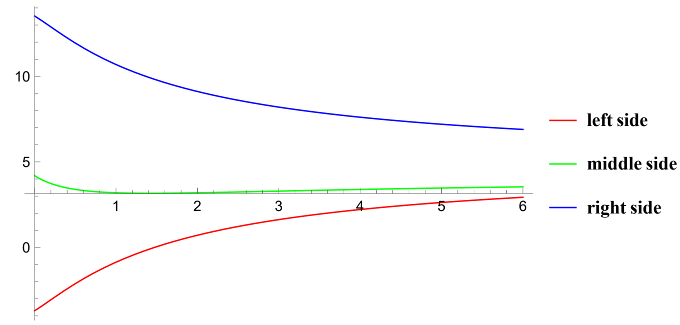

Figure 1: The 2D graphical view of the inequality (9) corresponding to $0 \leq \vartheta \leq 6$ is shown in Figure 1.

Table 1: The following table represents the comparison findings between the double inequality in Example 2.5.

Functions	1	2	3	4	5
$p_0(\vartheta)$	-0.861175	0.713323	1.62656	2.2167	2.62823
$p_1(\vartheta)$	3.19453	3.19453	3.29179	3.389056	3.472640
$p_2(\vartheta)$	10.6902	9.71573	8.20249	7.61236	7.20082

By setting the parameters $M = 1$, $k = 1$, $\vartheta = 1$, $q = 2$, $m = 1$, we get the following functions

$$
\begin{aligned}
&\frac{e^{2\nu_1} + e^{2\nu_2}}{2} - (\nu_2 - \nu_1)^2 \left(\frac{3(e^{4\nu_1} + e^{4\nu_2})}{5}\right)^{\frac{1}{2}} \\
&\leq \frac{1}{(\nu_2 - \nu_1)} \int_{\nu_1}^{\nu_2} e^{2\varphi} d\varphi \\
&\leq \frac{e^{2\nu_1} + e^{2\nu_2}}{2} + (\nu_2 - \nu_1)^2 \left(\frac{3(e^{4\nu_1} + e^{4\nu_2})}{5}\right)^{\frac{1}{2}}.
\end{aligned}
$$

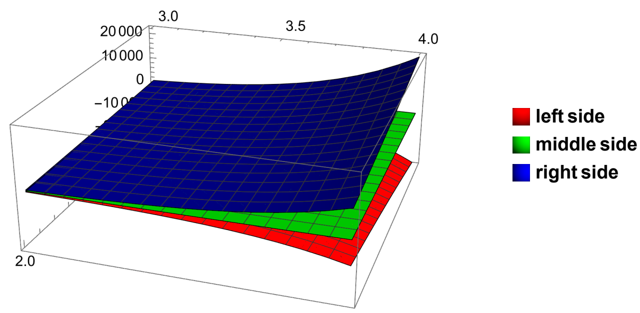

Figure 2: The 3D graphical view of the inequality (9) corresponding to $1 \leq \nu_1 \leq 2$, $3 \leq \nu_2 \leq 4$ is shown in Figure 2.

Remark 2.6. *By choosing $h(\varphi)=\varphi^s$ and $k=1$ in Theorem 2.4, we get*

$$\left|\frac{¥(mv_2)+¥(v_1)}{2}-\frac{\Gamma(\vartheta+1)}{2(mv_2-v_1)^{\vartheta}}\left[\beth^{\vartheta}_{v_1^+}¥(mv_2)+\beth^{\vartheta}_{mv_2^-}¥(v_1)\right]\right|$$
$$\leq \frac{(mv_2-v_1)^2}{2(\vartheta+1)}\left(1-\frac{2}{p(\vartheta+1)+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}$$
$$\times\left(\left|¥''(v_1)\right|^q+m\left|¥''(v_2)\right|^q\right)^{\frac{1}{q}}.$$

Theorem 2.7. *Suppose that $¥:[v_1,v_2]\to\Re$ is a twice differentiable function. If the function $|¥''|^q$ is integrable and (h,m)-convex on $[v_1,v_2]$ for a non-negative function $h:I\subseteq\Re\to\Re$, $q\geq1$ and $m\in[0,1]$, then for Riemann-type fractional integrals the following inequality*

$$\left|\frac{¥(v_1)+¥(mv_2)}{2}-\frac{\Gamma_k(\vartheta+k)}{2(mv_2-v_1)^{\frac{\vartheta}{k}}}\left[\beth^{\vartheta}_{v_1^+,k}¥(mv_2)+\beth^{\vartheta}_{mv_2^-,k}¥(v_1)\right]\right|$$
$$\leq \frac{kM(mv_2-v_1)^2}{2(\vartheta+k)}\left[\left(1-\frac{2}{q\left(\frac{\vartheta}{k}+1\right)+1}\right)\left(\left|¥''(v_1)\right|^q+m\left|¥''(v_2)\right|^q\right)\right]^{\frac{1}{q}}$$

is satisfied with $h(x)\leq M$.

Proof. Utilizing Lemma 2.1 and the Hölder's inequality, we obtain

$$\left|\frac{¥(v_1)+¥(mv_2)}{2}-\frac{\Gamma_k(\vartheta+k)}{2(mv_2-v_1)^{\frac{\vartheta}{k}}}\left[\beth^{\vartheta}_{v_1v_1^+,k}¥(mv_2)+\beth^{\vartheta}_{mv_2^-,k}¥(v_1)\right]\right|$$
$$\leq \frac{(mv_2-v_1)^2}{2}\int_0^1\left|\frac{1-(1-\varphi)^{\frac{\vartheta}{k}+1}-\varphi^{\frac{\vartheta}{k}+1}}{\frac{\vartheta}{k}+1}\right|\left|¥''(\varphi v_1+m(1-\varphi)v_2)\right|d\varphi$$
$$\leq \frac{k(mv_2-v_1)^2}{2(\vartheta+k)}\left(\int_0^1 1dt\right)^{\frac{1}{p}}\left(\int_0^1\left(1-(1-\varphi)^{\frac{\vartheta}{k}+1}-\varphi^{\frac{\vartheta}{k}+1}\right)^q\right.$$
$$\left.\times\left|¥''(\varphi v_1+m(1-\varphi)v_2)\right|^q d\varphi\right)^{\frac{1}{q}}.$$

By using the relation

$$\left(1-(1-\varphi)^{\frac{\vartheta}{k}+1}-\varphi^{\frac{\vartheta}{k}+1}\right)^q\leq 1-(1-\varphi)^{q\left(\frac{\vartheta}{k}+1\right)}-\varphi^{q\left(\frac{\vartheta}{k}+1\right)},$$

for any $\varphi\in(0,1)$ and $q\geq1$. Also using the Definition of (h,m)-convexity, we get

$$\leq \frac{k(mv_2-v_1)^2}{2(\vartheta+k)}\left[\int_0^1\left(1-(1-\varphi)^{q\left(\frac{\vartheta}{k}+1\right)}-\varphi^{q\left(\frac{\vartheta}{k}+1\right)}\right)\right.$$
$$\left.\times\left(h(\varphi)\left|¥''(v_1)\right|^q+mh(1-\varphi)\left|¥''\left(\frac{v_2}{m}\right)\right|^q\right)dt\right]^{\frac{1}{q}}$$
$$\leq \frac{k(mv_2-v_1)^2}{2(\vartheta+k)}\left[\left(1-\frac{2}{q\left(\frac{\vartheta}{k}+1\right)+1}\right)\left(M\left|¥''(v_1)\right|^q+mM\left|¥''(v_2)\right|^q\right)\right]^{\frac{1}{q}}$$
$$=\frac{kM(mv_2-v_1)^2}{2(\vartheta+k)}\left[\left(1-\frac{2}{q\left(\frac{\vartheta}{k}+1\right)+1}\right)\left(\left|¥''(v_1)\right|^q+m\left|¥''(v_2)\right|^q\right)\right]^{\frac{1}{q}}.$$

Hence, we obtained the required result. □

Remark 2.8. *By choosing $h(\varphi) = \varphi^s$ and $k = 1$ in Theorem 2.7, we get*

$$\left|\frac{¥(\nu_1) + ¥(m\nu_2)}{2} - \frac{\Gamma(\vartheta+1)}{2(m\nu_2-\nu_1)^{\vartheta}}\left[\beth^{\vartheta}_{\nu_1^+}¥(m\nu_2) + \beth^{\vartheta}_{m\nu_2^-}¥(\nu_1)\right]\right|$$
$$\leq \frac{(m\nu_2-\nu_1)^2}{2(\vartheta+1)}\left(\frac{1}{s+1} - \frac{1}{q(\vartheta+1)+s+1} - \beta\left(q(\vartheta+1)+1, s+1\right)\right)^{\frac{1}{q}}$$
$$\times\left(\left|¥''(\nu_1)\right|^q + m\left|¥''(\nu_2)\right|^q\right)^{\frac{1}{q}}.$$

Theorem 2.9. *Suppose that $¥ : [0, \nu_2^*] \to \Re$ is a twice differentiable function. If the function $|¥''|^q$ is integrable and (h, m)-convex on $[\nu_1, \frac{\nu_2}{m}]$, then for a non-negative function $h : I \subseteq \Re \to \Re$, $q \geq 1$, $0 \leq \nu_1 < \nu_2$ with $m \in (0, 1]$ and $\frac{\nu_2}{m} \leq \nu_2^*$, the following inequality*

$$\left|\frac{¥(\nu_1) + ¥(\nu_2)}{2} - \frac{\Gamma_k(\vartheta+k)}{2(\nu_2-\nu_1)^{\frac{\vartheta}{k}}}\left[\beth^{\vartheta}_{\nu_1+,k}¥(\nu_2) + \beth^{\vartheta}_{\nu_2^-,k}¥(\nu_1)\right]\right|$$
$$\leq \frac{\vartheta k M(\nu_2-\nu_1)^2}{2(\vartheta+k)(\vartheta+2k)}\left[\left|¥''(\nu_1)\right|^q + m\left|¥''\left(\frac{\nu_2}{m}\right)\right|^q\right]^{\frac{1}{q}} \tag{10}$$

is satisfied with $h(x) \leq M$.

Proof. First, we prove the result for the case $q = 1$. By using Lemma 1.9, we obtain

$$\left|\frac{¥(\nu_1) + ¥(\nu_2)}{2} - \frac{\Gamma_k(\vartheta+k)}{2(\nu_2-\nu_1)^{\frac{\vartheta}{k}}}\left[\beth^{\vartheta}_{\nu_1^+,k}¥(\nu_2) + \beth^{\vartheta}_{\nu_2^-,k}¥(\nu_1)\right]\right|$$
$$\leq \frac{(\nu_2-\nu_1)^2}{2}\int_0^1\left|\frac{1-(1-\varphi)^{\frac{\vartheta}{k}+1} - \varphi^{\frac{\vartheta}{k}+1}}{\frac{\vartheta}{k}+1}\right|\left|¥''(\varphi\nu_1 + (1-\varphi)\nu_2)\right|d\varphi. \tag{11}$$

Since $(1-\varphi)^{\frac{\vartheta}{k}+1} - \varphi^{\frac{\vartheta}{k}+1} \leq 1$ for each $\varphi \in [\nu_1, \nu_2]$. Since $\left|¥''\right|$ is (h, m)-convex, therefore we can write

$$\left|\frac{¥(\nu_1) + ¥(\nu_2)}{2} - \frac{\Gamma_k(\vartheta+k)}{2(\nu_2-\nu_1)^{\frac{\vartheta}{k}}}\left[\beth^{\vartheta}_{\nu_1^+,k}¥(\nu_2) + \beth^{\vartheta}_{\nu_2^-,k}¥(\nu_1)\right]\right|$$
$$\leq \frac{(\nu_2-\nu_1)^2}{2}\int_0^1\left|\frac{1-(1-\varphi)^{\frac{\vartheta}{k}+1} - \varphi^{\vartheta}+1}{\frac{\vartheta}{k}+1}\right|\left(h(\varphi)\left|¥''(\nu_1)\right| + mh(1-\varphi)\left|¥''\left(\frac{\nu_2}{m}\right)\right|\right)d\varphi$$
$$\leq \frac{(\nu_2-\nu_1)^2}{2}\int_0^1\left|\frac{1-(1-\varphi)^{\frac{\vartheta}{k}+1} - \varphi^{\frac{\vartheta}{k}+1}}{\frac{\vartheta}{k}+1}\right|\left(M\left|¥''(\nu_1)\right| + mM\left|¥''\left(\frac{\nu_2}{m}\right)\right|\right)d\varphi$$
$$= \frac{(\nu_2-\nu_1)^2}{2}\frac{\vartheta K M}{(\vartheta+K)(\vartheta+2K)}\left(\left|¥''(\nu_1)\right| + m\left|¥''\left(\frac{\nu_2}{m}\right)\right|\right).$$

This prove the result for $q = 1$.
Now, we prove the result for the case $q > 1$. By applying Lemma 1.9 and then power mean inequality, we obtain

$$\left|\frac{¥(\nu_1) + ¥(\nu_2)}{2} - \frac{\Gamma_k(\vartheta+k)}{2(\nu_2-\nu_1)^{\frac{\vartheta}{k}}}\left[\beth^{\vartheta}_{\nu_1^+,k}¥(\nu_2) + \beth^{\vartheta}_{\nu_2^-,k}¥(\nu_1)\right]\right|$$
$$\leq \frac{k(\nu_2-\nu_1)^2}{2(\vartheta+k)}\int_0^1\left|(1-(1-\varphi)^{\frac{\vartheta}{k}+1} - \varphi^{\frac{\vartheta}{k}+1})\right|\left|¥''(\varphi\nu_1 + (1-\varphi)\nu_2)\right|d\varphi$$
$$\leq \frac{k(\nu_2-\nu_1)^2}{2(\vartheta+k)}\left(\int_0^1|\left(1-(1-\varphi)^{\frac{\vartheta}{k}+1} - \varphi^{\frac{\vartheta}{k}+1}\right)|\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 \left|\left(1-(1-\varphi)^{\frac{\vartheta}{k}+1}-\varphi^{\frac{\vartheta}{k}+1}\right)\right|\left|¥''(\varphi\nu_1+(1-\varphi)\nu_2)\right|^q d\varphi\right)^{\frac{1}{q}}. \tag{12}$$

Since $\left|¥''\right|^q$ is (h,m)-convex, therefore we can have

$$\begin{aligned}
&\left|\frac{¥(\nu_1)+¥(\nu_2)}{2}-\frac{\Gamma_k(\vartheta+k)}{2(\nu_2-\nu_1)^{\frac{\vartheta}{k}}}\left[\beth^{\vartheta}_{\nu_1^+,k}¥(\nu_2)+\beth^{\vartheta}_{\nu_2^-,k}¥(\nu_1)\right]\right|\\
&\leq \frac{k(\nu_2-\nu_1)^2}{2(\vartheta+k)}\left(\int_0^1 |\left(1-(1-\varphi)^{\frac{\vartheta}{k}+1}-\varphi^{\frac{\vartheta}{k}}+1\right)\right)^{1-\frac{1}{q}}\left(\int_0^1\left(1-(1-\varphi)^{\frac{\vartheta}{k}}+1-\varphi^{\frac{\vartheta}{k}+1}\right)\right.\\
&\times\left.\left[h(\varphi)\left|¥''(\nu_1)\right|^q+mh(1-\varphi)\left|¥''\left(\frac{\nu_2}{m}\right)\right|^q\right]dt\right)^{\frac{1}{q}}\\
&\leq \frac{k(\nu_2-\nu_1)^2}{2(\vartheta+k)}\left(\frac{\vartheta}{\vartheta+2k}\right)^{1-\frac{1}{q}}\left(\frac{\vartheta}{\vartheta+2k}\right)^{\frac{1}{q}}\left[M\left|¥''(\nu_1)\right|^q+mM\left|¥''\left(\frac{\nu_2}{m}\right)\right|^q\right]^{\frac{1}{q}}\\
&=\frac{\vartheta k M(\nu_2-\nu_1)^2}{2(\vartheta+k)(\vartheta+2k)}\left[\left|¥''(\nu_1)\right|^q+m\left|¥''\left(\frac{\nu_2}{m}\right)\right|^q\right]^{\frac{1}{q}}.
\end{aligned}$$

The proof is done. □

Example 2.10. *Here, we confirm the validity of Theorem 2.9 via graphical representations. For this purpose, we substitute $¥(\varphi)=e^{\varphi}$ to get the following integral values*

$$\beth^{\vartheta}_{\nu_1^+,k}e^{\nu_2}=\frac{1}{k\Gamma_k(\vartheta)}\int_{\nu_1}^{\nu_2}(\nu_2-\varphi)^{\frac{\vartheta}{k}-1}e^{\varphi}d\varphi, \tag{13}$$

and

$$\beth^{\vartheta}_{\nu_2^-,k}e^{\nu_1}=\frac{1}{k\Gamma_k(\vartheta)}\int_{\nu_1}^{\nu_2}(\varphi-\nu_1)^{\frac{\vartheta}{k}-1}e^{\varphi}d\varphi. \tag{14}$$

By using the expressions (13) *and* (14) *in Theorem 2.9, we get*

$$\begin{aligned}
&\frac{e^{\nu_1}+e^{\nu_2}}{2}-\frac{\vartheta k M(\nu_2-\nu_1)^2}{2(\vartheta+k)(\vartheta+2k)}\left[|e^{\nu_1}|^q+m\left|e^{\frac{\nu_2}{m}}\right|^q\right]^{\frac{1}{q}}\\
&\leq \frac{\Gamma_k(\vartheta+k)}{2k\Gamma_k(\vartheta)(\nu_2-\nu_1)^{\frac{\vartheta}{k}}}\int_{\nu_1}^{\nu_2}\left[(\nu_2-\varphi)^{\frac{\vartheta}{k}-1}+(\varphi-\nu_1)^{\frac{\vartheta}{k}-1}\right]e^{\varphi}d\varphi\\
&\leq \frac{e^{\nu_1}+e^{\nu_2}}{2}+\frac{\vartheta k M(\nu_2-\nu_1)^2}{2(\vartheta+k)(\vartheta+2k)}\left[|e^{\nu_1}|^q+m\left|e^{\frac{\nu_2}{m}}\right|^q\right]^{\frac{1}{q}}.
\end{aligned} \tag{15}$$

By setting the parameters $M=1$, $k=1$, $\nu_1=0$, $\nu_2=1$, $q=2$, $m=1$ in the double inequality (15), *we obtain the following functions*

$$\begin{aligned}
p_o(\vartheta)&=1.8591-\frac{2.8964\vartheta}{2(\vartheta+1)(\vartheta+2)},\\
p_1(\vartheta)&=\frac{\vartheta}{2}\int_0^1\left[(1-\varphi)^{\vartheta-1}+\varphi^{\vartheta-1}\right]e^{\varphi}d\varphi,\\
p_2(\vartheta)&=1.8591+\frac{2.8964\vartheta}{2(\vartheta+1)(\vartheta+2)}.
\end{aligned}$$

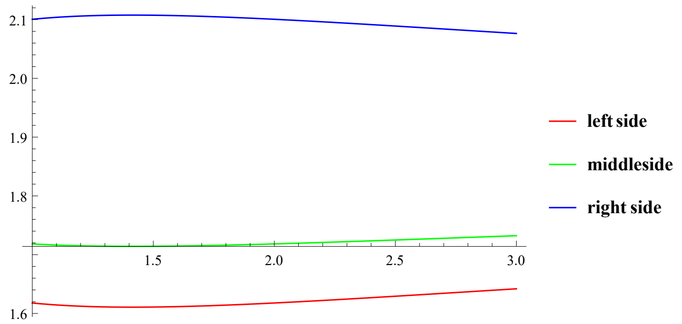

Figure 3: The 2D graphical view of the inequality (15) corresponding to $1 \leq \vartheta \leq 3$ is shown in Figure 3.

Table 2: The following table represents the comparison findings between the double inequality in Example 2.10.

Functions	1	1.5	2	2.5	3
$p_0(\vartheta)$	1.61777	1.61088	1.61777	1.62927	1.64191
$p_1(\vartheta)$	1.71828	1.71428	1.71828	1.72494	1.73227
$p_2(\vartheta)$	2.10091	2.1074	2.10091	2.08901	2.07637

By setting the parameters $M = 1, k = 1, \vartheta = 1, q = 2, m = 1$ we get the following functions

$$\begin{aligned}
&\frac{e^{\nu_1} + e^{\nu_2}}{2} - \frac{(\nu_2 - \nu_1)^2}{12}\left(|e^{\nu_1}|^2 + |e^{\nu_2}|^2\right)^{\frac{1}{2}} \\
&\leq \frac{1}{(\nu_2 - \nu_1)} \int_{\nu_1}^{\nu_2} e^{\varphi} d\varphi \\
&\leq \frac{e^{\nu_1} + e^{\nu_2}}{2} + \frac{(\nu_2 - \nu_1)^2}{12}\left(|e^{\nu_1}|^2 + |e^{\nu_2}|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

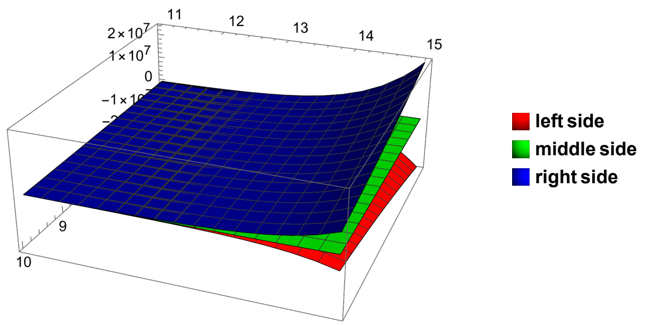

Figure 4: The 3D graphical view of the inequality (15) corresponding to $6 \leq \nu_1 \leq 10$, $11 \leq \nu_2 \leq 15$ is shown in Figure 4.

Remark 2.11. *By choosing $h(\varphi) = \varphi$ and $k = 1$ in Theorem 2.9, we attain [16, Theorem 2.2],*

$$\left|\frac{¥(\nu_1) + ¥(\nu_2)}{2} - \frac{\Gamma(\vartheta + 1)}{2(\nu_2 - \nu_1)^{\vartheta}}\left[\beth^{\vartheta}_{\nu_1^+} ¥(\nu_2) + \beth^{\vartheta}_{\nu_2^-} ¥(\nu_1)\right]\right|$$

$$\leq \frac{\vartheta(\nu_2-\nu_1)^2}{2(\vartheta+1)(\vartheta+2)}\left[\frac{|¥''(\nu_1)|^q + m\left|¥''\left(\frac{\nu_2}{m}\right)\right|^q}{2}\right]^{\frac{1}{q}}.$$

Theorem 2.12. *Suppose that* $¥ : [0, \nu_2^*] \to \Re$ *is a twice differentiable function. If the function* $|¥''|^q$ *is integrable and also* (h,m)*-convex on* $[\nu_1, \frac{\nu_2}{m}]$*, then for a non-negative function* $h : I \subseteq \Re \to \Re$*,* $q \geq 1$*,* $0 \leq \nu_1 < \nu_2$ *with* $m \in (0,1]$ *and* $\frac{\nu_2}{m} \leq \nu_2^*$*, the following inequality*

$$\left|\frac{¥(\nu_1)+¥(\nu_2)}{2} - \frac{\Gamma_k(\vartheta+k)}{2(\nu_2-\nu_1)^{\frac{\vartheta}{k}}}\left[\beth^{\vartheta}_{\nu_1^+,k}¥(\nu_2) + \beth^{\vartheta}_{\nu_2^-,k}¥(\nu_1)\right]\right|$$
$$\leq \frac{kM(\nu_2-\nu_1)^2}{2(\vartheta+k)}\left(1 - \frac{2}{p\left(\frac{\vartheta}{k}+1\right)+1}\right)^{\frac{1}{p}}\left(\left|¥''(\nu_1)\right|^q + m\left|¥''\left(\frac{\nu_2}{m}\right)\right|^q\right)^{\frac{1}{q}} \tag{16}$$

is satisfied with $h(x) \leq M$ *and* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. By employing Lemma 1.9 and Hölder's inequality, we obtain

$$\left|\frac{¥(\nu_1)+¥(\nu_2)}{2} - \frac{\Gamma_k(\vartheta+k)}{2(\nu_2-\nu_1)^{\frac{\vartheta}{k}}}\left[\beth^{\vartheta}_{\nu_1^+,k}¥(\nu_2) + \beth^{\vartheta}_{\nu_2^-,k}¥(\nu_1)\right]\right|$$
$$\leq \frac{(\nu_2-\nu_1)^2}{2}\int_0^1\left|\frac{1-(1-\varphi)^{\frac{\vartheta}{k}+1} - \varphi^{\frac{\vartheta}{k}+1}}{\frac{\vartheta}{k}+1}\right|\left|¥''(\varphi\nu_1 + (1-\varphi)\nu_2)\right|d\varphi$$
$$\leq \frac{k(\nu_2-\nu_1)^2}{2(\vartheta+k)}\left(\int_0^1\left(1-(1-\varphi)^{\frac{\vartheta}{k}+1} - \varphi^{\frac{\vartheta}{k}+1}\right)^p d\varphi\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^1\left|¥''(\varphi\nu_1 + (1-\varphi)\nu_2)\right|^q d\varphi\right)^{\frac{1}{q}}.$$

By using the relation

$$\left(1-(1-\varphi)^{\frac{\vartheta}{k}+1} - \varphi^{\frac{\vartheta}{k}+1}\right)^p \leq 1-(1-\varphi)^{p\left(\frac{\vartheta}{k}+1\right)} - \varphi^{p\left(\frac{\vartheta}{k}+1\right)},$$

for any $\varphi \in (0,1)$ and $p \geq 1$. Also using the Definition of (h,m)-convexity, we get

$$\left|\frac{¥(\nu_1)+¥(\nu_2)}{2} - \frac{\Gamma_k(\vartheta+k)}{2(\nu_2-\nu_1)^{\frac{\vartheta}{k}}}\left[\beth^{\vartheta}_{\nu_1^+,k}¥(\nu_2) + \beth^{\vartheta}_{\nu_2^-,k}¥(\nu_1)\right]\right|$$
$$\leq \frac{k(\nu_2-\nu_1)^2}{2(\vartheta+k)}\left(\int_0^1\left(1-(1-\varphi)^{p\left(\frac{\vartheta}{k}+1\right)} - \varphi^{p\left(\frac{\vartheta}{k}+1\right)}\right)d\varphi\right)^{\frac{1}{p}}$$
$$\times\left(\left|¥''(\nu_1)\right|^q\int_0^1 h(\varphi)d\varphi + m\left|¥''\left(\frac{\nu_2}{m}\right)\right|^q\int_0^1 h(1-\varphi)d\varphi\right)^{\frac{1}{q}}$$
$$\leq \frac{k(\nu_2-\nu_1)^2}{2(\vartheta+k)}\left(1 - \frac{2}{p\left(\frac{\vartheta}{k}+1\right)+1}\right)^{\frac{1}{p}}\left(M\left|¥''(\nu_1)\right|^q + mM\left|¥''\left(\frac{\nu_2}{m}\right)\right|^q\right)^{\frac{1}{q}}$$
$$= \frac{kM(\nu_2-\nu_1)^2}{2(\vartheta+k)}\left(1 - \frac{2}{p\left(\frac{\vartheta}{k}+1\right)+1}\right)^{\frac{1}{p}}\left(\left|¥''(\nu_1)\right|^q + m\left|¥''\left(\frac{\nu_2}{m}\right)\right|^q\right)^{\frac{1}{q}}.$$

The required result is proved. □

Example 2.13. *Here, we confirm the validity of the Theorem 2.12 via graphical representations. For this purpose, we substitute* $¥(\varphi) = \varphi^2$ *to get the following integral values*

$$\beth^{\vartheta}_{\nu_1^+,k}\varphi^{\nu_2} = \frac{1}{k\Gamma_k(\vartheta)}\int_{\nu_1}^{\nu_2}(\nu_2-\varphi)^{\frac{\vartheta}{k}-1}\varphi^2 d\varphi, \tag{17}$$

and

$$\beth^{\vartheta}_{\nu_2^-,k}\varphi^{\nu_1} = \frac{1}{k\Gamma_k(\vartheta)}\int_{\nu_1}^{\nu_2}(\varphi-\nu_1)^{\frac{\vartheta}{k}-1}\varphi^2 d\varphi. \tag{18}$$

By using the expressions (17) *and* (18) *in Theorem 2.12, we get*

$$\begin{aligned}
&\frac{\nu_1^2+\nu_2^2}{2} - \frac{kM(\nu_2-\nu_1)^2}{2(\vartheta+k)}\left[1-\frac{2}{p(\frac{\vartheta}{k}+1)}\right]^{\frac{1}{p}}\left[(2)^q + m\,(2)^q\right]^{\frac{1}{q}} \\
&\le \frac{\Gamma_k(\vartheta+k)}{2k\Gamma_k(\vartheta)(\nu_2-\nu_1)^{\frac{\vartheta}{k}}}\int_{\nu_1}^{\nu_2}\left[(\nu_2-\varphi)^{\frac{\vartheta}{k}-1}+(\varphi-\nu_1)^{\frac{\vartheta}{k}-1}\right]\varphi^2 d\varphi \\
&\le \frac{\nu_1^2+\nu_2^2}{2} + \frac{kM(\nu_2-\nu_1)^2}{2(\vartheta+k)}\left[1-\frac{2}{p(\frac{\vartheta}{k}+1)}\right]^{\frac{1}{p}}\left[(2)^q + m\,(2)^q\right]^{\frac{1}{q}}.
\end{aligned} \tag{19}$$

By setting the parameters $M = 1$, $k = 1$, $\nu_1 = 0$, $\nu_2 = 1$, $p = 2$, $q = 2$, $m = 1$ *in the double inequality* (19), *we obtain the following functions*

$$\begin{aligned}
p_o(\vartheta) &= \frac{1}{2} - \frac{1}{\vartheta+1}\left[\frac{2(2\vartheta+1)}{2\vartheta+3}\right]^{\frac{1}{2}}, \\
p_1(\vartheta) &= \frac{\vartheta}{2}\int_0^1\left[(1-\varphi)^{\vartheta-1}+\varphi^{\vartheta-1}\right]\varphi^2 d\varphi, \\
p_2(\vartheta) &= \frac{1}{2} + \frac{1}{\vartheta+1}\left[\frac{2(2\vartheta+1)}{2\vartheta+3}\right]^{\frac{1}{2}}.
\end{aligned}$$

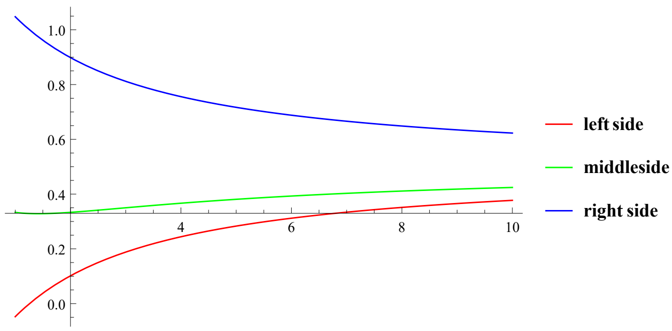

Figure 5: The 2D graphical view of the inequality (19) corresponding to $1 \le \vartheta \le 10$ is shown in Figure 5.

Table 3: The following table represents the comparison findings between the double inequality in Example 2.13.

Functions	1	2.5	4	5.5	7	8.5	10
$p_0(\vartheta)$	-0.04773	0.15007	0.24416	0.29857	0.33395	0.35878	0.37715
$p_1(\vartheta)$	0.33333	0.34127	0.36667	0.38718	0.40278	0.41479	0.42424
$p_2(\vartheta)$	1.04772	0.84993	0.75584	0.70143	0.66605	0.64123	0.62285

By setting the parameters $M = 1$, $k = 1$, $\vartheta = 1$, $q = 2$, $m = 1$, we get the following functions

$$\begin{aligned}&\frac{\nu_1^2+\nu_2^2}{2}-\frac{(\nu_2-\nu_1)^2}{2}\left(\frac{6}{5}\right)^{\frac{1}{2}}\\&\le \frac{1}{(\nu_2-\nu_1)}\int_{\nu_1}^{\nu_2}\varphi^2 d\varphi\\&\le \frac{\nu_1^2+\nu_2^2}{2}+\frac{(\nu_2-\nu_1)^2}{2}\left(\frac{6}{5}\right)^{\frac{1}{2}}.\end{aligned}$$

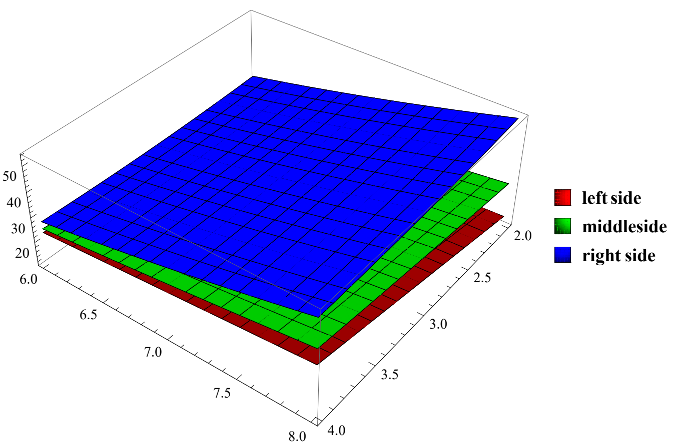

Figure 6: The 3D graphical view of the inequality (19) corresponding to $2 \le \nu_1 \le 4$, $6 \le \nu_2 \le 8$ is shown in Figure 6.

Remark 2.14. *By choosing $h(\varphi) = \varphi$ and $k = 1$ in Theorem 2.12, we attain [16, Theorem 2.4],*

$$\begin{aligned}&\left|\frac{¥(\nu_1)+¥(\nu_2)}{2}-\frac{\Gamma(\vartheta+1)}{2(\nu_2-\nu_1)^{\vartheta}}\left[\beth^{\vartheta}_{\nu_1^+}¥(\nu_2)+\beth^{\vartheta}_{\nu_2^-}¥(\nu_1)\right]\right|\\&\le \frac{(\nu_2-\nu_1)^2}{2(\vartheta+1)}\left(1-\frac{2}{p(\vartheta+1)+1}\right)^{\frac{1}{p}}\left(\frac{|¥''(\nu_1)|^q+m\left|¥''\left(\frac{\nu_2}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}}.\end{aligned}$$

Theorem 2.15. *Suppose that $¥ : [0, \nu_2^*] \to \mathfrak{R}$ is a twice differentiable mapping. If the function $|¥''|^q$ is integrable and (h, m)-convex on $[\nu_1, \frac{\nu_2}{m}]$ for a non-negative function $h : I \subseteq \mathfrak{R} \to \mathfrak{R}$, $q \ge 1$, $0 \le \nu_1 < \nu_2$ with $m \in (0, 1]$ and $\frac{\nu_2}{m} \le \nu_2^*$, then the following inequality*

$$\left|\frac{¥(\nu_1)+¥(\nu_2)}{2}-\frac{\Gamma_k(\vartheta+k)}{2(\nu_2-\nu_1)^{\frac{\vartheta}{k}}}\left[\beth^{\vartheta}_{\nu_1^+,k}¥(\nu_2)+\beth^{\vartheta}_{\nu_2^-,k}¥(\nu_1)\right]\right|$$

$$\leq \frac{kM(\nu_2-\nu_1)^2}{2(\vartheta+k)}\left[\left(1-\frac{2}{q\left(\frac{\vartheta}{k}+1\right)+1}\right)\left(\left|¥''(\nu_1)\right|^q+m\left|¥''(\frac{\nu_2}{m})\right|^q\right)\right]^{\frac{1}{q}}$$

is true with $h(x)\leq M$.

Proof. By employing Lemma 1.9 and Hölder's inequality, we obtain

$$\left|\frac{¥(\nu_1)+¥(\nu_2)}{2}-\frac{\Gamma_k(\vartheta+k)}{2(\nu_2-\nu_1)^{\frac{\vartheta}{k}}}\left[\beth^{\vartheta}_{\nu_1^+,k}¥(\nu_2)+\beth^{\vartheta}_{\nu_2^-,k}¥(\nu_1)\right]\right|$$
$$\leq \frac{(\nu_2-\nu_1)^2}{2}\int_0^1\left|\frac{1-(1-\varphi)^{\frac{\vartheta}{k}+1}-\varphi^{\vartheta}\frac{\vartheta}{k}+1}{\frac{\vartheta}{k}+1}\right|\left|¥''(\varphi\nu_1+(1-\varphi)\nu_2)\right|d\varphi$$
$$\leq \frac{k(\nu_2-\nu_1)^2}{2(\vartheta+k)}\left(\int_0^1 1d\varphi\right)^{\frac{1}{p}}\left(\int_0^1\left(1-(1-\varphi)^{\frac{\vartheta}{k}+1}-\varphi^{\frac{\vartheta}{k}+1}\right)^q\right.$$
$$\left.\times\left|¥''(\varphi\nu_1+(1-\varphi)\nu_2)\right|^q d\varphi\right)^{\frac{1}{q}}$$

By using the relation

$$\left(1-(1-\varphi)^{\frac{\vartheta}{k}+1}-\varphi^{\frac{\vartheta}{k}+1}\right)^q\leq 1-(1-\varphi)^{q\left(\frac{\vartheta}{k}+1\right)}-\varphi^{q\left(\frac{\vartheta}{k}+1\right)}$$

for any $\varphi\in(0,1)$ and $q\geq 1$. Also using the Definition of (h,m)-convexity, we get

$$\leq \frac{k(\nu_2-\nu_1)^2}{2(\vartheta+k)}\left[\int_0^1\left(1-(1-\varphi)^{q\left(\frac{\vartheta}{k}+1\right)}-\varphi^{q\left(\frac{\vartheta}{k}+1\right)}\right)\right.$$
$$\left.\times\left(h(\varphi)\left|¥''(\nu_1)\right|^q+mh(1-\varphi)\left|¥''\left(\frac{\nu_2}{m}\right)\right|^q\right)d\varphi\right]^{\frac{1}{q}}$$
$$\leq \frac{k(\nu_2-\nu_1)^2}{2(\vartheta+k)}\left[\left(1-\frac{2}{q\left(\frac{\vartheta}{k}+1\right)+1}\right)\left(M\left|¥''(\nu_1)\right|^q+mM\left|¥''(\frac{\nu_2}{m})\right|^q\right)\right]^{\frac{1}{q}}$$
$$=\frac{kM(\nu_2-\nu_1)^2}{2(\vartheta+k)}\left[\left(1-\frac{2}{q\left(\frac{\vartheta}{k}+1\right)+1}\right)\left(\left|¥''(\nu_1)\right|^q+m\left|¥''(\frac{\nu_2}{m})\right|^q\right)\right]^{\frac{1}{q}}.$$

Hence the proof is completed. □

Example 2.16. *To confirm the validity of the Theorem 2.15 via graphical representations. Let's substitute $¥(\varphi)=\varphi^3$ to get the following integral values*

$$\beth^{\vartheta}_{\nu_1^+,k}\varphi^{\nu_2}=\frac{1}{k\Gamma_k(\vartheta)}\int_{\nu_1}^{\nu_2}(\nu_2-\varphi)^{\frac{\vartheta}{k}-1}\varphi^3 d\varphi, \tag{20}$$

and

$$\beth^{\vartheta}_{\nu_2^-,k}\varphi^{\nu_1}=\frac{1}{k\Gamma_k(\vartheta)}\int_{\nu_1}^{\nu_2}(\varphi-\nu_1)^{\frac{\vartheta}{k}-1}\varphi^3 d\varphi. \tag{21}$$

By using the expressions (20) *and* (21) *in Theorem 2.15, we get*

$$\frac{\nu_1^3+\nu_2^3}{2}-\frac{kM(\nu_2-\nu_1)^2}{2(\vartheta+k)}\left[\left(1-\frac{2}{q(\frac{\vartheta}{k})+1}\right)\left((6a)^q+m(6\frac{b}{m})^q\right)\right]^{\frac{1}{q}}$$

$$\leq \frac{\Gamma_k(\vartheta+k)}{2k\Gamma_k(\vartheta)(\nu_2-\nu_1)^{\frac{\vartheta}{k}}}\int_{\nu_1}^{\nu_2}\left[(\nu_2-\varphi)^{\frac{\vartheta}{k}-1}+(\varphi-\nu_1)^{\frac{\vartheta}{k}-1}\right]\varphi^3 d\varphi$$
$$\leq \frac{\nu_1^3+\nu_2^3}{2}+\frac{kM(\nu_2-\nu_1)^2}{2(\vartheta+k)}\left[\left(1-\frac{2}{q(\frac{\vartheta}{k})+1}\right)\left((6a)^q+m(6\frac{b}{m})^q\right)\right]^{\frac{1}{q}}. \tag{22}$$

By setting the parameters $M=1$, $k=1$, $\nu_1=0$, $\nu_2=1$, $q=2$, $m=1$ in the graph of the double inequality (22)*, we obtain the following functions*

$$p_o(\vartheta)=\frac{1}{2}-\frac{3}{\vartheta+1}\left(\frac{2\vartheta+1}{2\vartheta+3}\right)^{\frac{1}{2}},$$
$$p_1(\vartheta)=\frac{\vartheta}{2}\int_0^1\left[(1-\varphi)^{\vartheta-1}+\varphi^{\vartheta-1}\right]\varphi^3 d\varphi,$$
$$p_2(\vartheta)=\frac{1}{2}+\frac{3}{\vartheta+1}\left(\frac{2\vartheta+1}{2\vartheta+3}\right)^{\frac{1}{2}}.$$

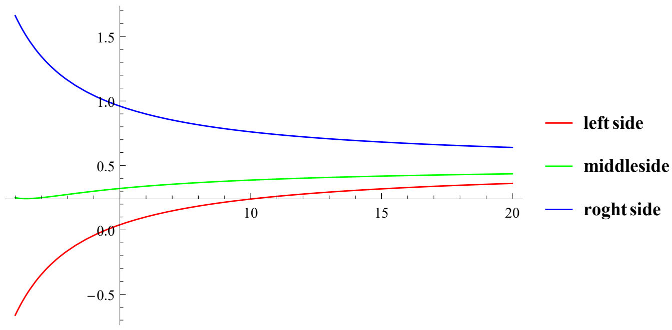

Figure 7: The 2D graphical view of the inequality (22) corresponding to $1 \leq \vartheta \leq 20$ is shown in Figure 7.

Table 4: The following table represents the comparison findings between the double inequality in Example 2.16.

Functions	1	2	3	4	5
$p_0(\vartheta)$	−0.66189	−0.34515	−0.16144	−0.04272	0.040067
$p_1(\vartheta)$	0.25	0.25	0.275	0.3	0.32143
$p_2(\vartheta)$	1.6619	1.34515	1.16144	1.04272	0.95993

By setting the parameters $M=1$, $k=1$, $\vartheta=1$, $q=2$, $m=1$, we get the following functions

$$\frac{\nu_1^3+\nu_2^3}{2}-\frac{3(\nu_2-\nu_1)^2}{2}\left(\frac{3(\nu_1^2+\nu_2^2)}{5}\right)^{\frac{1}{2}}$$
$$\leq \frac{1}{(\nu_2-\nu_1)}\int_{\nu_1}^{\nu_2}\varphi^3 d\varphi$$
$$\leq \frac{\nu_1^3+\nu_2^3}{2}+\frac{3(\nu_2-\nu_1)^2}{2}\left(\frac{3(\nu_1^2+\nu_2^2)}{5}\right)^{\frac{1}{2}}.$$

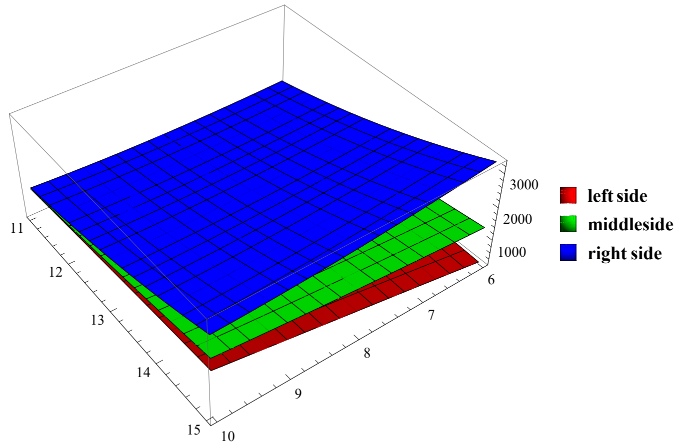

Figure 8: The 3D graphical view of the inequality (22) corresponding to $6 \leq \nu_1 \leq 10$, $11 \leq \nu_2 \leq 15$ is shown in Figure 8.

Remark 2.17. *By choosing $h(\varphi) = \varphi$ and $k = 1$ in Theorem 2.15, we attain [16, Theorem 2.6],*

$$\left|\frac{¥(\nu_1) + ¥(\nu_2)}{2} - \frac{\Gamma(\vartheta + 1)}{2(\nu_2 - \nu_1)^{\vartheta}}\left[\beth^{\vartheta}_{\nu_1^+}¥(\nu_2) + \beth^{\vartheta}_{\nu_2^-}¥(\nu_1)\right]\right|$$
$$\leq \frac{(\nu_2 - \nu_1)^2}{2(\vartheta + 1)}\left(\frac{q(\vartheta + 1) - 1}{q(\vartheta + 1) + 1}\right)^{\frac{1}{q}}\left(\frac{|¥''(\nu_1)|^q + m\left|¥''\left(\frac{\nu_2}{m}\right)\right|^q}{2}\right)^{\frac{1}{q}}.$$

3. Applications

In this section, we relate the main results with trapezoid type bounds. Trapezoidal inequalities for functions of diverse nature are useful in numerical computation. Trapezoid-type inequalities are fundamental in geometry and mathematics, offering a structural foundation for the comprehension and examination of trapezoids. They are essential tools for proving theorems, solving problems and laying the groundwork for more advanced mathematical concepts.

Proposition 3.1. *The following "trapezoid type inequality" is obtained by using the assumptions of Theorem 2.2 with $k = 1$, $M = 1$, and $\vartheta = 1$.*

$$\left|\frac{¥(\nu_1) + ¥(m\nu_2)}{2} - \frac{1}{(m\nu_2 - \nu_1)}\int_{\nu_1}^{m\nu_2} ¥(v)dv\right|$$
$$\leq \frac{(m\nu_2 - \nu_1)^2}{12}\left[|¥''(\nu_1)|^q + m|¥''(\nu_2)|^q\right]^{\frac{1}{q}}.$$

Proposition 3.2. *The following "trapezoid type inequality" is obtained by using the assumptions of Theorem 2.4 with $k = 1$, $M = 1$, and $\vartheta = 1$.*

$$\left|\frac{¥(\nu_1) + ¥(m\nu_2)}{2} - \frac{1}{(m\nu_2 - \nu_1)}\int_{\nu_1}^{m\nu_2} ¥(v)dv\right|$$
$$\leq \frac{(m\nu_2 - \nu_1)^2}{4}\left(\frac{2p - 1}{2p + 1}\right)^{\frac{1}{p}}\left(|¥''(\nu_1)|^q + m|¥''(\nu_2)|^q\right)^{\frac{1}{q}}.$$

Proposition 3.3. *The following "trapezoid type inequality" is obtained by using the assumptions of Theorem 2.7 with* $k = 1$, $M = 1$, *and* $\vartheta = 1$.

$$\left|\frac{¥(\nu_1) + ¥(m\nu_2)}{2} - \frac{1}{(m\nu_2 - \nu_1)} \int_{\nu_1}^{m\nu_2} ¥(v)dv\right|$$
$$\leq \frac{(m\nu_2 - \nu_1)^2}{4} \left[\left(\frac{2q-1}{2q+1}\right)\left(|¥''(\nu_1)|^q + m|¥''(\nu_2)|^q\right)\right]^{\frac{1}{q}}.$$

Proposition 3.4. *The following "trapezoid type inequality" is obtained by using the assumption of Theorem 2.9 with* $k = 1$, $M = 1$, $m = 1$ *and* $\vartheta = 1$.

$$\left|\frac{¥(\nu_1) + ¥(\nu_2)}{2} - \frac{1}{(\nu_2 - \nu_1)} \int_{\nu_1}^{\nu_2} ¥(v)dv\right|$$
$$\leq \frac{(\nu_2 - \nu_1)^2}{12} \left[|¥''(\nu_1)|^q + |¥''(\nu_2)|^q\right]^{\frac{1}{q}}.$$

Proposition 3.5. *The following "trapezoid type inequality" is obtained by using the assumption of Theorem 2.12 with* $k = 1$, $M = 1$, $m = 1$ *and* $\vartheta = 1$.

$$\left|\frac{¥(\nu_1) + ¥(\nu_2)}{2} - \frac{1}{(\nu_2 - \nu_1)} \int_{\nu_1}^{\nu_2} ¥(v)dv\right|$$
$$\leq \frac{(\nu_2 - \nu_1)^2}{4} \left(\frac{2p-1}{2p+1}\right)^{\frac{1}{p}} \left(|¥''(\nu_1)|^q + |¥''(\nu_2)|^q\right)^{\frac{1}{q}}.$$

Proposition 3.6. *The following "trapezoid type inequality" is obtained by using the assumption of Theorem 2.15 with* $k = 1$, $M = 1$, $m = 1$ *and* $\vartheta = 1$.

$$\left|\frac{¥(\nu_1) + ¥(\nu_2)}{2} - \frac{1}{(\nu_2 - \nu_1)} \int_{\nu_1}^{\nu_2} ¥(v)dv\right|$$
$$\leq \frac{(\nu_2 - \nu_1)^2}{4} \left[\left(\frac{2q-1}{2q+1}\right)\left(|¥''(\nu_1)|^q + |¥''(\nu_2)|^q\right)\right]^{\frac{1}{q}}.$$

4. Conclusion

The results of the work are derived through Hölder's inequality, which is related to the Lp norms of different functions. It is used in analysis, functional analysis, and probability theory. We also utilized the power mean inequality to explore the results. Both of these inequalities have strong applications in real-life problems. In this paper, we introduced novel developments of Hermite-Hadamard fractional integral inequalities via (h, m)-convexity. The graphs and corresponding tables of certain particular functions demonstrate the validity of the established findings. The results are coordinated with existing published works, presented as remarks. These remarks demonstrate that the presented results strongly generalize several previously reported works. The ideas and approaches presented in this paper may stimulate more investigation into this fascinating area.

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