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Convergence properties related to Bézier type of λ-Bernstein Kantorovich shifted knots operators

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Abstract. In this article, we introduce the L_p-spaces and create the Kantorovich-type operators of Schurer λ-Bernstein-Bézier basis functions, starting with shifted knots polynomials. We describe the convergence of our novel operators in the Lebesgue spaces and the continuous function space for any $1 \leq p < \infty$. The central moments for these operators are determined by computing the test functions. We then examine the properties of the Korovkin's type approximation with modulus of continuity of order one and two. We also derive the convergence theorems for these new operators using Peetre's K-functional and the fundamental conditions of Lipschitz continuous functions. Several direct approximation theorems are also derived by us. In last we given a numerical example with a graphical analysis.

1. Introduction and Preliminaries

One of the most famous Weierstrass approximation theorems was demonstrated most quickly and elegantly by S. N. Bernstein, who is one of the world's most famous mathematicians. Bernstein created the set of positive linear operators, suppose $\{B_s\}_{s\geq 1} : C[0,1] \to C[0,1]$, and known the Bernstein polynomial [6]. Bernstein, in his investigation, shows that the set of all continuous functions $[0,1]$ (suppose $C[0,1]$) is uniformly approximated by the sequences of polynomials $\{B_s\}_{s\geq 1}$. Thus, for any $y \in [0,1]$, the famous Bernstein yields the following operators:

$$B_s(g;y) = \sum_{i=0}^{s} g\left(\frac{i}{s}\right) b_{s,i}(y),$$

where the Bernstein polynomials $b_{s,i}(y)$ are the maximum degree of s for $s \in \mathbb{N}$, and $\mathbb{N}$ are the positive integers.

$$b_{s,i}(y) = \begin{cases} \binom{s}{i} y^i (1-y)^{s-i} & \text{for } \ i = 0, 1, \cdots, s, \\ 0 & \text{for } \ i > s \ \text{ or } \ i < 0. \end{cases}$$

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There is a very straightforward recursive connection to check for the Bernstein polynomials. The recursive relationship for Bernstein polynomials $b_{s,i}(y)$ is easily demonstrated as:

$$b_{s,i}(y) = (1-y)b_{s-1,i}(y) + yb_{s-1,i-1}(y).$$

Furthermore, the Kantorovich [13] definition of the traditional Bernstein operators is modified if measurable functions are used instead of continuous functions. For $s \in \mathbb{N}$ and $f \in L_p[0,1]$, Kantorovich defined the operators known in the mathematical literature as Kantorovich operators in his paper by $K_s : L_p[0,1] \to L_p[0,1]$ such that:

$$K_s(f;y) = (s+1)\sum_{k=0}^{s}\binom{s}{k}y^k(1-y)^{s-k}\int_{\frac{k}{s+1}}^{\frac{k+1}{s+1}} f(t)\mathrm{d}t, \tag{1}$$

and he demonstrated that the sequence of these operators converges almost everywhere to the function f on $[0,1]$ for any $f \in L_p[0,1], 1 \le p < \infty$.

Assuming $S^{\chi}_{s,k} : C[0,1+\chi] \to C[0,1]$, Schurer [27] introduced new Bernstein-type operators in 1962 using a positive integer, called χ, known as the Bernstein-Schurer operators. These operators are as follows:

$$S^{\chi}_{s,k}(g;y) = \sum_{k=0}^{s+\chi} g\left(\frac{k}{s}\right) r_{s,\chi,k}(y) \qquad (y \in [0,1]), \tag{2}$$

where $r_{s,\chi,k}(y)$ is known as the fundamental Bernstein-Schurer polynomials and χ is a constant positive integer so that

$$r_{s,\chi,k}(y) = \binom{s+\chi}{k} y^k(1-y)^{s+\chi-k} \qquad (k = 0,1,\cdots,s+\chi). \tag{3}$$

Cai et al. introduced the Bernstein polynomials in 2010 by introducing the shape parameter $\lambda \in [-1,1]$. According to [8], these bases are known as λ-Bernstein operators:

$$B_{s,\lambda}(g;y) = \sum_{i=0}^{s} g\left(\frac{i}{s}\right)\tilde{b}_{s,i}(\lambda;y), \tag{4}$$

where the new Bernstein basis function $\tilde{b}_{s,i}(\lambda;y)$ is used to express the Bernstein polynomial $b_{s,i}(y)$ defined by Ye et al., [33] as follows:

$$\begin{aligned}
\tilde{b}_{s,0}(\lambda;y) &= b_{s,0}(y) - \frac{\lambda}{s+1}b_{s+1,1}(y),\\
\tilde{b}_{s,i}(\lambda;y) &= b_{s,i}(y) + \lambda\left(\frac{s-2i+1}{s^2-1}b_{s+1,i}(y) - \frac{s-2i-1}{s^2-1}b_{s+1,i+1}(y)\right),\ \ 1 \le i \le s-1\\
\tilde{b}_{s,s}(\lambda;y) &= b_{s,s}(y) - \frac{\lambda}{s+1}b_{s+1,s}(y).
\end{aligned}$$

Additionally, Cai et al. proposed the shape-preserving features of generalized Bernstein operators [10] and the Kantorovich form of the λ-Bernstein Bézier operators [9]. Gadjiev et al. (2010) developed the following recent Bernstein-type Stancu polynomials, which are used to analyze the features of shifted knots [11]:

$$S_{s,\chi,\beta}(g;y) = \left(\frac{s+\rho_2}{s}\right)^s \sum_{i=0}^{s}\binom{s}{i}\left(y - \frac{\chi_2}{s+\rho_2}\right)^i\left(\frac{s+\chi_2}{s+\rho_2} - y\right)^{s-i} g\left(\frac{i+\chi_1}{s+\rho_1}\right), \tag{5}$$

assuming that, $y \in [\frac{\chi_2}{s+\rho_2}, \frac{s+\chi_2}{s+\rho_2}]$, and therefore $0 \le \chi_2 \le \chi_1 \le \rho_1 \le \rho_2$, χ_i, ρ_i, $i = 1,2$ are positive real numbers.

Recently, academics have constructed the Bernstein-type operators with the shape-preserving properties and the shifted knots properties approaches. These include the Bernstein-Kantorovich-Stancu type shifted knots operators [2], Szasz-Durrmeyer operators [3], Szász-Jakimovski-Leviatan Beta Type Integral Operators [4], Schurer-Stancu operators with shape parameter λ [5], the q-Bernstein operators related to the λ [7], Bernstein-Kantorovich operators by Stancu form [16], q-Bernstein shifted knots operators [17], the λ-Szász Kantorovich type operators [19], α type Bernstein-Schurer operators [20], Bézier bases with Schurer polynomials [22], Bernstein operators based in Bézier bases functions [29] and q-Bernstein operators based on the shape parameter λ [30]. The most recent works related to shape parameters we discuss [14, 23–26, 31, 32].

The shifted knots of λ-Bernstein-operators were developed by [18] by using the Bézier basis functions. The λ-Bernstein shifted knots operators $B_{s,\lambda}^{\varpi_1,\varpi_2}$ for any $\frac{\varpi_1}{s+\varpi_2} \leq y \leq \frac{s+\varpi_1}{s+\varpi_2}$ and the real number $0 \leq \varpi_1 \leq \varpi_2$ are defined as follows:

$$B_{s,\lambda}^{\varpi_1,\varpi_2}(g;y) = \left(\frac{s+\varpi_2}{s}\right)^s \sum_{i=0}^{s} \tilde{b}_{s,i}^{\varpi_1,\varpi_2}(\lambda;y) g\left(\frac{i}{s}\right), \tag{6}$$

where the Bézier bases functions $\tilde{b}_{s,i}^{\varpi_1,\varpi_2}(\lambda;y)$ defined by:

$$\begin{aligned}
\tilde{b}_{s,0}^{\varpi_1,\varpi_2}(\lambda;y) &= b_{s,0}^{\varpi_1,\varpi_2}(y) - \frac{\lambda}{s+1} b_{s+1,1}^{\varpi_1,\varpi_2}(y),\\
\tilde{b}_{s,i}^{\varpi_1,\varpi_2}(\lambda;y) &= b_{s,i}^{\varpi_1,\varpi_2}(y) + \lambda\left(\frac{s-2i+1}{s^2-1} b_{s+1,i}^{\varpi_1,\varpi_2}(y) - \frac{s-2i-1}{s^2-1} b_{s+1,i+1}^{\varpi_1,\varpi_2}(y)\right), \text{ for } 1 \leq i \leq s-1\\
\tilde{b}_{s,s}^{\varpi_1,\varpi_2}(\lambda;y) &= b_{s,s}^{\varpi_1,\varpi_2}(y) - \frac{\lambda}{s+1} b_{s+1,s}^{\varpi_1,\varpi_2}(y);
\end{aligned}$$

and

$$b_{s,i}^{\varpi_1,\varpi_2}(y) = \binom{s}{i}\left(y - \frac{\varpi_1}{s+\varpi_2}\right)^i \left(\frac{s+\varpi_1}{s+\varpi_2} - y\right)^{s-i}. \tag{7}$$

F. Özger [22] defined the Schurer variant of λ-Bernstein Bézier bases functions $\tilde{b}_{s,i,\aleph}^{\varpi_1,\varpi_2}(\lambda;y)$ and built the Schurer variant of the traditional λ-Bernstein Bézier bases functions. Most recently, A. Alotaibi [1] constructed the Schurer form of λ. Bernstein shifted knots operators in terms of Bézier bases functions with the help of a recently published article [18, 22] and constructed the operators as follows:

$$\mathcal{B}_{s,\lambda,\aleph}^{\varpi_1,\varpi_2}(g;y) = (s+1)\left(\frac{s+\aleph+\varpi_2}{s+\aleph}\right)^{s+\aleph} \sum_{i=0}^{s+\aleph} \tilde{b}_{s,i,\aleph}^{\varpi_1,\varpi_2}(\lambda;y) g\left(\frac{i}{s}\right), \tag{8}$$

where,

$$\begin{aligned}
\tilde{b}_{s,0,\aleph}^{\varpi_1,\varpi_2}(\lambda;y) &= b_{s,0,\aleph}^{\varpi_1,\varpi_2}(y) - \frac{\lambda}{s+\aleph+1} b_{s+1,1,\aleph}^{\varpi_1,\varpi_2}(y),\\
\tilde{b}_{s,j,\aleph}^{\varpi_1,\varpi_2}(\lambda;y) &= b_{s,j,\aleph}^{\varpi_1,\varpi_2}(y) + \lambda\left(\frac{s+\aleph-2j+1}{(s+\aleph)^2-1} b_{s+1,j,\aleph}^{\varpi_1,\varpi_2}(y) - \frac{s+\aleph-2j-1}{(s+\aleph)^2-1} b_{s+1,j+1,\aleph}^{\varpi_1,\varpi_2}(y)\right), \text{ for } 1 \leq j \leq s+\aleph-1\\
\tilde{b}_{s,s,\aleph}^{\varpi_1,\varpi_2,\aleph}(\lambda;y) &= b_{s,s,\aleph}^{\varpi_1,\varpi_2}(y) - \frac{\lambda}{s+\aleph+1} b_{s+1,s,\aleph}^{\varpi_1,\varpi_2}(y),
\end{aligned}$$

and the Schurer variation of the Bernstein basis function $b_{s,i}^{\varpi_1,\varpi_2}$ in terms of shifted knots defined by: for a positive integer $\aleph$

$$b_{s,i,\aleph}^{\varpi_1,\varpi_2}(y) = \binom{s+\aleph}{i}\left(y - \frac{\varpi_1}{s+\aleph+\varpi_2}\right)^i \left(\frac{s+\aleph+\varpi_1}{s+\aleph+\varpi_2} - y\right)^{s+\aleph-i}. \tag{9}$$

Clearly, for $\aleph = 0$, the Bernstein basis function $b^{\varpi_1,\varpi_2}_{s,i,\aleph}$ reduced to $b^{\varpi_1,\varpi_2}_{s,i}$ by equality (7). For $\varpi_1 = \varpi_2 = 0$ and $\aleph = 0$, then the classical Bernstein polynomial and classical Bézier basis functions are obtained.

Lemma 1.1. *Operators $\mathcal{B}^{\varpi_1,\varpi_2}_{s,\lambda,\aleph}$ having following equalities:*

$$\begin{aligned}
\mathcal{B}^{\varpi_1,\varpi_2}_{s,\lambda,\aleph}(1;y) &= 1;\\
\mathcal{B}^{\varpi_1,\varpi_2}_{s,\lambda,\aleph}\left(\frac{i}{s};y\right) &= \left(\frac{s+\aleph+\varpi_2}{s+\aleph}-\frac{2\lambda}{s(s+\aleph-1)}\right)\left(y-\frac{\varpi_1}{s+\aleph+\varpi_2}\right)\\
&\quad+\frac{\lambda}{s(s+\aleph-1)}\left(\frac{s+\aleph+\varpi_2}{s+\aleph}\right)^{s+\aleph}\left(y-\frac{\varpi_1}{s+\aleph+\varpi_2}\right)^{s+\aleph+1}\\
&\quad-\frac{\lambda}{s(s+\aleph-1)}\left(\frac{s+\aleph+\varpi_2}{s+\aleph}\right)^{s+\aleph}\left(\frac{s+\aleph+\varpi_1}{s+\aleph+\varpi_2}-y\right)^{s+\aleph+1}+\frac{\lambda}{s(s+\aleph-1)}\left(\frac{s+\aleph+\varpi_2}{s+\aleph}\right);\\
\mathcal{B}^{\varpi_1,\varpi_2}_{s,\lambda,\aleph}\left(\frac{i^2}{s^2};y\right) &= \left(\frac{s+\aleph}{s}\right)^2\left[\left(\frac{s+\aleph+\varpi_2}{s+\aleph}\right)^{s+\aleph}+\frac{2\lambda}{s+\aleph-1}\right]\left(y-\frac{\varpi_1}{s+\aleph+\varpi_2}\right)\\
&\quad+\left(\frac{s+\aleph+\varpi_2}{s+\aleph}\right)\left[\frac{s+\aleph-1}{s}\frac{s+\aleph+\varpi_2}{s}-\frac{4\lambda}{s^2}\right]\left(y-\frac{\varpi_1}{s+\varpi_2}\right)^2\\
&\quad+\lambda\left(\frac{s+\aleph+\varpi_2}{s+\aleph}\right)^{s+\aleph}\left[\frac{(s+\aleph+1)^2}{s^2(s+\aleph-1)}+\frac{1}{s+\aleph+1}\left(\frac{s+\aleph}{s}\right)^2\right]\left(y-\frac{\varpi_1}{s+\aleph+\varpi_2}\right)^{s+\aleph+1}\\
&\quad+\frac{\lambda}{s^2(s+\aleph-1)}\left(\frac{s+\aleph+\varpi_2}{s+\aleph}\right)^{s+\aleph}\left(\frac{s+\aleph+\varpi_1}{s+\aleph+\varpi_2}-y\right)^{s+\aleph+1}-\frac{\lambda}{s^2(s+\aleph-1)}\left(\frac{s+\aleph+\varpi_2}{s+\aleph}\right).
\end{aligned}$$

2. Operators and basic results

In this section, we consider the recently published article by Abdullah Alotaibi [1], the Schurer form of λ-Bernstein shifted knots operators, and their applications in various mathematical contexts. Alotaibi's work provides new insights into the behavior of these operators and demonstrates revealing potential for solving many problems in the approximation theory and related fields. Our motive is here to construct the Kantorovich form of the operators $\mathcal{B}^{\varpi_1,\varpi_2}_{s,\lambda,\aleph}$ and to obtain the approximation in $L_p[0,1+\aleph]$ spaces. We take $s+\aleph=\gamma$ and $\aleph$ to be a fixed positive constant. Moreover, let $p\in[1,\infty)$, $\frac{\varpi_1}{\gamma+\varpi_2}\le y\le\frac{\gamma+\varpi_1}{\gamma+\varpi_2}$ and $Q_s=\left[\frac{\varpi_1}{\gamma+\varpi_2},\frac{\gamma+\varpi_1}{\gamma+\varpi_2}\right]$, then for any $g\in L_p[0,1+\aleph]$ and $s\in\mathbb{N}$, we define the operators $B^{\varpi_1,\varpi_2}_{s,\lambda,\aleph}:L_p[0,1+\aleph]\to L_p(Q_s)$ such that:

$$G^{\varpi_1,\varpi_2}_{s,\lambda,\aleph}(g;y)=(s+1)\left(\frac{\gamma+\varpi_2}{\gamma}\right)^{\gamma}\sum_{i=0}^{\gamma}\tilde{b}^{\varpi_1,\varpi_2}_{s,i,\aleph}(\lambda;y)\int_{\frac{i}{s+1}}^{\frac{i+1}{s+1}}g(F)\,\mathrm{d}F,\tag{10}$$

where, $0\le\varpi_1\le\varpi_2$,$\mathbb{N}$ be the positive integers, Bézier basis functions $\tilde{b}^{\varpi_1,\varpi_2}_{s,i,\aleph}(\lambda;y)$ defined by equality (8) in terms of bernstein polynomial $b^{\varpi_1,\varpi_2}_{s,i,\aleph}(y)$ by equality (9).

In this study, the Bézier basis functions are used to construct the Kantorovich form of Schurer-type λ-Bernstein operators. The Schurer form of λ-Bernstein shifted knots operators and the classical form of λ-Bernstein Bézier basis functions are used and the novel Kantorovich type operators are introduced. For any function g that belongs to $L_p[0,1+\aleph]$, we compute the convergence of operators $B^{\varpi_1,\varpi_2}_{s,\lambda,\aleph}$ in L_p-spaces. Lastly, we provide a theorem of convergence for Lipschitz continuous functions, a theorem of local approximation, a computation of Korovkin's theorem, and some direct approximations.

Remark 2.1.

$$\int_{\frac{i}{s+1}}^{\frac{i+1}{s+1}} (F)^{\hbar-1}\,\mathrm{d}F = \begin{cases} \frac{1}{s+1} & \text{for } \hbar = 1, \\ \frac{1}{2(s+1)^2} + \frac{i}{(s+1)^2}, & \text{for } \hbar = 2, \\ \frac{1}{3(s+1)^3} + \frac{i^2}{(s+1)^3} + \frac{i}{(s+1)^3}, & \text{for } \hbar = 3. \end{cases} \tag{11}$$

Lemma 2.2. *Take $g = 1, t, t^2$, then for all $y \in Q_s$ and $s \in \mathbb{N} \setminus \{1\}$, the operators $G^{\varpi_1,\varpi_2}_{s,\lambda,\aleph}$ have the following identities:*

$$G^{\varpi_1,\varpi_2}_{s,\lambda,\aleph}(1;y) = 1;$$

$$\begin{aligned} G^{\varpi_1,\varpi_2}_{s,\lambda}(F;y) &= \frac{s}{(s+1)}\left(\frac{\gamma+\varpi_2}{\gamma} - \frac{2\lambda}{s(\gamma-1)}\right)\left(y - \frac{\varpi_1}{\gamma+\varpi_2}\right) \\ &+\frac{1}{(s+1)}\frac{\lambda}{(\gamma-1)}\left(\frac{\gamma+\varpi_2}{\gamma}\right)^{\gamma}\left(y - \frac{\varpi_1}{\gamma+\varpi_2}\right)^{\gamma+1} \\ &-\frac{1}{(s+1)}\frac{\lambda}{(\gamma-1)}\left(\frac{\gamma+\varpi_2}{\gamma}\right)^{\gamma}\left(\frac{\gamma+\varpi_1}{\gamma+\varpi_2} - y\right)^{\gamma+1} \\ &+\frac{1}{(s+1)}\frac{\lambda}{(\gamma-1)}\left(\frac{\gamma+\varpi_2}{\gamma}\right) + \frac{1}{2(s+1)}; \end{aligned}$$

$$\begin{aligned} G^{\varpi_1,\varpi_2}_{s,\lambda}\left(F^2;y\right) &= \frac{\gamma^2}{(s+1)^2}\left(y - \frac{\varpi_1}{\gamma+\varpi_2}\right)\left[\frac{2\lambda}{\gamma-1} + \left(\frac{\gamma+\varpi_2}{\gamma}\right)^{\gamma}\right] \\ &+\frac{s^2}{(s+1)^2}\left(\frac{\gamma+\varpi_2}{\gamma}\right)\left[\frac{\gamma-1}{s}\frac{\gamma+\varpi_2}{s} - \frac{4\lambda}{s^2}\right]\left(y - \frac{\varpi_1}{s+\varpi_2}\right)^2 \\ &+\frac{s^2}{(s+1)^2}\lambda\left(\frac{\gamma+\varpi_2}{\gamma}\right)^{\gamma}\left[\frac{(\gamma+1)^2}{s^2(\gamma-1)} + \frac{1}{\gamma+1}\left(\frac{\gamma}{s}\right)^2\right]\left(y - \frac{\varpi_1}{\gamma+\varpi_2}\right)^{\gamma+1} \\ &+\frac{1}{(s+1)^2}\frac{\lambda}{(\gamma-1)}\left(\frac{\gamma+\varpi_2}{\gamma}\right)^{\gamma}\left(\frac{\gamma+\varpi_1}{\gamma+\varpi_2} - y\right)^{\gamma+1} \\ &-\frac{1}{(s+1)^2}\frac{\lambda}{(\gamma-1)}\left(\frac{\gamma+\varpi_2}{\gamma}\right) \\ &+\frac{s}{(s+1)^2}\left(\frac{\gamma+\varpi_2}{\gamma} - \frac{2\lambda}{s(\gamma-1)}\right)\left(y - \frac{\varpi_1}{\gamma+\varpi_2}\right) \\ &+\frac{1}{(s+1)^2}\frac{\lambda}{(\gamma-1)}\left(\frac{\gamma+\varpi_2}{\gamma}\right)^{\gamma}\left(y - \frac{\varpi_1}{\gamma+\varpi_2}\right)^{\gamma+1} \\ &-\frac{1}{(s+1)^2}\frac{\lambda}{(\gamma-1)}\left(\frac{\gamma+\varpi_2}{\gamma}\right)^{\gamma}\left(\frac{\gamma+\varpi_1}{\gamma+\varpi_2} - y\right)^{\gamma+1} \\ &+\frac{1}{(s+1)^2}\frac{\lambda}{(\gamma-1)}\left(\frac{\gamma+\varpi_2}{\gamma}\right) + \frac{1}{3(s+1)^2}. \end{aligned}$$

Proof. We proof the identities in the light of Lemma 1.1 and equality 11, thus

$$\begin{aligned} G^{\varpi_1,\varpi_2}_{s,\lambda}(1;y) &= (s+1)\mathcal{B}^{\varpi_1,\varpi_2}_{s,\lambda}\int_{\frac{i}{s+1}}^{\frac{i+1}{s+1}} \mathrm{d}F \\ &= (s+1)\mathcal{B}^{\varpi_1,\varpi_2}_{s,\lambda}(1;y) \\ &= 1; \end{aligned}$$

$$
\begin{aligned}
G_{s,\lambda}^{\varpi_1,\varpi_2}(F;y) &= (s+1)\mathcal{B}_{s,\lambda}^{\varpi_1,\varpi_2}\int_{\frac{i}{s+1}}^{\frac{i+1}{s+1}}(F)\,\mathrm{d}F \\
&= \frac{1}{2(s+1)}\mathcal{B}_{s,\lambda}^{\varpi_1,\varpi_2}(1;y) + \frac{s}{(s+1)}\mathcal{B}_{s,\lambda}^{\varpi_1,\varpi_2}(F;y)\,; \\
B_{s,\lambda}^{\varpi_1,\varpi_2}\left(F^2;y\right) &= (s+1)\mathcal{B}_{s,\lambda}^{\varpi_1,\varpi_2}\int_{\frac{i}{s+1}}^{\frac{i+1}{s+1}}\left(F^2\right)\mathrm{d}F \\
&= \frac{1}{3(s+1)^2}\mathcal{B}_{s,\lambda}^{\varpi_1,\varpi_2}(1;y) + \frac{s^2}{(s+1)^2}\mathcal{B}_{s,\lambda}^{\varpi_1,\varpi_2}\left(F^2;y\right) + \frac{s}{(s+1)^2}\mathcal{B}_{s,\lambda}^{\varpi_1,\varpi_2}(F;y)\,.
\end{aligned}
$$

Thus we get the results of Lemma 2.2. □

Lemma 2.3. *For the operators $B_{s,\lambda,\aleph}^{\varpi_1,\varpi_2}$, we get the following central moments of order one and two:*

$$
\begin{aligned}
G_{s,\lambda}^{\varpi_1,\varpi_2}(F-y;y) \;=\;& \left(\frac{s(\gamma+\varpi_2)}{(s+1)(\gamma)} - \frac{2\lambda}{(s+1)(\gamma-1)} - 1\right)y - \frac{s}{(s+1)}\left(\frac{\gamma+\varpi_2}{\gamma} - \frac{2\lambda}{s(\gamma-1)}\right)\frac{\varpi_1}{\gamma+\varpi_2} \\
&+\frac{\lambda}{(s+1)(\gamma-1)}\left(\frac{\gamma+\varpi_2}{\gamma}\right)^{\gamma}\left(y-\frac{\varpi_1}{\gamma+\varpi_2}\right)^{\gamma+1} - \frac{\lambda}{(s+1)(\gamma-1)}\left(\frac{\gamma+\varpi_2}{\gamma}\right)^{\gamma}\left(\frac{\gamma+\varpi_1}{\gamma+\varpi_2}-y\right)^{\gamma+1} \\
&+\frac{\lambda}{(s+1)(\gamma-1)}\left(\frac{\gamma+\varpi_2}{\gamma}\right) + \frac{1}{2(s+1)};
\end{aligned}
$$

$$
\begin{aligned}
G_{s,\lambda}^{\varpi_1,\varpi_2}\left((F-y)^2;y\right) &= \left(\frac{\gamma}{s+1}\right)^2\left[\left(\frac{\gamma+\varpi_2}{\gamma}\right)^{\gamma} + \frac{2\lambda}{\gamma-1}\right]\left(y-\frac{\varpi_1}{\gamma+\varpi_2}\right) \\
&+\frac{s^2}{(s+1)^2}\left(\frac{\gamma+\varpi_2}{\gamma}\right)\left(y-\frac{\varpi_1}{s+\varpi_2}\right)^2\left[\frac{\gamma-1}{s}\frac{\gamma+\varpi_2}{s} - \frac{4\lambda}{s^2}\right] \\
&+\lambda\left(\frac{\gamma+\varpi_2}{\gamma}\right)^{\gamma}\left[\frac{(\gamma+1)^2}{(s+1)^2(\gamma-1)} + \frac{1}{\gamma+1}\left(\frac{\gamma}{s+1}\right)^2\right]\left(y-\frac{\varpi_1}{\gamma+\varpi_2}\right)^{\gamma+1} \\
&+\frac{\lambda}{(s+1)^2(\gamma-1)}\left(\frac{\gamma+\varpi_2}{\gamma}\right)^{\gamma}\left(\frac{\gamma+\varpi_1}{\gamma+\varpi_2}-y\right)^{\gamma+1} - \frac{\lambda}{(s+1)^2(\gamma-1)}\left(\frac{\gamma+\varpi_2}{\gamma}\right) \\
&+\frac{s}{(s+1)^2}\left(\frac{\gamma+\varpi_2}{\gamma} - \frac{2\lambda}{s(\gamma-1)}\right)\left(y-\frac{\varpi_1}{\gamma+\varpi_2}\right) + \frac{\lambda}{(s+1)^2(\gamma-1)}\left(\frac{\gamma+\varpi_2}{\gamma}\right)^{\gamma}\left(y-\frac{\varpi_1}{\gamma+\varpi_2}\right)^{\gamma+1} \\
&-\frac{\lambda}{(s+1)^2(\gamma-1)}\left(\frac{\gamma+\varpi_2}{\gamma}\right)^{\gamma}\left(\frac{\gamma+\varpi_1}{\gamma+\varpi_2}-y\right)^{\gamma+1} + \frac{\lambda}{(s+1)^2(\gamma-1)}\left(\frac{\gamma+\varpi_2}{\gamma}\right) + \frac{1}{3(s+1)^2} \\
&-\left(\frac{2s(\gamma+\varpi_2)}{(s+1)(\gamma)} - \frac{4\lambda}{(s+1)(\gamma-1)} - 1\right)y^2 + \frac{2s\varpi_1 y}{(s+1)(\gamma+\varpi_2)}\left(\frac{\gamma+\varpi_2}{\gamma} - \frac{2\lambda}{s(\gamma-1)}\right) \\
&-\frac{2\lambda y}{(s+1)(\gamma-1)}\left(\frac{\gamma+\varpi_2}{\gamma}\right)^{\gamma}\left(y-\frac{\varpi_1}{\gamma+\varpi_2}\right)^{\gamma+1} + \frac{2\lambda y}{(s+1)(\gamma-1)}\left(\frac{\gamma+\varpi_2}{\gamma}\right)^{\gamma}\left(\frac{\gamma+\varpi_1}{\gamma+\varpi_2}-y\right)^{\gamma+1} \\
&-\frac{2\lambda y}{(s+1)(\gamma-1)}\left(\frac{\gamma+\varpi_2}{\gamma}\right) - \frac{y}{(s+1)}.
\end{aligned}
$$

Remark 2.4. *We define the auxiliary operators $H_{s,\lambda}^{\varpi_1,\varpi_2}$ given by*

$$
H_{s,\lambda}^{\varpi_1,\varpi_2}(g;y) = \begin{cases} G_{s,\lambda}^{\varpi_1,\varpi_2}(g;y) & \text{if } y \in \mathcal{Q}_s, \\ g(y) & \text{if } y \in [0,1+\aleph] \setminus \mathcal{Q}_s. \end{cases} \tag{12}
$$

Theorem 2.5. *For all $g \in C[0, 1 + \aleph]$ and $\mathcal{Q}_s = \left[\frac{\varpi_1}{\gamma+\varpi_2}, \frac{\gamma+\varpi_1}{\gamma+\varpi_2}\right]$ the operators $G_{s,\lambda}^{\varpi_1,\varpi_2}$ satisfying:*

$$\lim_{s\to\infty} \|G_{s,\lambda}^{\varpi_1,\varpi_2}(g; y) - g(y)\|_{C(\mathcal{Q}_s)} = 0.$$

Proof. We take $g(F) = (F)^{\ell-1}$ for all $\ell = 1, 2, 3$, then from Lemma 2.2 it is easy to get that:

$$\lim_{s\to\infty} \max_{y\in\mathcal{Q}_s} |G_{s,\lambda}^{\varpi_1,\varpi_2}\left((F)^{\ell-1}; y\right) - y^{\ell-1}| = 0, \tag{13}$$

from (12), it is obvious to write:

$$\|H_{s,\lambda}^{\varpi_1,\varpi_2}(g; y) - g(y)\|_{C[0,1+\aleph]} = \max_{y\in\mathcal{Q}_s} |G_{s,\lambda}^{\varpi_1,\varpi_2}(g; y) - g(y)|. \tag{14}$$

From (13), (14) we get

$$\lim_{s\to\infty} \|H_{s,\lambda}^{\varpi_1,\varpi_2}\left((F)^{\ell-1}; y\right) - y^{\ell-1}\|_{C[0,1+\aleph]} = 0, \quad \ell = 1, 2, 3.$$

By Korovkin's theorem to operators $H_{s,\lambda}^{\varpi_1,\varpi_2}$, we see

$$\lim_{s\to\infty} \|H_{s,\lambda}^{\varpi_1,\varpi_2}(g; y) - g(y)\|_{C[0,1+\aleph]} = 0,$$

for all $f \in C[0, 1 + \aleph]$, (14) gives

$$\lim_{s\to\infty} \max_{y\in\mathcal{Q}_s} |G_{s,\lambda}^{\varpi_1,\varpi_2}(g; y) - g(y)| = 0.$$

□

Theorem 2.6. *Let $p \in [1, \infty)$, then for any $g \in L_p[0, 1 + \aleph]$, the operators $B_{s,\lambda}^{\varpi_1,\varpi_2}$ satisfying:*

$$\lim_{s\to\infty} \|G_{s,\lambda}^{\varpi_1,\varpi_2}(g; y) - g(y)\|_{L_p(\mathcal{Q}_s)} = 0.$$

Proof. Using Theorem 2.5 and the operators $H_{s,\lambda}^{\varpi_1,\varpi_2}$ by (12), we establish the result. We use Luzin's Theorem [15], then for a continuous function f on $[0, 1 + \aleph]$, a positive ϵ exists such that $\|g - f\|_{L_p[0,1+\aleph]} < \epsilon$. We use Theorem 2.5, then for any $\epsilon > 0$, there exists $s_0 \in \mathbb{N}$ such that $s \geq s_0$, and $\|H_{s,\lambda}^{\varpi_1,\varpi_2}(f; y) - f(y)\|_{C[0,1+\aleph]} < \epsilon$.For $s \in \mathbb{N}$, let the operator of $H_{s,\lambda}^{\varpi_1,\varpi_2}$ be $\|H_{s,\lambda}^{\varpi_1,\varpi_2}\|$ and be considered as$\|H_{s,\lambda}^{\varpi_1,\varpi_2}\| : L_p[0, 1 + \aleph] \to L_p[0, 1 + \aleph]$. To establish the results of Theorem 2.6, it is sufficient to show that a positive constant M exists such that $\|H_{s,\lambda}^{\varpi_1,\varpi_2}\| \leq M$. Thus, from this discussion, we immediately write:

$$\begin{aligned}\|H_{s,\lambda}^{\varpi_1,\varpi_2}(g; y) - g(y)\|_{L_p[0,1+\aleph]} &\leq \|H_{s,\lambda}^{\varpi_1,\varpi_2}(g; y) - H_{s,\lambda}^{\varpi_1,\varpi_2}(f; y)\|_{L_p[0,1+\aleph]} \\ &+\|H_{s,\lambda}^{\varpi_1,\varpi_2}(f; y) - f(y)\|_{C[0,1+\aleph]} \\ &+\|g - f\|_{L_p[0,1+\aleph]}.\end{aligned} \tag{15}$$

By Jensen's inequality we can get,

$$\begin{aligned}|G_{s,\lambda}^{\varpi_1,\varpi_2}(g; y)|^p &\leq \left\{(s+1)\left(\frac{\gamma+\varpi_2}{\gamma}\right)^{\gamma} \sum_{i=0}^{\gamma} \tilde{b}_{s,i,\aleph}^{\varpi_1,\varpi_2}(\lambda; y) \int_{\frac{i}{s+1}}^{\frac{i+1}{s+1}} g(F)\,dF\right\}^p \\ &\leq \left\{(\gamma+1+\varpi_2)\left(\frac{\gamma+\varpi_2}{\gamma}\right)^{\gamma} \sum_{i=0}^{\gamma} \tilde{b}_{s,i,\aleph}^{\varpi_1,\varpi_2}(\lambda; y) \int_{\frac{i}{s+1}}^{\frac{i+1}{s+1}} g(F)\,dF\right\}^p\end{aligned}$$

$$\leq \left\{(\gamma+1)\left(\frac{\gamma+\varpi_2}{\gamma}\right)^{\gamma}\sum_{i=0}^{\gamma}\left(\int_{\frac{i}{s+1}}^{\frac{i+1}{s+1}}\left|g(F)\right|\mathrm{d}F\right)\left(\frac{\gamma+1+\varpi_2}{\gamma+1}\right)\tilde{b}_{s,i,\aleph}^{\varpi_1,\varpi_2}(\lambda;y)\right\}^{p}$$
$$\leq \sum_{i=0}^{\gamma}\left(\frac{\gamma+\varpi_2}{\gamma}\right)^{\gamma}\tilde{b}_{s,i,\aleph}^{\varpi_1,\varpi_2}(\lambda;y)(\gamma+1)\left\{\frac{\gamma+1+\varpi_2}{\gamma+1}\int_{\frac{i}{s+1}}^{\frac{i+1}{s+1}}\left|g(F)\right|\mathrm{d}F\right\}^{p}$$
$$\leq \sum_{i=0}^{\gamma}\left(\frac{\gamma+\varpi_2}{\gamma}\right)^{\gamma}\tilde{b}_{s,i,\aleph}^{\varpi_1,\varpi_2}(\lambda;y)(\gamma+1)\left(\frac{\gamma+1+\varpi_2}{\gamma+1}\right)^{p}\int_{\frac{i}{s+1}}^{\frac{i+1}{s+1}}\left|g(F)\right|^{p}\mathrm{d}F.$$

By taking integral on both sides over $\left[\frac{\varpi_1}{\gamma+\varpi_2},\frac{\gamma+\varpi_1}{\gamma+\varpi_2}\right]$, we can obtain

$$\begin{aligned}
\int_{\frac{\varpi_1}{\gamma+\varpi_2}}^{\frac{\gamma+\varpi_1}{\gamma+\varpi_2}}\left|G_{s,\lambda}^{\varpi_1,\varpi_2}(g;y)\right|^{p}\mathrm{d}y \quad &\leq \sum_{i=0}^{\gamma}\left(\frac{\gamma+\varpi_2}{\gamma}\right)^{\gamma}\sum_{i=0}^{\gamma}\tilde{b}_{s,i,\aleph}^{\varpi_1,\varpi_2}(\lambda;y)(\gamma+1)\left(\frac{\gamma+1+\varpi_2}{\gamma+1}\right)^{p}\int_{\frac{i}{s+1}}^{\frac{i+1}{s+1}}\left|g(F)\right|^{p}\mathrm{d}F\\
&= \sum_{i=0}^{\gamma}\left(\frac{\gamma+\varpi_2}{\gamma}\right)^{\gamma}\sum_{i=0}^{\gamma}\left\{b_{s,j,\aleph}^{\varpi_1,\varpi_2}(y)\right.\\
&\left.+\lambda\left(\frac{\gamma-2j+1}{(\gamma)^2-1}b_{s+1,j,\aleph}^{\varpi_1,\varpi_2}(y)-\frac{\gamma-2j-1}{(\gamma)^2-1}b_{s+1,j+1,\aleph}^{\varpi_1,\varpi_2}(y)\right)\right\}\\
&\times(\gamma+1)\left(\frac{\gamma+1+\varpi_2}{\gamma+1}\right)^{p}\int_{\frac{i}{s+1}}^{\frac{i+1}{s+1}}\left|g(F)\right|^{p}\mathrm{d}F\\
&= \Sigma_1+\Sigma_2+\Sigma_3(suppose),
\end{aligned}$$

where

$$\Sigma_1 \quad = \sum_{i=0}^{\gamma}\left(\frac{\gamma+\varpi_2}{\gamma}\right)^{\gamma}\sum_{i=0}^{\gamma}b_{s,j,\aleph}^{\varpi_1,\varpi_2}(y)(\gamma+1)\left(\frac{\gamma+1+\varpi_2}{\gamma+1}\right)^{p}\int_{\frac{i}{s+1}}^{\frac{i+1}{s+1}}\left|g(F)\right|^{p}\mathrm{d}F,$$

$$\Sigma_2 \quad = \sum_{i=0}^{\gamma}\left(\frac{\gamma+\varpi_2}{\gamma}\right)^{\gamma}\sum_{i=0}^{\gamma}\lambda\frac{\gamma-2j+1}{(\gamma)^2-1}b_{s+1,j,\aleph}^{\varpi_1,\varpi_2}(y)(\gamma+1)\left(\frac{\gamma+1+\varpi_2}{\gamma+1}\right)^{p}\int_{\frac{i}{s+1}}^{\frac{i+1}{s+1}}\left|g(F)\right|^{p}\mathrm{d}F,$$

$$\Sigma_3 \quad = \sum_{i=0}^{\gamma}\left(\frac{\gamma+\varpi_2}{\gamma}\right)^{\gamma}\sum_{i=0}^{\gamma}(-\lambda)\frac{\gamma-2j-1}{(\gamma)^2-1}b_{s+1,j+1,\aleph}^{\varpi_1,\varpi_2}(y)(\gamma+1)\left(\frac{\gamma+1+\varpi_2}{\gamma+1}\right)^{p}\int_{\frac{i}{s+1}}^{\frac{i+1}{s+1}}\left|g(F)\right|^{p}\mathrm{d}F.$$

Now we can conclude that:

$$\begin{aligned}
\Sigma_1 \quad &= \sum_{i=0}^{\gamma}\left(\frac{\gamma+\varpi_2}{\gamma}\right)^{\gamma}\sum_{i=0}^{\gamma}b_{s,j,\aleph}^{\varpi_1,\varpi_2}(y)(\gamma+1)\left(\frac{\gamma+1+\varpi_2}{\gamma+1}\right)^{p}\int_{\frac{i}{s+1}}^{\frac{i+1}{s+1}}\left|g(F)\right|^{p}\mathrm{d}F\\
&= \sum_{i=0}^{\gamma}\left(\frac{\gamma+\varpi_2}{\gamma}\right)^{\gamma}\sum_{i=0}^{\gamma}\binom{\gamma}{i}\left(y-\frac{\varpi_1}{\gamma+\varpi_2}\right)^{i}\left(\frac{\gamma+\varpi_1}{\gamma+\varpi_2}-y\right)^{\gamma-i}(\gamma+1)\left(\frac{\gamma+1+\varpi_2}{\gamma+1}\right)^{p}\int_{\frac{i}{s+1}}^{\frac{i+1}{s+1}}\left|g(F)\right|^{p}\mathrm{d}F\\
&= \sum_{i=0}^{\gamma}\left(\frac{\gamma+\varpi_2}{\gamma}\right)^{\gamma}\left(\frac{\gamma}{\gamma+\varpi_2}\right)^{\gamma}(\gamma+1)\left(\frac{\gamma+1+\varpi_2}{\gamma+1}\right)^{p}\int_{\frac{i}{s+1}}^{\frac{i+1}{s+1}}\left|g(F)\right|^{p}\mathrm{d}F\\
&\leq \left(\frac{\gamma+1+\varpi_2}{\gamma+1}\right)^{p}\|g\|_{L_p[0,1+\aleph]}^{p}\\
&\leq (1+\varpi_2)\|g\|_{L_p[0,1+\aleph]}^{p}
\end{aligned}$$

$$\leq M_1 \|g\|^p_{L_p[0,1+\aleph]}, \quad M_1 > 0.$$

In similar way we can conclude that

$$\begin{aligned} \Sigma_2 &= \sum_{i=0}^{\gamma} \left(\frac{\gamma+\varpi_2}{\gamma}\right)^{\gamma} \sum_{i=0}^{\gamma} \lambda \frac{\gamma-2j+1}{(\gamma)^2-1} b^{\varpi_1,\varpi_2}_{s+1,j,\aleph}(y)(\gamma+1) \left(\frac{\gamma+1+\varpi_2}{\gamma+1}\right)^p \int_{\frac{i}{s+1}}^{\frac{i+1}{s+1}} \left|g\left(F\right)\right|^p \mathrm{d}F \\ &\leq M_2 \|g\|^p_{L_p[0,1+\aleph]}, \quad M_2 > 0, \end{aligned}$$

$$\begin{aligned} \Sigma_3 &= \sum_{i=0}^{\gamma} \left(\frac{\gamma+\varpi_2}{\gamma}\right)^{\gamma} \sum_{i=0}^{\gamma} (-\lambda) \frac{\gamma-2j-1}{(\gamma)^2-1} b^{\varpi_1,\varpi_2}_{s+1,j+1,\aleph}(y)(\gamma+1) \left(\frac{\gamma+1+\varpi_2}{\gamma+1}\right)^p \int_{\frac{i}{s+1}}^{\frac{i+1}{s+1}} \left|g\left(F\right)\right|^p \mathrm{d}F \\ &\leq M_3 \|g\|^p_{L_p[0,1+\aleph]}, \quad M_3 > 0, \end{aligned}$$

If we consider $\int_{[0,1+\aleph]\setminus Q_s} |g(y)|^p \mathrm{d}y \leq \|g\|^p_{L_p[0,1+\aleph]}$ and the above-mentioned inequality then we immediately can find

$$\int_0^{1+\aleph} |H^{\varpi_1,\varpi_2}_{s,\lambda}(g;y)|^p \mathrm{d}y \leq \Big[1 + M_1 + M_2 + M_3\Big] \|g\|^p_{L_p[0,1+\aleph]}. \tag{16}$$

In the view of above expression we can write:

$$\begin{aligned} \|H^{\varpi_1,\varpi_2}_{s,\lambda}(g;y)\|_{L_p[0,1+\aleph]} &\leq (1 + M_1 + M_2 + M_3) \|g\|_{L_p[0,1+\aleph]} \\ &\leq M \|g\|_{L_p[0,1+\aleph]}. \end{aligned}$$

Thus, there exists a positive number M such that $\|H^{\varpi_1,\varpi_2}_{s,\lambda}\|_{L_p[0,1+\aleph]} \leq M$.
By taking into account the equality (15), we can write here:

$$\begin{aligned} \|H^{\varpi_1,\varpi_2}_{s,\lambda}(g;y) - g(y)\|_{L_p[0,1+\aleph]} &\leq \left(1 + \|H^{\varpi_1,\varpi_2}_{s,\lambda}\|_{L_p[0,1+\aleph]}\right) \|g - f\|_{L_p[0,1+\aleph]} + \|H^{\varpi_1,\varpi_2}_{s,\lambda}(f;y) - f(y)\|_{L_p[0,1+\aleph]} \\ &\leq M\epsilon + 2\epsilon. \end{aligned}$$

Thus, in the view of above inequality we have

$$\begin{aligned} \|H^{\varpi_1,\varpi_2}_{s,\lambda}(g;y) - g(y)\|_{L_p[0,1+\aleph]} &= \left(\int_0^{1+\aleph} |H^{\varpi_1,\varpi_2}_{s,\lambda}(g;y) - g(y)|^p \mathrm{d}y\right)^{\frac{1}{p}} \\ &= \left(\int_{Q_s} |G^{\varpi_1,\varpi_2}_{s,\lambda}(g;y) - g(y)|^p \mathrm{d}y\right)^{\frac{1}{p}} \\ &= \|G^{\varpi_1,\varpi_2}_{s,\lambda}(g;y) - g(y)\|_{L_p(Q_s)} \\ &\leq M\epsilon + 2\epsilon. \end{aligned}$$

Thus we can conclude $\lim_{s\to\infty} \|G^{\varpi_1,\varpi_2}_{s,\lambda}(g;y) - g(y)\|_{L_p(Q_s)} = 0$. This gives the complete proof of our Theorem 2.6. □

3. Rate of Convergence and approximation in Lipschitz-spaces

In the present section, we calculate the convergence properties, derive various Lipschitz-type approximations, and provide some direct approximation properties. Assume that all uniformly continuous functions on $[0,1]$ belong to the class $g \in \tilde{C}[0,1]$. Let $\tilde{\omega}(g;\tilde{\delta})$ be used to represent the classical modulus of continuity for every positive $\tilde{\delta}$, and $\lim_{\tilde{\delta}\to 0+}\tilde{\omega}(g;\tilde{\delta}) = 0$, and

$$\tilde{\omega}(g;\tilde{\delta}) = \sup_{|y_1-y_2|\leq\tilde{\delta}} |\, g(y_1) - g(y_2)\,|; \quad y_1, y_2 \in [0,1], \tag{17}$$

$$|\, g(y_1) - g(y_2)\,| \leq \left(1 + \frac{|\, y_1 - y_2\,|}{\tilde{\delta}}\right)\tilde{\omega}(g;\tilde{\delta}). \tag{18}$$

Theorem 3.1. *[28] Let $[\alpha_1,\alpha_2] \subseteq [\beta_1,\beta_2]$, then operators $\{K_r\}_{r\geq 1} : [\alpha_1,\alpha_2] \to [\beta_1,\beta_2]$ one have*

1. *if $g \in C[\beta_1,\beta_2]$ and $y \in [\alpha_1,\alpha_2]$*

$$\begin{aligned}|K_r(g;y) - g(y)| &\leq |g(y)||K_r(1;y) - 1| \\ &+\Big\{K_r(1;y) + \frac{1}{\tilde{\delta}}\sqrt{K_r((t-y)^2;y)}\sqrt{K_r(1;y)}\Big\}\tilde{\omega}(g;\tilde{\delta}),\end{aligned}$$

2. *if $g' \in C[\beta_1,\beta_2]$ and $y \in [\alpha_1,\alpha_2]$*

$$\begin{aligned}|K_r(g;y) - g(y)| &\leq |g(y)||K_r(1;y) - 1| + |g'(y)||K_r(t-y;y)| \\ &+K_r((t-y)^2;y)\Big\{\sqrt{K_r(1;y)} + \frac{1}{\tilde{\delta}}\sqrt{K_r((t-y)^2;y)}\Big\}\tilde{\omega}(g';\tilde{\delta}).\end{aligned}$$

Theorem 3.2. *For any $g \in \tilde{C}[0,1+\aleph]$ and $y \in \mathcal{Q}_s$ we have*

$$|G_{s,\lambda}^{\varpi_1,\varpi_2}(g;y) - g(y)| \leq 2\tilde{\omega}\left(f; \sqrt{\tilde{\delta}_{s,\lambda}^{\varpi_1,\varpi_2}(y)}\right),$$

where $\tilde{\delta}_{s,\lambda}^{\varpi_1,\varpi_2}(y) = G_{s,\lambda}^{\varpi_1,\varpi_2}\left((F-y)^2;y\right)$ which defined by Lemma 2.3.

Proof. It is simple to write when taking into account the (1) of Theorem 3.1 and applying the Lemma 2.2:

$$\begin{aligned}|G_{s,\lambda}^{\varpi_1,\varpi_2}(g;y) - g(y)| &\leq |g(y)||G_{s,\lambda}^{\varpi_1,\varpi_2}(1;y) - 1| + \Big\{G_{s,\lambda}^{\varpi_1,\varpi_2}(1;y) \\ &+\frac{1}{\tilde{\delta}}\sqrt{G_{s,\lambda}^{\varpi_1,\varpi_2}\left((F-y)^2;y\right)}\sqrt{G_{s,\lambda}^{\varpi_1,\varpi_2}(1;y)}\Big\}\tilde{\omega}(g;\tilde{\delta}).\end{aligned}$$

By choosing, $\tilde{\delta} = \sqrt{\tilde{\delta}_{s,\lambda}^{\varpi_1,\varpi_2}(y)} = \sqrt{G_{s,\lambda}^{\varpi_1,\varpi_2}\left((F-y)^2;y\right)}$, then easy to get result. □

Corollary 3.3. *Let $g \in \tilde{C}[0,1+\aleph]$ and $y \in \mathcal{Q}_s$ then*

$$|G_{s,\lambda}^{\varpi_1,\varpi_2}(g;y) - g(y)| \leq 2\tilde{\omega}\left(g; \sqrt{\tilde{\delta}}\right),$$

where $\tilde{\delta} = \max_{y\in\mathcal{Q}_s}\tilde{\delta}_{s,\lambda}^{\varpi_1,\varpi_2}(y)$.

Theorem 3.4. *For any $f' \in \tilde{C}[0,1+\aleph]$ and $y \in \mathcal{Q}_s$ we have*

$$|G_{s,\lambda}^{\varpi_1,\varpi_2}(f;y) - f(y)| \leq \Phi_{s,\lambda}^{\varpi_1,\varpi_2}(y)|f'(y)| + 2\Psi_{s,\lambda}^{\varpi_1,\varpi_2}(y)\tilde{\omega}\left(f'; \sqrt{\tilde{\delta}_{s,\lambda}^{\varpi_1,\varpi_2}(y)}\right),$$

where $\Phi_{s,\lambda}^{\varpi_1,\varpi_2}(y) = \max_{y\in\mathcal{Q}_s}\left|G_{s,\lambda}^{\varpi_1,\varpi_2}(F-y;y)\right|$, $\Psi_{s,\lambda}^{\varpi_1,\varpi_2}(y) = \max_{y\in\mathcal{Q}_s}\left|\tilde{\delta}_{s,\lambda}^{\varpi_1,\varpi_2}(y)\right|$ and $\tilde{\delta}_{s,\lambda}^{\varpi_1,\varpi_2}(y) = G_{s,\lambda}^{\varpi_1,\varpi_2}\left((F-y)^2;y\right)$.

Proof. From the (2) of Theorem 3.1 and applying the Lemma 2.2:

$$\begin{aligned}|G_{s,\lambda}^{\varpi_1,\varpi_2}(f;y)-f(y)| \quad &\leq |g(y)||G_{s,\lambda}^{\varpi_1,\varpi_2}(1;y)-1|+|f'(y)|\left|G_{s,\lambda}^{\varpi_1,\varpi_2}(F-y;y)\right| \\ &+G_{s,\lambda}^{\varpi_1,\varpi_2}\left((F-y)^2;y\right)\left\{\sqrt{G_{s,\lambda}^{\varpi_1,\varpi_2}(1;y)}\right. \\ &\left.+\frac{1}{\tilde{\delta}}\sqrt{G_{s,\lambda}^{\varpi_1,\varpi_2}\left((F-y)^2;y\right)}\right\}\tilde{\omega}(f';\tilde{\delta}).\end{aligned}$$

We choose $\Phi_{s,\lambda}^{\varpi_1,\varpi_2}(y)=\max_{y\in Q_s}\left|G_{s,\lambda}^{\varpi_1,\varpi_2}(F-y;y)\right|$ and taking into account Theorem 3.2 we get result. □

Here, we derive the approximations in terms of Lipschitz spaces and Peetre's K-functional. Therefore, we remember the integral modification of the modulus of continuity provided for every $g\in L_p(\Theta_\Lambda)$ for $p\in[1,\infty)$.

$$\tilde{\omega}_{1,p}(g,t)=\sup_{y\in[0,1]}\sup_{0<\Lambda\leq t}\|\, g(y+\Lambda)-g(y)\,\|_{L_p(\Theta_\Lambda)}, \tag{19}$$

where the L_p-norm over $\Theta_\Lambda=[0,1-\Lambda]$ is the equipped by $\|\ .\ \|_{Lp(\Theta_\Lambda)}$.Furthermore, we construct f as an absolutely continuous function such that $\mathcal{W}_{1,p}(\Theta_\Lambda)=\{f,f'\in L_p(\Theta_\Lambda)\}$ in order to quantify the quantitative estimations using Peetre's K-functional. Given $g\in L_p(\Theta_\Lambda)$, Peetre's K-functional for every $p\in[1,\infty)$ is as follows:

$$\mathcal{K}_{1,p}(g;t)=\inf_{\phi\in\mathcal{W}_{1,p}(\Theta_\Lambda)}\Big(\|\, g-\phi\,\|_{L_p(\Theta_\Lambda)}+t\,\|\,\phi'\,\|_{L_p(\Theta_\Lambda)}\Big). \tag{20}$$

The relationship between the Peetre's K-functional and integral modulus of continuity is then given by the inequality [12].

$$C_1\tilde{\omega}_{1,p}(g,t)\leq K_{1,p}(g,t)\leq C_2\tilde{\omega}_{1,p}(g,t). \tag{21}$$

Theorem 3.5. *Let $g\in\mathcal{W}_{1,p}(\Theta_\Lambda)$, then for all $p>1$ we get*

$$\left\|G_{s,\lambda}^{\varpi_1,\varpi_2}(g;y)-g(y)\right\|_{L_p(Q_s)}\leq 2^{\frac{1}{p}}\left(1+\frac{1}{p-1}\right)\max_{y\in Q_s}\left(\tilde{\delta}_{s,\lambda}^{\varpi_1,\varpi_2}(y)\right)^{\frac{1}{2}}\left\|g'\right\|_{L_p[0,1+\aleph]}$$

where $\tilde{\delta}_{s,\lambda}^{\varpi_1,\varpi_2}(y)$ given by Theorem 3.2 and $G_{s,\lambda}^{\varpi_1,\varpi_2}\left((F-y)^2;y\right)$ is defined by Lemma 2.3.

Proof. For any $y\in Q_s$, we can write

$$\begin{aligned}\left|G_{s,\lambda}^{\varpi_1,\varpi_2}(g;y)-g(y)\right| \quad &=(s+1)\left|\sum_{i=0}^{\gamma}\left(\frac{\gamma+\varpi_2}{\gamma}\right)^{\gamma}\tilde{b}_{s,i,\aleph}^{\varpi_1,\varpi_2}(\lambda;y)\int_{Q_s}(g(F)-g(y))\,\mathrm{d}F\right| \\ &\leq(s+1)\sum_{i=0}^{\gamma}\left(\frac{\gamma+\varpi_2}{\gamma}\right)^{\gamma}\tilde{b}_{s,i,\aleph}^{\varpi_1,\varpi_2}(\lambda;y)\int_{Q_s}\int_y^F\left|g'(\zeta)\right|\mathrm{d}\zeta\mathrm{d}F \\ &\leq\Omega_{g'}(y)(s+1)\sum_{i=0}^{\gamma}\left(\frac{\gamma+\varpi_2}{\gamma}\right)^{\gamma}\tilde{b}_{s,i,\aleph}^{\varpi_1,\varpi_2}(\lambda;y)\int_{Q_s}\left|F-y\right|\mathrm{d}F,\end{aligned}$$

where $\Omega_{g'}(y)$ is the Hardy-Littlewood's majorant of g', given by

$$\Omega_{g'}(y)=\sup_{F\in[0,1]}\frac{1}{F-y}\int_y^F\left|g'(\zeta)\right|\mathrm{d}\zeta \qquad (F\neq y).$$

It is obvious to write from Cauchy-Schwarz's inequality that

$$\left|G^{\varpi_1,\varpi_2}_{s,\lambda}(g;y)-g(y)\right| \leq \Omega_{g'}(y)(s+1)^{\frac{1}{2}}\left(\sum_{i=0}^{\gamma}\left(\frac{\gamma+\varpi_2}{\gamma}\right)^{\gamma}\tilde{b}^{\varpi_1,\varpi_2}_{s,i,\aleph}(\lambda;y)\right)^{\frac{1}{2}}$$

$$\times\left(\sum_{i=0}^{\gamma}\left(\frac{\gamma+\varpi_2}{\gamma}\right)^{\gamma}\tilde{b}^{\varpi_1,\varpi_2}_{s,i,\aleph}(\lambda;y)\int_{Q_s}(F-y)^2\,\mathrm{d}t\right)^{\frac{1}{2}}$$

$$\leq \Omega_{g'}(y)\max_{y\in Q_s}\left(G^{\varpi_1,\varpi_2}_{s,\lambda}\left((F-y)^2;y\right)\right)^{\frac{1}{2}}.$$

We apply the Hardy-Littlewood's Theorem by [34] for all $1<p<\infty$, then

$$\int_0^{1+\aleph}\Omega_{g'}(y)\mathrm{d}y \leq 2\left(\frac{p}{p-1}\right)^{p}\int_0^{1+\aleph}\left|g'(y)\right|^{p}\mathrm{d}y.$$

Thus we get

$$\left\|G^{\varpi_1,\varpi_2}_{s,\lambda}(g;y)-g(y)\right\|_{L_p(Q_s)} \leq \max_{y\in Q_s}\left(G^{\varpi_1,\varpi_2}_{s,\lambda}\left((F-y)^2;y\right)\right)^{\frac{1}{2}}\left\|g'\right\|_{L_p[0,1+\aleph]}2^{\frac{1}{p}}\left(1+\frac{1}{p-1}\right)$$

□

Theorem 3.6. *Let $p\in[1,\infty)$, then for any $g\in L_p[0,1+\aleph]$ we get*

$$\left\|G^{\varpi_1,\varpi_2}_{s,\lambda}(g;y)-g(y)\right\|_{L_p(Q_s)} \leq M\tilde{\omega}_{1,p}\left(g;\rho^{\varpi_1,\varpi_2}_{s,\lambda}(y)\right),$$

with $K=2^{\frac{1}{p}}K_2\left(2^{\frac{p-1}{p}}+\left(1+\frac{1}{p-1}\right)\right)$, $\rho^{\varpi_1,\varpi_2}_{s,\lambda}(y)=\max_{y\in Q_s}\sqrt{G^{\varpi_1,\varpi_2}_{s,\lambda}\left((F-y)^2;y\right)}$ and K be positive real number.

Proof. We Consider

$$\left\|G^{\varpi_1,\varpi_2}_{s,\lambda}(h;y)-h(y)\right\|_{L_p[0,1+\aleph]} \leq \begin{cases} 2\,\|h\|_{L_p[0,1+\aleph]}, \\ \text{for } \ h\in L_p[0,1+\aleph], \\ 2^{\frac{1}{p}}\left(\frac{p}{p-1}\right)\rho^{\varpi_1,\varpi_2}_{s,\lambda}(y)\,\|h\|_{L_p[0,1+\aleph]}, \\ \text{for } \ h\in\mathcal{W}_{1,p}[0,1+\aleph], \end{cases} \tag{22}$$

where, we denote $\max_{y\in Q_s}\left(G^{\varpi_1,\varpi_2}_{s,\lambda}\left((F-y)^2;y\right)\right)^{\frac{1}{2}}=\rho^{\varpi_1,\varpi_2}_{s,\lambda}(y)$.

For an arbitrary function $f\in\mathcal{W}_{1,p}[0,1+\aleph]$, we can write:

$$\left\|G^{\varpi_1,\varpi_2}_{s,\lambda}(f;y)-f(y)\right\|_{L_p(Q_s)} \leq \left\|G^{\varpi_1,\varpi_2}_{s,\lambda}(g-f;y)-(g-f)(y)\right\|_{L_p(Q_s)}+\left\|G^{\varpi_1,\varpi_2}_{s,\lambda}(f;y)-f(y)\right\|_{L_p(Q_s)}$$

$$\leq 2\left\{2^{\frac{1-p}{p}}\left(1+\frac{1}{p-1}\right)\rho^{\varpi_1,\varpi_2}_{s,\lambda}(y)\left\|f'\right\|_{L_p[0,1]}+\left\|g-f\right\|_{L_p[0,1+\aleph]}\right\}$$

$$\leq 2K_{1,p}\left\{g;2^{\frac{1-p}{p}}\left(\frac{p}{p-1}\right)\rho^{\varpi_1,\varpi_2}_{s,\lambda}(y)\right\}$$

$$\leq 2M_2\tilde{\omega}_{1,p}\left\{g;2^{\frac{1-p}{p}}\left(\frac{1}{p-1}\right)\rho^{\varpi_1,\varpi_2}_{s,\lambda}(y)\right\}$$

$$\leq 2M_2\left\{1+2^{\frac{1-p}{p}}\left(\frac{p}{p-1}\right)\right\}\tilde{\omega}_{1,p}\left(g;\rho^{\varpi_1,\varpi_2}_{s,\lambda}(y)\right)$$

$$\leq M\tilde{\omega}_{1,p}\left(g;\rho_{s,\lambda}^{\varpi_1,\varpi_2}(y)\right).$$

□

The operators $G_{s,\lambda}^{\varpi_1,\varpi_2}$ are now approximated using Lipschitz-type maximum functions and local direct estimates. We assume that $\rho_1,\ \rho_2>0$ and $\theta\in(0,1]$ are real constants. From [21], we recall the Lipschitz-type maximal function:

$$Lip_{s,\lambda}^{\varpi_1,\varpi_2}[0,1]:=\left\{\Phi\in C[0,1]:|\Phi(t)-\Phi(y)|\leq K\frac{|t-y|^{\theta}}{(\rho_1y^2+\rho_2y+t)^{\frac{\theta}{2}}};\ y,\ t\in[0,1]\right\},$$

where K be any positive constant.

Theorem 3.7. *Let $f\in Lip_{s,\lambda}^{\varpi_1,\varpi_2}[0,1+\aleph]$, then for all $\theta\in(0,1]$ there exists positive K*

$$|G_{s,\lambda}^{\varpi_1,\varpi_2}(f;y)-f(y)|\leq K(\rho_1y^2+\rho_2y)^{-\theta/2}\left[G_{s,\lambda}^{\varpi_1,\varpi_2}\left((F-y)^2;y\right)\right]^{\frac{\theta}{2}},$$

where $G_{s,\lambda}^{\varpi_1,\varpi_2}\left((F-y)^2;y\right)$ is defined by Lemma 2.3.

Proof. Suppose that $f\in Lip_{s,\lambda}^{\varpi_1,\varpi_2}[0,1+\aleph]$ as well as $\theta\in(0,1]$. First, we wish to demonstrate that our equivalence is true for $\theta=1$. Consequently, we can say that

$$\begin{aligned}|G_{s,\lambda}^{\varpi_1,\varpi_2}(f;y)-f(y)| &\leq |G_{s,\lambda}^{\varpi_1,\varpi_2}(|f(t)-f(y)|;y)|+f(y)\,|G_{s,\lambda}^{\varpi_1,\varpi_2}(1;y)-1|\\ &\leq (s+1)\sum_{i=0}^{\gamma}\left(\frac{\gamma+\varpi_2}{\gamma}\right)^{\gamma}\tilde{b}_{s,i,\aleph}^{\varpi_1,\varpi_2}(\lambda;y)\int_{\frac{i}{s+1}}^{\frac{i+1}{s+1}}\left|f(F)-f(y)\right|\mathrm{d}F\\ &\leq K(s+1)\sum_{i=0}^{\gamma}\left(\frac{\gamma+\varpi_2}{\gamma}\right)^{\gamma}\tilde{b}_{s,i,\aleph}^{\varpi_1,\varpi_2}(\lambda;y)\int_{\frac{i}{s+1}}^{\frac{i+1}{s+1}}\frac{|F-y|}{(\rho_1y^2+\rho_2y+t)^{\frac{1}{2}}}\mathrm{d}F,\end{aligned}$$

where a positive constant K is used. Therefore, we utilize $(\rho_1y^2+\rho_2y+t)^{-1/2}\leq(\rho_1y^2+\rho_2y)^{-1/2}$ for constants $\rho_1,\ \rho_2\geq 0,$. The Cauchy-Schwarz inequality can be applied, and then it is easy to get

$$\begin{aligned}|G_{s,\lambda}^{\varpi_1,\varpi_2}(f;y)-f(y)| &\leq K(s+1)(\rho_1y^2+\rho_2y)^{-1/2}\sum_{i=0}^{\gamma}\left(\frac{\gamma+\varpi_2}{\gamma}\right)^{\gamma}\tilde{b}_{s,i,\aleph}^{\varpi_1,\varpi_2}(\lambda;y)\int_{\frac{i}{s+1}}^{\frac{i+1}{s+1}}\left|F-y\right|\mathrm{d}F\\ &\leq K(\rho_1y^2+\rho_2y)^{-1/2}\left|G_{s,\lambda}^{\varpi_1,\varpi_2}(F-y;y)\right|\\ &\leq K(\rho_1y^2+\rho_2y)^{-1/2}\left|G_{s,\lambda}^{\varpi_1,\varpi_2}\left((F-y)^2;y\right)\right|^{1/2}.\end{aligned}$$

As a result, we can conclude that equality is true for $\theta=1$. However, we now demonstrate that our findings apply to $\theta\in(0,1)$. Using the Hölder's inequality and the monotonicity property on $G_{s,\lambda}^{\varpi_1,\varpi_2}$, we observe here

$$\begin{aligned}\left|G_{s,\lambda}^{\varpi_1,\varpi_2}(f;y)-f(y)\right| &\leq (s+1)\sum_{i=0}^{\gamma}\left(\frac{\gamma+\varpi_2}{\gamma}\right)^{\gamma}\tilde{b}_{s,i,\aleph}^{\varpi_1,\varpi_2}(\lambda;y)\int_{\frac{i}{s+1}}^{\frac{i+1}{s+1}}\left|f(F)-f(y)\right|\mathrm{d}F\\ &\leq \Bigg((s+1)\sum_{i=0}^{\gamma}\left(\frac{\gamma+\varpi_2}{\gamma}\right)^{\gamma}\tilde{b}_{s,i,\aleph}^{\varpi_1,\varpi_2}(\lambda;y)\\ &\times\int_{\frac{i}{s+1}}^{\frac{i+1}{s+1}}\left|f(F)-f(y)\right|^{\frac{2}{\theta}}\mathrm{d}F\Bigg)^{\frac{\theta}{2}}\left((s+1)\sum_{i=0}^{\gamma}\left(\frac{\gamma+\varpi_2}{\gamma}\right)^{\gamma}\tilde{b}_{s,i,\aleph}^{\varpi_1,\varpi_2}(\lambda;y)\int_{\frac{i}{s+1}}^{\frac{i+1}{s+1}}\mathrm{d}F\right)^{\frac{2-\sigma}{2}}\\ &\leq K\left(\sum_{i=0}^{\gamma}\left(\frac{\gamma+\varpi_2}{\gamma}\right)^{\gamma}\tilde{b}_{s,i,\aleph}^{\varpi_1,\varpi_2}(\lambda;y)\int_{\frac{i}{s+1}}^{\frac{i+1}{s+1}}(F-y)^2\,\mathrm{d}F\right)^{\frac{\theta}{2}}\left(\frac{(s+1)}{t+\rho_1y^2+\rho_2y}\right)^{\frac{\theta}{2}}\end{aligned}$$

$$\leq K(\rho_1 y^2 + \rho_2 y)^{-\theta/2}\Big\{\sum_{i=0}^{\gamma}\left(\frac{\gamma+\varpi_2}{\gamma}\right)^{\gamma}\tilde{b}^{\varpi_1,\varpi_2}_{s,i,\aleph}(\lambda;y)\int_{\frac{i}{s+1}}^{\frac{i+1}{s+1}}(F-y)^2\,\mathrm{d}F\Big\}^{\frac{\theta}{2}}$$

$$\leq K(\rho_1 y^2 + \rho_2 y)^{-1/2}\left|G^{\varpi_1,\varpi_2}_{s,\lambda}\left((F-y)^2;y\right)\right|^{1/2}.$$

Thus we complete the our proof. □

4. Numerical Comparison with Classical Kantorovich-Bernstein Operators

To highlight the efficiency and flexibility of the newly constructed Kantorovich-type Schurer λ-Bernstein operators with shifted knots, we compare them with the Schurer λ-Bernstein shifted knots operators of Bézier bases functions.

We consider the function:

$$g(y) = e^y, \quad y \in [0,1],$$

For comparison, we fix the parameters:

$$s = 5, \quad \aleph = 2, \quad \lambda = 0.5, \quad \varpi_1 = 1, \quad \varpi_2 = 2.$$

The following figure presents the plots of the original function $g(y)$, its approximation using the Schurer λ-Bernstein shifted knots operators of Bézier bases functions $K_s(g;y)$, and the newly developed operator $G^{\varpi_1,\varpi_2}_{s,\lambda,\aleph}(g;y)$.

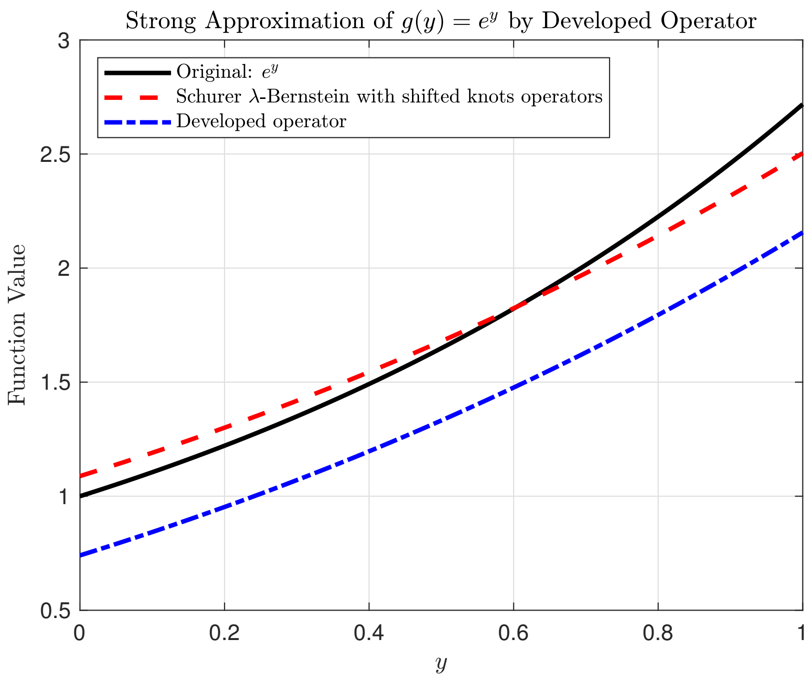

Figure 1: Comparison of $g(y) = e^y$ with Schurer λ-Bernstein shifted knots operators of Bézier bases functions and the developed Kantorovich-Schurer λ-Bernstein operator with shifted knots for $s = 5, 10, 15$, $\aleph = 2$, $\lambda = 0.5$.

As shown, the developed operator provides a closer approximation to the original function $g(y)$ over the interval $[0,1]$, exhibiting enhanced convergence behavior due to the involvement of the shape parameter λ, the Schurer variation, and the shifted knot properties.

5. Conclusion & Observation

Based on the published article [1], we deduce that the new operators (10) are the Kantorovich formulation of the Schurer λ-Bernstein shifted knots operators of Bézier bases functions. For any $1 \leq p < \infty$ and $\aleph$ is a fixed positive integer, we showed in our work that the sequence of these operators converges nearly everywhere to the function $f \in L_p[0, 1+\aleph]$ on $[0, 1+\aleph]$. The operators $\mathcal{B}^{\varpi_1,\varpi_2}_{s,\lambda,\aleph}$ provide the nearly convergent results in the continuous function spaces, as we can finally show using Korovkin's theorem, the modulus of continuity, Lipschitz spaces, and Peetre's K-functional. They are a generalized version of the previously published article [1, 6, 8, 18, 22] and Lebesgue spaces.

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