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New integral inequalities on the co-ordinates for geometrically exponentially $s-$convex functions in the second sense

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Abstract. The primary objective of this study is to introduce novel definitions for geometrically exponentially $s-$ convex functions in the second sense and establish new integral inequalities associated with them. To derive the main results, we employed classical mathematical inequalities, including Hölder's and Young's inequalities. These fundamental tools played a crucial role in obtaining refined and generalized forms of integral inequalities, thereby contributing to the existing body of knowledge in this field.

1. Introduction

The concept of convexity holds a significant position in inequality theory and has been a focal point of research due to its broad applicability in various mathematical disciplines. In particular, convex functions serve as essential tools in optimization, mathematical analysis, and functional inequalities, making them a subject of great interest among scholars. Their properties and generalizations have been widely explored, leading to numerous advancements in the field. A formal definition of convex functions is provided in [3] and the definition of exponentially convex functions is expressed as follows.

Definition 1.1. *[2] A function $\Omega : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be exponentially convex function, if*

$$\Omega\left((1-\xi)\,\varrho_1+\xi\varrho_2\right) \leq (1-\xi)\frac{\Omega(\varrho_1)}{e^{\alpha\varrho_1}}+\xi\frac{\Omega(\varrho_2)}{e^{\alpha\varrho_2}}$$

for all $\varrho_1, \varrho_2 \in I, \alpha \in \mathbb{R}$ and $\xi \in [0,1]$.

The concept of geometrically convex functions, which plays a significant role in mathematical analysis and optimization, was first introduced in [4]. This notion extends the classical idea of convexity by incorporating geometric considerations, leading to a broader class of functions with diverse applications. A formal definition of geometrically convex functions, as presented in [4], is given below.

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Definition 1.2. *A function $\Omega : I \subseteq \mathbb{R}^{+} \longrightarrow \mathbb{R}^{+}$ is said to be a geometrically convex function, if*

$$\Omega(\varrho_1^{\xi}\varrho_2^{1-\xi}) \leq [\Omega(\varrho_1)]^{\xi}\,[\Omega(\varrho_2)]^{1-\xi}$$

for all $\varrho_1, \varrho_2 \in I$ and $\xi \in [0,1]$.

Several recent studies have explored various properties and applications of geometrically convex functions, contributing to the ongoing development of this field. For a detailed discussion and related results, refer to [5],[6],[7],[8],[9],[10], [23]-[31].

Aslan and Akdemir introduced the definition of exponential convex functions on the coordinates, which extends the classical concept of convexity to a broader framework. Their formal definition is given as follows.

Definition 1.3. *[11] Let us consider the bidimensional interval $\Delta = [a,b] \times [c,d]$ in R^2 with $a < b$ and $c < d$. The mapping $\Omega : \Delta \longrightarrow R$ is exponentially convex on the co-ordinates on Δ, if the following inequality holds,*

$$\Omega\left(\xi\varrho_1 + (1-\xi)\varrho_3, \xi\varrho_2 + (1-\xi)\varrho_4\right) \leq \xi\frac{\Omega(\varrho_1,\varrho_2)}{e^{\alpha(\varrho_1+\varrho_2)}} + (1-\xi)\frac{\Omega(\varrho_3,\varrho_4)}{e^{\alpha(\varrho_3+\varrho_4)}}$$

for all $(\varrho_1,\varrho_2),(\varrho_3,\varrho_4) \in \Delta, \alpha \in R$,and $\xi \in [0,1]$.

Aslan and Akdemir provided an alternative but equivalent formulation of the definition of exponentially convex functions on the coordinates on Δ. Their refined definition is presented as follows.

Definition 1.4. *[11] The mapping $\Omega : \Delta \longrightarrow \mathbb{R}$ is exponentially convex on the co-ordinates on Δ, if the following inequality holds,*

$$\begin{aligned} &\Omega\left(\xi\varrho_1 + (1-\xi)\varrho_2, \omega\varrho_3 + (1-\omega)\varrho_4\right) \\ \leq\ & \xi\omega\frac{\Omega(\varrho_1,\varrho_3)}{e^{\alpha(\varrho_1+\varrho_3)}} + \xi(1-\omega)\frac{\Omega(\varrho_1,\varrho_4)}{e^{\alpha(\varrho_1+\varrho_4)}} + (1-\xi)\omega\frac{\Omega(\varrho_2,\varrho_3)}{e^{\alpha(\varrho_2+\varrho_3)}} + (1-\xi)(1-\omega)\frac{\Omega(\varrho_2,\varrho_4)}{e^{\alpha(\varrho_2+\varrho_4)}} \end{aligned}$$

for all $(\varrho_1,\varrho_3),(\varrho_1,\varrho_4),(\varrho_2,\varrho_3),(\varrho_2,\varrho_4) \in \Delta, \alpha \in \mathbb{R}$ and $\xi,\omega \in [0,1]$.

The representation of convex functions on the coordinates naturally led to the question of whether the Hermite-Hadamard inequality could be extended to the coordinates. This intriguing problem was addressed in Dragomir's work [12], where a significant generalization of the Hermite-Hadamard inequality was established. As a result, this extension, which broadens the inequality from the plane $\mathbb{R}^2$ to a rectangular domain, has become a well-recognized result in the literature. The formal statement of this extension is given below.

Theorem 1.5. *[12] Suppose that $\Omega : \Delta = [\varrho_1,\varrho_2] \times [\varrho_3,\varrho_4] \to \mathbb{R}$ is convex on the co-ordinates on Δ. Then, one has the inequalities:*

$$\begin{aligned} &\Omega\left(\frac{\varrho_1+\varrho_2}{2}, \frac{\varrho_3+\varrho_4}{2}\right) \\ &\leq \frac{1}{(\varrho_2-\varrho_1)(\varrho_4-\varrho_3)}\int_{\varrho_1}^{\varrho_2}\int_{\varrho_3}^{\varrho_4}\Omega(x,y)dxdy \\ &\leq \frac{\Omega(\varrho_1,\varrho_3)+\Omega(\varrho_1,\varrho_4)+\Omega(\varrho_2,\varrho_3)+\Omega(\varrho_2,\varrho_4)}{4}. \end{aligned}$$

The above inequalities are sharp.

Theorem 1.6. *[11] Let $\Omega : \Delta = [\varrho_1,\varrho_2] \times [\varrho_3,\varrho_4] \to \mathbb{R}$ be partially differentiable mapping on $\Delta = [\varrho_1,\varrho_2] \times [\varrho_3,\varrho_4]$ and $\Omega \in L(\Delta)$, $\alpha \in \mathbb{R}$. If Ω is exponentially convex function on the co-ordinates on Δ, then the following inequality holds:*

$$\begin{aligned} &\frac{1}{(\varrho_2-\varrho_1)(\varrho_4-\varrho_3)}\int_{\varrho_1}^{\varrho_2}\int_{\varrho_3}^{\varrho_4}\Omega(x,y)dxdy \\ &\leq \frac{\frac{\Omega(\varrho_1,\varrho_3)}{e^{\alpha(\varrho_1+\varrho_3)}} + \frac{\Omega(\varrho_1,\varrho_4)}{e^{\alpha(\varrho_1+\varrho_4)}} + \frac{\Omega(\varrho_2,\varrho_3)}{e^{\alpha(\varrho_2+\varrho_3)}} + \frac{\Omega(\varrho_2,\varrho_4)}{e^{\alpha(\varrho_2+\varrho_4)}}}{4}. \end{aligned}$$

Numerous recent studies have investigated the properties and applications of exponentially convex functions, along with their extensions on the coordinates. These works have contributed significantly to the understanding and further development of this area. For a thorough exploration and additional results, refer to [13],[14],[15],[16],[17],[18],[19], [32]-[39].

In their paper [20], Anderson et al. introduced the following definition, which significantly advanced the theoretical framework in this field and contributed to the ongoing development of the topic.

Definition 1.7. *A function $M : (0,\infty) \times (0,\infty) \to (0,\infty)$ is called a Mean function if*

(1) $M(\varrho_1, \varrho_2) = M(\varrho_2, \varrho_1)$,
(2) $M(\varrho_1, \varrho_1) = \varrho_1$,
(3) $\varrho_1 < M(\varrho_1, \varrho_2) < \varrho_2$, whenever $\varrho_1 < \varrho_2$,
(4) $M(a\varrho_1, a\varrho_2) = aM(\varrho_1, \varrho_2)$ for all $a > 0$.
Let us recall special means as in [20],[21],[22] as followings.
1. Arithmetic Mean: $M(\varrho_1, \varrho_2) = A(\varrho_1, \varrho_2) = \frac{\varrho_1+\varrho_2}{2}$.
2. Geometric Mean: $M(\varrho_1, \varrho_2) = G(\varrho_1, \varrho_2) = \sqrt{\varrho_1\varrho_2}$.
3. Harmonic Mean: $M(\varrho_1, \varrho_2) = H(\varrho_1, \varrho_2) = 1/A\left(\frac{1}{\varrho_1}, \frac{1}{\varrho_2}\right)$.
4. Logarithmic Mean: $M(\varrho_1, \varrho_2) = L(\varrho_1, \varrho_2) = (\varrho_1 - \varrho_2) / (\log \varrho_1 - \log \varrho_2)$ for $\varrho_1 \neq \varrho_2$ and $L(\varrho_1, \varrho_1) = \varrho_1$.
5. Identric Mean: $M(\varrho_1, \varrho_2) = I(\varrho_1, \varrho_2) = (1/e)\left(\varrho_1^{\varrho_1}/\varrho_2^{\varrho_2}\right)^{1/(\varrho_1-\varrho_2)}$ for $\varrho_1 \neq \varrho_2$ and $I(\varrho_1, \varrho_1) = \varrho_1$.
Now, we are in a position to put in order as:

$$H(\varrho_1, \varrho_2) \leq G(\varrho_1, \varrho_2) \leq L(\varrho_1, \varrho_2) \leq I(\varrho_1, \varrho_2) \leq A(\varrho_1, \varrho_2) \leq K(\varrho_1, \varrho_2).$$

In [20], the authors also introduced a concept known as $MN-$convexity, providing the following definition to characterize this class of functions.

Definition 1.8. *Let $\Omega : I \to (0,\infty)$ be continuous, where I is subinterval of $(0,\infty)$. Let M and N be any two Mean functions. We say Ω is MN-convex (concave) if*

$$\Omega(M(\varrho_1, \varrho_2)) \leq (\geq) N(\Omega(\varrho_1), \Omega(\varrho_2))$$

for all $\varrho_1, \varrho_2 \in I$.

Aslan provided the following definition for geometrically exponentially convex functions on the coordinates:

Definition 1.9. *[1] Let us consider the bidimensional interval $\Delta = [\varrho_1, \varrho_2] \times [\varrho_3, \varrho_4]$ in R^2 with $\varrho_1 < \varrho_2$ and $\varrho_3 < \varrho_4$. The mapping $\Omega : \Delta \longrightarrow R^+$ is geometrically-exponentially convex on the co-ordinates on Δ, if the following inequality holds,*

$$\Omega\left(\varrho_1^{\xi}\varrho_3^{(1-\xi)}, \varrho_2^{\xi}\varrho_4^{(1-\xi)}\right) \leq \frac{\Omega^{\xi}(\varrho_1, \varrho_2)}{e^{\alpha(\varrho_1+\varrho_2)}}\frac{\Omega^{(1-\xi)}(\varrho_3, \varrho_4)}{e^{\alpha(\varrho_3+\varrho_4)}}$$

for all $(\varrho_1, \varrho_2), (\varrho_3, \varrho_4) \in \Delta, \alpha \in R$ and $\xi \in [0,1]$.

Aslan presented an alternative but equivalent formulation of the definition of geometrically-exponentially convex functions on the coordinates, as detailed below:

Definition 1.10. *[1] The mapping $\Omega : \Delta \longrightarrow R_+$ is geometrically exponentially convex on the co-ordinates on Δ, if the following inequality holds,*

$$\begin{aligned}&\Omega\left(\varrho_1^{\xi}\varrho_2^{(1-\xi)}, \varrho_3^{\omega}\varrho_4^{(1-\omega)}\right)\\ &\leq \frac{\Omega^{\xi\omega}(\varrho_1, \varrho_3)}{e^{\alpha(\varrho_1+\varrho_3)}}\frac{\Omega^{\xi(1-\omega)}(\varrho_1, \varrho_4)}{e^{\alpha(\varrho_1+\varrho_4)}}\frac{\Omega^{(1-\xi)\omega}(\varrho_2, \varrho_3)}{e^{\alpha(\varrho_2+\varrho_3)}}\frac{\Omega^{(1-\xi)(1-\omega)}(\varrho_2, \varrho_4)}{e^{\alpha(\varrho_2+\varrho_4)}}\end{aligned}$$

for all $(\varrho_1, \varrho_3), (\varrho_1, \varrho_4), (\varrho_2, \varrho_3), (\varrho_2, \varrho_4) \in \Delta, \alpha \in R$ and $\xi, \omega \in [0,1]$.

In this work, we introduce the concept of geometrically-exponentially $s-$convex functions in the second sense on the coordinates and establish a fundamental Hadamard-type integral inequality specifically for these functions. This result expands the theoretical framework for geometrically exponentially $s-$convex functions in the second sense and provides new insights into their integral properties.

2. Main Results

Definition 2.1. *Let us consider the bidimensional interval $\Delta = [\varrho_1, \varrho_2] \times [\varrho_3, \varrho_4]$ in R^2 with $\varrho_1 < \varrho_2$ and $\varrho_3 < \varrho_4$. The mapping $\Omega : \Delta \longrightarrow R^+$ is geometrically-exponentially $s-$convex function in the second sense on the co-ordinates on Δ, if the following inequality holds,*

$$\Omega\left(\varrho_1^{\xi}\varrho_3^{(1-\xi)}, \varrho_2^{\xi}\varrho_4^{(1-\xi)}\right) \leq \frac{\Omega^{\xi^s}(\varrho_1, \varrho_2)}{e^{\alpha(\varrho_1+\varrho_2)}} \frac{\Omega^{(1-\xi)^s}(\varrho_3, \varrho_4)}{e^{\alpha(\varrho_3+\varrho_4)}}$$

for all $(\varrho_1, \varrho_2), (\varrho_3, \varrho_4) \in \Delta, \alpha \in R, s \in (0, 1]$ and $\xi \in [0, 1]$.

An alternative but equivalent formulation of the definition of geometrically exponentially $s-$ convex function in the second sense on the coordinates can be expressed as follows:

Definition 2.2. *Let us consider the bidimensional interval $\Delta = [\varrho_1, \varrho_2] \times [\varrho_3, \varrho_4]$ in R^2 with $\varrho_1 < \varrho_2$ and $\varrho_3 < \varrho_4$. The mapping $\Omega : \Delta \longrightarrow R^+$ is geometrically exponentially $s-$convex function in the second sense on the co-ordinates on Δ, if the following inequality holds,*

$$\begin{aligned}
&\Omega\left(\varrho_1^{\xi}\varrho_2^{(1-\xi)}, \varrho_3^{\omega}\varrho_4^{(1-\omega)}\right) \\
&\leq \frac{\Omega^{(\xi^s)(\omega^s)}(\varrho_1, \varrho_3)}{e^{\alpha(\varrho_1+\varrho_3)}} \frac{\Omega^{(\xi^s)(1-\omega)^s}(\varrho_1, \varrho_4)}{e^{\alpha(\varrho_1+\varrho_4)}} \frac{\Omega^{(1-\xi)^s(\omega^s)}(\varrho_2, \varrho_3)}{e^{\alpha(\varrho_2+\varrho_3)}} \frac{\Omega^{(1-\xi)^s(1-\omega)^s}(\varrho_2, \varrho_4)}{e^{\alpha(\varrho_2+\varrho_4)}}
\end{aligned}$$

for all $(\varrho_1, \varrho_3), (\varrho_1, \varrho_4), (\varrho_2, \varrho_3), (\varrho_2, \varrho_4) \in \Delta, \alpha \in R, s \in (0, 1]$ and $\xi, \omega \in [0, 1]$.

Lemma 2.3. *A function $\Omega : \Delta \longrightarrow R^+$ will be called geometrically exponentially $s-$convex function in the second sense on the co-ordinates on Δ, if the partial mappings $\Omega_{\rho_2} : [\varrho_1, \varrho_2] \longrightarrow R$, $\Omega_{\rho_2}(u) = e^{\alpha\rho_2} f(u, \rho_2)$ and $\Omega_{\rho_1} : [\varrho_3, \varrho_4] \longrightarrow R$, $\Omega_{\rho_1}(v) = e^{\alpha\rho_1} f(\rho_1, v)$ are geometrically exponentially $s-$convex function in the second sense on the co-ordinates on Δ, where defined for all $\rho_2 \in [\varrho_3, \varrho_4]$ and $\rho_1 \in [\varrho_1, \varrho_2]$.*

Proof. From the definition of partial mapping Ω_{ρ_1} we can write

$$\begin{aligned}
\Omega_{\rho_1}\left(v_1^{\xi}v_2^{(1-\xi)}\right) &= e^{\alpha\rho_1}\Omega\left(\rho_1, v_1^{\xi}v_2^{(1-\xi)}\right) \\
&= e^{\alpha\rho_1}\Omega\left(\rho_1^{\xi}\rho_1^{(1-\xi)}, v_1^{\xi}v_2^{(1-\xi)}\right) \\
&\leq e^{\alpha\rho_1}\left[\frac{\Omega^{(\xi^s)}(\rho_1, v_1)}{e^{\alpha(\rho_1+v_1)}} \frac{\Omega^{(1-\xi)^s}(\rho_1, v_2)}{e^{\alpha(\rho_1+v_2)}}\right] \\
&= \frac{\Omega^{(\xi^s)}(\rho_1, v_1)}{e^{\alpha v_1}} \frac{\Omega^{(1-\xi)^s}(\rho_1, v_2)}{e^{\alpha v_2}} \\
&= \frac{\Omega_{\rho_1}^{(\xi^s)}(v_1)}{e^{\alpha v_1}} \frac{\Omega_{\rho_1}^{(1-\xi)^s}(v_2)}{e^{\alpha v_2}}.
\end{aligned}$$

Similarly, one can easily see that

$$\Omega_{\rho_2}\left(u_1^{\xi}u_2^{(1-\xi)}\right) \leq \frac{\Omega_{\rho_2}^{(\xi^s)}(u_1)}{e^{\alpha u_1}} \frac{\Omega_{\rho_2}^{(1-\xi)^s}(u_2)}{e^{\alpha u_2}}.$$

The proof is completed. □

Theorem 2.4. *Let $\Omega : \Delta = [\varrho_1, \varrho_2] \times [\varrho_3, \varrho_4] \to R^+$ be partially differentiable mapping on $\Delta = [\varrho_1, \varrho_2] \times [\varrho_3, \varrho_4]$ and $\Omega \in L(\Delta)$, $\alpha \in R, s \in (0, 1]$ and $0 < \Omega \leq 1$. If Ω is geometrically exponentially $s-$convex function in the second sense on the co-ordinates on Δ, then the following inequality holds:*

$$\begin{aligned}
&\frac{1}{(\ln \varrho_2 - \ln \varrho_1)(\ln \varrho_4 - \ln \varrho_3)} \int_{\varrho_1}^{\varrho_2} \int_{\varrho_3}^{\varrho_4} \frac{\Omega(\rho_1, \rho_2)}{\rho_1 \rho_2} d\rho_1 d\rho_2 \\
&\leq \frac{L\left(\Omega^{(s^2)}(\varrho_1, \varrho_3), \Omega^{(s^2)}(\varrho_1, \varrho_4)\right) + L\left(\Omega^{(s^2)}(\varrho_2, \varrho_3) \Omega^{(s^2)}(\varrho_2, \varrho_4)\right)}{2e^{2\alpha(\varrho_1 + \varrho_3 + \varrho_1 + \varrho_4)}}.
\end{aligned}$$

where $\rho_1 \in [\varrho_1, \varrho_2]$ and $\rho_2 \in [\varrho_3, \varrho_4]$.

Proof. By the definition of the geometrically exponentially $s-$convex function in the second sense on the co-ordinates on Δ, we can write

$$\begin{aligned}
&\Omega\left(\varrho_1^{\xi} \varrho_2^{(1-\xi)}, \varrho_3^{\omega} \varrho_4^{(1-\omega)}\right) \\
&\leq \frac{\Omega^{(\xi^s)(\omega^s)}(\varrho_1, \varrho_3)}{e^{\alpha(\varrho_1 + \varrho_3)}} \frac{\Omega^{(\xi^s)(1-\omega)^s}(\varrho_1, \varrho_4)}{e^{\alpha(\varrho_1 + \varrho_4)}} \frac{\Omega^{(1-\xi)^s \omega^s}(\varrho_2, \varrho_3)}{e^{\alpha(\varrho_2 + \varrho_3)}} \frac{\Omega^{(1-\xi)^s (1-\omega)^s}(\varrho_2, \varrho_4)}{e^{\alpha(\varrho_2 + \varrho_4)}}
\end{aligned}$$

By integrating both sides of the above inequality with respect to ξ, ω on $[0, 1]^2$, we have

$$\begin{aligned}
&\int_0^1 \int_0^1 \Omega\left(\varrho_1^{\xi} \varrho_2^{(1-\xi)}, \varrho_3^{\omega} \varrho_4^{(1-\omega)}\right) d\xi d\omega \\
\leq\ &\int_0^1 \int_0^1 \frac{\Omega^{(\xi^s)(\omega^s)}(\varrho_1, \varrho_3)}{e^{\alpha(\varrho_1 + \varrho_3)}} \frac{\Omega^{(\xi^s)(1-\omega)^s}(\varrho_1, \varrho_4)}{e^{\alpha(\varrho_1 + \varrho_4)}} \frac{\Omega^{(1-\xi)^s (\omega^s)}(\varrho_2, \varrho_3)}{e^{\alpha(\varrho_2 + \varrho_3)}} \frac{\Omega^{(1-\xi)^s (1-\omega)^s}(\varrho_2, \varrho_4)}{e^{\alpha(\varrho_2 + \varrho_4)}} d\xi d\omega
\end{aligned}$$

If $0 < \mu \leq 1, 0 < \alpha, s \leq 1$

$$\mu^{(\alpha^s)} \leq \mu^{\alpha s}$$

$$\begin{aligned}
&\int_0^1 \int_0^1 \Omega\left(\varrho_1^{\xi} \varrho_2^{(1-\xi)}, \varrho_3^{\omega} \varrho_4^{(1-\omega)}\right) d\xi d\omega \\
\leq\ &\frac{1}{e^{2\alpha(\varrho_1 + \varrho_3 + \varrho_1 + \varrho_4)}} \\
\times\ &\int_0^1 \int_0^1 \Omega^{s\xi(\omega^s)}(\varrho_1, \varrho_3) \Omega^{s\xi(1-\omega)^s}(\varrho_1, \varrho_4) \Omega^{s(1-\xi)(\omega^s)}(\varrho_2, \varrho_3) \Omega^{s(1-\xi)(1-\omega)^s}(\varrho_2, \varrho_4) d\xi d\omega \\
=\ &\frac{1}{e^{2\alpha(\varrho_1 + \varrho_3 + \varrho_1 + \varrho_4)}} \\
\times\ &\int_0^1 \frac{\Omega^{s(\omega^s)}(\varrho_1, \varrho_3) \Omega^{s(1-\omega)^s}(\varrho_1, \varrho_4) - \Omega^{s(\omega^s)}(\varrho_2, \varrho_3) \Omega^{s(1-\omega)^s}(\varrho_2, \varrho_4)}{\ln \Omega^{\omega}(\varrho_1, \varrho_3) \Omega^{(1-\omega)}(\varrho_1, \varrho_4) - \ln \Omega^{\omega}(\varrho_2, \varrho_3) \Omega^{m(1-\omega)}(\varrho_2, \varrho_4)} d\omega \\
=\ &\frac{1}{e^{2\alpha(\varrho_1 + \varrho_3 + \varrho_1 + \varrho_4)}} \\
\times\ &\int_0^1 L\left(\Omega^{s(\omega^s)}(\varrho_1, \varrho_3) \Omega^{s(1-\omega)^s}(\varrho_1, \varrho_4), \Omega^{s(\omega^s)}(\varrho_2, \varrho_3) \Omega^{s(1-\omega)^s}(\varrho_2, \varrho_4)\right) d\omega
\end{aligned}$$

If we perform a change of variables as $\rho_1 = \varrho_1^{\xi} \varrho_2^{1-\xi}, \rho_2 = \varrho_3^{\omega} \varrho_4^{(1-\omega)}$ and the $L(a, b) < A(a, b)$ feature is taken into account, the following result is obtained.

$$\frac{1}{(\ln \varrho_2 - \ln \varrho_1)(\ln \varrho_4 - \ln \varrho_3)} \int_{\varrho_1}^{\varrho_2} \int_{\varrho_3}^{\varrho_4} \frac{\Omega(\rho_1, \rho_2)}{\rho_1 \rho_2} d\rho_1 d\rho_2$$

$$\leq \frac{1}{e^{2\alpha\left(\varrho_1+\varrho_3+\varrho_1+\varrho_4\right)}} \int_0^1 A\left(\Omega^{s(\omega^s)}\left(\varrho_1,\varrho_3\right)\Omega^{s(1-\omega)^s}\left(\varrho_1,\varrho_4\right),\Omega^{s(\omega^s)}\left(\varrho_2,\varrho_3\right)\Omega^{s(1-\omega)^s}\left(\varrho_2,\varrho_4\right)\right)d\omega$$

If $0<\mu\leq 1, 0<\alpha, s\leq 1$

$$\mu^{(\alpha^s)} \leq \mu^{\alpha s}$$

$$\begin{aligned}
& \frac{1}{\left(\ln\varrho_2-\ln\varrho_1\right)\left(\ln\varrho_4-\ln\varrho_3\right)}\int_{\varrho_1}^{\varrho_2}\int_{\varrho_3}^{\varrho_4}\frac{\Omega\left(\rho_1,\rho_2\right)}{\rho_1\rho_2}d\rho_1 d\rho_2 \\
\leq\ & \frac{1}{e^{2\alpha\left(\varrho_1+\varrho_3+\varrho_1+\varrho_4\right)}} \\
\times\ & \int_0^1 A\left(\Omega^{(s^2)\omega}\left(\varrho_1,\varrho_3\right)\Omega^{(s^2)(1-\omega)}\left(\varrho_1,\varrho_4\right),\Omega^{(s^2)\omega}\left(\varrho_2,\varrho_3\right)\Omega^{(s^2)(1-\omega)}\left(\varrho_2,\varrho_4\right)\right)d\omega \\
=\ & \frac{1}{e^{2\alpha\left(\varrho_1+\varrho_3+\varrho_1+\varrho_4\right)}} \\
\times\ & \int_0^1 \frac{\Omega^{(s^2)\omega}\left(\varrho_1,\varrho_3\right)\Omega^{(s^2)(1-\omega)}\left(\varrho_1,\varrho_4\right)+\Omega^{(s^2)\omega}\left(\varrho_2,\varrho_3\right)\Omega^{(s^2)(1-\omega)}\left(\varrho_2,\varrho_4\right)}{2}d\omega \\
=\ & \frac{L\left(\Omega^{(s^2)}\left(\varrho_1,\varrho_3\right),\Omega^{(s^2)}\left(\varrho_1,\varrho_4\right)\right)+L\left(\Omega^{(s^2)}\left(\varrho_2,\varrho_3\right)\Omega^{(s^2)}\left(\varrho_2,\varrho_4\right)\right)}{2e^{2\alpha\left(\varrho_1+\varrho_3+\varrho_1+\varrho_4\right)}}.
\end{aligned}$$

The proof is completed. □

Remark 2.5. *If we choose $s=1$ and $\alpha=0$ in Theorem 2.4, the result is consistent with geometrically convex functions on the coordinates.*

$$\begin{aligned}
& \frac{1}{\left(\ln\varrho_2-\ln\varrho_1\right)\left(\ln\varrho_4-\ln\varrho_3\right)}\int_{\varrho_1}^{\varrho_2}\int_{\varrho_3}^{\varrho_4}\frac{\Omega\left(\rho_1,\rho_2\right)}{\rho_1\rho_2}d\rho_1 d\rho_2 \\
& \leq \frac{L\left(\Omega\left(\varrho_1,\varrho_3\right),\Omega\left(\varrho_1,\varrho_4\right)\right)+L\left(\Omega\left(\varrho_2,\varrho_3\right)\Omega\left(\varrho_2,\varrho_4\right)\right)}{2}.
\end{aligned}$$

where $\rho_1\in[\varrho_1,\varrho_2]$ and $\rho_2\in[\varrho_3,\varrho_4]$.

Remark 2.6. *If we choose $s=1$ in Theorem 2.4, the result is consistent with geometrically convex functions*

$$\begin{aligned}
& \frac{1}{\left(\ln\varrho_2-\ln\varrho_1\right)\left(\ln\varrho_4-\ln\varrho_3\right)}\int_{\varrho_1}^{\varrho_2}\int_{\varrho_3}^{\varrho_4}\frac{\Omega\left(\rho_1,\rho_2\right)}{\rho_1\rho_2}d\rho_1 d\rho_2 \\
& \leq \frac{L\left(\Omega\left(\varrho_1,\varrho_3\right),\Omega\left(\varrho_1,\varrho_4\right)\right)+L\left(\Omega\left(\varrho_2,\varrho_3\right)\Omega\left(\varrho_2,\varrho_4\right)\right)}{2e^{2\alpha\left(\varrho_1+\varrho_3+\varrho_1+\varrho_4\right)}}.
\end{aligned}$$

where $\rho_1\in[\varrho_1,\varrho_2]$ and $\rho_2\in[\varrho_3,\varrho_4]$.

Theorem 2.7. *Let $\Omega:\Delta=[\varrho_1,\varrho_2]\times[\varrho_3,\varrho_4]\rightarrow R^+$ be partially differentiable mapping on $\Delta=[\varrho_1,\varrho_2]\times[\varrho_3,\varrho_4]$ and $\Omega\in L(\Delta)$, $\alpha\in R$ and $0<\Omega\leq 1$. If $|\Omega|$ is geometrically exponentially $s-$convex function in the second sense on the co-ordinates on Δ, $p>1$ and $s\in(0,1]$, then the following inequality holds:*

$$\begin{aligned}
& \left|\frac{1}{\left(\ln\varrho_2-\ln\varrho_1\right)\left(\ln\varrho_4-\ln\varrho_3\right)}\int_{\varrho_1}^{\varrho_2}\int_{\varrho_3}^{\varrho_4}\frac{\Omega\left(\rho_1,\rho_2\right)}{\rho_1\rho_2}d\rho_1 d\rho_2\right| \\
\leq\ & \frac{L^{\frac{1}{q}}\left(\left|\Omega\left(\varrho_1,\varrho_3\right)\right|^{q(s^2)},\left|\Omega\left(\varrho_1,\varrho_4\right)\right|^{q(s^2)}\right)+L^{\frac{1}{q}}\left(\left|\Omega\left(\varrho_1,\varrho_3\right)\right|^{q(s^2)},\left|\Omega\left(\varrho_1,\varrho_4\right)\right|^{q(s^2)}\right)}{2e^{2\alpha\left(\varrho_1+\varrho_3+\varrho_1+\varrho_4\right)}}
\end{aligned}$$

where $\rho_1\in[\varrho_1,\varrho_2]$, $\rho_2\in[\varrho_3,\varrho_4]$ and $p^{-1}+q^{-1}=1$.

Proof. By the definition of the geometrically exponentially $s-$convex function in the second sense on the co-ordinates on Δ, we can write

$$\begin{aligned}&\Omega\left(\varrho_1^{\xi}\varrho_2^{(1-\xi)},\varrho_3^{\omega}\varrho_4^{(1-\omega)}\right)\\&\leq \frac{\Omega^{(\xi^s)(\omega^s)}(\varrho_1,\varrho_3)}{e^{\alpha(\varrho_1+\varrho_3)}}\frac{\Omega^{(\xi^s)(1-\omega)^s}(\varrho_1,\varrho_4)}{e^{\alpha(\varrho_1+\varrho_4)}}\frac{\Omega^{(1-\xi)^s(\omega^s)}(\varrho_2,\varrho_3)}{e^{\alpha(\varrho_2+\varrho_3)}}\frac{\Omega^{(1-\xi)^s(1-\omega)^s}(\varrho_2,\varrho_4)}{e^{\alpha(\varrho_2+\varrho_4)}}\end{aligned}$$

By integrating both sides of the above inequality with respect to ξ,ω on $[0,1]^2$, we have

$$\begin{aligned}&\int_0^1\int_0^1\Omega\left(\varrho_1^{\xi}\varrho_2^{(1-\xi)},\varrho_3^{\omega}\varrho_4^{(1-\omega)}\right)d\xi d\omega\\\leq\ &\int_0^1\int_0^1\frac{\Omega^{(\xi^s)(\omega^s)}(\varrho_1,\varrho_3)}{e^{\alpha(\varrho_1+\varrho_3)}}\frac{\Omega^{(\xi^s)(1-\omega)^s}(\varrho_1,\varrho_4)}{e^{\alpha(\varrho_1+\varrho_4)}}\frac{\Omega^{(1-\xi)^s(\omega^s)}(\varrho_2,\varrho_3)}{e^{\alpha(\varrho_2+\varrho_3)}}\frac{\Omega^{(1-\xi)^s(1-\omega)^s}(\varrho_2,\varrho_4)}{e^{\alpha(\varrho_2+\varrho_4)}}d\xi d\omega\end{aligned}$$

If $0<\mu\leq 1, 0<\alpha,s\leq 1$

$$\mu^{(\alpha^s)}\leq\mu^{\alpha s}$$

$$\begin{aligned}&\int_0^1\int_0^1\Omega\left(\varrho_1^{\xi}\varrho_2^{(1-\xi)},\varrho_3^{\omega}\varrho_4^{(1-\omega)}\right)d\xi d\omega\\\leq\ &\frac{1}{e^{2\alpha(\varrho_1+\varrho_3+\varrho_1+\varrho_4)}}\\\times\ &\int_0^1\int_0^1\Omega^{s\xi(\omega^s)}(\varrho_1,\varrho_3)\,\Omega^{s\xi(1-\omega)^s}(\varrho_1,\varrho_4)\,\Omega^{s(1-\xi)(\omega^s)}(\varrho_2,\varrho_3)\,\Omega^{s(1-\xi)(1-\omega)^s}(\varrho_2,\varrho_4)\,d\xi d\omega\\=\ &\frac{1}{e^{2\alpha(\varrho_1+\varrho_3+\varrho_1+\varrho_4)}}\\&\times\int_0^1\frac{\Omega^{s(\omega^s)}(\varrho_1,\varrho_3)\,\Omega^{s(1-\omega)^s}(\varrho_1,\varrho_4)-\Omega^{s(\omega^s)}(\varrho_2,\varrho_3)\,\Omega^{s(1-\omega)^s}(\varrho_2,\varrho_4)}{\ln\Omega^{\omega}(\varrho_1,\varrho_3)\,\Omega^{(1-\omega)}(\varrho_1,\varrho_4)-\ln\Omega^{\omega}(\varrho_2,\varrho_3)\,\Omega^{(1-\omega)}(\varrho_2,\varrho_4)}d\omega\\=\ &\frac{1}{e^{2\alpha(\varrho_1+\varrho_3+\varrho_1+\varrho_4)}}\\&\times\int_0^1 L\left(\Omega^{s(\omega^s)}(\varrho_1,\varrho_3)\,\Omega^{s(1-\omega)^s}(\varrho_1,\varrho_4),\Omega^{s(\omega^s)}(\varrho_2,\varrho_3)\,\Omega^{s(1-\omega)^s}(\varrho_2,\varrho_4)\right)d\omega\end{aligned}$$

If we perform a change of variables as $\rho_1=\varrho_1^{\xi}\varrho_2^{1-\xi},\rho_2=\varrho_3^{\omega}\varrho_4^{(1-\omega)}$ and the $L(a,b)<A(a,b)$ feature is taken into account, the following result is obtained.

$$\begin{aligned}&\frac{1}{(\ln\varrho_2-\ln\varrho_1)(\ln\varrho_4-\ln\varrho_3)}\int_{\varrho_1}^{\varrho_2}\int_{\varrho_3}^{\varrho_4}\frac{\Omega(\rho_1,\rho_2)}{\rho_1\rho_2}d\rho_1 d\rho_2\\\leq\ &\frac{1}{e^{2\alpha(\varrho_1+\varrho_3+\varrho_1+\varrho_4)}}\\&\times\int_0^1 A\left(\Omega^{s(\omega^s)}(\varrho_1,\varrho_3)\,\Omega^{s(1-\omega)^s}(\varrho_1,\varrho_4),\Omega^{s(\omega^s)}(\varrho_2,\varrho_3)\,\Omega^{s(1-\omega)^s}(\varrho_2,\varrho_4)\right)d\omega\end{aligned}$$

If $0<\mu\leq 1, 0<\alpha,s\leq 1$

$$\mu^{(\alpha^s)}\leq\mu^{\alpha s}$$

$$
\begin{aligned}
& \frac{1}{(\ln \varrho_2-\ln \varrho_1)(\ln \varrho_4-\ln \varrho_3)} \int_{\varrho_1}^{\varrho_2} \int_{\varrho_3}^{\varrho_4} \frac{\Omega(\rho_1, \rho_2)}{\rho_1 \rho_2} d\rho_1 d\rho_2 \\
\leq\ & \frac{1}{e^{2\alpha(\varrho_1+\varrho_3+\varrho_1+\varrho_4)}} \int_0^1 A\left(\Omega^{(s^2)\omega}(\varrho_1, \varrho_3)\, \Omega^{(s^2)(1-\omega)}(\varrho_1, \varrho_4), \Omega^{(s^2)\omega}(\varrho_2, \varrho_3)\, \Omega^{(s^2)(1-\omega)}(\varrho_2, \varrho_4)\right) d\omega \\
=\ & \frac{1}{e^{2\alpha(\varrho_1+\varrho_3+\varrho_1+\varrho_4)}} \int_0^1 \frac{\Omega^{(s^2)\omega}(\varrho_1, \varrho_3)\, \Omega^{(s^2)(1-\omega)}(\varrho_1, \varrho_4)+\Omega^{(s^2)\omega}(\varrho_2, \varrho_3)\, \Omega^{(s^2)(1-\omega)}(\varrho_2, \varrho_4)}{2} d\omega
\end{aligned}
$$

If we take the absolute value of both sides of the inequality and apply Hölder's inequality to the right-hand side, we obtain the following expression.

$$
\begin{aligned}
& \left|\frac{1}{(\ln \varrho_2-\ln \varrho_1)(\ln \varrho_4-\ln \varrho_3)} \int_{\varrho_1}^{\varrho_2} \int_{\varrho_3}^{\varrho_4} \frac{\Omega(\rho_1, \rho_2)}{\rho_1 \rho_2} d\rho_1 d\rho_2\right| \\
\leq\ & \left(\frac{1}{2e^{2p\alpha(\varrho_1+\varrho_3+\varrho_1+\varrho_4)}}\right)^{\frac{1}{p}} \times\left[\left(\int_0^1 \left|\Omega(\varrho_1, \varrho_3)\right|^{q(s^2)\omega} \left|\Omega(\varrho_1, \varrho_4)\right|^{q(s^2)(1-\omega)} d\omega\right)^{\frac{1}{q}}\right. \\
& \left.+\left(\int_0^1 \left|\Omega(\varrho_2, \varrho_3)\right|^{q(s^2)\omega} \left|\Omega(\varrho_2, \varrho_4) d\omega\right|^{q(s^2)(1-\omega)} d\omega\right)^{\frac{1}{q}}\right] \\
=\ & \frac{1}{2e^{2\alpha(\varrho_1+\varrho_3+\varrho_1+\varrho_4)}}\left[L^{\frac{1}{q}}\left(\left|\Omega(\varrho_1, \varrho_3)\right|^{q(s^2)}, \left|\Omega(\varrho_1, \varrho_4)\right|^{q(s^2)}\right)+L^{\frac{1}{q}}\left(\left|\Omega(\varrho_1, \varrho_3)\right|^{q(s^2)}, \left|\Omega(\varrho_1, \varrho_4)\right|^{q(s^2)}\right)\right]
\end{aligned}
$$

□

Corollary 2.8. *If we choose $s = 1$ and $\alpha = 0$ in Theorem 2.7, the result is consistent with the geometric convexity on the co-ordinates*

$$
\begin{aligned}
& \left|\frac{1}{(\ln \varrho_2-\ln \varrho_1)(\ln \varrho_4-\ln \varrho_3)} \int_{\varrho_1}^{\varrho_2} \int_{\varrho_3}^{\varrho_4} \frac{\Omega(\rho_1, \rho_2)}{\rho_1 \rho_2} d\rho_1 d\rho_2\right| \\
\leq\ & \frac{L^{\frac{1}{q}}\left(\left|\Omega(\varrho_1, \varrho_3)\right|^q \left|\Omega(\varrho_1, \varrho_4)\right|^q\right)+L^{\frac{1}{q}}\left(\left|\Omega(\varrho_2, \varrho_3)\right|^q \left|\Omega(\varrho_2, \varrho_4)\right|^q\right)}{2}
\end{aligned}
$$

where $\rho_1 \in [\varrho_1, \varrho_2]$, $\rho_2 \in [\varrho_3, \varrho_4]$ and $p^{-1} + q^{-1} = 1$.

Corollary 2.9. *If we choose $s = 1$ in Theorem 2.7, the result is consistent with the geometric exponentially convexity on the co-ordinates*

$$
\begin{aligned}
& \left|\frac{1}{(\ln \varrho_2-\ln \varrho_1)(\ln \varrho_4-\ln \varrho_3)} \int_{\varrho_1}^{\varrho_2} \int_{\varrho_3}^{\varrho_4} \frac{\Omega(\rho_1, \rho_2)}{\rho_1 \rho_2} d\rho_1 d\rho_2\right| \\
\leq\ & \frac{L^{\frac{1}{q}}\left(\left|\Omega(\varrho_1, \varrho_3)\right|^q, \left|\Omega(\varrho_1, \varrho_4)\right|^q\right)+L^{\frac{1}{q}}\left(\left|\Omega(\varrho_1, \varrho_3)\right|^q, \left|\Omega(\varrho_1, \varrho_4)\right|^q\right)}{2e^{2\alpha(\varrho_1+\varrho_3+\varrho_1+\varrho_4)}}
\end{aligned}
$$

where $\rho_1 \in [\varrho_1, \varrho_2]$, $\rho_2 \in [\varrho_3, \varrho_4]$ and $p^{-1} + q^{-1} = 1$.

Theorem 2.10. *Let $\Omega : \Delta = [\varrho_1, \varrho_2] \times [\varrho_3, \varrho_4] \to R^+$ be partially differentiable mapping on $\Delta = [\varrho_1, \varrho_2] \times [\varrho_3, \varrho_4]$ and $\Omega \in L(\Delta)$, $\alpha \in R$ and $0 < \Omega \leq 1$. If $|\Omega|$ is geometrically exponentially $s-$convex function in the second sense on the co-ordinates on Δ, $p > 1$ and $s \in (0, 1]$, then the following inequality holds:*

$$
\left|\frac{1}{(\ln \varrho_2-\ln \varrho_1)(\ln \varrho_4-\ln \varrho_3)} \int_{\varrho_1}^{\varrho_2} \int_{\varrho_3}^{\varrho_4} \frac{\Omega(\rho_1, \rho_2)}{\rho_1 \rho_2} d\rho_1 d\rho_2\right|
$$

$$\leq \quad \frac{1}{2pe^{2p\alpha(\varrho_1+\varrho_3+\varrho_1+\varrho_4)}} + \frac{L\left(\left|\Omega\left(\varrho_1,\varrho_3\right)\right|^{q(s^2)},\left|\Omega\left(\varrho_1,\varrho_4\right)\right|^{q(s^2)}\right) + L\left(\left|\Omega\left(\varrho_1,\varrho_3\right)\right|^{q(s^2)},\left|\Omega\left(\varrho_1,\varrho_4\right)\right|^{q(s^2)}\right)}{q}$$

where $\rho_1 \in [\varrho_1,\varrho_2]$, $\rho_2 \in [\varrho_3,\varrho_4]$, *and* $p^{-1}+q^{-1}=1$.

Proof. By the definition of the geometrically exponentially $s-$convex function in the second sense on the co-ordinates on Δ, we can write

$$\begin{aligned}&\Omega\left(\varrho_1^{\xi}\varrho_2^{(1-\xi)},\varrho_3^{\omega}\varrho_4^{(1-\omega)}\right)\\ \leq\;& \frac{\Omega^{(\xi^s)(\omega^s)}\left(\varrho_1,\varrho_3\right)}{e^{\alpha(\varrho_1+\varrho_3)}}\frac{\Omega^{(\xi^s)(1-\omega)^s}\left(\varrho_1,\varrho_4\right)}{e^{\alpha(\varrho_1+\varrho_4)}}\frac{\Omega^{(1-\xi)^s(\omega^s)}\left(\varrho_2,\varrho_3\right)}{e^{\alpha(\varrho_2+\varrho_3)}}\frac{\Omega^{(1-\xi)^s(1-\omega)^s}\left(\varrho_2,\varrho_4\right)}{e^{\alpha(\varrho_2+\varrho_4)}}\end{aligned}$$

By integrating both sides of the above inequality with respect to ξ,ω on $[0,1]^2$, we have

$$\begin{aligned}&\int_0^1\int_0^1\Omega\left(\varrho_1^{\xi}\varrho_2^{(1-\xi)},\varrho_3^{\omega}\varrho_4^{(1-\omega)}\right)d\xi d\omega\\ \leq\;&\int_0^1\int_0^1\frac{\Omega^{(\xi^s)(\omega^s)}\left(\varrho_1,\varrho_3\right)}{e^{\alpha(\varrho_1+\varrho_3)}}\frac{\Omega^{(\xi^s)(1-\omega)^s}\left(\varrho_1,\varrho_4\right)}{e^{\alpha(\varrho_1+\varrho_4)}}\frac{\Omega^{(1-\xi)^s(\omega^s)}\left(\varrho_2,\varrho_3\right)}{e^{\alpha(\varrho_2+\varrho_3)}}\frac{\Omega^{(1-\xi)^s(1-\omega)^s}\left(\varrho_2,\varrho_4\right)}{e^{\alpha(\varrho_2+\varrho_4)}}d\xi d\omega\end{aligned}$$

If $0<\mu\leq 1, 0<\alpha,s\leq 1$

$$\mu^{(\alpha^s)}\leq\mu^{\alpha s}$$

$$\begin{aligned}&\int_0^1\int_0^1\Omega\left(\varrho_1^{\xi}\varrho_2^{(1-\xi)},\varrho_3^{\omega}\varrho_4^{(1-\omega)}\right)d\xi d\omega\\ \leq\;&\frac{1}{e^{2\alpha(\varrho_1+\varrho_3+\varrho_1+\varrho_4)}}\\ \times\;&\int_0^1\int_0^1\Omega^{s\xi(\omega^s)}\left(\varrho_1,\varrho_3\right)\Omega^{s\xi(1-\omega)^s}\left(\varrho_1,\varrho_4\right)\Omega^{s(1-\xi)(\omega^s)}\left(\varrho_2,\varrho_3\right)\Omega^{s(1-\xi)(1-\omega)^s}\left(\varrho_2,\varrho_4\right)d\xi d\omega\\ =\;&\frac{1}{e^{2\alpha(\varrho_1+\varrho_3+\varrho_1+\varrho_4)}}\int_0^1\frac{\Omega^{s(\omega^s)}\left(\varrho_1,\varrho_3\right)\Omega^{s(1-\omega)^s}\left(\varrho_1,\varrho_4\right)-\Omega^{s(\omega^s)}\left(\varrho_2,\varrho_3\right)\Omega^{s(1-\omega)^s}\left(\varrho_2,\varrho_4\right)}{\ln\Omega^{\omega}\left(\varrho_1,\varrho_3\right)\Omega^{(1-\omega)}\left(\varrho_1,\varrho_4\right)-\ln\Omega^{\omega}\left(\varrho_2,\varrho_3\right)\Omega^{(1-\omega)}\left(\varrho_2,\varrho_4\right)}d\omega\\ =\;&\frac{1}{e^{2\alpha(\varrho_1+\varrho_3+\varrho_1+\varrho_4)}}\int_0^1 L\left(\Omega^{s(\omega^s)}\left(\varrho_1,\varrho_3\right)\Omega^{s(1-\omega)^s}\left(\varrho_1,\varrho_4\right),\Omega^{s(\omega^s)}\left(\varrho_2,\varrho_3\right)\Omega^{s(1-\omega)^s}\left(\varrho_2,\varrho_4\right)\right)d\omega\end{aligned}$$

If we perform a change of variables as $\rho_1=\varrho_1^{\xi}\varrho_2^{1-\xi}$, $\rho_2=\varrho_3^{\omega}\varrho_4^{(1-\omega)}$ and the $L(a,b)<A(a,b)$ feature is taken into account, the following result is obtained.

$$\begin{aligned}&\frac{1}{(\ln\varrho_2-\ln\varrho_1)(\ln\varrho_4-\ln\varrho_3)}\int_{\varrho_1}^{\varrho_2}\int_{\varrho_3}^{\varrho_4}\frac{\Omega\left(\rho_1,\rho_2\right)}{\rho_1\rho_2}d\rho_1 d\rho_2\\ \leq\;&\frac{1}{e^{2\alpha(\varrho_1+\varrho_3+\varrho_1+\varrho_4)}}\int_0^1 A\left(\Omega^{s(\omega^s)}\left(\varrho_1,\varrho_3\right)\Omega^{s(1-\omega)^s}\left(\varrho_1,\varrho_4\right),\Omega^{s(\omega^s)}\left(\varrho_2,\varrho_3\right)\Omega^{s(1-\omega)^s}\left(\varrho_2,\varrho_4\right)\right)d\omega\end{aligned}$$

If $0<\mu\leq 1, 0<\alpha,s\leq 1$

$$\mu^{(\alpha^s)}\leq\mu^{\alpha s}$$

$$\frac{1}{(\ln\varrho_2-\ln\varrho_1)(\ln\varrho_4-\ln\varrho_3)}\int_{\varrho_1}^{\varrho_2}\int_{\varrho_3}^{\varrho_4}\frac{\Omega\left(\rho_1,\rho_2\right)}{\rho_1\rho_2}d\rho_1 d\rho_2$$

$$\begin{aligned}
&\leq \frac{1}{e^{2\alpha(\varrho_1+\varrho_3+\varrho_1+\varrho_4)}}\int_0^1 A\left(\Omega^{(s^2)\omega}(\varrho_1,\varrho_3)\,\Omega^{(s^2)(1-\omega)}(\varrho_1,\varrho_4),\Omega^{(s^2)\omega}(\varrho_2,\varrho_3)\,\Omega^{(s^2)(1-\omega)}(\varrho_2,\varrho_4)\right)d\omega\\
&= \frac{1}{e^{2\alpha(\varrho_1+\varrho_3+\varrho_1+\varrho_4)}}\int_0^1 \frac{\Omega^{(s^2)\omega}(\varrho_1,\varrho_3)\,\Omega^{(s^2)(1-\omega)}(\varrho_1,\varrho_4)+\Omega^{(s^2)\omega}(\varrho_2,\varrho_3)\,\Omega^{(s^2)(1-\omega)}(\varrho_2,\varrho_4)}{2}d\omega
\end{aligned}$$

If we take the absolute value of both sides of the inequality and apply Young's inequality to the right-hand side, we obtain the following expression.

$$\begin{aligned}
&\left|\frac{1}{(\ln\varrho_2-\ln\varrho_1)(\ln\varrho_4-\ln\varrho_3)}\int_{\varrho_1}^{\varrho_2}\int_{\varrho_3}^{\varrho_4}\frac{\Omega(\rho_1,\rho_2)}{\rho_1\rho_2}d\rho_1 d\rho_2\right|\\
&\leq \frac{1}{p}\left(\frac{1}{2e^{2p\alpha(\varrho_1+\varrho_3+\varrho_1+\varrho_4)}}\right)+\left[\frac{1}{q}\left(\int_0^1\left|\Omega(\varrho_1,\varrho_3)\right|^{q(s^2)\omega}\left|\Omega(\varrho_1,\varrho_4)\right|^{q(s^2)(1-\omega)}d\omega\right)\right.\\
&\quad\left.+\frac{1}{q}\left(\int_0^1\left|\Omega(\varrho_2,\varrho_3)\right|^{q(s^2)\omega}\left|\Omega(\varrho_2,\varrho_4)\,d\omega\right|^{q(s^2)(1-\omega)}d\omega\right)\right]\\
&= \frac{1}{2pe^{2p\alpha(\varrho_1+\varrho_3+\varrho_1+\varrho_4)}}\\
&\quad+\frac{1}{q}L\left(\left|\Omega(\varrho_1,\varrho_3)\right|^{q(s^2)},\left|\Omega(\varrho_1,\varrho_4)\right|^{q(s^2)}\right)+\frac{1}{q}L\left(\left|\Omega(\varrho_1,\varrho_3)\right|^{q(s^2)},\left|\Omega(\varrho_1,\varrho_4)\right|^{q(s^2)}\right).
\end{aligned}$$

The proof is completed. □

Corollary 2.11. *If we choose $s = 1$ and $\alpha = 0$ in Theorem 2.10, the result is consistent with geometrically convex functions on the co-ordinates.*

$$\begin{aligned}
&\left|\frac{1}{(\ln\varrho_2-\ln\varrho_1)(\ln\varrho_4-\ln\varrho_3)}\int_{\varrho_1}^{\varrho_2}\int_{\varrho_3}^{\varrho_4}\frac{\Omega(\rho_1,\rho_2)}{\rho_1\rho_2}d\rho_1 d\rho_2\right|\\
&\leq \frac{1}{2p}+\frac{L\left(\left|\Omega(\varrho_1,\varrho_3)\right|^{q},\left|\Omega(\varrho_1,\varrho_4)\right|^{q}\right)+L\left(\left|\Omega(\varrho_1,\varrho_3)\right|^{q},\left|\Omega(\varrho_1,\varrho_4)\right|^{q}\right)}{q}
\end{aligned}$$

where $\rho_1 \in [\varrho_1,\varrho_2]$, $\rho_2 \in [\varrho_3,\varrho_4]$, and $p^{-1}+q^{-1} = 1$.

Corollary 2.12. *If we choose $s = 1$ in Theorem 2.10, the result is consistent with geometrically convex functions on the co-ordinates.*

$$\begin{aligned}
&\left|\frac{1}{(\ln\varrho_2-\ln\varrho_1)(\ln\varrho_4-\ln\varrho_3)}\int_{\varrho_1}^{\varrho_2}\int_{\varrho_3}^{\varrho_4}\frac{\Omega(\rho_1,\rho_2)}{\rho_1\rho_2}d\rho_1 d\rho_2\right|\\
&\leq \frac{1}{2pe^{2p\alpha(\varrho_1+\varrho_3+\varrho_1+\varrho_4)}}\\
&\quad+\frac{L\left(\left|\Omega(\varrho_1,\varrho_3)\right|^{q},\left|\Omega(\varrho_1,\varrho_4)\right|^{q}\right)+L\left(\left|\Omega(\varrho_1,\varrho_3)\right|^{q},\left|\Omega(\varrho_1,\varrho_4)\right|^{q}\right)}{q}
\end{aligned}$$

where $\rho_1 \in [\varrho_1,\varrho_2]$, $\rho_2 \in [\varrho_3,\varrho_4]$ and $p^{-1}+q^{-1} = 1$.

Proposition 2.13. *If $\Omega,\Psi : \Delta \to R$ are two geometrically exponentially $s-$convex functions in the second sense on the co-ordinates on Δ, then $\Omega\Psi$ is geometrically exponential $s-$convex in the second sense on the co-ordinates on Δ.*

Proof. From the definition of geometrically exponentially $s-$convex function, we can write

$$\Omega\left(\varrho_1^{\xi}\varrho_2^{(1-\xi)},\varrho_3^{\omega}\varrho_4^{(1-\omega)}\right)\times\Psi\left(\varrho_1^{\xi}\varrho_2^{(1-\xi)},\varrho_3^{\omega}\varrho_4^{(1-\omega)}\right)$$

$$
\begin{aligned}
&\leq \frac{\Omega^{(\xi^s)(\omega^s)}(\varrho_1,\varrho_3)}{e^{\alpha(\varrho_1+\varrho_3)}}\frac{\Omega^{(\xi^s)(1-\omega)^s}(\varrho_1,\varrho_4)}{e^{\alpha(\varrho_1+\varrho_4)}}\frac{\Omega^{(1-\xi)^s(\omega^s)}(\varrho_2,\varrho_3)}{e^{\alpha(\varrho_2+\varrho_3)}}\frac{\Omega^{(1-\xi)^s(1-\omega)^s}(\varrho_2,\varrho_4)}{e^{\alpha(\varrho_2+\varrho_4)}}\\
&\quad\times\frac{\Psi^{(\xi^s)(\omega^s)}(\varrho_1,\varrho_3)}{e^{\alpha(\varrho_1+\varrho_3)}}\frac{\Psi^{(\xi^s)(1-\omega)^s}(\varrho_1,\varrho_4)}{e^{\alpha(\varrho_1+\varrho_4)}}\frac{\Psi^{(1-\xi)^s(\omega^s)}(\varrho_2,\varrho_3)}{e^{\alpha(\varrho_2+\varrho_3)}}\frac{\Psi^{(1-\xi)^s(1-\omega)^s}(\varrho_2,\varrho_4)}{e^{\alpha(\varrho_2+\varrho_4)}}\\
&= \frac{(\Omega\Psi)^{(\xi^s)(\omega^s)}(\varrho_1,\varrho_3)}{e^{\alpha(\varrho_1+\varrho_3)}}\frac{(\Omega\Psi)^{(\xi^s)(1-\omega)^s}(\varrho_1,\varrho_4)}{e^{\alpha(\varrho_1+\varrho_4)}}\\
&\quad\times\frac{(\Omega\Psi)^{(1-\xi)^s(\omega^s)}(\varrho_2,\varrho_3)}{e^{\alpha(\varrho_2+\varrho_3)}}\frac{(\Omega\Psi)^{(1-\xi)^s(1-\omega)^s}(\varrho_2,\varrho_4)}{e^{\alpha(\varrho_2+\varrho_4)}}
\end{aligned}
$$

Therefore $\Omega\Psi$ is geometrically exponentially $s-$convex function in the second sense on the co-ordinates on Δ. $\square$

3. Conclusion

In this study, we propose and formalize the notion of geometrically-exponentially s-convex functions in the second sense defined on the coordinates of a domain. We then proceed to derive a Hadamard-type integral inequality that is particularly suited for this new class of functions. The inequality we establish not only generalizes existing results in the literature concerning convex and s-convex functions, but also enriches the theoretical structure surrounding geometrically-exponentially s-convexity in the second sense on the co-ordinates. Furthermore, our findings contribute to a deeper understanding of the integral behaviors and properties of such functions, potentially opening new avenues for applications in mathematical analysis and related fields.

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