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Convergence of Bézier type (λ, q)-Bernstein-Kantorovich shifted knots operators

Fahad Maqbul Alamrani[a], Ebrahem A. Algehyne[a], Md. Nasiruzzaman[a,*], M. Mursaleen[b,c]

[a]*Department of Mathematics, Faculty of Science, University of Tabuk, PO Box 4279, Tabuk-71491, Saudi Arabia*
[b]*Department of Mathematical Sciences, Saveetha School of Engineering, Saveetha Institute of Medical and Technical Sciences, Chennai 602105, Tamilnadu, India*
[c]*Department of Medical Research, China Medical University Hospital, China Medical University (Taiwan), Taichung, Taiwan*

Abstract. This article presents the Kantorovich formulation of (λ, q)-Bernstein operators related by Bézier basis functions using the features of shifted knots. We derive these operators' convergence and other associated approximation features. We use the modulus of continuity to discuss the degree of convergence of our operators. Furthermore, we derive the approximation in Lipschitz spaces and provide several fundamental direct theorems. The Voronovskaja type asymptotic formulas and some graphical examples are also presented.

1. Introduction and preliminaries

The series of positive linear operators implied by $\{B_m\}_{m\geq 1}$ was initially created by one of the world's most renowned mathematicians, S. N. Bernstein, who also provided the simplest and most elegant proof of one of the most well-known Weierstrass approximation theorems. The class of all continuous functions on $[0,1]$ can be uniformly approximated by a function known as the famous Bernstein polynomial, which is defined in [5]. This was demonstrated in Bernstein's work for every $g \in C[0,1]$. Thus, for any $\eth \in [0,1]$, the well-known Bernstein polynomial can be defined as follows:

$$B_m(g;\eth) = \sum_{i=0}^{m} g\left(\frac{i}{m}\right) b_{m,i}(\eth),$$

where the Bernstein polynomials of degree at most m are defined as $b_{m,i}(\eth)$ and $m \in \mathbb{N}$ (positive integers),

$$b_{m,i}(\eth) = \binom{m}{i} \eth^i (1-\eth)^{m-i} \qquad (i = 0, 1, \cdots, m; \eth \in [0,1])$$

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* Corresponding author: Md. Nasiruzzaman
Email addresses: fm.alomrani@ut.edu.sa (Fahad Maqbul Alamrani), e.algehyne@ut.edu.sa (Ebrahem A. Algehyne), nasir3489@gmail.com; mfarooq@ut.edu.sa (Md. Nasiruzzaman), mursaleenm@gmail.com (M. Mursaleen)
ORCID iDs: https://orcid.org/0009-0006-9816-1719 (Fahad Maqbul Alamrani), https://orcid.org/0000-0002-5022-1157 (Ebrahem A. Algehyne), https://orcid.org/0000-0003-4823-0588 (Md. Nasiruzzaman), https://orcid.org/0000-0003-4128-0427 (M. Mursaleen)

and

$$b_{m,i}(ð) = 0 \qquad (i < 0 \text{ or } i > m).$$

Cai and colleagues (2010) introduced new Bézier bases with shape parameter $\lambda \in [-1, 1]$, which they called the λ-Bernstein operators.

$$B_{m,\lambda}(g; ð) = \sum_{j=0}^{m} g\left(\frac{j}{m}\right) \tilde{b}_{m,j}(\lambda; ð), \tag{1}$$

where Ye et al. [38] defined the Bernstein polynomial $b_{m,j}(ð)$ in terms of the new Bernstein basis function $\tilde{b}_{m,j}(\lambda; y)$ as follows:

$$\begin{aligned}
\tilde{b}_{m,0}(\lambda; ð) &= b_{m,0}(ð) - \frac{\lambda}{m+1} b_{m+1,1}(ð),\\
\tilde{b}_{m,j}(\lambda; ð) &= b_{m,j}(ð) + \lambda\Big(\frac{m-2j+1}{m^2-1} b_{m+1,j}(ð) - \frac{m-2j-1}{m^2-1} b_{m+1,j+1}(ð)\Big), \text{ for } 1 \le j \le m-1\\
\tilde{b}_{m,m}(\lambda; ð) &= b_{m,m}(ð) - \frac{\lambda}{m+1} b_{m+1,m}(ð).
\end{aligned}$$

Because of the approximation process, researchers have obtained the Bernstein-type Stancu polynomials using shifted knots [12], q-Phillips operators [1], Bernstein-Kantorovich-Stancu shifted knots operators [2], Stancu variant of Bernstein-Kantorovich operators [21], a new family of Bernstein-Kantorovich operators [22], Bézier bases' q-Bernstein shifted operators [24], Bernstein operators based on Bézier bases [35], Bézier bases with Schurer polynomials [26], etc. M. Ayman-Mursaleen et al. employ Bézier bases to create the shifted knots type Bernstein operators in terms of the Bézier bases function (see [3]):

$$B_{m,\lambda}^{\wp_1,\wp_2}(g; ð) = \left(\frac{m+\wp_2}{m}\right)^m \sum_{i=0}^{m} \tilde{b}_{m,i}^{\wp_1,\wp_2}(\lambda; ð) g\left(\frac{i}{m}\right), \tag{2}$$

where Bézier bases $\tilde{b}_{m,i}^{\wp_1,\wp_2}$ and Bernstein basis function $b_{m,j}^{\wp_1,\wp_2}$ are defined as:

$$\begin{aligned}
\tilde{b}_{m,0}^{\wp_1,\wp_2}(\lambda; ð) &= b_{m,0}^{\wp_1,\wp_2}(ð) - \frac{\lambda}{m+1} b_{m+1,1}^{\wp_1,\wp_2}(ð),\\
\tilde{b}_{m,j}^{\wp_1,\wp_2}(\lambda; ð) &= b_{m,j}^{\wp_1,\wp_2}(ð) + \lambda\Big(\frac{m-2j+1}{m^2-1} b_{m+1,j}^{\wp_1,\wp_2}(ð) - \frac{m-2j-1}{m^2-1} b_{m+1,j+1}^{\wp_1,\wp_2}(ð)\Big), \text{ for } 1 \le j \le m-1\\
\tilde{b}_{m,m}^{\wp_1,\wp_2}(\lambda; ð) &= b_{m,m}^{\wp_1,\wp_2}(ð) - \frac{\lambda}{m+1} b_{m+1,m}^{\wp_1,\wp_2}(ð),
\end{aligned}$$

$$b_{m,j}^{\wp_1,\wp_2}(ð) = \binom{m}{j}\left(ð - \frac{\wp_1}{m+\wp_2}\right)^j \left(\frac{m+\wp_1}{m+\wp_2} - ð\right)^{m-j}. \tag{3}$$

Several fundamental features of q-integers and their properties are readily obtained (see to [15, 16]). For instance, for every $q \in (0, 1)$, and $0 \le \mu \le \nu$, the binomial coefficient for q-integers is provided by $\begin{bmatrix} \mu \\ \nu \end{bmatrix}_q = \frac{[\mu]_q!}{[\mu-\nu]_q!\,[\nu]_q!}$. For any $\mu, \nu \in \mathbb{N}$, the q-binomial polynomial is given by $(1+ð)_q^{\mu-\nu} = (1+ð)(1+qð)\cdots(1+q^{\mu-\nu-1}ð)$, and if $\mu = \nu = 0$, then $(1+ð)_q^{\mu-\nu} = 1$. Using the q-analogue, Lupaş [20] introduced the first Bernstein polynomials, calculating various approximation properties and shape-preserving properties. According to Phillips [32] in 1997, the classical-Bernstein polynomials in another q-analogue are defined as follows:

$$\tilde{B}_{m,q}(g; ð) = \sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix}_q ð^j \prod_{j=0}^{m-j-1} (1 - q^j ð)\ \ g\left(\frac{[j]_q}{[m]_q}\right), \ \ ð \in [0, 1].$$

M. Mursaleen et al. revealed the q-analogue of Lupas Bernstein operators with shifted knots (see [24]) by:

$$B_{m,j}^{\wp_1,\wp_2}(f;ð,q)=\frac{1}{(\mathcal{Z})_q^m}\sum_{j=0}^{m}\left[\begin{array}{c} m \\ j \end{array}\right]_q (ð-\mathcal{X})^j\,(\mathcal{Y}-ð)^{m-j}\,f\left(\frac{[j]_q}{[m]_q}\right), \tag{4}$$

where, $\mathcal{X}=\frac{\wp_1}{[m]_q+\wp_2}$, $\mathcal{Y}=\frac{[m]_q+\wp_1}{[m]_q+\wp_2}$ and $\mathcal{Z}=\frac{[m]_q}{[m]_q+\wp_2}$.

Using shifted knots defined as follows, the Bernstein basis function $b_{m,j}^{\wp_1,\wp_2}$ is obtained (see [12, 24]):

$$b_{m,j}^{\wp_1,\wp_2}(ð;q)=\left[\begin{array}{c} m \\ j \end{array}\right]_q q^{\frac{j(j-1)}{2}}\,(ð-\mathcal{X})_q^j\,(\mathcal{Y}-ð)^{m-j}\,. \tag{5}$$

The equality (5) can be also written as:

$$b_{m,j}^{\wp_1,\wp_2}(ð;q)=\left[\begin{array}{c} m \\ j \end{array}\right]_q (ð-\mathcal{X})_q^j \prod_{w=0}^{m-j-1}(\mathcal{Y}-q^w ð)\,. \tag{6}$$

The Bézier bases function $\tilde{b}_{m,j}^{\wp_1,\wp_2}(ð;q,\lambda)$ by utilizing the Bernstein basis function $b_{m,j}^{\wp_1,\wp_2}(ð;q)$ is defined as (see [6, 38]):

$$\begin{aligned}
\tilde{b}_{m,0}^{\wp_1,\wp_2}(ð;q,\lambda) &= b_{m,0}^{\wp_1,\wp_2}(ð;q)-\frac{\lambda}{[m]_q+1}b_{m+1,1}^{\wp_1,\wp_2}(ð;q),\\
\tilde{b}_{m,j}^{\wp_1,\wp_2}(ð;q,\lambda) &= b_{m,j}^{\wp_1,\wp_2}(ð;q)+\lambda\Big(\frac{[m]_q-2[j]_q+1}{[m]_q^2-1}b_{m+1,j}^{\wp_1,\wp_2}(ð;q)\\
&\qquad -\frac{[m]_q-2q[j]_q-1}{[m]_q^2-1}b_{m+1,j+1}^{\wp_1,\wp_2}(ð;q)\Big),\ \text{for } 1\le j\le m-1\\
\tilde{b}_{m,m}^{\wp_1,\wp_2}(ð;q,\lambda) &= b_{m,m}^{\wp_1,\wp_2}(ð;q)-\frac{\lambda}{[m]_q+1}b_{m+1,m}^{\wp_1,\wp_2}(ð;q).
\end{aligned}$$

Most recently, by using Bézier base functions, the q-analogue of the shifted knots type of Bernstein operators is defined as follows (see [25]):

$$\tilde{U}_{m,\lambda}^{\wp_1,\wp_2}(f;ð,q)=\frac{1}{(\mathcal{Z})_q^m}\sum_{j=0}^{m}\tilde{b}_{m,j}^{\wp_1,\wp_2}(ð;q,\lambda)f\left(\frac{[j]_q}{[m]_q}\right), \tag{7}$$

From [25], we can see that the lemma is:

Lemma 1.1.

$$\begin{aligned}
\tilde{U}_{m,\lambda}^{\wp_1,\wp_2}(1;ð,q) &= 1;\\
\tilde{U}_{m,\lambda}^{\wp_1,\wp_2}(t;ð,q) &= \frac{1}{(\mathcal{Z})_q^m}(ð-\mathcal{X})+\frac{1}{(\mathcal{Z})_q^m}\frac{\lambda\,(ð-\mathcal{X})\,[m+1]_q}{[m]_q([m]_q-1)}\Big\{(\mathcal{Z})_q^m-(ð-\mathcal{X})^m\Big\}\\
&\quad+\frac{1}{(\mathcal{Z})_q^m}\frac{-2\lambda\,(ð-\mathcal{X})\,[m+1]_q}{[m]_q^2-1}\Big[\frac{1}{[m]_q}\Big\{(\mathcal{Z})_q^m-(ð-\mathcal{X})_q^m\Big\}+q\,(ð-\mathcal{X})\Big\{(\mathcal{Z})_q^{m-1}-(ð-\mathcal{X})^{m-1}\Big\}\Big]\\
&\quad+\frac{1}{(\mathcal{Z})_q^m}\frac{\lambda}{q[m]_q([m]_q+1)}\Big[(\mathcal{Z})_q^{m+1}-\prod_{w=0}^{m}(\mathcal{Y}-q^w ð)\\
&\quad-(ð-\mathcal{X})_q^{m+1}-[m+1]_q\,(ð-\mathcal{X})\Big\{(\mathcal{Z})_q^m-(ð-\mathcal{X})^m\Big\}\Big]
\end{aligned}$$

$$+\frac{1}{(\mathcal{Z})_q^m}\frac{\lambda}{[m]_q([m]_q^2-1)}\Big[2[m]_q[m+1]_q\,(\eth-\mathcal{X})_q^2\Big\{(\mathcal{Z})_q^{m-1}-(\eth-\mathcal{X})^{m-1}\Big\}$$
$$-\frac{2}{q}[m+1]_q\,(\eth-\mathcal{X})\Big\{(\mathcal{Z})_q^m-(\eth-\mathcal{X})^m\Big\}$$
$$+\frac{2}{q}\Big\{(\mathcal{Z})_q^{m+1}-\prod_{w=0}^{m}(\mathcal{Y}-q^w\eth)-(\eth-\mathcal{X})_q^{m+1}\Big\}\Big].$$

$$\begin{aligned}\tilde{U}_{m,\lambda}^{\wp_1,\wp_2}(t^2;\eth,q) &= \frac{1}{(\mathcal{Z})_q^m}(\eth-\mathcal{X})_q^2+\frac{1}{(\mathcal{Z})_q^m}\frac{\lambda\,(\eth-\mathcal{X})\,[m+1]_q}{[m]_q([m]_q-1)}\Big[q\,(\eth-\mathcal{X})\Big\{(\mathcal{Z})_q^{m-1}-(\eth-\mathcal{X})^{m-1}\Big\}\Big]\\
&+\frac{1}{(\mathcal{Z})_q^m}\frac{-2\lambda\,(\eth-\mathcal{X})\,[m+1]_q}{[m]_q^2-1}\Big[\frac{1}{[m]_q}\Big\{(\mathcal{Z})_q^m-(\eth-\mathcal{X})_q^m\Big\}\\
&+q(2+q)\,(\eth-\mathcal{X})\Big\{(\mathcal{Z})_q^{m-1}-(\eth-\mathcal{X})^{m-1}\Big\}\\
&+q^3[m-1]_q\,(\eth-\mathcal{X})_q^2\Big\{(\mathcal{Z})_q^{m-2}-(\eth-\mathcal{X})^{m-2}\Big\}\Big]\\
&+\frac{1}{(\mathcal{Z})_q^m}\frac{-\lambda}{q[m]_q([m]_q+1)}\Big[[m+1]_q\,(\eth-\mathcal{X})_q^2\Big\{(\mathcal{Z})_q^{m-1}-(\eth-\mathcal{X})^{m-1}\Big\}\\
&-\frac{[m+1]_q}{q[m]_q}(\eth-\mathcal{X})\Big\{(\mathcal{Z})_q^m-(\eth-\mathcal{X})^m\Big\}\\
&+\frac{1}{q[m]_q}\Big\{(\mathcal{Z})_q^{m+1}-\prod_{w=0}^{m}(\mathcal{Y}-q^w\eth)-(\eth-\mathcal{X})_q^{m+1}\Big]\\
&+\frac{1}{(\mathcal{Z})_q^m}\frac{2\lambda}{[m]_q([m]_q^2-1)}\Big[q[m-1]_q[m+1]_q\,(\eth-\mathcal{X})_q^3\Big\{(\mathcal{Z})_q^{m-2}-(\eth-\mathcal{X})_q^{m-2}\Big\}\\
&-\frac{(1-q)[m+1]_q}{q}(\eth-\mathcal{X})_q^2\Big\{(\mathcal{Z})_q^{m-1}-(\eth-\mathcal{X})_q^{m-1}\Big\}\\
&+\frac{[m+1]_q}{q^2[m]_q}(\eth-\mathcal{X})\Big\{(\mathcal{Z})_q^m-(\eth-\mathcal{X})_q^m\Big\}\\
&-\frac{1}{q^2[m]_q}\Big\{(\mathcal{Z})_q^{m+1}-\prod_{w=0}^{m}(\mathcal{Y}-q^w\eth)-(\eth-\mathcal{X})_q^{m+1}\Big\}\Big].\end{aligned}$$

F. Özger et al. constructed the various approximating operators and obtained the convergence properties. For instance, we see the Combining the Volterra integral equations [28], Class of Szász-Mirakjan-Kantorovich operators [29], and Generalized Weighted Statistical Convergence [30]. Additionally, we used shape-preserving qualities and other relevant convergence results: Bézier-Baskakov-Beta [17], Volterra integral equations [8], modified Bernstein-Schurer operators [23], (α,λ,s)-Bernstein Basis [18], Szász–Kantorovich Operators [33], λ- Szász– Operators [34], shape properties of Kantorovich [36], and Bézier-Baskakov-Jain type operators [37].

2. Operators and basic formula

The purpose of this section is to use the Bézier bases function to give an extension of the operators (7) for integrable functions on $\mathfrak{R}=[0,1]$. We utilize the shifted knots and Bézier bases function to generalize the operators (7) in the Kantorovich sense. The Lebesgue integrable functions can be approximated using the Kantorovich modifications of sequences of linear positive operators. The fundamental idea behind

Kantorovich modifications is to replace the sample values $f\left(\frac{j}{m}\right)$ with the mean values of f in the intervals $\left[\frac{j}{m+1}, \frac{j+1}{m+1}\right]$. To learn more about Kantorovich operators, readers can consult additional publications [21, 22]. We take $\mathcal{X} \leq ð \leq \mathcal{Y}$ for all $0 \leq \wp_1 \leq \wp_2$. Moreover, for all $1 \leq p < \infty$, we suppose $\mathcal{J}_{m,q}^{\wp_1,\wp_2} = [\mathcal{X}, \mathcal{Y}]$. Thus, in terms of Bézier bases function $\tilde{b}_{m,j}^{\wp_1,\wp_2}(ð; q, \lambda)$. We define the Kantorovich variant of q-Bernstein shifted knots operators. Suppose $\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}$. Therefore, we take $\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2} : L_p\mathfrak{R} \to L_p(\mathcal{J}_{m,q}^{\wp_1,\wp_2})$ and for all $f \in L_p\mathfrak{R}$, we define operators as follows:

$$\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(f; ð, q) = \frac{[m+1]_q}{(\mathcal{Z})_q^m} \sum_{j=0}^{m} \tilde{b}_{m,j}^{\wp_1,\wp_2}(ð; q, \lambda) q^{-j} \int_{\frac{[j]_q}{[m+1]_q}}^{\frac{[j+1]_q}{[m+1]_q}} f(t) \mathrm{d}_q t, \tag{8}$$

where $m \in \mathbb{N}$ (the set of positive integers) and $C\mathfrak{R}$ (the set of all continuous functions defined on $\mathfrak{R}$) are represented.

This paper is generally structured as follows: We study the moments and central moments of our new operators, (8). We give a convergence theorem for Lipschitz continuous functions, evaluate the Korovkin's approximation theorem, prove the local approximation theorem, and study an asymptotic formula for Voronovskaja.

Lemma 2.1. *For $f(t) = 1, t, t^2$, we can see*

1. $\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(1; ð, q) = \tilde{U}_{m,\lambda}^{\wp_1,\wp_2}(1; ð, q) = 1;$
2. $\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(t; ð, q) = \dfrac{[m]_q}{[m+1]_q} \tilde{U}_{m,\lambda}^{\wp_1,\wp_2}(t; ð, q) + \dfrac{[1]_q}{[2]_q[m+1]_q}$
3. $\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(t^2; ð, q) = \dfrac{[m]_q^2}{[m+1]_q^2} \tilde{U}_{m,\lambda}^{\wp_1,\wp_2}\left(t^2; ð, q\right) + \dfrac{2q+1}{[3]_q} \dfrac{[m]_q}{[m+1]_q^2} \tilde{U}_{m,\lambda}^{\wp_1,\wp_2}(t; ð, q) + \dfrac{[1]_q}{[3]_q[m+1]_q^2},$

where operators $\tilde{U}_{m,\lambda}^{\wp_1,\wp_2}$ are defined by Lemma 1.1.

Proof. To prove our equality, we take into account the Lemma (1.1) and use the equality $[j+1]_q = q^j + [j]_q$ and $[j+1]_q = 1 + q[j]_q$. Thus, by applying the famous q-Jackson integral, we conclude that

$$\begin{aligned}
\int_{\frac{[j]_q}{[m+1]_q}}^{\frac{[j+1]_q}{[m+1]_q}} t^\gamma \mathrm{d}_q t &= \int_0^{\frac{[j+1]_q}{[m+1]_q}} t^\gamma d_q t - \int_0^{\frac{[j]_q}{[m+1]_q}} t^\gamma d_q t \\
&= (1-q) \frac{[j+1]_q}{[m+1]_q} \sum_{m=0}^{\infty} \left(\frac{[j+1]_q}{[m+1]_q} q^m\right)^\gamma q^m - (1-q) \frac{[j]_q}{[m+1]_q} \sum_{m=0}^{\infty} \left(\frac{[j]_q}{[m+1]_q}\right)^\gamma q^m \\
&= \frac{(1-q)}{([1+m]_q)^{\gamma+1}} \Big(([j+1]_q)^{\gamma+1} - ([j]_q)^{\gamma+1}\Big) \sum_{m=0}^{\infty} q^{m(1+\gamma)}.
\end{aligned}$$

Therefore easy to conclude that

$$\int_{\frac{[j]_q}{[m+1]_q}}^{\frac{[j+1]_q}{[m+1]_q}} t^\gamma \mathrm{d}_q t = \begin{cases} \frac{q^j}{[m+1]_q} & \text{for } \gamma = 0; \\ \frac{q^j}{([m+1]_q)^2} \left([j]_q + \frac{1}{[2]_q}\right) & \text{for } \gamma = 1; \\ \frac{q^j}{([m+1]_q)^3} \left([j]_q^2 + \frac{2q+1}{[3]_q}[j]_q + \frac{1}{[3]_q}\right) & \text{for } \gamma = 2. \end{cases} \tag{9}$$

From the Lemma (1.1) and equality (9), it is easy to get

$$\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(1; ð, q) = \frac{[m+1]_q}{(\mathcal{Z})_q^m} \sum_{j=0}^{m} \tilde{b}_{m,j}^{\wp_1,\wp_2}(ð; q, \lambda) q^{-j} \frac{q^j}{[m+1]_q}$$

$$
\begin{aligned}
&= \tilde{U}^{\wp_1,\wp_2}_{m,\lambda}(1;ð,q)\\
&= 1;
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{BK}^{\wp_1,\wp_2}_{m,\lambda}(t;ð,q) &= \frac{[m+1]_q}{(\mathcal{Z})^m_q}\sum_{j=0}^{m}\tilde{b}^{\wp_1,\wp_2}_{m,j}(ð;q,\lambda)q^{-j}\frac{q^j}{\left([m+1]_q\right)^2}\left([j]_q+\frac{1}{[2]_q}\right)\\
&= \frac{[m]_q}{[m+1]_q}\tilde{U}^{\wp_1,\wp_2}_{m,\lambda}\left(\frac{[j]_q}{[m]_q};ð,q\right)+\frac{[1]_q}{[2]_q[m+1]_q}\tilde{U}^{\wp_1,\wp_2}_{m,\lambda}(1;ð,q)\\
&= \frac{[m]_q}{[m+1]_q}\tilde{U}^{\wp_1,\wp_2}_{m,\lambda}(t;ð,q)+\frac{[1]_q}{[2]_q[m+1]_q};
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{BK}^{\wp_1,\wp_2}_{m,\lambda}(t^2;ð,q) &= \frac{[m+1]_q}{(\mathcal{Z})^m_q}\sum_{j=0}^{m}\tilde{b}^{\wp_1,\wp_2}_{m,j}(ð;q,\lambda)q^{-j}\frac{q^j}{\left([m+1]_q\right)^3}\left([j]_q^2+\frac{2q+1}{[3]_q}[j]_q+\frac{1}{[3]_q}\right)\\
&= \frac{[m]_q^2}{[m+1]_q^2}\tilde{U}^{\wp_1,\wp_2}_{m,\lambda}\left(\frac{[j]_q^2}{[m]_q^2};ð,q\right)+\frac{2q+1}{[3]_q}\frac{[m]_q}{[m+1]_q^2}\tilde{U}^{\wp_1,\wp_2}_{m,\lambda}\left(\frac{[j]_q}{[m]_q};ð,q\right)\\
&\quad+\frac{[1]_q}{[3]_q[m+1]_q^2}\tilde{U}^{\wp_1,\wp_2}_{m,\lambda}(1;ð,q)\\
&= \frac{[m]_q^2}{[m+1]_q^2}\tilde{U}^{\wp_1,\wp_2}_{m,\lambda}\left(t^2;ð,q\right)+\frac{2q+1}{[3]_q}\frac{[m]_q}{[m+1]_q^2}\tilde{U}^{\wp_1,\wp_2}_{m,\lambda}(t;ð,q)+\frac{[1]_q}{[3]_q[m+1]_q^2}.
\end{aligned}
$$

□

Lemma 2.2. *Take* $t-ð=t_ð$. *Operators* (8) *have the following equalities:*

1. $$\begin{aligned}\mathcal{BK}^{\wp_1,\wp_2}_{m,\lambda}(t_ð;ð,q) &= \frac{[m]_q}{[m+1]_q}\tilde{U}^{\wp_1,\wp_2}_{m,\lambda}(t;ð,q)+\frac{[1]_q}{[2]_q[m+1]_q}-ð\\ &:= \Psi^{\wp_1,\wp_2}_m(\lambda;q,ð)\ (suppose).\end{aligned}$$
2. $$\begin{aligned}\mathcal{BK}^{\wp_1,\wp_2}_{m,\lambda}((t_ð)^2;ð,q) &= \frac{[m]_q^2}{[m+1]_q}\tilde{U}^{\wp_1,\wp_2}_{m,\lambda}\left(t^2;ð,q\right)+\frac{1}{[3]_q}\frac{1}{[m+1]_q^2}+ð^2\\ &\quad+\left(\frac{2q+1}{[3]_q}\frac{[m]_q}{[m+1]_q^2}-2ð\frac{[m]_q}{[m+1]_q}\right)\tilde{U}^{\wp_1,\wp_2}_{m,\lambda}(t;ð,q)\\ &:= \Phi^{\wp_1,\wp_2}_m(\lambda;q,ð)\ (suppose).\end{aligned}$$

3. Some approximation results for operators $\mathcal{BK}^{\wp_1,\wp_2}_{m,\lambda}(t;ð,q)$

For the operators $\mathcal{BK}^{\wp_1,\wp_2}_{m,\lambda}(t;ð,q)$, we get numerous global and local approximation theorems by (8). First, we use the Ditzian-Totik uniform modulus of smoothness to define the uniform convergence property for our operators, which allows us to obtain the local and global approximations. Next, we build a number of elementary theorems based on the maximal approximation property of Lipschitz type and Peetre's K-functional property. For a continuous function g in $C\mathfrak{R}$ on $\mathfrak{R}$, we can replace it with a real-valued function with the norm $\| g \|_{C\mathfrak{R}}=\sup_{ð\in\mathfrak{R}}\left|g(ð)\right|$.

Theorem 3.1. *([11, 19]) The formula* $\lim\limits_{m\to\infty}K_m(t^\rho;y)=ð^\rho$, *will be satisfied for any* $C[u,v]$ *by any sequence of positive linear operators* K_m *acting uniformly on* $[u,v]$ *for all* $\rho=0,1,2$. *The operators* $\lim\limits_{m\to\infty}K_m(g)=g$ *are then uniformly convergent for any* $g\in C[u,v]$ *for every compact subset of* $[u,v]$.

Theorem 3.2. *Assuming $q = q_m$ to be any real number that satisfies $0 < q_m < 1$, we obtain $\lim_{m\to\infty} \mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(g;ð,q) = g(ð)$ for all $g \in C\mathfrak{R}$, converges uniformly on $\mathfrak{R}$; in particular, the set of all continuous functions on $\mathfrak{R}$ is represented by $C\mathfrak{R}$.*

Proof. Lemma 2.1 makes it clear that $\lim_{m\to\infty} \mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(t^\rho;ð,q) = ð^\rho \quad (\rho = 0,1,2)$ will be obtained. According to the Bohman-Korovkin-Popoviciu theorem, the operators $\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(g;ð,q)$ are uniformly convergent to set $g \in C\mathfrak{R}$. We complete the necessary proof of the theorem. □

Theorem 3.3. *[13, 14] For each operator acting from $C\mathfrak{R}$ to $C\mathfrak{R}$, $\{P_m\}_{m\geq 1}$. For every $g \in C\mathfrak{R}$, let $\lim_{m\to\infty} \|P_m(t^\rho) - ð^\rho\|_{C\mathfrak{R}} = 0,\ \rho = 0,1,2$. This will yield*

$$\lim_{m\to\infty} \|P_m(g) - g\|_{C\mathfrak{R}} = 0.$$

Theorem 3.4. *Take operators $\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}$ which acts $C\mathfrak{R}$ to $C\mathfrak{R}$ and satisfy the property $\lim_{m\to\infty} \|\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(t^\rho;ð,q) - ð^\rho\|_{C\mathfrak{R}} = 0$. Additionally, consider the sequences of positive numbers $q = q_m$ such that $q_m \in (0,1)$. For any $g \in C\mathfrak{R}$, we then obtain*

$$\lim_{m\to\infty} \| \mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(g;ð,q) - g \|_{C\mathfrak{R}} = 0.$$

Proof. Considering Theorem 3.3 and Korovkin's theorem, we can readily demonstrate that

$$\lim_{m\to\infty} \|\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(t^\rho;ð,q) - ð^\rho\|_{C\mathfrak{R}} = 0, \quad \rho = 0,1,2.$$

According to Lemma 2.1, easily obtained $\|\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(1;ð,q) - 1\|_{C\mathfrak{R}} = \sup_{ð\in\mathfrak{R}} \left|\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(1;ð,q) - 1\right| = 0$. For $\rho = 1$, it is evident

$$\begin{aligned}\|\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(t;ð,q) - ð\|_{C\mathfrak{R}} &= \sup_{ð\in\mathfrak{R}} \left|\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(t;ð,q) - ð\right| \\ &= \sup_{ð\in\mathfrak{R}} \Psi_m^{\wp_1,\wp_2}(\lambda;q,ð).\end{aligned}$$

We obtain $\|\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(t;ð,q) - ð\|_{C\mathfrak{R}} \to 0$ since $m \to \infty$ implies that $\frac{1}{[m]_q} \to 0,\ \frac{[m]_q+[\wp_1]_q}{[m]_q+\wp_2} \to 1$. Likewise, for $\rho = 2$, we observe

$$\begin{aligned}\|\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(t^2;ð,q) - ð^2\|_{C\mathfrak{R}} &= \sup_{ð\in\mathfrak{R}} \left|\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(t^2;ð,q) - ð^2\right| \\ &= \sup_{ð\in\mathfrak{R}} \Phi_m^{\wp_1,\wp_2}(\lambda;q,ð).\end{aligned}$$

Accordingly, $\|\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(t^2;ð,q) - ð^2\|_{C\mathfrak{R}} \to 0$ as $m \to \infty$ is obtained. We obtained our proof from these findings. □

We show some results on global approximations using the Ditzian-Totik uniform modulus of smoothness. We discuss the basic property of the uniform modulus of smoothness for first and second orders, which is

$$\omega_1^\gamma(g,\delta) := \sup_{0<|\beta|\leq\delta} \ \sup_{ð,ð+\beta\gamma(ð)\in\mathfrak{R}} \{|g(ð+\beta\gamma(ð)) - g(ð)|\};$$

$$\omega_2^\gamma(g,\delta) := \sup_{0<|\beta|\leq\delta} \ \sup_{ð,ð\pm\beta\gamma(ð)\in\mathfrak{R}} \{|g(ð+\beta\gamma(ð)) - 2g(ð) + g(ð-\beta\gamma(ð))|\}.$$

Assume $\gamma(ð) = [(ð-u)(v-ð)]^{1/2}$ and the step-weight function γ on $[u,v]$. If $ð \in [u,v]$, then (see [10]). Assuming that AC^* is the set of all absolutely continuous functions, Peetre's K-functional property can be expressed as follows:

$$K_2^{\gamma}(g,\delta) = \inf_{\zeta\in\Xi^2(\gamma)} \Big\{ \|g-\zeta\|_{C\mathfrak{R}} + \delta\|\gamma^2\zeta''\|_{C\mathfrak{R}} : \zeta \in C^2[0,1] \Big\}.$$

For every $\delta > 0$, $\Xi^2(\gamma) = \{\zeta \in C\mathfrak{R} : \zeta' \in C\mathfrak{R},\ \gamma^2\zeta'' \in C\mathfrak{R}\}$ and $C^2\mathfrak{R} = \{\zeta \in C\mathfrak{R} : \zeta', \zeta'' \in C\mathfrak{R}\}$.

Remark 3.5. *([9]) One has for every absolute positive constant M*

$$M\omega_2^{\gamma}(g, \sqrt{\delta}) \le K_2^{\gamma}(g,\delta) \le M^{-1}\omega_2^{\gamma}(g, \sqrt{\delta}). \tag{10}$$

Theorem 3.6. *For any $g \in C\mathfrak{R}$ and $m \in \mathfrak{R}$, let $\gamma(ð)$ ($\gamma \neq 0$) be any step-weight function such that γ^2 is concave. Then, operators $\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}$ fulfill that*

$$|\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(g;ð,q) - g(ð)| \le M\,\omega_2^{\gamma}\Big(g, \frac{[\Phi_m^{\wp_1,\wp_2}(\lambda;q,ð) + \Psi_m^{\wp_1,\wp_2}(\lambda;q,ð)]^{1/2}}{2\gamma(ð)}\Big) + \omega_1^{\gamma}\Big(g, \frac{\Psi_m^{\wp_1,\wp_2}(\lambda;q,ð)}{\gamma(ð)}\Big),$$

where $0 < q < 1$ is the number and $\Psi_m^{\wp_1,\wp_2}(\lambda;q,ð) = \mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(t_ð;ð,q)$, and $\Phi_m^{\wp_1,\wp_2}(\lambda;q,ð) = \mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}\big((t_ð)^2;ð,q\big)$.

Proof. Taking into account an auxiliary operator

$$\tilde{\Omega}_{m,\lambda}^{\wp_1,\wp_2}(g;ð,q) = \mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(g;ð,q) + g(ð) - g\Big(\Psi_m^{\wp_1,\wp_2}(\lambda;q,ð) + ð\Big), \tag{11}$$

Lemma 2.1 makes it easy to obtain the following relations given $g \in C\mathfrak{R}$ and $m \in \mathfrak{R}$.

$$\tilde{\Omega}_{m,\lambda}^{\wp_1,\wp_2}(1;ð,q) = 1 \text{ and } \tilde{\Omega}_{m,\lambda}^{\wp_1,\wp_2}(t;ð,q) = ð,$$

$$\tilde{\Omega}_{m,\lambda}^{\wp_1,\wp_2}(t_ð;ð,q) = 0.$$

For $\beta \in \mathfrak{R}$, let $ð = \beta ð + (1-\beta)t$. Given that γ^2 is concave on $\mathfrak{R}$ and that $\gamma^2(ð) \ge \rho\gamma^2(ð) + (1-\beta)\gamma^2(t)$

$$\frac{|t_ð|}{\gamma^2(ð)} \le \frac{\beta|ð - t|}{\rho\gamma^2(ð) + (1-\beta)\gamma^2(t)} \le \frac{|t_ð|}{\gamma^2(ð)}. \tag{12}$$

The following identities are obtained:

$$\begin{aligned} |\tilde{\Omega}_{m,\lambda}^{\wp_1,\wp_2}(g;ð,q) - g(ð)| &\le |\tilde{\Omega}_{m,\lambda}^{\wp_1,\wp_2}(g-\zeta;ð,q)| + |\tilde{\Omega}_{m,\lambda}^{\wp_1,\wp_2}(\zeta;ð,q) - \zeta(ð)| + |g(ð) - \zeta(ð)| \\ &\le 4\|g-\zeta\|_{C[0,1]} + |\tilde{\Omega}_{m,\lambda}^{\wp_1,\wp_2}(\zeta;ð,q) - \zeta(ð)|. \end{aligned} \tag{13}$$

Utilizing Taylor's series, we can determine that

$$\begin{aligned} |\tilde{\Omega}_{m,\lambda}^{\wp_1,\wp_2}(\zeta;ð,q) - \zeta(ð)| \;\le\; & \mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}\Big(\Big|\int_ð^t |t_ð|\,|\zeta''(ð)| d_qð\Big|;ð,q\Big) \\ & + \Big|\int_ð^{\tilde{\Omega}_{m,\lambda}^{\wp_1,\wp_2}(t;ð,q)} \Big|\tilde{\Omega}_{m,\lambda}^{\wp_1,\wp_2}(t;ð,q) - ð\Big|\,|\zeta''(ð)|\,d_qð\Big| \\ \le\; & \|\gamma^2\zeta''\|_{C\mathfrak{R}}\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}\Big(\Big|\int_ð^t \frac{|t_ð|}{\gamma^2(ð)} d_qð\Big|;ð,q\Big) + \|\gamma^2\zeta''\|_{C\mathfrak{R}} \\ & \times\Big|\int_ð^{\tilde{\Omega}_{m,\lambda}^{\wp_1,\wp_2}(t;ð,q)} \Big|\tilde{\Omega}_{m,\lambda}^{\wp_1,\wp_2}(t;ð,q) - ð\Big| \frac{d_qð}{\gamma^2(ð)}\Big| \\ \le\; & \gamma^{-2}(ð))\|\gamma^2\zeta''\|_{C\mathfrak{R}}\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}((t_ð)^2;ð,q) \end{aligned}$$

$$+\gamma^{-2}(ð)\Psi_m^{\wp_1,\wp_2}(\lambda;q,ð)\|\gamma^2\zeta''\|_{C\mathfrak{R}}. \tag{14}$$

Using the relations (10), (13), and (14) with Peetre's K-functional properties, it is simple to obtain

$$\begin{aligned}\left|\tilde{\Omega}_{m,\lambda}^{\wp_1,\wp_2}(g;ð,q)-g(ð)\right| &\le 4\|g-\zeta\|_{C[0,1]}+\gamma^{-2}(ð)\|\gamma^2\zeta''\|_{C\mathfrak{R}}\Big(\Phi_m^{\wp_1,\wp_2}(\lambda;q,ð)+\Psi_m^{\wp_1,\wp_2}(\lambda;q,ð)\Big)\\ &\le M\,\omega_2^{\gamma}\Bigg(g,\frac{1}{2}\sqrt{\frac{\Phi_m^{\wp_1,\wp_2}(\lambda;q,ð)+\Psi_m^{\wp_1,\wp_2}(\lambda;q,ð)}{\gamma(ð)}}\Bigg).\end{aligned}$$

Therefore, it is clear from the order one uniform smoothness characteristic that

$$\left|g\left(\Psi_m^{\wp_1,\wp_2}(\lambda;q,ð)+ð\right)-g(ð)\right|=\left|g\Big(\gamma(ð)\frac{\Psi_m^{\wp_1,\wp_2}(\lambda;q,ð)}{\gamma(ð)}+ð\Big)-g(ð)\right|\le\omega_1^{\gamma}\Big(g,\frac{\Psi_m^{\wp_1,\wp_2}(\lambda;q,ð)}{\gamma(ð)}\Big).$$

Consequently, we ultimately obtain the inequality

$$\begin{aligned}|\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(g;ð,q)-g(ð)| &\le |\tilde{\Omega}_{m,\lambda}^{\wp_1,\wp_2}(g;ð,q)-g(ð)|+\left|g\left(\Psi_m^{\wp_1,\wp_2}(\lambda;q,ð)+ð\right)-g(ð)\right|\\ &\le M\,\omega_2^{\gamma}\Bigg(g,\frac{1}{2}\sqrt{\frac{\Phi_m^{\wp_1,\wp_2}(\lambda;q,ð)+\Psi_m^{\wp_1,\wp_2}(\lambda;q,ð)}{(ð-u)(v-ð)}}\Bigg)+\omega_1^{\gamma}\Big(g,\frac{\Psi_m^{\wp_1,\wp_2}(\lambda;q,ð)}{\gamma(ð)}\Big).\end{aligned}$$

It completes Theorem 3.6's desired proof. □

Theorem 3.7. *Assuming the real number $0<q<1$, then for any $f'\in C\mathfrak{R}$ and $ð\in\mathfrak{R}$, we obtain the inequality:*

$$|\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(f;ð,q)-f(ð)|\le\Phi_m^{\wp_1,\wp_2}(\lambda;q,ð)\,|f'(ð)|+2\sqrt{\Phi_m^{\wp_1,\wp_2}(\lambda;q,ð)}\,\omega_1^{\gamma}\Big(f',\sqrt{\Phi_m^{\wp_1,\wp_2}(\lambda;q,ð)}\Big).$$

Proof. We are aware of the relationship.

$$f(t)=f(ð)+f'(ð)(t_ð)+\int_ð^t(f'(z)-f'(ð))\mathrm{d}_qz, \tag{15}$$

as long as $t,ð\in\mathfrak{R}$. Applying $\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}$ to equality (15) yields

$$\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(f(t)-f(ð);ð,q)=f'(ð)\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(t_ð;ð,q)+\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}\Big(\int_ð^t(f'(z)-f'(ð))\mathrm{d}_qz;ð,q\Big).$$

One has for each $f\in C\mathfrak{R}$ and $ð\in\mathfrak{R}$

$$|f(t)-f(ð)|\le\Big(1+\frac{|t_ð|}{\delta}\Big)\omega_1^{\gamma}(f,\delta),\quad \delta>0.$$

Given the inequality above, we have

$$\left|\int_t^ð(f'(ð)-f'(z))\mathrm{d}_qz\right|\le\Big(|t_ð|+\frac{(t_ð)^2}{\delta}\Big)\omega_1^{\gamma}(f',\delta).$$

Thus, simple to get

$$|\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(f;ð,q)-f(ð)|\le|f'(ð)|\,|\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(t_ð;ð,q)|+\omega_1^{\gamma}(f',\delta)\Big\{\frac{1}{\delta}\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}((t_ð)^2;ð,q)+\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(|t_ð|;ð,q)\Big\}.$$

The result of the Cauchy-Schwarz inequality is

$$\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(|t_ð|;ð,q) \leq \mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(1;ð,q)^{\frac{1}{2}}\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}((t_ð)^2;ð,q)^{\frac{1}{2}} = \mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}((t_ð)^2;ð,q)^{\frac{1}{2}}.$$

Thus, we have

$$\begin{aligned}|\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(f;ð,q)-f(ð)| &\leq f'(ð)\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}((t_ð)^2;ð,q)\\ &\quad+\left\{\frac{1}{\delta}\sqrt{\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}((t_ð)^2;ð,q)}+1\right\}\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}((t_ð)^2;ð,q)^{\frac{1}{2}}\omega_1^{\gamma}(f',\delta),\end{aligned}$$

The desired results are obtained by using $j=\sqrt{\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}((t_ð)^2;ð,q)}$. □

We then utilize the Lipschitz-type maximum function to estimate the local direct approximation, which we remember from [31].

$$Lip_M(\kappa) := \left\{g \in C\mathfrak{R} : |g(t)-g(ð)| \leq M\frac{|t_ð|^{\kappa}}{(\beta_1 ð^2+\beta_2 ð+t)^{\frac{\kappa}{2}}};\ ð,t\in\mathfrak{R}\right\}$$

This is where $M>0$ and $\beta_1 \geq 0,\ \beta_2>0,\ \kappa\in(0,1]$ be any constants (see [31]).

Theorem 3.8. *For all $\kappa\in(0,1]$ and $0<q<1$, if $g\in Lip_M(\kappa)$, we get*

$$|\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(g;ð,q)-g(ð)| \leq M\sqrt{\frac{[\Phi_m^{\wp_1,\wp_2}(\lambda;q,ð)]^{\kappa}}{(\beta_1ð^2+\beta_2ð)^{\kappa}}}.$$

Proof. Take any $\kappa\in(0,1]$ and take $g\in Lip_M(\kappa)$. Initially, we wish to demonstrate that our result holds true for $\kappa=1$. For each $g\in Lip_M(1)$, we therefore have

$$\begin{aligned}|\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(g;ð,q)-g(ð)| &\leq |\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(|g(t)-g(ð)|;ð,q)|+g(ð)\,|\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(1;ð,q)-1|\\ &\leq \frac{1}{(\mathcal{Z})_q^m}\sum_{j=0}^{m}\tilde{b}_{m,j}^{\wp_1,\wp_2}(ð;q,\lambda)\Big|g(t)-g(ð)\Big|\\ &\leq M\frac{1}{(\mathcal{Z})_q^m}\sum_{j=0}^{m}\frac{\tilde{b}_{m,j}^{\wp_1,\wp_2}(ð;q,\lambda)|t_ð|}{(\beta_1ð^2+\beta_2ð+t)^{\frac{1}{2}}}.\end{aligned}$$

By using

$$(\beta_1ð^2+\beta_2ð+t)^{-1/2} \leq (\beta_1ð^2+\beta_2ð)^{-1/2} \quad (\beta_1\geq 0,\beta_2>0)$$

and we observe through the Cauchy-Schwarz inequality

$$\begin{aligned}|\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(g;ð,q)-g(x)| &\leq M(\beta_1ð^2+\beta_2ð)^{-1/2}\frac{1}{(\mathcal{Z})_q^m}\sum_{j=0}^{m}\tilde{b}_{m,j}^{\wp_1,\wp_2}(ð;q,\lambda)\Big|t_ð\Big|\\ &= M(\beta_1ð^2+\beta_2ð)^{-1/2}|\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(t_ð;ð,q)|\\ &\leq M(\beta_1ð^2+\beta_2ð)^{-1/2}\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(|t_ð|;ð,q)\\ &\leq M\left(\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}((t_ð)^2;ð,q)\right)^{\frac{1}{2}}(\beta_1ð^2+\beta_2ð)^{-1/2}.\end{aligned}$$

Therefore, for $\kappa=1$, our result is correct. Additionally, the required assertion is also satisfied for $\kappa\in(0,1]$ by applying the monotonicity property to operators $\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(g;ð,q)$ and introducing the Hölder's property.

$$\Big|\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(g;ð,q)-g(ð)\Big| \quad\leq\quad \frac{1}{(\mathcal{Z})_q^m}\sum_{j=0}^{m}\tilde{b}_{m,j}^{\wp_1,\wp_2}(ð;q,\lambda)\Big|g(t)-g(ð)\Big|$$

$$\le \left(\frac{1}{(\mathcal{Z})_q^m} \sum_{j=0}^{m} \tilde{b}_{m,j}^{\wp_1,\wp_2}(ð; q, \lambda) \Big| g(t) - g(ð) \Big| \right)^{\frac{\kappa}{2}} \left(\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(1; ð, q) \right)^{\frac{2-\kappa}{2}}$$

$$\le M \Bigg(\frac{1}{(\mathcal{Z})_q^m} \frac{\sum_{j=0}^{m} \tilde{b}_{m,j}^{\wp_1,\wp_2}(ð; q, \lambda) \Big(t_ð \Big)^2}{\beta_1 ð^2 + \beta_2 ð + t} \Bigg)^{\frac{\kappa}{2}}$$

$$\le M \Bigg\{ \frac{1}{(\mathcal{Z})_q^m} \sum_{j=0}^{m} \tilde{b}_{m,j}^{\wp_1,\wp_2}(ð; q, \lambda) \Big(t_ð \Big)^2 \Bigg\}^{\frac{\kappa}{2}} (\beta_1 ð^2 + \beta_2 ð + t)^{-\frac{\kappa}{2}}$$

$$\le M(\beta_1 ð^2 + \beta_2 ð)^{-\kappa/2} \left[\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2} \Big((t_ð)^2; ð, q \Big) \right]^{\frac{\kappa}{2}}$$

$$= M \sqrt{\frac{[\Phi_m^{\wp_1,\wp_2}(\lambda; q, ð)]^\kappa}{(\beta_1 ð^2 + \beta_2 ð)^\kappa}}.$$

□

4. Voronovskaja type asymptotic formula

We study the quantitative Voronovskaja-type approximation theorem and derive the Voronovskaja-type approximation properties for our novel operators $\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}$, based on the paper [4, 27]. This is made possible by using the definition of the modulus of smoothness that was covered in the previous section. This smoothness modulus can be explained as

$$\omega_\Xi(\zeta, \delta) := \sup_{0<|\rho|\le\delta} \left\{ \left| f\Big(ð + \frac{\rho \Xi(ð)}{2} \Big) - \zeta\Big(ð - \frac{\rho \Xi(ð)}{2} \Big) \right|, ð \pm \frac{\rho \Xi(ð)}{2} \in \mathfrak{R} \right\}.$$

$\zeta \in C[0,1]$ and $\Xi(ð) = (ð - ð^2)^{1/2}$ are used here, along with the associated Peetre's K-functional, which is provided by:

$$K_\Xi(\zeta, j) = \inf_{f \in \omega_\Xi \mathfrak{R}} \Big\{ \|f - \zeta\| + j \|\Xi f'\| : f' \in C\mathfrak{R}, j > 0 \Big\},$$

where $\omega_\Xi \mathfrak{R} = \{ f : f' \in AC^* \mathfrak{R}, \ \|\Xi f'\| < \infty \}$ and $AC^* \mathfrak{R}$ denote fully absolutely continuous functions on the intervals $[a, b] \subset \mathfrak{R}$. There is a positive constant M such that

$$K_\Xi(f, \delta) \le M \, \omega_\Xi(f, \delta).$$

Theorem 4.1. *For all $\zeta, \zeta', \zeta'' \in C\mathfrak{R}$, it verify that*

$$\left| \mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(\zeta; ð, q) - \zeta(ð) - \psi_{m,\lambda}^{\wp_1,\wp_2}(ð)\zeta'(ð) - \frac{\delta_{m,\lambda}^{\wp_1,\wp_2}(ð) + 1}{2} \zeta''(ð) \right| \le M \frac{\Xi^2(ð)}{m} \omega_\Xi\Big(\zeta'', \frac{1}{\sqrt{m}} \Big),$$

where $C > 0$ is a constant, $ð \in \mathfrak{R}$ and $\Psi_m^{\wp_1,\wp_2}(\lambda; q, ð) = \mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(t_ð; ð, q)$, and $\delta_{m,\lambda}^{\wp_1,\wp_2}(ð) = \mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}\Big((t_ð)^2; ð, q \Big)$.

Proof. Assuming $\zeta \in C\mathfrak{R}$, we analyze the following outcomes from the Taylor series expansion:

$$\zeta(t) - \zeta(ð) - \zeta'(ð)(t_ð) = \int_ð^t \zeta''(\theta)(t - \theta) \mathrm{d}_q \theta,$$

then it is easy to get

$$\zeta(t) - \zeta(ð) - (t_ð)\zeta'(ð) - \frac{\zeta''(ð)}{2}\Big((t_ð)^2 + 1 \Big) \le \int_ð^t (t - \theta)[\zeta''(\theta) - \zeta''(ð)] \mathrm{d}_q \theta. \tag{16}$$

Consequently, (16) provides us with

$$\left|\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(\zeta;\eth,q)-\zeta(\eth)-\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(t_\eth;\eth,q)\zeta'(\eth)-\frac{\zeta''(\eth)}{2}\Big(\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}((t_\eth)^2;\eth,q)+\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(1;\eth,q)\Big)\right|$$
$$\leq \mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}\Big(\Big|\int_\eth^t|\zeta''(\theta)|t-\theta|-\zeta''(\eth)|\,\mathrm{d}_q\theta\Big|;\eth,q\Big). \tag{17}$$

Based on the right part of equality (17), we can estimate the following:

$$\left|\int_\eth^t|t-\theta|\,|\zeta''(\theta)-\zeta''(\eth)|\,\mathrm{d}_q\theta\right|\leq 2\|\zeta''-f\|(t_\eth)^2+2\|\Xi f'\|\Xi^{-1}(\eth)|t_\eth|^3, \tag{18}$$

where $\zeta\in\omega_\Xi\mathfrak{R}$. Assume there is a positive constant M such that

$$\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}((t_\eth)^2;\eth,q)\leq\frac{M}{2[m]_q}\Xi^2(\eth)\qquad\text{and}\qquad\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}((t_\eth)^4;\eth,q)\leq\frac{M}{2[m]_q^2}\Xi^4(\eth). \tag{19}$$

By using the Cauchy-Schwarz inequality, we may ascertain that

$$\left|\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(\zeta;\eth,q)-\zeta(\eth)-\zeta'(\eth)\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(t_\eth;\eth,q)-\frac{\zeta''(\eth)}{2}\Big(\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}((t_\eth)^2;\eth,q)+\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(1;\eth,q)\Big)\right|$$
$$\leq 2\|\zeta''-f\|\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}((t_\eth)^2;\eth,q)+2\|\Xi(\eth)f'\|\Xi^{-1}(\eth)\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(|t_\eth|^3;\eth,q)$$
$$\leq\frac{M}{[m]_q}\Xi^2(\eth)\|\zeta''-f\|+2\|\Xi(\eth)f'\|\Xi^{-1}(\eth)\{\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}((t_\eth)^2;\eth,q)\}^{1/2}\{\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}((t_\eth)^4;\eth,q)\}^{1/2}$$
$$\leq M\frac{\Xi^2(\eth)}{[m]_q}\Big\{\|\zeta''-f\|+[m]_q^{-1/2}\|\Xi(\eth)f'\|\Big\}.$$

By taking the infimum over all $f\in\omega_\Xi\mathfrak{R}$, we may ascertain that.

$$\left|\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(\zeta;\eth,q)-\zeta(\eth)-\Psi_m^{\wp_1,\wp_2}(\lambda;q,\eth)\zeta'(\eth)-\frac{\delta_{m,\lambda}^{\wp_1,\wp_2}(\eth)+1}{2}\zeta''(\eth)\right|\leq\frac{M}{[m]_q}\Xi^2(\eth)\omega_\Xi\Big(\zeta'',\frac{1}{\sqrt{[m]_q}}\Big),$$

which brings the proof. □

Theorem 4.2. *Considering any* $\zeta\in C_B[0,1]$, *we get*

$$\lim_{m\to\infty}[m]_q\left[\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(\zeta;\eth,q)-\zeta(\eth)-\Psi_m^{\wp_1,\wp_2}(\lambda;q,\eth)\zeta'(\eth)-\frac{\delta_{m,\lambda}^{\wp_1,\wp_2}(\eth)}{2}\zeta''(\eth)\right]=0.$$

Proof. The following can be written using Taylor's series expansion if $\zeta\in C_B\mathfrak{R}$:

$$\zeta(t)=\zeta(\eth)+(t_\eth)\zeta'(\eth)+\frac{1}{2}(t_\eth)^2\zeta''(\eth)+(t_\eth)^2\,Q_\eth(t). \tag{20}$$

Additionally, $t\to\eth$ is equal to $Q_\eth(t)\to 0$, where $Q_\eth(t)\in C\mathfrak{R}$ and defined for the Peano form of remainder. The equality (20) can be easily seen by using the operators $\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(\cdot;\eth,q)$.

$$\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(\zeta;\eth,q)-\zeta(\eth)=\zeta'(\eth)\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(t_\eth;\eth,q)+\frac{\zeta''(\eth)}{2}\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}((t_\eth)^2;\eth,q)+\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}((t_\eth)^2Q_\eth(t);\eth,q).$$

Cauchy-Schwarz inequality gives us

$$\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}((t_\eth)^2Q_\eth(t);\eth,q)\leq\sqrt{\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(Q_\eth^2(t);\eth,q)}\sqrt{\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}((t_\eth)^4;\eth,q)}. \tag{21}$$

Here, we can plainly see that $q \to 1$ and $\lim_{m\to\infty} \mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(Q_{\eth}^2(t);\eth,q) = 0$

$$\lim_{m\to\infty}[m]_q\{\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}((t_{\eth})^2Q_{\eth}(t);\eth,q)\} = 0.$$

Thus, we have

$$\begin{aligned}\lim_{m\to\infty}[m]_q\{\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(\zeta;\eth,q)-\zeta(\eth)\} = \lim_{m\to\infty}[m]_q\Big\{&\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}(t_{\eth};\eth,q)\zeta'(\eth)+\frac{\zeta''(\eth)}{2}\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}((t_{\eth})^2;\eth,q)\\ &+\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}((t_{\eth})^2Q_{\eth}(t);\eth,q)\Big\}.\end{aligned}$$

□

5. Graphical examples

In this section, we will present numerical examples using MATLAB, accompanied by illustrative graphics.

Example 5.1. *The formula $g(\eth) = (\eth - \frac{4}{9})(\eth - \frac{2}{5})$ can be examined for $\chi_1 = 3$, $\chi_2 = 4$, $s \in \{11, 25, 91\}$, and $\lambda = 2.5$. The convergence of the operator towards $g(\eth)$ is shown in Figure 1 (here, we denote $s = m$, $\chi_1 = \wp_1$, $\chi_2 = \wp_2$).*

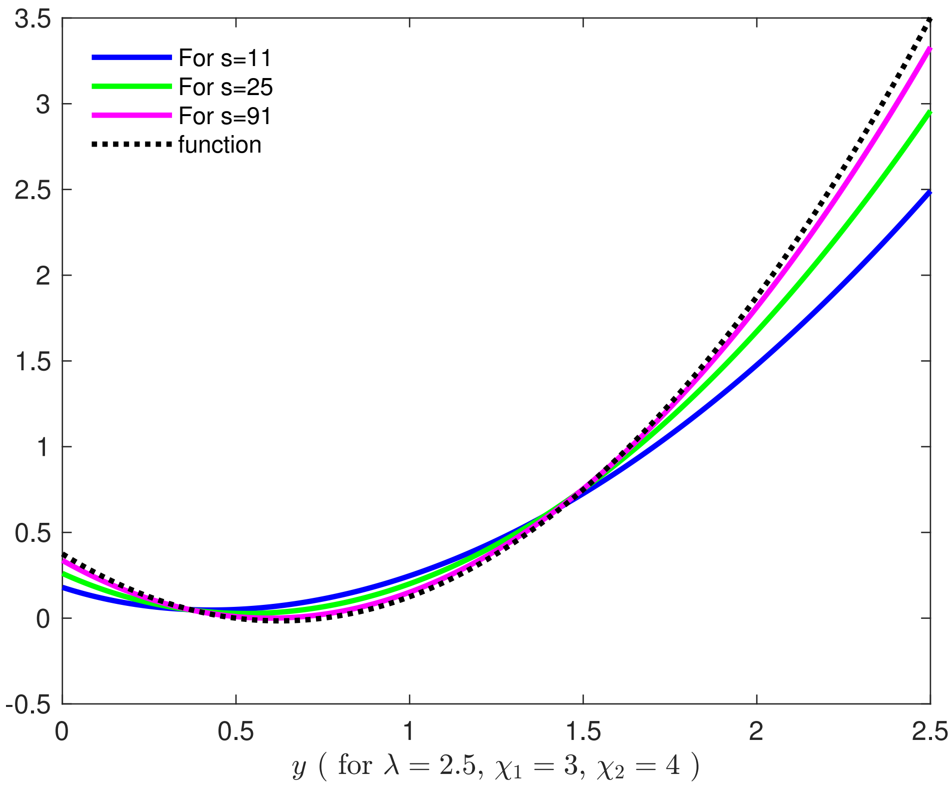

Figure 1: Convergence of the operator towards $g(\eth) = (\eth - \frac{4}{9})(\eth - \frac{2}{5})$

Example 5.2. *Suppose that $g(\eth) = \eth^2 + \frac{20}{6}$, $s \in \{18, 34, 80\}$, and $\chi_1 = 4$, $\chi_2 = 5.5$ and $\lambda = 0.7$. The operator's convergence to the function $g(\eth)$ is shown in Figure 2 (here, we denote $s = m$, $\chi_1 = \wp_1$, $\chi_2 = \wp_2$).*

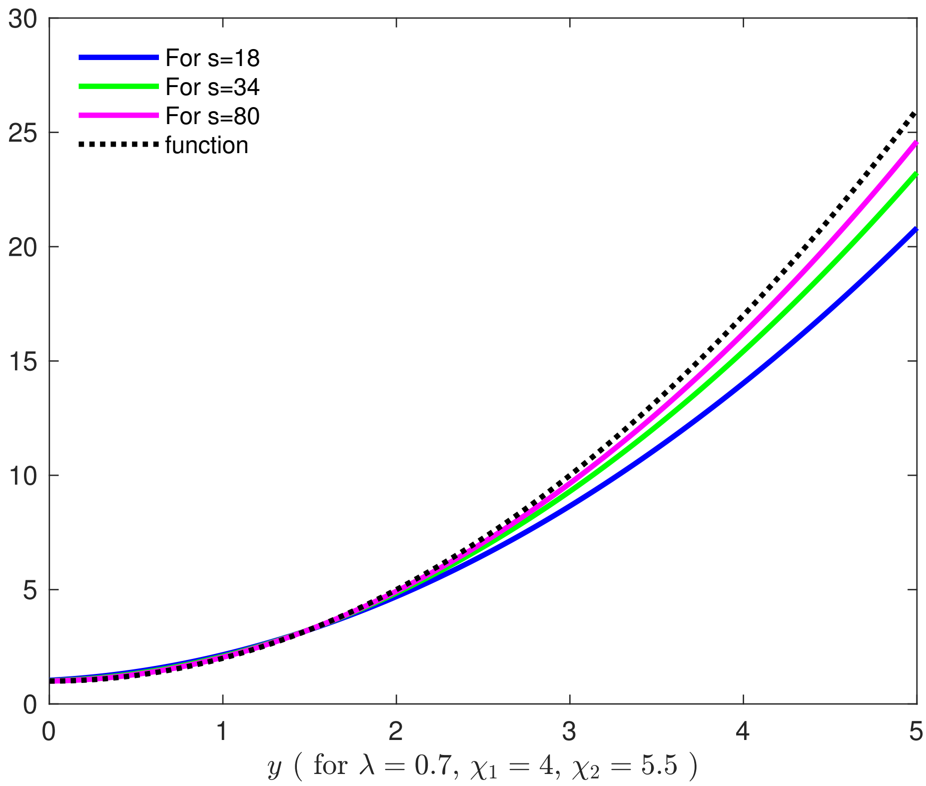

Figure 2: Operator's convergence with respect to the function $g(ð) = ð^2 + \frac{20}{6}$

These examples demonstrate how the operators' approximation of the function improves when we take larger values of m.

Example 5.3. *Consider the function $f(ð) = ð^2$ evaluated at $ð = 1$. We use the Kantorovich form of q-Bernstein shifted knots operators $\mathcal{BK}_{m,\lambda}^{\wp_1,\wp_2}$ defined in (2.1) of the paper with parameters $\lambda = 0$, $\wp_1 = 0$, $\wp_2 = 0$, and $q_m = 1 - \frac{1}{m}$. The approximation is given by:*

$$\mathcal{BK}_{m,0}^{0,0}(f;ð,q) = \frac{[m]_q^2}{[m+1]_q^2}\left(ð - \frac{1}{[m]_q}\right)^2 \frac{(2q+1)[m]_q}{[3]_q[m+1]_q}\left(ð - \frac{1}{[m]_q}\right) + \frac{1}{[3]_q[m+1]_q^2},$$

where we take $ð = 1$, $[m]_q = \frac{1-q^m}{1-q}$, $[2]_q = 1+q = -\frac{1}{m}$ *and* $[3]_q = 3 - \frac{3}{m} + \frac{1}{m^2}$. *The absolute error is* $|1 - \mathcal{BK}_{m,0}^{0,0}(f;1,q)|$.

Table 1: Approximation of $f(1) = 1$ by $\mathcal{BK}_{m,0}^{0,0}(f;1,q)$

m	$q_m = 1 - \frac{1}{m}$	$[m]_q$	$\mathcal{BK}_{m,0}^{0,0}(f;1,q)$	Absolute Error
1	0.0000	1.0000	1.0000	0.0000
5	0.4000	1.4000	0.5063	0.2937
10	0.7000	2.3516	0.7613	0.1377
15	0.8000	5.6532	0.8695	0.0605
20	0.9500	11.8303	0.9886	0.0414
50	0.9900	35.8916	0.9992	0.0017

Observations:

- *As m increases, $q_m \to 1$ and $[m]_q \to \infty$.*
- *The approximation $\mathcal{BK}^{0,0}_{m,0}(f; 1, q)$ converges to $f(1) = 1$.*
- *The absolute error decreases monotonically, confirming theoretical convergence.*

6. Conclusion & Observation

Our new operators (8) are the Kantorovich form of operators [25], as we explicitly conclude in this paper. According to [3], our operators $\mathcal{BK}^{\wp_1,\wp_2}_{m,\lambda}(f; \eth, q)$ have approximation properties to the operators(2) for $q = 1$ in the equality (8).Our operators $\mathcal{BK}^{\wp_1,\wp_2}_{m,\lambda}(f; \eth, q)$ reduced to the approximation results of operators(6) defined by Cai et al. [7] for $q = 1$ and $\wp_1 = \wp_2 = 0$ in the equality (8). Consequently, we can characterize our operators (8) as unique instances of traditional Bernstein-operators [5], Lupaş q-Bernstein-operators [20], λ-Bernstein operators [7], (λ, q)-Bernstein operators [6] and (λ, q)-Bernstein operators with Bézier basis [3].

Declarations

- **Conflicts of Interest.** The authors declare that they have no conflicts of interest.
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