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Some characterizations of normal curves under isometry using Darboux frame

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Abstract. This paper deals with the study of characterizations of a normal curve in 3-dimensional Euclidean space and presents a sufficient condition for the invariance of a normal curve on regular surfaces under isometric transformations using the Darboux frame $\{T_1, P_1, U_1\}$. Furthermore, we compute the deviations of the position vector of the normal curve on regular surfaces along the unit tangent vector T_1, the unit normal U_1 perpendicular to the oriented surface, and the cross product of U_1 and T_1 due to isometric transformations.

1. Introduction

Differential geometry is a branch of mathematics that plays an important role in the study of curves and surfaces. When studying curves in three dimensional Euclidean space $\mathbb{E}^3$, a connection can be established between every point on the curve and a moving set of three perpendicular axes known as the Serret-Frenet frame, represented as $\{T_1, N_1, B_1\}$. This frame consists of the tangent vector T_1, the binormal B_1, and the principal normal N_1. At each point on the curve, this frame generates three perpendicular planes, namely, the rectifying plane, the normal plane, and the osculating plane. These three planes are formed by using the vectors $\{B_1, T_1\}$, $\{N_1, B_1\}$, and $\{T_1, N_1\}$, respectively. The curves that lie within these planes are known as rectifying curves, normal curves, and osculating curves, respectively. The importance of these concepts in the field of differential geometry of curves and surfaces is thoroughly explained in [1–4].

In $\mathbb{E}^3$, a lot of research has been conducted on characterizing curves by constraining their position vectors within specific planes associated with surfaces. For example, Chen [5] introduced the concept of a rectifying curve, as a spatial curve whose position vector consistently stays within its corresponding rectifying plane. Furthermore, Chen [6], and Chen and Dillen [7] conducted a thorough investigation of rectifying curves, exploring their dynamic behaviour and examining various attributes associated with

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these curves. They further studied a rectifying curve as an extremal curve by analyzing the dilation of unit speed parameterized curve on the unit sphere S^2 in $\mathbb{E}^3$. Ilarslan and Nesovic [8, 9] introduced the concept of timelike and null normal and rectifying curves in Minkowski 3-space.

Camci et al. [10] investigated the characterizations of a surface curve by restricting its position vector to three perpendicular planes on the surface and established the existence of such a curve. Motivated by Chen [5], Ilarslan and Nesovic [11, 12] researched rectifying curves and osculating curves, and investigated their characterizations in $\mathbb{E}^3$. Likewise, Deshmukh et al. [13] studied the characterizations of rectifying curves through centrodes of unit speed curves in Euclidean spaces. For recent work related to these types of curves, where the position vectors are restricted to the normal, rectifying and osculating planes on surfaces under isometric and conformal transformations, we can refer to [14, 15, 17, 18].

The investigation of osculating curves, normal curves, and rectifying curves within $\mathbb{E}^3$ has been studied by various researchers [11, 16, 17]. The authors of [11], investigated some specific characteristics of osculating curves within $\mathbb{E}^3$ and defined osculating curves in $\mathbb{E}^4$. In $\mathbb{E}^4$, they defined the position vector for osculating curves which are consistently confined to their corresponding osculating plane, satisfying the equation $\beta(r) = \mu_1 T_1 + \mu_2 N_1 + \mu_3 B_2$, where μ_1, μ_2, and μ_3 are smooth functions. Lone [16] investigated some geometric properties of normal curves that are invariant under conformal transformations using the Frenet frame. Shaikh et al. [17] investigated a sufficient condition for normal curves on isometric smooth surfaces with the help of the Frenet frame.

Furthermore, the authors in [12] introduced a novel method for defining rectifying curves in $\mathbb{E}^4$. They characterized rectifying curves through a profound geometric property; the position vector of the curve always resides in the orthogonal complement $N_1^{\perp}$ of its principal normal vector field N_1. This approach enhances our comprehension of rectifying curves by situating them in a higher-dimensional space and underscoring their intrinsic relationship with the orthogonal aspects of their principal normal vectors.

The work presented in [14–19] primarily focuses on normal, rectifying, and osculating curves on smooth immersed surfaces. The researchers conducted an in-depth analysis to understand the conditions under which such curves maintain their original characteristics when the surface undergoes isometric changes. The Frenet frame serves as the analytical framework for this investigation. On a related note, when examining a space curve lying on a regular surface within three-dimensional space, a dynamic orthonormal frame known as the Darboux frame $\{T_1, P_1, U_1\}$ naturally emerges at each point along the curve. In this frame, T_1 represents the unit tangent vector at a specific point on the curve, U_1 stands for the unit surface normal, and P_1 is the result of the cross product between U_1 and T_1. This Darboux frame provides valuable insights into the geometric properties of the curve in relation to the underlying surface.

The authors of [8, 10] used the Darboux frame to determine the position vector of a curve with unit speed parameterization in $\mathbb{E}^3$. This position vector consistently lies within the planes defined by $\{T_1, U_1\}$, $\{T_1, P_1\}$, and $\{P_1, U_1\}$. To gain deeper insights, let's explore the concept of a normal curve lying on a regular surface within $\mathbb{E}^3$ with the help of the Darboux frame rather than the Frenet frame.

The motivation for the study and its findings are quite interesting because the study integrates the characteristics of normal curves on regular immersed surfaces under isometry using the Darboux frame instead of the Frenet frame. The main objective of this study is to obtain sufficient conditions for the normal curve to remain unchanged by using the Darboux frame. By employing these obtained conditions, we compute the deviations of the position vector of the normal curve along the unit tangent vector T_1, the unit normal U_1 oriented perpendicular to the surface, and P_1 (the cross product of U_1 and T_1) due to isometric transformations.

This paper is structured as follows. In section 2, we discuss some fundamental concepts and definitions related to normal curves, the Frenet frame and the Darboux frame. In Section 3, we obtain sufficient conditions for the invariance of a normal curve under isometric transformations. Section 4 provides the conclusion and outlines the future scope of this study.

2. Preliminaries

In this section, we present preliminary concepts, definitions and notations used in this sequel.
Consider two regular surfaces, M and $\tilde{M}$, immersed in $\mathbb{E}^3$. Let $\beta : I \to \mathbb{E}^3$ be a unit-speed parameterized curve, where $I = (a,b) \subset \mathbb{R}$, having at least fourth order continuous derivatives. The tangent vector T_1 can be expressed as $T_1(r) = \beta'(r)$ for each $r \in I$, where β' represents the derivative of β with respect to the arc length parameter r. The principal normal vector to the curve β is denoted by N_1. Consequently, the binormal vector B_1 is derived as the cross product of T_1 and N_1, i.e., $B_1 = T_1 \times N_1$. In [1, 5], the Serret-Frenet equations can be written as follows

$$\begin{aligned} T_1'(r) &= \kappa_1(r)N_1(r), \\ N_1'(r) &= -\kappa_1(r)T_1(r) + \tau_1(r)B_1(r), \\ B_1'(r) &= -\tau_1(r)N_1(r), \end{aligned}$$

where, κ_1 and τ_1 represent the curvature function and torsion function of the curve β, respectively. These functions satisfy the following conditions

$$T_1(r) = \beta'(\mathrm{r}),\ N_1(r) = \tfrac{T_1'(r)}{\kappa_1(r)},\ B_1(r) = T_1(r) \times N_1(r).$$

Starting from any point $\beta(r)$ along the curve β, the plane formed by $\{T_1, N_1\}$ is known as the osculating plane, and the plane spanned by $\{T_1, B_1\}$ is called the rectifying plane. Similarly, the plane spanned by $\{N_1, B_1\}$ is referred to as the normal plane [6, 7]. In the analysis of a curve's position vector, which distinguishes between different types of curves [3, 8, 9], a curve is categorized as a normal curve if its position vector lies within the normal plane. Similarly, a curve is referred to as a rectifying curve if its position vector lies within the rectifying plane. Furthermore, when a curve's position vector is situated in the osculating plane, it is identified as an osculating curve.

Let M be a regular surface immersed in $\mathbb{E}^3$, with $\sigma : V \subset \mathbb{R}^2 \to M$ representing the coordinate map. Suppose $\beta : I \to M \in \mathbb{R}^3$ is a parameterized curve situated within the range of the surface patch σ of M. Then we can write $\beta(r)$ as

$$\beta(r) = \sigma(x(r), y(r)), \qquad \forall r \in I. \tag{1}$$

When the curve $\beta(r)$ lies on the surface M, a unique moving orthonormal frame, referred to as the Darboux frame $\{T_1, P_1, U_1\}$, can be associated with every point along the curve $\beta(r)$. It's important to note that the unit tangent vector T_1 remains consistent in both the Darboux frame and the Frenet frame, and the vectors N_1, B_1, P_1, U_1 all lie within the same plane. As a result, the relationship between these frames is described by the following equation

$$\begin{bmatrix} T_1 \\ P_1 \\ U_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} T_1 \\ N_1 \\ B_1 \end{bmatrix}, \tag{2}$$

where α represents the angle between the vectors N_1 and P_1.

Since $\beta(r)$ is a unit-speed parameterized curve on the surface M, we can conclude that β'' is perpendicular to β, forming a linear combination involving U_1 and P_1, where "$\prime$" denotes the derivative with respect to the arc parameter r. Thus, we can express this as

$$\beta''(r) = k_n(r)U_1(r) + k_g(r)P_1(r). \tag{3}$$

Here, k_n and k_g represent the normal curvature and geodesic curvature of β. Since we know that both vectors U_1 and P_1 are perpendicular, from equation (3) we can derive the following :

$$k_n(r) = \beta''(r) \cdot U_1(r) \quad \text{and} \quad k_g(r) = \beta''(r) \cdot P_1(r). \tag{4}$$

Also, Frenet equation and (4), together yields

$$k_n(r) = \kappa_1(r)N_1(r) \cdot U_1(r) \quad \text{and} \quad k_g(r) = \kappa_1(r)N_1(r) \cdot P_1(r), \tag{5}$$

which implies that

$$k_n(r) = \kappa_1(r) sin\alpha \quad \text{and} \quad k_g(r) = \kappa_1(r) cos\alpha. \tag{6}$$

Consequently, the curve β is a geodesic if and only if the curvature k_g is zero, and similarly, β is an asymptotic curve if and only if the curvature k_n is zero.

Taking the derivative of equation (1) with respect to r, we obtain

$$T_1(r) = \beta'(r) = x'\sigma_x + y'\sigma_y, \tag{7}$$

where $x' = \frac{dx}{dr}$, $y' = \frac{dy}{dr}$ and $\beta' = \frac{d\beta}{dr}$.
Now, the normal U_1 to the surface M is defined as follows

$$U_1(r) = \frac{\sigma_x \times \sigma_y}{\|\sigma_x \times \sigma_y\|} = \frac{\sigma_x \times \sigma_y}{\sqrt{EG - F^2}}, \tag{8}$$

where E, F and G are the coefficients of the first fundamental form of M. It is given that $P_1 = U_1 \times T_1$, by using equations (7) and (8), and after simplification, we obtain

$$P_1(r) = \frac{1}{\sqrt{EG - F^2}}\Big(Ex'\sigma_y + F(y'\sigma_y - x'\sigma_x) - Gy'\sigma_x\Big). \tag{9}$$

Definition 2.1. *If M and $\tilde{M}$ are regular surfaces immersed in $\mathbb{E}^3$, then a diffeomorphism $J : M \to \tilde{M}$ is called an isometry if it preserves the lengths of curves.*

Definition 2.2. *The quadratic form that represents the first fundamental form of a regular immersed surface M at a point q is denoted as $J_q : T_q(M) \to \mathbb{R}$, and can be written as*

$$J_q(\beta'(r)) = \langle \beta'(r), \beta'(r) \rangle = Ex'^2 + 2Fx'y' + Gy'^2.$$

where $'$ represents the derivative with respect to the arc length parameter.

Let $R \subset M$ be a neighborhood of a point $q \in M$ and $S \subset \tilde{M}$ be a neighborhood of the point $J_q \in \tilde{M}$ such that $J_q : R \to S$ is an isometry. Then the mapping $J_q : R \to \tilde{M}$ is called a local isometry at q. The surfaces M and $\tilde{M}$ are said to be locally isometric if there exists an isometry at each point of M and J_q is said to be global isometry if it is local isometry at every point of M.

We note that under isometry between two surfaces, geodesics remains invariant. If E, F, G and $\tilde{E}$, $\tilde{F}$, $\tilde{G}$ are the coefficients of first fundamental form of M and $\tilde{M}$ respectively, then M and $\tilde{M}$ are isometric iff

$$\tilde{E} = E, \; \tilde{F} = F \text{ and } \; \tilde{G} = G. \tag{10}$$

3. Normal Curves in the Context of the Darboux Frame

This section involves the study of normal curves on a regular surface using the Darboux frame. A curve $\beta(r)$ on a regular surface M in $\mathbb{E}^3$ is classified as a normal curve when its position vector consistently lies within the normal plane. Consequently, the position vector of β can be expressed as follows

$$\beta(r) = \mu_1(r)N_1(r) + \mu_2(r)B_1(r), \tag{11}$$

where $\mu_1(r)$ and $\mu_2(r)$ are smooth functions. Using equation (2) in (11), we get

$$\beta(r) \;=\; \mu_1(r)P_1(r)cos\alpha - \mu_1(r)U_1(r)sin\alpha + \mu_2(r)P_1(r)sin\alpha + \mu_2(r)U_1(r)cos\alpha. \tag{12}$$

Thus, in view of equations (6), (8), (9) and (12), we obtain

$$\begin{aligned} \beta(r) &= \left\{\mu_1(r)\frac{k_g(r)}{k_1(r)} + \mu_2(r)\frac{k_n(r)}{k_1(r)}\right\} P_1(r) + \left\{\mu_2(r)\frac{k_g(r)}{k_1(r)} - \mu_1(r)\frac{k_n(r)}{k_1(r)}\right\} U_1(r), \\ &= \left\{\mu_1(r)\frac{k_g(r)}{k_1(r)} + \mu_2(r)\frac{k_n(r)}{k_1(r)}\right\} \cdot \frac{1}{\sqrt{EG-F^2}}\Big(Ex'\sigma_y + F(y'\sigma_y - x'\sigma_x) - Gy'\sigma_x\Big) \\ &\quad + \left\{\mu_2(r)\frac{k_g(r)}{k_1(r)} - \mu_1(r)\frac{k_n(r)}{k_1(r)}\right\} \frac{\sigma_x \times \sigma_y}{\sqrt{EG-F^2}}. \end{aligned} \tag{13}$$

This equation represents the fundamental basis of the definition of a normal curve on a regular surface, one that does not fall into the category of being either a geodesic or an asymptotic curve on the surface. Here, we consider two distinct cases, as follows:

Case 1: When the normal curve on the regular surface takes the form of a geodesic curve (i.e., $k_g(r) = 0$), the angle becomes $\pi/2$. Additionally, when $k_n(r) = \kappa_1(r)$, the equation for normal curves can be written as

$$\beta(r) = \frac{\mu_2(r)}{\sqrt{EG-F^2}}\Big(Ex'\sigma_y + F(y'\sigma_y - x'\sigma_x) - Gy'\sigma_x\Big) - \mu_1(r)\frac{\sigma_x \times \sigma_y}{\sqrt{EG-F^2}}. \tag{14}$$

Case 2: When the normal curve on the regular surface takes the form of an asymptotic curve (i.e., $k_n(r) = 0$), the angle becomes 0. Furthermore, when $k_g(r) = \kappa_1(r)$, the equation for normal curves takes the following form:

$$\beta(r) = \frac{\mu_1(r)}{\sqrt{EG-F^2}}\Big(Ex'\sigma_y + F(y'\sigma_y - x'\sigma_x) - Gy'\sigma_x\Big) + \mu_2(r)\frac{\sigma_x \times \sigma_y}{\sqrt{EG-F^2}}. \tag{15}$$

Theorem 3.1. *Let M and $\tilde{M}$ be two regular surfaces and $J : M \to \tilde{M}$ be an isometry. If $\beta(r)$ is a normal curve on M such that $k_n \neq 0$, then $\tilde{\beta} = J \circ \beta$ is a normal curve on $\tilde{M}$, provided any one of the following conditions are hold:*

(i) $\tilde{\beta}$ represents a geodesic curve on $\tilde{M}$ and $\tilde{\beta}(r) = J_*(\beta(r))$,

(ii) $\tilde{\beta}$ represents an asymptotic curve on $\tilde{M}$ and

$$\tilde{\beta}(r) = J_*(\beta(r)) - \frac{\mu_2(r)k_n(r)}{\kappa_1(r)}\tilde{P}_1(r) + \frac{\mu_1(r)k_n(r)}{\kappa_1(r)}\tilde{U}_1(r),$$

(iii) $\tilde{\beta}$ neither represents a geodesic curve nor an asymptotic curve on $\tilde{M}$ and $\tilde{\beta}(r) = J_*(\beta(r))$.

Proof. It is given that $J : M \to \tilde{M}$ is an isometry, and $\beta(r)$ represents a normal curve on M such that $k_n \neq 0$. Assume that condition (i) holds. Then by definition of geodesic curve, we have $\tilde{k}_g(r) = 0$, and $\tilde{\beta}(r) = J_*(\beta(r))$. Now, we can write

$$\begin{aligned} \tilde{\beta}(r) &= J_*(\beta(r)), \\ &= \left\{\mu_1(r)\frac{k_g(r)}{k_1(r)} + \mu_2(r)\frac{k_n(r)}{k_1(r)}\right\} \frac{1}{\sqrt{EG-F^2}}\Big(Ex'J_*\sigma_y + F(y'J_*\sigma_y - x'J_*\sigma_x) - Gy'J_*\sigma_x\Big) \\ &\quad + \left\{\mu_2(r)\frac{k_g(r)}{k_1(r)} - \mu_1(r)\frac{k_n(r)}{k_1(r)}\right\} (J_*U_1(r)). \end{aligned} \tag{16}$$

Now, from equation (10) and (16), we find that

$$\begin{aligned}\tilde{\beta}(r) &= \left\{\tilde{\mu}_1(r)\frac{\tilde{k}_g(r)}{\tilde{k}_1(r)} + \tilde{\mu}_2(r)\frac{\tilde{k}_n(r)}{\tilde{k}_1(r)}\right\}\frac{1}{\sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}}\Big(\tilde{E}x'\tilde{\sigma}_y + \tilde{F}(y'\tilde{\sigma}_y - x'\tilde{\sigma}_x) - \tilde{G}y'\tilde{\sigma}_x\Big)\\ &\quad + \left\{\tilde{\mu}_2(r)\frac{\tilde{k}_g(r)}{\tilde{k}_1(r)} - \tilde{\mu}_1(r)\frac{\tilde{k}_n(r)}{\tilde{k}_1(r)}\right\}(\tilde{U}_1(r)), \qquad (17)\end{aligned}$$

where $\tilde{\mu}_1(r)\frac{\tilde{k}_g(r)}{\tilde{k}_1(r)} = \mu_1(r)\frac{k_g(r)}{k_1(r)}$, $\tilde{\mu}_2(r)\frac{\tilde{k}_n(r)}{\tilde{k}_1(r)} = \mu_2(r)\frac{k_n(r)}{k_1(r)}$, $\tilde{\mu}_2(r)\frac{\tilde{k}_g(r)}{\tilde{k}_1(r)} = \mu_2(r)\frac{k_g(r)}{k_1(r)}$ and $\tilde{\mu}_1(r)\frac{\tilde{k}_n(r)}{\tilde{k}_1(r)} = \mu_1(r)\frac{k_n(r)}{k_1(r)}$.

Since $\tilde{k}_g(r) = 0$ and $\tilde{k}_n(r) = \tilde{k}_1(r)$, from equation (17), we get

$$\tilde{\beta}(r) = \frac{\tilde{\mu}_2(r)}{\sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}}\Big(\tilde{E}x'\tilde{\sigma}_y + \tilde{F}(y'\tilde{\sigma}_y - x'\tilde{\sigma}_x) - \tilde{G}y'\tilde{\sigma}_x\Big) - \tilde{\mu}_1(r)(\tilde{U}_1(r)). \qquad (18)$$

This equation defines a normal curve on the surface $\tilde{M}$ that exhibits geodesic behaviour on the same surface. Assume that condition (ii) holds. Then $\tilde{k}_n(r) = 0$ and $\tilde{\beta}(r) = J_*(\beta(r)) - \frac{\mu_2(r)k_n(r)}{\kappa_1(r)}\tilde{P}_1(r) + \frac{\mu_1(r)k_n(r)}{\kappa_1(r)}\tilde{U}_1(r)$.
Now

$$\begin{aligned}\tilde{\beta}(r) &= J_*(\beta(r)) - \frac{\mu_2(r)k_n(r)}{\kappa_1(r)}\tilde{P}_1(r) + \frac{\mu_1(r)k_n(r)}{\kappa_1(r)}\tilde{U}_1(r),\\ &= \left\{\mu_1(r)\frac{k_g(r)}{k_1(r)} + \mu_2(r)\frac{k_n(r)}{k_1(r)}\right\}\frac{1}{\sqrt{EG-F^2}}\Big(Ex'J_*\sigma_y + F(y'J_*\sigma_y - x'J_*\sigma_x) - Gy'J_*\sigma_x\Big)\\ &\quad + \left\{\mu_2(r)\frac{k_g(r)}{k_1(r)} - \mu_1(r)\frac{k_n(r)}{k_1(r)}\right\}(J_*U_1(r)) - \frac{\mu_2(r)k_n(r)}{\kappa_1(r)}\tilde{P}_1(r) + \frac{\mu_1(r)k_n(r)}{\kappa_1(r)}\tilde{U}_1(r). \qquad (19)\end{aligned}$$

Again, from equation (10) and (19), we find that

$$\begin{aligned}\tilde{\beta}(r) &= \left\{\mu_1(r)\frac{k_g(r)}{k_1(r)} + \mu_2(r)\frac{k_n(r)}{k_1(r)}\right\}\frac{1}{\sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}}\Big(\tilde{E}x'\tilde{\sigma}_y + \tilde{F}(y'\tilde{\sigma}_y - x'\tilde{\sigma}_x) - \tilde{G}y'\tilde{\sigma}_x\Big)\\ &\quad + \left\{\mu_2(r)\frac{k_g(r)}{k_1(r)} - \mu_1(r)\frac{k_n(r)}{k_1(r)}\right\}(\tilde{U}_1(r)) - \frac{\mu_2(r)k_n(r)}{\kappa_1(r)}\tilde{P}_1(r) + \frac{\mu_1(r)k_n(r)}{\kappa_1(r)}\tilde{U}_1(r),\\ &= \left\{\tilde{\mu}_1(r)\frac{\tilde{k}_g(r)}{\tilde{k}_1(r)} + \tilde{\mu}_2(r)\frac{\tilde{k}_n(r)}{\tilde{k}_1(r)}\right\}\frac{1}{\sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}}\Big(\tilde{E}x'\tilde{\sigma}_y + \tilde{F}(y'\tilde{\sigma}_y - x'\tilde{\sigma}_x) - \tilde{G}y'\tilde{\sigma}_x\Big)\\ &\quad + \tilde{\mu}_2(r)\frac{\tilde{k}_g(r)}{\tilde{k}_1(r)}(\tilde{U}_1(r)) - \frac{\tilde{\mu}_2(r)\tilde{k}_n(r)}{\tilde{\kappa}_1(r)}\tilde{P}_1(r), \qquad (20)\end{aligned}$$

where $\tilde{\mu}_1(r)\frac{\tilde{k}_g(r)}{\tilde{k}_1(r)} = \mu_1(r)\frac{k_g(r)}{k_1(r)}$, $\tilde{\mu}_2(r)\frac{\tilde{k}_n(r)}{\tilde{k}_1(r)} = \mu_2(r)\frac{k_n(r)}{k_1(r)}$ and $\tilde{\mu}_2(r)\frac{\tilde{k}_g(r)}{\tilde{k}_1(r)} = \mu_2(r)\frac{k_g(r)}{k_1(r)}$.

Since $\tilde{k}_n(r) = 0$ and $\tilde{k}_g(r) = \tilde{k}_1(r)$, from equation (20), we get

$$\tilde{\beta}(r) = \frac{\tilde{\mu}_1(r)}{\sqrt{\tilde{E}\tilde{G}-\tilde{F}^2}}\Big(\tilde{E}x'\tilde{\sigma}_y + \tilde{F}(y'\tilde{\sigma}_y - x'\tilde{\sigma}_x) - \tilde{G}y'\tilde{\sigma}_x\Big) + \tilde{\mu}_2(r)(\tilde{U}_1(r)).$$

This is the required equation of a normal curve on the surface $\tilde{M}$, which is an asymptotic curve on that surface.

Next, assume that condition (iii) holds. Then $\tilde{k}_n(r) \neq 0$, $\tilde{k}_g(r) \neq 0$ and $\tilde{\beta}(r) = J_*(\beta(r))$.
Now we can write

$$\begin{aligned}
\tilde{\beta}(r) &= J_*(\beta(r)), \\
&= \left\{\mu_1(r)\frac{k_g(r)}{k_1(r)} + \mu_2(r)\frac{k_n(r)}{k_1(r)}\right\} \frac{1}{\sqrt{EG - F^2}}\Big(Ex'J_*\sigma_y + F(y'J_*\sigma_y - x'J_*\sigma_x) - Gy'J_*\sigma_x\Big) \\
&\quad + \left\{\mu_2(r)\frac{k_g(r)}{k_1(r)} - \mu_1(r)\frac{k_n(r)}{k_1(r)}\right\}(J_*U_1(r)). \qquad (21)
\end{aligned}$$

Now, from equation (10) and (21), we obtain

$$\begin{aligned}
\tilde{\beta}(r) &= \left\{\mu_1(r)\frac{k_g(r)}{k_1(r)} + \mu_2(r)\frac{k_n(r)}{k_1(r)}\right\} \frac{1}{\sqrt{\tilde{E}\tilde{G} - \tilde{F}^2}}\Big(\tilde{E}x'\tilde{\sigma}_y + \tilde{F}(y'\tilde{\sigma}_y - x'\tilde{\sigma}_x) - \tilde{G}y'\tilde{\sigma}_x\Big) \\
&\quad + \left\{\mu_2(r)\frac{k_g(r)}{k_1(r)} - \mu_1(r)\frac{k_n(r)}{k_1(r)}\right\}(\tilde{U}_1(r)), \\
&= \left\{\tilde{\mu}_1(r)\frac{\tilde{k}_g(r)}{\tilde{k}_1(r)} + \tilde{\mu}_2(r)\frac{\tilde{k}_n(r)}{\tilde{k}_1(r)}\right\} \frac{1}{\sqrt{\tilde{E}\tilde{G} - \tilde{F}^2}}\Big(\tilde{E}x'\tilde{\sigma}_y + \tilde{F}(y'\tilde{\sigma}_y - x'\tilde{\sigma}_x) - \tilde{G}y'\tilde{\sigma}_x\Big) \\
&\quad + \left\{\tilde{\mu}_2(r)\frac{\tilde{k}_g(r)}{\tilde{k}_1(r)} - \tilde{\mu}_1(r)\frac{\tilde{k}_n(r)}{\tilde{k}_1(r)}\right\}(\tilde{U}_1(r)) \qquad (22)
\end{aligned}$$

where $\tilde{\mu}_1(r)\frac{\tilde{k}_g(r)}{\tilde{k}_1(r)} = \mu_1(r)\frac{k_g(r)}{k_1(r)}$, $\tilde{\mu}_2(r)\frac{\tilde{k}_n(r)}{\tilde{k}_1(r)} = \mu_2(r)\frac{k_n(r)}{k_1(r)}$, $\tilde{\mu}_2(r)\frac{\tilde{k}_g(r)}{\tilde{k}_1(r)} = \mu_2(r)\frac{k_g(r)}{k_1(r)}$ and $\tilde{\mu}_1(r)\frac{\tilde{k}_n(r)}{\tilde{k}_1(r)} = \mu_1(r)\frac{k_n(r)}{k_1(r)}$.

This equation represents the essential form of a normal curve on the surface $\tilde{M}$, which is neither an asymptotic nor a geodesic curve on that surface. □

Theorem 3.2. *Let M and $\tilde{M}$ be two regular surfaces and $J : M \to \tilde{M}$ be an isometry. If $\beta(r)$ is a normal curve on M such that $k_n = 0$, then $\tilde{\beta} = J \circ \beta$ is also a normal curve on $\tilde{M}$, provided any one of the following conditions are hold:*

(i) $\tilde{\beta}$ does not represent an asymptotic curve on the surface $\tilde{M}$ and

$$\tilde{\beta}(r) = J_*(\beta(r)) + \frac{\mu_2(r)k_n(r)}{\kappa_1(r)}\tilde{P}_1(r) - \frac{\mu_1(r)k_n(r)}{\kappa_1(r)}\tilde{U}_1(r),$$

(ii) $\tilde{\beta}$ represents asymptotic curve on the surface $\tilde{M}$ and $\tilde{\beta}(r) = J_*(\beta(r))$.

Proof. The proof of this result follows from that of Theorem 3.1 by setting $k_n = 0$. □

Theorem 3.3. *Let M and $\tilde{M}$ be two regular surfaces immersed in $\mathbb{E}^3$ and $J : M \to \tilde{M}$ be an isometry. If $\beta(r)$ and $\tilde{\beta}(r)$ are respectively normal curves on M and $\tilde{M}$, such that $k_n \neq 0$, and $T_1(r) = a\sigma_x + b\sigma_y$ represents any tangent vector at the point $\beta(r)$ on the surface M, then the following holds:*

(i) If the surface $\tilde{M}$ has a geodesic curve $\tilde{\beta}$, then $\tilde{\beta}(r) \cdot \tilde{T}_1(r) = \beta(r) \cdot T_1(r)$.

(ii) If the surface $\tilde{M}$ has an asymptotic curve $\tilde{\beta}$, then

$$\tilde{\beta}(r) \cdot \tilde{T}_1(r) - \beta(r) \cdot T_1(r) = \sqrt{EG - F^2}(bx' - ay')\left\{\mu_1(r) - \frac{\mu_1(r)k_g(r)}{k_1(r)} - \frac{\mu_2(r)k_n(r)}{k_1(r)}\right\}.$$

(iii) If the curve $\tilde{\beta}$ is neither an asymptotic nor a geodesic curve on the surface $\tilde{M}$, then

$$\tilde{\beta}(r) \cdot \tilde{T}_1(r) = \beta(r) \cdot T_1(r).$$

Proof. Let β and $\tilde{\beta}$ be the normal curves on the surface M and $\tilde{M}$ respectively, with $k_n \neq 0$, and J represents an isometry between M and $\tilde{M}$. Also, $T_1(r) = a\sigma_x + b\sigma_y$ represents any tangent vector at the point $\beta(r)$ on the surface M. Then

$$\begin{aligned}\tilde{\beta}(r)\cdot\tilde{T}_1(r) - \beta(r)\cdot T_1(r) &= \tilde{\beta}(r)\cdot(a\tilde{\sigma}_x + b\tilde{\sigma}_y) - \beta(r)\cdot(a\sigma_x + b\sigma_y),\\ &= a(\tilde{\beta}(r)\cdot\tilde{\sigma}_x - \beta(r)\cdot\sigma_x) + b(\tilde{\beta}(r)\cdot\tilde{\sigma}_y - \beta(r)\cdot\sigma_y). \qquad (23)\end{aligned}$$

Now

$$\begin{aligned}\beta(r)\cdot\sigma_x &= \left\{\mu_1(r)\frac{k_g(r)}{k_1(r)} + \mu_2(r)\frac{k_n(r)}{k_1(r)}\right\}\frac{1}{\sqrt{EG-F^2}}\Big(Ex'\sigma_y + F(y'\sigma_y - x'\sigma_x) - Gy'\sigma_x\Big)\cdot\sigma_x\\ &\quad + \left\{\mu_2(r)\frac{k_g(r)}{k_1(r)} - \mu_1(r)\frac{k_n(r)}{k_1(r)}\right\}\frac{\sigma_x\times\sigma_y}{\sqrt{EG-F^2}}\cdot\sigma_x,\\ &= \left\{\mu_1(r)\frac{k_g(r)}{k_1(r)} + \mu_2(r)\frac{k_n(r)}{k_1(r)}\right\}\frac{y'(F^2-EG)}{\sqrt{EG-F^2}},\\ &= -y'\left\{\mu_1(r)\frac{k_g(r)}{k_1(r)} + \mu_2(r)\frac{k_n(r)}{k_1(r)}\right\}\sqrt{EG-F^2}. \qquad (24)\end{aligned}$$

Similarly, we can obtain

$$\beta(r)\cdot\sigma_y = x'\left\{\mu_1(r)\frac{k_g(r)}{k_1(r)} + \mu_2(r)\frac{k_n(r)}{k_1(r)}\right\}\sqrt{EG-F^2}. \qquad (25)$$

For (i), if $\tilde{\beta}$ is a geodesic curve on the surface $\tilde{M}$, then $\tilde{k}_g = 0$, and hence from equation (10) and (14), we get

$$\tilde{\beta}(r)\cdot\tilde{\sigma}_x = -y'\tilde{\mu}_2(r)\sqrt{EG-F^2}. \qquad (26)$$

Similarly, we can obtain

$$\tilde{\beta}(r)\cdot\tilde{\sigma}_y = x'\tilde{\mu}_2(r)\sqrt{EG-F^2}. \qquad (27)$$

Again, by applying the condition of geodesic and then by virtue of equation (24) and (26), we get

$$\tilde{\beta}(r)\cdot\tilde{\sigma}_x - \beta(r)\cdot\sigma_x = y'(\mu_2(r) - \tilde{\mu}_2(r))\sqrt{EG-F^2}. \qquad (28)$$

Similarly, (25) and (27) together yields

$$\tilde{\beta}(r)\cdot\tilde{\sigma}_y - \beta(r)\cdot\sigma_y = x'(\tilde{\mu}_2(r) - \mu_2(r))\sqrt{EG-F^2}. \qquad (29)$$

Using equation (28) and (29) in (23), we obtain

$$\tilde{\beta}(r)\cdot\tilde{T}_1(r) - \beta(r)\cdot T_1(r) = (\mu_2(r) - \tilde{\mu}_2(r))\sqrt{EG-F^2}(ay' - bx').$$

Since $\beta(r)$ and $\tilde{\beta}(r)$ are normal curves on the surface M and $\tilde{M}$ respectively, so $\mu_2(r) = \tilde{\mu}_2(r)$, and hence $\tilde{\beta}(r)\cdot\tilde{T}_1(r) = \beta(r)\cdot T_1(r)$. This proves (i).
Moving on to point (ii), let's assume that $\tilde{\beta}$ represents an asymptotic curve on the surface $\tilde{M}$. Then $\tilde{k}_n = 0$ and $\tilde{k}_g(r) = \tilde{\kappa}_1(r)$. Thus (10) and (15) respectively entails

$$\begin{aligned}\tilde{\beta}(r)\cdot\tilde{\sigma}_x &= -y'\tilde{\mu}_1(r)\sqrt{\tilde{E}\tilde{G}-\tilde{F}^2},\\ &= -y'\mu_1(r)\sqrt{EG-F^2} \qquad (30)\end{aligned}$$

and

$$\tilde{\beta}(r)\cdot\tilde{\sigma}_y = x'\mu_1(r)\sqrt{EG-F^2}. \qquad (31)$$

In view of (24) and (30) we have

$$\tilde{\beta}(r)\cdot\tilde{\sigma}_x-\beta(r)\cdot\sigma_x = -y'(\sqrt{EG-F^2})\left\{\mu_1(r)-\mu_1(r)\frac{k_g(r)}{k_1(r)}-\mu_2(r)\frac{k_n(r)}{k_1(r)}\right\}. \tag{32}$$

Similarly, from (25) and (31) we get

$$\tilde{\beta}(r)\cdot\tilde{\sigma}_y-\beta(r)\cdot\sigma_y = x'(\sqrt{EG-F^2})\left\{\mu_1(r)-\mu_1(r)\frac{k_g(r)}{k_1(r)}-\mu_2(r)\frac{k_n(r)}{k_1(r)}\right\}. \tag{33}$$

Also (23) entails

$$\begin{aligned}\tilde{\beta}(r)\cdot\tilde{T}_1(r) - \beta(r)\cdot T_1(r) &= a(\tilde{\beta}(r)\cdot\tilde{\sigma}_x-\beta(r)\cdot\sigma_x)+b(\tilde{\beta}(r)\cdot\tilde{\sigma}_y-\beta(r)\cdot\sigma_y),\\ &= \sqrt{EG-F^2}(bx'-ay')\left\{\mu_1(r)-\frac{\mu_1(r)k_g(r)}{k_1(r)}-\frac{\mu_2(r)k_n(r)}{k_1(r)}\right\}.\end{aligned}$$

This proves (ii).
Moving on to point (iii), let's assume that $\tilde{\beta}$ is neither an asymptotic curve nor a geodesic curve on the surface $\tilde{M}$. Then from (10) and (13), we get

$$\tilde{\beta}(r)\cdot\tilde{\sigma}_x = -y'\left\{\tilde{\mu}_1(r)\frac{\tilde{k}_g(r)}{\tilde{k}_1(r)}+\tilde{\mu}_2(r)\frac{\tilde{k}_n(r)}{\tilde{k}_1(r)}\right\}\sqrt{EG-F^2}$$

and

$$\tilde{\beta}(r)\cdot\tilde{\sigma}_y = x'\left\{\tilde{\mu}_1(r)\frac{\tilde{k}_g(r)}{\tilde{k}_1(r)}+\tilde{\mu}_2(r)\frac{\tilde{k}_n(r)}{\tilde{k}_1(r)}\right\}\sqrt{EG-F^2}.$$

By virtue of (24) and (25), we obtain

$$\tilde{\beta}(r)\cdot\tilde{\sigma}_x-\beta(r)\cdot\sigma_x = y'(\sqrt{EG-F^2})\left\{\left(\mu_1(r)\frac{k_g(r)}{k_1(r)}-\tilde{\mu}_1(r)\frac{\tilde{k}_g(r)}{\tilde{k}_1(r)}\right)+\left(\mu_2(r)\frac{k_n(r)}{k_1(r)}-\tilde{\mu}_2(r)\frac{\tilde{k}_n(r)}{\tilde{k}_1(r)}\right)\right\},$$

and

$$\tilde{\beta}(r)\cdot\tilde{\sigma}_y-\beta(r)\cdot\sigma_y = x'(\sqrt{EG-F^2})\left\{\left(\tilde{\mu}_1(r)\frac{\tilde{k}_g(r)}{\tilde{k}_1(r)}-\mu_1(r)\frac{k_g(r)}{k_1(r)}\right)+\left(\tilde{\mu}_2(r)\frac{\tilde{k}_n(r)}{\tilde{k}_1(r)}-\mu_2(r)\frac{k_n(r)}{k_1(r)}\right)\right\}.$$

Substituting these values in (23), we have

$$\begin{aligned}\tilde{\beta}(r)\cdot\tilde{T}_1(r)-\beta(r)\cdot T_1(r) &= \left\{\left(\mu_1(r)\frac{k_g(r)}{k_1(r)}-\tilde{\mu}_1(r)\frac{\tilde{k}_g(r)}{\tilde{k}_1(r)}\right)+\left(\mu_2(r)\frac{k_n(r)}{k_1(r)}-\tilde{\mu}_2(r)\frac{\tilde{k}_n(r)}{\tilde{k}_1(r)}\right)\right\}\\ &\quad(\sqrt{EG-F^2})(ay'-bx').\end{aligned}$$

Since $\beta(r)$ and $\tilde{\beta}(r)$ are normal curves on the surfaces M and $\tilde{M}$, respectively, so we have $\mu_1(r)\frac{k_g(r)}{k_1(r)}=\tilde{\mu}_1(r)\frac{\tilde{k}_g(r)}{\tilde{k}_1(r)}$, and $\mu_2(r)\frac{k_n(r)}{k_1(r)}=\tilde{\mu}_2(r)\frac{\tilde{k}_n(r)}{\tilde{k}_1(r)}$. Thus, $\tilde{\beta}(r)\cdot\tilde{T}_1(r)=\beta(r)\cdot T_1(r)$. This proves (iii). □

Theorem 3.4. *Let M and $\tilde{M}$ be two regular surfaces immersed in $\mathbb{E}^3$ and $J: M\to\tilde{M}$ be an isometry. If $\beta(r)$ and $\tilde{\beta}(r)$ are normal curves on M and $\tilde{M}$, such that $k_n=0$, and $T_1(r)=a\sigma_x+b\sigma_y$ represents any tangent vector at the point $\beta(r)$ on the surface M, then the following holds;*

(i) If the surface $\tilde{M}$ has an asymptotic curve $\tilde{\beta}$, then $\tilde{\beta}(r)\cdot\tilde{T}_1(r)=\beta(r)\cdot T_1(r)$.
(ii) If the surface $\tilde{M}$ has not an asymptotic curve $\tilde{\beta}$, then

$$\tilde{\beta}(r)\cdot\tilde{T}_1(r)-\beta(r)\cdot T_1(r)=\sqrt{EG-F^2}(bx'-ay')\left\{\tilde{\mu}_2(r)\frac{\tilde{k}_n(r)}{\tilde{k}_1(r)}\right\}.$$

Proof. The proof of this theorem follows from that of Theorem 3.3 by setting $k_n = 0$. $\square$

Theorem 3.5. *Let M and $\tilde{M}$ be two regular surfaces immersed in $\mathbb{E}^3$ and $J : M \to \tilde{M}$ be an isometry. If $\beta(r)$ and $\tilde{\beta}(r)$ are normal curves on M and $\tilde{M}$ respectively such that $k_n \neq 0$ and $P_1 = U_1 \times T_1$, where $T_1(r) = a\sigma_x + b\sigma_y$ represents any tangent vector at the point $\beta(r)$ on the surface M, then the following holds:*

(i) If the surface $\tilde{M}$ has a geodesic curve $\tilde{\beta}$, then $\tilde{\beta}(r) \cdot \tilde{P}_1(r) = \beta(r) \cdot P_1(r)$.

(ii) If the surface $\tilde{M}$ has an asymptotic curve $\tilde{\beta}$, then

$$\tilde{\beta}(r) \cdot \tilde{P}_1(r) - \beta(r) \cdot P_1(r) = \left\{\mu_1(r) - \frac{\mu_1(r)k_g(r)}{k_1(r)} - \frac{\mu_2(r)k_n(r)}{k_1(r)}\right\}\Big(a(x'E + y'F) + b(x'F + y'G)\Big).$$

(iii) If the curve $\tilde{\beta}$ is neither an asymptotic nor a geodesic curve on the surface $\tilde{M}$, then

$$\tilde{\beta}(r) \cdot \tilde{P}_1(r) = \beta(r) \cdot P_1(r).$$

Proof. Let β and $\tilde{\beta}$ be the normal curves on the surfaces M and $\tilde{M}$, with $k_n \neq 0$, and an isometric transformation J from M to $\tilde{M}$. We know that $P_1 = U_1 \times T_1$, where $T_1(r) = a\sigma_x + b\sigma_y$ represents any tangent vector at the point $\beta(r)$ on the surface M. Then, by using (8), we have

$$\begin{aligned}\beta(r) \cdot P_1(r) = \beta(r) \cdot (U_1(r) \times T_1(r)) \quad &= \quad \beta(r) \cdot \left\{\left(\frac{\sigma_x \times \sigma_y}{\sqrt{EG - F^2}}\right) \times (a\sigma_x + b\sigma_y)\right\}, \\ &= \quad \frac{1}{\sqrt{EG - F^2}}\Big\{(aE + bF)\beta(r) \cdot \sigma_y - (aF + bG)\beta(r) \cdot \sigma_x\Big\}. \end{aligned} \tag{34}$$

Similarly, by using (10), we obtain

$$\tilde{\beta}(r) \cdot \tilde{P}_1(r) = \frac{1}{\sqrt{EG - F^2}}\Big\{(aE + bF)\tilde{\beta}(r) \cdot \tilde{\sigma}_y - (aF + bG)\tilde{\beta}(r) \cdot \tilde{\sigma}_x\Big\}. \tag{35}$$

Now

$$\begin{aligned}\tilde{\beta}(r) \cdot \tilde{P}_1(r) - \beta(r) \cdot P_1(r) \quad = \quad & \frac{1}{\sqrt{EG - F^2}}\Big\{(aE + bF)(\tilde{\beta}(r) \cdot \tilde{\sigma}_y - \beta(r) \cdot \sigma_y) \\ & -(aF + bG)(\tilde{\beta}(r) \cdot \tilde{\sigma}_x - \beta(r) \cdot \sigma_x)\Big\}. \end{aligned} \tag{36}$$

For (i), if $\tilde{\beta}$ is a geodesic curve on the surface $\tilde{M}$, then by using (28) and (29) in (36), we obtain

$$\begin{aligned}\tilde{\beta}(r) \cdot \tilde{P}_1(r) - \beta(r) \cdot P_1(r) \quad &= \quad \frac{1}{\sqrt{EG - F^2}}\Big\{(aE + bF)(x'(\tilde{\mu}_2(r) - \mu_2(r))\sqrt{EG - F^2}) \\ & \quad\quad -(aF + bG)(y'(\mu_2(r) - \tilde{\mu}_2(r))\sqrt{EG - F^2})\Big\}, \\ &= \quad (\tilde{\mu}_2(r) - \mu_2(r))(a(x'E + y'F) + b(x'F + y'G)). \end{aligned}$$

Since $\beta(r)$ and $\tilde{\beta}(r)$ are normal curves on the surfaces M and $\tilde{M}$ respectively, so $\mu_2(r) = \tilde{\mu}_2(r)$, and hence $\tilde{\beta}(r) \cdot \tilde{P}_1(r) = \beta(r) \cdot P_1(r)$. This proves (i).

To prove (ii), let's assume that $\tilde{\beta}$ represents an asymptotic curve on the surface $\tilde{M}$. Then, by using (32) and (33) in (36), we obtain

$$\begin{aligned}\tilde{\beta}(r) \cdot \tilde{P}_1(r) - \beta(r) \cdot P_1(r) \quad &= \quad \frac{1}{\sqrt{EG - F^2}}\Big[(aE + bF)x'(\sqrt{EG - F^2})\left\{\mu_1(r) - \mu_1(r)\frac{k_g(r)}{k_1(r)} - \mu_2(r)\frac{k_n(r)}{k_1(r)}\right\} \\ & \quad\quad +(aF + bG)y'(\sqrt{EG - F^2})\left\{\mu_1(r) - \mu_1(r)\frac{k_g(r)}{k_1(r)} - \mu_2(r)\frac{k_n(r)}{k_1(r)}\right\}\Big], \\ &= \quad \left\{\mu_1(r) - \mu_1(r)\frac{k_g(r)}{k_1(r)} - \mu_2(r)\frac{k_n(r)}{k_1(r)}\right\}\Big(a(x'E + y'F) + b(x'F + y'G)\Big). \end{aligned}$$

This proves (ii).
For (iii), suppose $\tilde{\beta}$ is neither an asymptotic curve nor a geodesic curve on the surface $\tilde{M}$. Then (36) yields

$$\begin{aligned}\tilde{\beta}(r)\cdot\tilde{P}_1(r)-\beta(r)\cdot P_1(r) &= \frac{1}{\sqrt{EG-F^2}}\Big[(aE+bF)\Big\{x'(\sqrt{EG-F^2})\Big\{(\tilde{\mu}_1(r)\frac{\tilde{k}_g(r)}{\tilde{k}_1(r)}-\mu_1(r)\frac{k_g(r)}{k_1(r)})\\ &\quad+(\tilde{\mu}_2(r)\frac{\tilde{k}_n(r)}{\tilde{k}_1(r)}-\mu_2(r)\frac{k_n(r)}{k_1(r)})\Big\}\Big\}-(aF+bG)\Big\{y'(\sqrt{EG-F^2})\\ &\quad\Big\{(\mu_1(r)\frac{k_g(r)}{k_1(r)}-\tilde{\mu}_1(r)\frac{\tilde{k}_g(r)}{\tilde{k}_1(r)})+(\mu_2(r)\frac{k_n(r)}{k_1(r)}-\tilde{\mu}_2(r)\frac{\tilde{k}_n(r)}{\tilde{k}_1(r)})\Big\}\Big\}\Big],\\ &= \left\{(\mu_1(r)\frac{k_g(r)}{k_1(r)}-\tilde{\mu}_1(r)\frac{\tilde{k}_g(r)}{\tilde{k}_1(r)})+(\mu_2(r)\frac{k_n(r)}{k_1(r)}-\tilde{\mu}_2(r)\frac{\tilde{k}_n(r)}{\tilde{k}_1(r)})\right\}\\ &\quad((aE+bF)x'-(aF+bG)y').\end{aligned}$$

Since $\beta(r)$ and $\tilde{\beta}(r)$ are normal curves on the surface M and $\tilde{M}$ respectively, so $\mu_1(r)\frac{k_g(r)}{k_1(r)}=\tilde{\mu}_1(r)\frac{\tilde{k}_g(r)}{\tilde{k}_1(r)}$, and $\mu_2(r)\frac{k_n(r)}{k_1(r)}=\tilde{\mu}_2(r)\frac{\tilde{k}_n(r)}{\tilde{k}_1(r)}$. Therefore $\tilde{\beta}(r)\cdot\tilde{P}_1(r)=\beta(r)\cdot P_1(r)$. This proves (iii). □

Theorem 3.6. *Let M and $\tilde{M}$ be two regular surfaces immersed in $\mathbb{E}^3$ and $J: M\to\tilde{M}$ be an isometry. If $\beta(r)$ and $\tilde{\beta}(r)$ are normal curves on M and $\tilde{M}$ respectively such that $k_n = 0$ and $P_1 = U_1\times T_1$, where $T_1(r)=a\sigma_x+b\sigma_y$ represents any tangent vector at the point $\beta(r)$ on the surface M, then the following holds:*

(i) If the surface $\tilde{M}$ has an asymptotic curve $\tilde{\beta}$, then $\tilde{\beta}(r)\cdot\tilde{P}_1(r)=\beta(r)\cdot P_1(r)$.
(ii) If the surface $\tilde{M}$ has not an asymptotic curve $\tilde{\beta}$, then

$$\tilde{\beta}(r)\cdot\tilde{P}_1(r)-\beta(r)\cdot P_1(r)=\Big((aE+bF)x'-(aF+bG)y'\Big)\left\{\tilde{\mu}_2(r)\frac{\tilde{k}_n(r)}{\tilde{k}_1(r)}\right\}.$$

Proof. The proof of this result follows from that of Theorem 3.5 by setting $k_n = 0$. □

Theorem 3.7. *Let M and $\tilde{M}$ be two regular surfaces immersed in $\mathbb{E}^3$ and $J: M\to\tilde{M}$ be an isometry. If $\beta(r)$ and $\tilde{\beta}(r)$ are normal curves on M and $\tilde{M}$ respectively, then $\tilde{\beta}(r)\cdot\tilde{U}_1(r)=\beta(r)\cdot U_1(r)$.*

Proof. Let β and $\tilde{\beta}$ be the normal curves on the surfaces M and $\tilde{M}$ respectively and J denotes an isometric transformation between M and $\tilde{M}$. Now, using (8) and (13), we get

$$\beta(r)\cdot U_1(r)=\beta(r)\cdot\frac{\sigma_x\times\sigma_y}{\sqrt{EG-F^2}} = \left\{\mu_2(r)\frac{k_g(r)}{k_1(r)}-\mu_1(r)\frac{k_n(r)}{k_1(r)}\right\}, \tag{37}$$

and

$$\tilde{\beta}(r)\cdot\tilde{U}_1(r) = \left\{\tilde{\mu}_2(r)\frac{\tilde{k}_g(r)}{\tilde{k}_1(r)}-\tilde{\mu}_1(r)\frac{\tilde{k}_n(r)}{\tilde{k}_1(r)}\right\}. \tag{38}$$

Again in view of (37) and (38), we obtain

$$\tilde{\beta}(r)\cdot\tilde{U}_1(r)-\beta(r)\cdot U_1(r) = \left\{\tilde{\mu}_2(r)\frac{\tilde{k}_g(r)}{\tilde{\kappa}_1(r)}-\mu_2(r)\frac{k_g(r)}{k_1(r)}\right\}-\left\{\tilde{\mu}_1(r)\frac{\tilde{k}_n(r)}{\tilde{\kappa}_1(r)}-\mu_1(r)\frac{k_n(r)}{k_1(r)}\right\}. \tag{39}$$

Thus, if $k_g = 0$ in (39), then $\tilde{\beta}(r)\cdot\tilde{U}_1(r)=\beta(r)\cdot U_1(r)$. Further, for $k_g\neq 0$, since $\beta(r)$ and $\tilde{\beta}(r)$ are normal curves on the surfaces M and $\tilde{M}$ respectively, so $\mu_1(r)\frac{k_n(r)}{k_1(r)}=\tilde{\mu}_1(r)\frac{\tilde{k}_n(r)}{\tilde{k}_1(r)}$, and $\mu_2(r)\frac{k_g(r)}{k_1(r)}=\tilde{\mu}_2(r)\frac{\tilde{k}_g(r)}{\tilde{k}_1(r)}$, and hence from (39), we get

$$\tilde{\beta}(r)\cdot\tilde{U}_1(r) = \beta(r)\cdot U_1(r).$$

This completes the proof. □

4. Conclusion

In this paper, we investigated the requirements for the invariance of a normal curve on regular surfaces under isometric transformations using the Darboux frame instead of the Frenet frame. We found that the components of the position vector of a normal curve $\beta(r)$ along various directions remain unchanged under isometric transformations, provided either the conditions of being geodesic or asymptotic are satisfied. These directions include the tangent vector T_1, the unit normal U_1 and $P_1 = U_1 \times T_1$.

In future research, one could introduce the concepts of Darboux rectifying and Darboux normal curves under conformal and isometric transformations on smooth surfaces in $\mathbb{E}^3$ using the Darboux frame. Furthermore, these findings can also be extended to $\mathbb{E}^4$ by introducing the concepts of the Darboux frame and the Frenet frame.

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No experiments involving human or animal subjects were conducted by any of the authors and are included in this article.

References

[1] M. P. Do Carmo, *Differential geometry of curves and surfaces: revised and updated second edition*, Courier Dover Publications, (2016).
[2] D. Fuchs, I. Izmestiev, M. Raffaelli, G. Szewieczek, and S. Tabachnikov, *Differential geometry of space curves: forgotten chapters*, Math. Intell., (2023), 1-13.
[3] A. Pressley, *Elementary differential geometry*, Springer-Verlag, (2001).
[4] D. S. Kim, H. S. Chung, and K. H. Cho, *Space curves satisfying* $\tau/\kappa = as + b$, Honam Math. J., **15** (1993), 5-9.
[5] B. Y. Chen, *When does the position vector of a space curve always lie in its rectifying plane*?, Amer. Math. Monthly, **110**(2) (2003), 147-152.
[6] B. Y. Chen, *Rectifying curves and geodesics on a cone in the Euclidean 3-space*, Tamkang J. Math., **48** (2017), 209-214.
[7] B. Y. Chen and F. Dillen, *Rectfying curve as centrode and extremal curve*, Bull. Inst. Math. Acad. Sinica, **33**(2) (2005), 77-90.
[8] K. Ilarslan and E. Nešović, *Timelike and null normal curves in Minkowski space* $\mathbb{E}^3_1$, Indian J. Pure Appl. Math., **35** (2004), 881-888.
[9] K. Ilarslan, and E. Nesovic, *Some characterizations of null, pseudo null and partially null rectifying curves in Minkowski space-time*, Taiwan. J. Math., **12**(5) (2008), 1035-1044.
[10] C. Camci, L. Kula, and K. Ilarslan, *Characterizations of the position vector of a surface curve in Euclidean 3-space*, An. St. Univ. Ovidius Constanta, **19**(3) (2011), 59-70.
[11] K. Ilarslan, and E. Nesovic, *Some characterizations of osculating curves in the euclidean spaces*, Demonstr. Math., **41**(4) (2008), 931-939.
[12] K. Ilarslan, and E. Nesovic, *Some characterizations of rectifying curves in the euclidean spaces* E^4, Turk. J. Math., **32**(1) (2008), 21-30.
[13] S. Deshmukh, B. Y. Chen, and S. H. Alshammari, *On rectifying curves in Euclidean 3-space*, Turkish J. Math., **42**(2) (2018), 609-620.

[14] A. A. Shaikh and P. R. Ghosh, *Rectifying and osculating curves on a smooth surface*, Indian J. Pure Appl. Math., **51**(1) (2020), 67-75.
[15] S. Sharma and K. Singh, *Some aspects of rectifying curves on regular surfaces under different transformations*, Int. J. Appl., **21** (2023), 78.
[16] M. S. Lone, *Geometric invariants of normal curves under conformal transformation in E^3*, Tamkang J. Math., **53**(1) (2022), 75-87.
[17] A. A. Shaikh, M. S. Lone, and P. R. Ghosh, *Normal curves on a smooth immersed surface*, Indian J. Pure Appl. Math., **51** (2020), 1343-1355.
[18] A. A. Shaikh, M. S. Lone, and P. R. Ghosh, *Rectifying curves under conformal transformation*, J. Geom Phys., **163** (2021), 104117.
[19] A. A. Shaikh and P. R. Ghosh, *Curves on a smooth surface with position vectors lie in the tangent plane*, Indian J. Pure Appl. Math., **51**(3) (2020), 1097-1104.
[20] M. He, D. B. Goldgof, and C. Kambhamettu, *Variation of Gaussian curvature under conformal mapping and its application*, Comput. Math. with Appl., **26**(1) (1993), 63-74.
[21] F. Schwarz, *Transformation to canonical form*, Algorithmic Lie theory solving ordinary Differ. Equ., (2007), 257-320.