



Geometry of the statistical spacelike hypersurfaces of the GRW spacetimes

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Abstract. We explore the geometry of the Generalized Robertson-Walker (GRW) spacetime under the assumption that its fiber is a statistical manifold. This approach allows us to endow a statistical structure on the GRW spacetime. Later, we derive necessary and sufficient conditions for the existence of a statistical Ricci soliton on the statistical spacelike hypersurface of a GRW spacetime. We explore the condition for a GRW spacetime to have an almost Ricci soliton structure, when the fiber of the GRW spacetime constitutes a statistical Ricci soliton. Furthermore, we examine the case where both the fiber and the GRW spacetime are Ricci solitons, and we establish the necessary and sufficient conditions for the statistical spacelike hypersurface to exhibit a Ricci soliton structure.

1. Introduction

In the context of information geometry, Amari [4] introduced an interesting class of manifolds named *statistical manifolds*. The study of the statistical manifolds has emerged as a significant tool in modern differential geometry to understand complex geometric structures infused with statistical properties. These manifolds, characterized by the interplay between a Riemannian metric and an affine connection, provide deep insight into various domains such as information geometry [3, 4], neural networks [2], null geometry [16], and many others. It is known that the conjugate symmetric statistical structures are significant in the geometry of statistical structures as affine spheres in the geometry of affine hypersurfaces. Therefore, Opozda [20] generalized some important results about complete affine spheres to the statistical structures

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on a manifold. In [21], Opozda studied the completeness of affine connections on the statistical manifolds or on affine hypersurfaces.

Let Σ be an open subset of $\mathbb{R}^n$ and Θ be a sample space with parameters $\zeta = (\zeta^1, \dots, \zeta^n)$. Then, the set of probability density functions

$$\mathcal{S} = \{p(x; \zeta) : \int_{\Theta} p(x; \zeta) dx = 1, p(x; \zeta) > 0, \zeta \in \Sigma \subset \mathbb{R}^n\}$$

is called a *statistical model*. On the statistical model $\mathcal{S}$, a *semi-definite Fisher information matrix* $g(\zeta) = [g_{ij}(\zeta)]$ is given by

$$g_{ij}(\zeta) = \int_{\Theta} \partial_i \ell_{\zeta} \partial_j \ell_{\zeta} p(x; \zeta) dx = E_p[\partial_i \ell_{\zeta} \partial_j \ell_{\zeta}],$$

where $\ell_{\zeta} = \ell(x; \zeta) = \log p(x; \zeta)$, $\partial_i = \frac{\partial}{\partial \zeta^i}$ and $E_p[f]$ is the expectation of f with respect to $p(x; \zeta)$. Then, the statistical model $\mathcal{S}$ equipped with the semi-definite Fisher information matrices is called a *statistical manifold*. Furthermore, in the context of differential geometry, the statistical manifolds [4] are inspired from the statistical model, where the density function, the Fisher information matrix, the dual connections $\nabla^{(-1)}$ and $\nabla^{(1)}$ are replaced by an arbitrary Riemannian manifold $\bar{M}$, the Riemannian metric $\bar{g}$ of $\bar{M}$, the dual connections $\bar{\nabla}$ and $\bar{\nabla}^*$, respectively. Since the geometry of statistical manifolds includes the dual connections, analogous to the conjugate connections of the affine geometry, then the geometry of statistical manifolds is closely related to affine differential geometry. Moreover, the geometry of statistical manifolds has potential applications in the various fields of science and engineering.

On the other hand, in our attempt to expand our knowledge of the interplay between geometry and physics, the study of spacetime manifolds has emerged as a pivotal area of research. Among the various models that have been developed, the Generalized Robertson-Walker (GRW) spacetime [1] has gained significant attention due to its versatility and relevance in cosmological contexts. The GRW spacetime also serves as a natural generalization of the classical Robertson-Walker model, extending its applicability in both mathematics and physics. Moreover, the GRW spacetimes include Minkowski spacetime, de Sitter spacetime, Friedmann cosmological models, and static Einstein spacetime [23]. Recently, Singh et al. [25] used the natural correspondence between totally umbilical null hypersurfaces of the GRW spacetimes and twisted decomposition of the fibres, to study the rigged scalar curvature of totally umbilical null graphs of the GRW spacetimes.

Ricci solitons, on the other hand, play a critical role in the study of geometric flows, particularly of the Ricci flow. The Ricci solitons generalize the concept of Einstein metrics and serve as self-similar solutions to the Ricci flow, making them essential in understanding the evolution of geometric structures over time, see [15]. The concept of Ricci solitons gained significant importance and popularity following Perelman's use of them to resolve the long-standing Poincaré conjecture. Since then, Ricci solitons have become a prominent area of research, valued for their geometric significance and broader applications in theoretical physics. Barros et al. [5] investigated the immersions of Ricci solitons into Riemannian manifolds, showing that when a shrinking Ricci soliton is immersed in a space form with constant mean curvature, it becomes a Gaussian soliton. In [6], Blaga and Chen provided necessary and sufficient conditions for some particular couples (g, ∇) of semi-Riemannian metrics and affine connections to be statistical structures if one has gradient almost Einstein, almost Ricci, almost Yamabe solitons, or a more general type of solitons on the manifold.

We present below few important known results.

In [17], the authors showed that a Ricci simple manifold with vanishing divergence of the conformal curvature tensor admits a proper concircular vector field and it is necessarily a GRW spacetime. They also proved that a stiff matter perfect fluid spacetime or a mass-less scalar field with time-like gradient and with divergence-free Weyl tensor are GRW spacetimes.

In [10], De et al. examined gradient type Ricci solitons and (m, τ) -quasi Einstein solitons in GRW spacetimes. Besides, it was demonstrated that in this scenario the GRW spacetime presents the Robertson-Walker spacetime and the perfect fluid spacetime presents the phantom era. Consequently, it was proved

that if a GRW spacetime permits a gradient τ -Einstein solitons, then it also represents a perfect fluid spacetime under certain condition.

In [24], Ricci–Yamabe solitons and gradient Ricci–Yamabe solitons in GRW spacetimes were investigated. The authors proved that if a GRW spacetime admits a Ricci–Yamabe soliton, then it becomes a perfect fluid spacetime and the divergence of the Weyl tensor vanishes.

In [11], GRW spacetimes were investigated in light of perfect fluid spacetimes. It was established that a perfect fluid spacetime with nonvanishing vorticity whose associated scalars are constant along the velocity vector field becomes a GRW spacetime. Among others, it was also shown that a Ricci parallel perfect fluid spacetime is either a GRW spacetime or a static spacetime. Also, it was proved that in a conformally semisymmetric GRW spacetime of dimension 4, the scalar curvature vanishes and the spacetime is locally isometric to the Minkowski spacetime, provided the electric part of the Weyl tensor vanishes.

In [12], U.C. De and F. Farrell proved that a Ricci symmetric spacetime is a perfect fluid spacetime if and only if it is a GRW spacetime. Consequently, they established that under certain constraints on the associated scalar, such a spacetime turns into a static spacetime. Also, it was proved that a semi-symmetric as well as pseudo-Ricci symmetric GRW spacetime is a perfect fluid spacetime. The conformally flat GRW spacetime as a solution of $f(R^*)$ -gravity theory was examined and the physical relevance was described.

In [18], the authors proved that a perfect fluid spacetime of dimension $n \geq 4$ with irrotational velocity vector field and null divergence of the Weyl tensor is a GRW spacetime with an Einstein fiber. First condition is verified whenever pressure and energy density are related by an equation of state. The contraction of the Weyl tensor with the velocity vector field is zero. Conversely, a GRW spacetime with null divergence of the Weyl tensor is a perfect fluid spacetime.

In this paper, we explore an approach by considering the fiber of the GRW spacetime as a statistical manifold. This approach allows us to endow a statistical structure on the GRW spacetime, thereby broadening the scope of its geometric and physical interpretation. The fusion of statistical manifolds with the GRW spacetime opens up new ways for analyzing the complicated relationships between curvature, entropy, and information geometry in the context of cosmological models. Our study proceeds by investigating the geometry of spacelike hypersurfaces of the GRW spacetime and examine the Ricci soliton structures on the spacelike hypersurfaces of the GRW spacetime. By analyzing the Ricci soliton in the setting of a statistical structure, we uncover new insights into the behavior of these solitons in the GRW spacetimes. In particular, we derive the necessary and sufficient conditions for the existence of a statistical Ricci soliton on the statistical spacelike hypersurface of the GRW spacetime. When the fiber of the GRW spacetime constitutes a statistical Ricci soliton, we obtain the necessary and sufficient conditions for the GRW spacetime to constitute an almost Ricci soliton. Additionally, when both the fiber and the GRW spacetime constitute Ricci solitons, we derive the necessary and sufficient conditions for the statistical spacelike hypersurface to exhibit a Ricci soliton structure.

2. Statistical Manifolds

Let $(\bar{M}, \bar{g})$ be an $(n + 2)$ -dimensional semi-Riemannian manifold equipped with a semi-Riemannian metric $\bar{g}$ of constant index q , and let $\bar{\nabla}$ be a torsion free affine connection on $\bar{M}$. A pair $(\bar{\nabla}, \bar{g})$ is referred to as a statistical structure on $\bar{M}$ if the Codazzi equation

$$(\bar{\nabla}_U \bar{g})(V, W) = (\bar{\nabla}_V \bar{g})(U, W) \quad (1)$$

holds for any vector fields $U, V, W \in \Gamma(T\bar{M})$. In this case, the triplet $(\bar{M}, \bar{g}, \bar{\nabla})$ is referred to be an *indefinite statistical manifold*. A torsion free affine connection $\bar{\nabla}^*$ on $\bar{M}$ is called the *dual connection* of $\bar{\nabla}$ with respect to $\bar{g}$ if it satisfies

$$U\bar{g}(V, W) = \bar{g}(\bar{\nabla}_U V, W) + \bar{g}(V, \bar{\nabla}_U^* W), \quad (2)$$

for any vector fields $U, V, W \in \Gamma(T\bar{M})$. If the pair $(\bar{\nabla}, \bar{g})$ constitutes a statistical structure on the semi-Riemannian manifold $\bar{M}$, then the pair $(\bar{\nabla}^*, \bar{g})$ also forms a statistical structure on $\bar{M}$. Furthermore $(\bar{\nabla}^*)^* = \bar{\nabla}$.

Therefore, we denote an indefinite statistical manifold also as $(\overline{M}, \overline{g}, \overline{\nabla}, \overline{\nabla}^*)$. It is easy to verify the following identity

$$(\overline{\nabla}_U \overline{g})(V, W) + (\overline{\nabla}_U^* \overline{g})(V, W) = 0,$$

for any vector fields $U, V, W \in \Gamma(T\overline{M})$.

Let $\overline{\nabla}^{\overline{g}}$ be the Levi-Civita connection of $\overline{g}$. Then $\overline{\nabla}^{\overline{g}} = \frac{1}{2}(\overline{\nabla} + \overline{\nabla}^*)$, and from (1) it is evident that $(\overline{M}, \overline{g}, \overline{\nabla}^{\overline{g}})$ always forms a statistical manifold, and furthermore $(\overline{\nabla}^{\overline{g}})^* = \overline{\nabla}^{\overline{g}}$.

Let $(\overline{\nabla}, \overline{g})$ be a statistical structure on $\overline{M}$. Furuhashi [14] defined a difference tensor field $\overline{\mathcal{K}} \in \Gamma(T\overline{M}^{(1,2)})$ on $\overline{M}$ as

$$\overline{\mathcal{K}}(U, V) = \overline{\nabla}_U V - \overline{\nabla}_U^{\overline{g}} V,$$

which satisfies the following identities

$$\overline{\mathcal{K}}(U, V) = \overline{\mathcal{K}}(V, U), \quad \overline{g}(\overline{\mathcal{K}}(U, V), W) = \overline{g}(V, \overline{\mathcal{K}}(U, W)), \quad (3)$$

for any $U, V, W \in \Gamma(T\overline{M})$. Conversely, for a semi-Riemannian metric $\overline{g}$, if $\overline{\mathcal{K}} \in \Gamma(T\overline{M}^{(1,2)})$ satisfies (3), then the pair $(\overline{\nabla} = \overline{\nabla}^{\overline{g}} + \overline{\mathcal{K}}, \overline{g})$ forms a statistical structure. Furthermore

$$\overline{\mathcal{K}} = \overline{\nabla}^{\overline{g}} - \overline{\nabla}^* = \frac{1}{2}(\overline{\nabla} - \overline{\nabla}^*).$$

We denote the curvature tensor fields of $\overline{\nabla}$ and $\overline{\nabla}^*$ by $\overline{R}$ and $\overline{R}^*$, respectively. The *statistical curvature tensor field* $\overline{S}$ of the statistical structure $(\overline{\nabla}, \overline{g})$ is defined by

$$\overline{S}(U, V)W = \frac{1}{2}(\overline{R}(U, V)W + \overline{R}^*(U, V)W), \quad (4)$$

for any $U, V, W \in \Gamma(T\overline{M})$. It is easy to verify that

$$\overline{g}(\overline{S}(U, V)W, Z) = -\overline{g}(W, \overline{S}(U, V)Z)$$

and

$$\overline{S}(U, V)W + \overline{S}(V, W)U + \overline{S}(W, U)V = 0,$$

hold for any $U, V, W, Z \in \Gamma(T\overline{M})$.

Consider an orthonormal basis $\{e_1, \dots, e_{n+2}\}$ for $T_p \overline{M}$ at a point $p \in \overline{M}$. Now, for a 2-dimensional non-degenerate plane Π spanned by $\{e_i, e_j\}$ in $T_p \overline{M}$, the statistical sectional curvature of the plane Π is denoted by $K_{\Pi}^{\overline{S}}$ and is defined as

$$K_{\Pi}^{\overline{S}} = \frac{\overline{g}(\overline{S}(e_i, e_j)e_j, e_i)}{\overline{g}(e_i, e_i)\overline{g}(e_j, e_j) - \overline{g}(e_i, e_j)^2}.$$

A statistical manifold $(\overline{M}, \overline{g}, \overline{\nabla}, \overline{\nabla}^*)$ is said to have constant statistical sectional curvature $c \in \mathbb{R}$ if c is constant for all points p and planes Π . Moreover, the statistical sectional curvature of a statistical manifold $(\overline{M}, \overline{g}, \overline{\nabla}, \overline{\nabla}^*)$ is constant c [13] if and only if

$$\overline{S}(U, V)W = c\{\overline{g}(V, W)U - \overline{g}(U, W)V\},$$

for any $U, V, W \in \Gamma(T\overline{M})$.

3. Spacelike Hypersurfaces of GRW spacetimes

An $(n + 2)$ -dimensional Generalized Robertson-Walker (GRW) spacetime can be described as a warped product manifold denoted as $\bar{M} = I \times_{\varrho} F$. In this construction, the base is represented by an open interval I of the real line $\mathbb{R}$, the fiber F is an $(n + 1)$ -dimensional Riemannian manifold with the metric g_F , and the warping function ϱ is a smooth, positive function defined on I . The GRW spacetime itself is a Lorentzian manifold equipped with a Lorentzian metric denoted by $\bar{g}$, which can be expressed as

$$\bar{g} = -dt^2 + \varrho^2(t)g_F. \quad (5)$$

Here, t serves as a natural parameter on the real line $\mathbb{R}$. Let π_I and π_F denote the natural projections of $I \times F$ onto I and F , respectively. Define $\mathfrak{L}(I)$ and $\mathfrak{L}(F)$ as the sets of horizontal and vertical lifts of vector fields on I and F to $I \times F$, respectively. Let $\partial_t \in \mathfrak{L}(I)$ be the horizontal lift of the standard vector field $\frac{d}{dt}$ on I . For any vector field U tangent to $\bar{M}$, we have the following unique orthogonal decomposition

$$U = \phi_U \partial_t + U^F, \quad (6)$$

where $\phi_U = -\bar{g}(U, \partial_t)$ and U^F is the lift of the projection of U onto the fiber F .

In this article, we consistently employ an overline to signify the lifting of vector fields and functions from F to $\bar{M}$, without making a distinction between objects in I and their corresponding lifts.

Let $\bar{\nabla}^{\bar{g}}$ be the Levi-Civita connection of the GRW spacetime $\bar{M} = I \times_{\varrho} F$ and ∇^{g_F} be the Levi-Civita connection of the fiber (F, g_F) . The following lemma from [19] is well known regarding this Levi-Civita connection of the GRW spacetime $\bar{M}$.

Lemma 3.1. *Let $\bar{\nabla}^{\bar{g}}$ be the Levi-Civita connection of the GRW spacetime $\bar{M} = I \times_{\varrho} F$. Then, for the vector fields X, Y in $\mathfrak{L}(F)$, we have*

- (i) $\bar{\nabla}_{\partial_t}^{\bar{g}} \partial_t = 0$,
- (ii) $\bar{\nabla}_{\partial_t}^{\bar{g}} X = \bar{\nabla}_X^{\bar{g}} \partial_t = (\ln \varrho)' X$,
- (iii) Horizontal component of $\bar{\nabla}_X^{\bar{g}} Y = \bar{g}(X, Y)(\ln \varrho)' \partial_t$,
- (iv) Vertical component of $\bar{\nabla}_X^{\bar{g}} Y$ is the lift of the Levi-Civita connection $\nabla_X^{g_F} Y$ on the fiber F .

Let the fiber $(F, \nabla^F = \nabla^{g_F} + \mathcal{K}^F, \nabla^{*F}, g_F)$ be a statistical manifold. Define $\bar{\mathcal{K}} \in \Gamma(T\bar{M})$ by

$$\bar{\mathcal{K}}(U^F, V^F) = \mathcal{K}^F(U^F, V^F), \quad \bar{\mathcal{K}}(\partial_t, U^F) = \bar{\mathcal{K}}(U^F, \partial_t) = 0, \quad \bar{\mathcal{K}}(\partial_t, \partial_t) = v \partial_t, \quad (7)$$

where $U^F, V^F \in \Gamma(TF)$, v is a smooth function defined on I . Set $\bar{\nabla} = \bar{\nabla}^{\bar{g}} + \bar{\mathcal{K}}$. Using (6) and (7), we derive

$$\bar{g}(\bar{\mathcal{K}}(U, V), W) = \varrho^2 g_F(\mathcal{K}^F(U^F, V^F), W^F) - v \phi_U \phi_V \phi_W, \quad (8)$$

for any $U, V, W \in \Gamma(T\bar{M})$. Since $\mathcal{K}^F$ satisfies the identities in (3), therefore from (8), it is easy to verify that $\bar{\mathcal{K}}$ also satisfies the identities in (3) and, consequently, $(\bar{\nabla}, \bar{g})$ is a statistical structure on the GRW spacetime $\bar{M}$. Let $\bar{\nabla}^*$ be a dual connection of the statistical connection $\bar{\nabla}$ with respect to $\bar{g}$. Then $(\bar{M} = I \times_{\varrho} F, \bar{g}, \bar{\nabla} = \bar{\nabla}^{\bar{g}} + \bar{\mathcal{K}}, \bar{\nabla}^*)$ is an indefinite statistical manifold, where $\bar{\mathcal{K}}$ holds the identities in (7).

Example 3.2. Consider the upper half-space model of hyperbolic space $F = \{(x, y, z) \in \mathbb{R}^3 : z > 0\}$ with the Riemannian metric

$$g_F = \frac{1}{z^2}(dx^2 + dy^2 + dz^2).$$

By straightforward calculations, we obtain

$$\begin{aligned}\nabla_{\partial x}^{g_F} \partial x &= \frac{1}{z} \partial z, & \nabla_{\partial x}^{g_F} \partial y &= 0, & \nabla_{\partial x}^{g_F} \partial z &= -\frac{1}{z} \partial x, \\ \nabla_{\partial y}^{g_F} \partial x &= 0, & \nabla_{\partial y}^{g_F} \partial y &= \frac{1}{z} \partial z, & \nabla_{\partial y}^{g_F} \partial z &= -\frac{1}{z} \partial y, \\ \nabla_{\partial z}^{g_F} \partial x &= -\frac{1}{z} \partial x, & \nabla_{\partial z}^{g_F} \partial y &= -\frac{1}{z} \partial y, & \nabla_{\partial z}^{g_F} \partial z &= -\frac{1}{z} \partial z.\end{aligned}$$

For the Riemannian metric g_F of (F, g_F) , we define the difference tensor field $\mathcal{K}^F \in \Gamma(TF)$ on F by

$$\begin{aligned}\mathcal{K}^F(\partial x, \partial x) &= \frac{1}{z} \partial z, & \mathcal{K}^F(\partial x, \partial y) &= 0, & \mathcal{K}^F(\partial x, \partial z) &= \frac{1}{z} \partial x, \\ \mathcal{K}^F(\partial y, \partial x) &= 0, & \mathcal{K}^F(\partial y, \partial y) &= \frac{1}{z} \partial z, & \mathcal{K}^F(\partial y, \partial z) &= \frac{1}{z} \partial y, \\ \mathcal{K}^F(\partial z, \partial x) &= \frac{1}{z} \partial x, & \mathcal{K}^F(\partial z, \partial y) &= \frac{1}{z} \partial y, & \mathcal{K}^F(\partial z, \partial z) &= \frac{2}{z} \partial z.\end{aligned}$$

It is straightforward to verify that the difference tensor field $\mathcal{K}^F$ satisfies the following identities

$$\mathcal{K}^F(U^F, V^F) = \mathcal{K}^F(V^F, U^F), \quad g_F(\mathcal{K}^F(U^F, V^F), W^F) = g_F(V^F, \mathcal{K}^F(U^F, W^F)), \quad (9)$$

for any $U^F, V^F, W^F \in \Gamma(TF)$. Thus, $(\nabla^F = \nabla^{g_F} + \mathcal{K}^F, g_F)$ is a statistical structure on (F, g_F) and it is given by

$$\begin{aligned}\nabla_{\partial x}^F \partial x &= \frac{2}{z} \partial z, & \nabla_{\partial x}^F \partial y &= 0, & \nabla_{\partial x}^F \partial z &= 0, \\ \nabla_{\partial y}^F \partial x &= 0, & \nabla_{\partial y}^F \partial y &= \frac{2}{z} \partial z, & \nabla_{\partial y}^F \partial z &= 0, \\ \nabla_{\partial z}^F \partial x &= 0, & \nabla_{\partial z}^F \partial y &= 0, & \nabla_{\partial z}^F \partial z &= \frac{1}{z} \partial z.\end{aligned}$$

Using $\nabla^{*F} = \nabla^{g_F} - \mathcal{K}^F$, the dual statistical connection ∇^{*F} of ∇^F with respect to g_F is given by

$$\begin{aligned}\nabla_{\partial x}^{*F} \partial x &= 0, & \nabla_{\partial x}^{*F} \partial y &= 0, & \nabla_{\partial x}^{*F} \partial z &= -\frac{2}{z} \partial x, \\ \nabla_{\partial y}^{*F} \partial x &= 0, & \nabla_{\partial y}^{*F} \partial y &= 0, & \nabla_{\partial y}^{*F} \partial z &= -\frac{2}{z} \partial y, \\ \nabla_{\partial z}^{*F} \partial x &= -\frac{2}{z} \partial x, & \nabla_{\partial z}^{*F} \partial y &= -\frac{2}{z} \partial y, & \nabla_{\partial z}^{*F} \partial z &= -\frac{3}{z} \partial z.\end{aligned}$$

Thus, $(F, g_F, \nabla^F = \nabla^{g_F} + \mathcal{K}^F, \nabla^{*F})$ is a statistical manifold.

Let $\overline{M} = I \times_{\cosh t} F$ be a warped product manifold, where the base I is an open interval of the real line $\mathbb{R}$, the fiber (F, g_F) is a Riemannian manifold, and the warping function $\cosh t$ is defined on I . Let the warped product manifold $\overline{M} = I \times_{\cosh t} F$ be endowed with metric tensor $\overline{g}$ which is given by

$$\overline{g} = -dt^2 + (\cosh t)^2 g_F, \quad (10)$$

where t serves as a natural parameter on the base I . Let $\bar{\nabla}^{\bar{g}}$ denote the Levi-Civita connection of the warped product manifold $(\bar{M} = I \times_{\cosh t} F, \bar{g})$ with respect to the metric $\bar{g}$. Then, by straightforward calculations, we obtain

$$\begin{aligned}\bar{\nabla}_{\partial t}^{\bar{g}} \partial t &= 0, \quad \bar{\nabla}_{\partial t}^{\bar{g}} \partial x = \frac{\sinh t}{\cosh t} \partial x, \quad \bar{\nabla}_{\partial t}^{\bar{g}} \partial y = \frac{\sinh t}{\cosh t} \partial y, \quad \bar{\nabla}_{\partial t}^{\bar{g}} \partial z = \frac{\sinh t}{\cosh t} \partial z, \\ \bar{\nabla}_{\partial x}^{\bar{g}} \partial t &= \frac{\sinh t}{\cosh t} \partial x, \quad \bar{\nabla}_{\partial x}^{\bar{g}} \partial x = \frac{\sinh t \cosh t}{z^2} \partial t + \frac{1}{z} \partial z, \quad \bar{\nabla}_{\partial x}^{\bar{g}} \partial y = 0, \quad \bar{\nabla}_{\partial x}^{\bar{g}} \partial z = -\frac{1}{z} \partial x, \\ \bar{\nabla}_{\partial y}^{\bar{g}} \partial t &= \frac{\sinh t}{\cosh t} \partial y, \quad \bar{\nabla}_{\partial y}^{\bar{g}} \partial x = 0, \quad \bar{\nabla}_{\partial y}^{\bar{g}} \partial y = \frac{\sinh t \cosh t}{z^2} \partial t + \frac{1}{z} \partial z, \quad \bar{\nabla}_{\partial y}^{\bar{g}} \partial z = -\frac{1}{z} \partial y, \\ \bar{\nabla}_{\partial z}^{\bar{g}} \partial t &= \frac{\sinh t}{\cosh t} \partial z, \quad \bar{\nabla}_{\partial z}^{\bar{g}} \partial x = -\frac{1}{z} \partial x, \quad \bar{\nabla}_{\partial z}^{\bar{g}} \partial y = -\frac{1}{z} \partial y, \quad \bar{\nabla}_{\partial z}^{\bar{g}} \partial z = \frac{\sinh t \cosh t}{z^2} \partial t - \frac{1}{z} \partial z.\end{aligned}$$

For the Lorentzian metric $\bar{g}$ of the warped product manifold $(\bar{M} = I \times_{\cosh t} F, \bar{g})$, we define the difference tensor field $\bar{\mathcal{K}} \in \Gamma(T\bar{M})$ on $\bar{M}$ by

$$\bar{\mathcal{K}}(U^F, V^F) = \mathcal{K}^F(U^F, V^F), \quad \bar{\mathcal{K}}(\partial_t, U^F) = \bar{\mathcal{K}}(U^F, \partial_t) = 0, \quad \bar{\mathcal{K}}(\partial_t, \partial_t) = v \partial_t, \quad (11)$$

where $U^F, V^F \in \Gamma(TF)$ and v is a smooth function defined on I . For any vector field $U \in \Gamma(T\bar{M})$, we can decompose $U = U^F + \phi_U \partial_t$, where $\phi_U = -\bar{g}(U, \partial_t)$. Utilizing (10) and (11), we obtain

$$\bar{g}(\bar{\mathcal{K}}(U, V), W) = (\cosh t)^2 g_F(\mathcal{K}^F(U^F, V^F), W^F) - v \phi_U \phi_V \phi_W,$$

for any $U, V, W \in \Gamma(T\bar{M})$. Since $\mathcal{K}^F$ satisfies the identities in (9), $\bar{\mathcal{K}}$ also satisfies the identities in (9) and consequently $(\bar{\nabla} = \bar{\nabla}^{\bar{g}} + \bar{\mathcal{K}}, \bar{g})$ is a statistical structure on $(\bar{M}, \bar{g})$. Therefore

$$\begin{aligned}\bar{\nabla}_{\partial t} \partial t &= v \partial t, \quad \bar{\nabla}_{\partial t} \partial x = \frac{\sinh t}{\cosh t} \partial x, \quad \bar{\nabla}_{\partial t} \partial y = \frac{\sinh t}{\cosh t} \partial y, \quad \bar{\nabla}_{\partial t} \partial z = \frac{\sinh t}{\cosh t} \partial z, \\ \bar{\nabla}_{\partial x} \partial t &= \frac{\sinh t}{\cosh t} \partial x, \quad \bar{\nabla}_{\partial x} \partial x = \frac{\sinh t \cosh t}{z^2} \partial t + \frac{2}{z} \partial z, \quad \bar{\nabla}_{\partial x} \partial y = 0, \quad \bar{\nabla}_{\partial x} \partial z = 0, \\ \bar{\nabla}_{\partial y} \partial t &= \frac{\sinh t}{\cosh t} \partial y, \quad \bar{\nabla}_{\partial y} \partial x = 0, \quad \bar{\nabla}_{\partial y} \partial y = \frac{\sinh t \cosh t}{z^2} \partial t + \frac{2}{z} \partial z, \quad \bar{\nabla}_{\partial y} \partial z = 0, \\ \bar{\nabla}_{\partial z} \partial t &= \frac{\sinh t}{\cosh t} \partial z, \quad \bar{\nabla}_{\partial z} \partial x = 0, \quad \bar{\nabla}_{\partial z} \partial y = 0, \quad \bar{\nabla}_{\partial z} \partial z = \frac{\sinh t \cosh t}{z^2} \partial t + \frac{1}{z} \partial z.\end{aligned}$$

The dual statistical connection $\bar{\nabla}^*$ of $\bar{\nabla}$ with respect to $\bar{g}$ is given by

$$\begin{aligned}\bar{\nabla}_{\partial t}^* \partial t &= -v \partial t, \quad \bar{\nabla}_{\partial t}^* \partial x = \frac{\sinh t}{\cosh t} \partial x, \quad \bar{\nabla}_{\partial t}^* \partial y = \frac{\sinh t}{\cosh t} \partial y, \quad \bar{\nabla}_{\partial t}^* \partial z = \frac{\sinh t}{\cosh t} \partial z, \\ \bar{\nabla}_{\partial x}^* \partial t &= \frac{\sinh t}{\cosh t} \partial x, \quad \bar{\nabla}_{\partial x}^* \partial x = \frac{\sinh t \cosh t}{z^2} \partial t, \quad \bar{\nabla}_{\partial x}^* \partial y = 0, \quad \bar{\nabla}_{\partial x}^* \partial z = -\frac{2}{z} \partial x, \\ \bar{\nabla}_{\partial y}^* \partial t &= \frac{\sinh t}{\cosh t} \partial y, \quad \bar{\nabla}_{\partial y}^* \partial x = 0, \quad \bar{\nabla}_{\partial y}^* \partial y = \frac{\sinh t \cosh t}{z^2} \partial t, \quad \bar{\nabla}_{\partial y}^* \partial z = -\frac{2}{z} \partial y, \\ \bar{\nabla}_{\partial z}^* \partial t &= \frac{\sinh t}{\cosh t} \partial z, \quad \bar{\nabla}_{\partial z}^* \partial x = -\frac{2}{z} \partial x, \quad \bar{\nabla}_{\partial z}^* \partial y = -\frac{2}{z} \partial y, \quad \bar{\nabla}_{\partial z}^* \partial z = \frac{\sinh t \cosh t}{z^2} \partial t - \frac{3}{z} \partial z.\end{aligned}$$

Thus, $(\bar{M} = I \times_{\cosh t} F, \bar{g}, \bar{\nabla} = \bar{\nabla}^{\bar{g}} + \bar{\mathcal{K}}, \bar{\nabla}^*)$ is an indefinite statistical manifold.

Next, by applying (6), (7) and Lemma 3.1, we have following observations.

Lemma 3.3. Let $(\overline{M} = I \times_{\varrho} F, \overline{g}, \overline{\nabla} = \overline{\nabla}^{\overline{g}} + \overline{\mathcal{K}}, \overline{\nabla}^*)$ be an indefinite statistical manifold such that the fiber (F, g_F) of the GRW spacetime $\overline{M}$ has the statistical structure $(g_F, \nabla^F = \nabla^{g_F} + \mathcal{K}^F, \nabla^{*F})$. Then

$$\overline{\nabla}_U V = \nabla_{U^F}^F V^F + \frac{\varrho'}{\varrho} \overline{g}(U^F, V^F) \partial_t + \frac{\varrho'}{\varrho} (\phi_U V^F + \phi_V U^F) + (U(\phi_V) + v \phi_U \phi_V) \partial_t \quad (12)$$

and

$$\overline{\nabla}_U^* V = \nabla_{U^F}^{*F} V^F + \frac{\varrho'}{\varrho} \overline{g}(U^F, V^F) \partial_t + \frac{\varrho'}{\varrho} (\phi_U V^F + \phi_V U^F) + (U(\phi_V) - v \phi_U \phi_V) \partial_t, \quad (13)$$

for any $U, V \in \Gamma(T\overline{M})$. In particular

$$\begin{aligned} \overline{\nabla}_{\partial_t} \partial_t &= v \partial_t, \\ \overline{\nabla}_{U^F} \partial_t &= \overline{\nabla}_{\partial_t} U^F = \frac{\varrho'}{\varrho} U^F, \\ \overline{\nabla}_{U^F} V^F &= \nabla_{U^F}^F V^F + \frac{\varrho'}{\varrho} \overline{g}(U^F, V^F) \partial_t \end{aligned} \quad (14)$$

and

$$\begin{aligned} \overline{\nabla}_{\partial_t}^* \partial_t &= -v \partial_t, \\ \overline{\nabla}_{U^F}^* \partial_t &= \overline{\nabla}_{\partial_t}^* U^F = \frac{\varrho'}{\varrho} U^F, \\ \overline{\nabla}_{U^F}^* V^F &= \nabla_{U^F}^{*F} V^F + \frac{\varrho'}{\varrho} \overline{g}(U^F, V^F) \partial_t. \end{aligned} \quad (15)$$

Remark 3.4. From this point forward, we will refer to “an indefinite statistical manifold $(\overline{M} = I \times_{\varrho} F, \overline{g}, \overline{\nabla} = \overline{\nabla}^{\overline{g}} + \overline{\mathcal{K}}, \overline{\nabla}^*)$ where the fiber (F, g_F) of the GRW spacetime $\overline{M}$ has the statistical structure $(\nabla^F = \nabla^{g_F} + \mathcal{K}^F, \nabla^{*F}, g_F)$ ” simply as “a GRW spacetime $(\overline{M} = I \times_{\varrho} F, \overline{g}, \overline{\nabla} = \overline{\nabla}^{\overline{g}} + \overline{\mathcal{K}}, \overline{\nabla}^*)$ endowed with a statistical structure”.

We denote the curvature tensor fields of $\overline{\nabla}, \overline{\nabla}^*, \nabla^F$ and ∇^{*F} by $\overline{R}, \overline{R}^*, R^F$ and R^{*F} , respectively. Then, using Lemma 3.3, we have following results which will be used later.

Lemma 3.5. Let $(\overline{M} = I \times_{\varrho} F, \overline{g}, \overline{\nabla} = \overline{\nabla}^{\overline{g}} + \overline{\mathcal{K}}, \overline{\nabla}^*)$ be a GRW spacetime endowed with a statistical structure. Then

$$\begin{aligned} \overline{R}(U^F, V^F) \partial_t &= 0, \quad \overline{R}(U^F, \partial_t) \partial_t = \frac{v \varrho' - \varrho''}{\varrho} U^F, \\ \overline{R}(U^F, \partial_t) V^F &= -\frac{\varrho'' + v \varrho'}{\varrho} \overline{g}(U^F, V^F) \partial_t, \\ \overline{R}(U^F, V^F) W^F &= R^F(U^F, V^F) W^F + \left(\frac{\varrho'}{\varrho}\right)^2 (\overline{g}(V^F, W^F) U^F - \overline{g}(U^F, W^F) V^F) \end{aligned}$$

and

$$\begin{aligned} \overline{R}^*(U^F, V^F) \partial_t &= 0, \quad \overline{R}^*(U^F, \partial_t) \partial_t = -\frac{v \varrho' + \varrho''}{\varrho} U^F, \\ \overline{R}^*(U^F, \partial_t) V^F &= -\frac{\varrho'' - v \varrho'}{\varrho} \overline{g}(U^F, V^F) \partial_t, \\ \overline{R}^*(U^F, V^F) W^F &= R^{*F}(U^F, V^F) W^F + \left(\frac{\varrho'}{\varrho}\right)^2 (\overline{g}(V^F, W^F) U^F - \overline{g}(U^F, W^F) V^F), \end{aligned}$$

for any $U, V, W \in \Gamma(T\overline{M})$.

We denote by $\bar{S}$ the statistical curvature tensor field of a GRW spacetime $(\bar{M} = I \times_{\varrho} F, \bar{g}, \bar{\nabla} = \bar{\nabla}^{\bar{g}} + \bar{\mathcal{K}}, \bar{\nabla}^*)$ endowed with a statistical structure. Consequently, from Lemma 3.5, we obtain the following results.

Lemma 3.6. *Let $(\bar{M} = I \times_{\varrho} F, \bar{g}, \bar{\nabla} = \bar{\nabla}^{\bar{g}} + \bar{\mathcal{K}}, \bar{\nabla}^*)$ be a GRW spacetime endowed with a statistical structure. Then*

$$\begin{aligned}\bar{S}(\partial_t, \partial_t)\partial_t &= 0, \quad \bar{S}(\partial_t, \partial_t)U^F = 0, \quad \bar{S}(U^F, V^F)\partial_t = 0, \\ \bar{S}(U^F, \partial_t)\partial_t &= -\frac{\varrho''}{\varrho}U^F, \\ \bar{S}(U^F, \partial_t)V^F &= -\frac{\varrho''}{\varrho}\bar{g}(U^F, V^F)\partial_t, \\ \bar{S}(U^F, V^F)W^F &= S^F(U^F, V^F)W^F + \left(\frac{\varrho'}{\varrho}\right)^2(\bar{g}(V^F, W^F)U^F - \bar{g}(U^F, W^F)V^F),\end{aligned}$$

for any $U, V, W \in \Gamma(T\bar{M})$, where S^F is the statistical curvature tensor field of the statistical structure (∇^F, g_F) .

Furthermore, using Lemma 3.6, we get

$$\begin{aligned}\bar{S}(U, V)W &= S^F(U^F, V^F)W^F + \left(\frac{\varrho'}{\varrho}\right)^2(\bar{g}(V^F, W^F)U^F - \bar{g}(U^F, W^F)V^F) \\ &\quad + \frac{\varrho''}{\varrho}(\bar{g}(V^F, W^F)\phi_U - \bar{g}(U^F, W^F)\phi_V)\partial_t \\ &\quad + \frac{\varrho''}{\varrho}(\phi_U\phi_WV^F - \phi_V\phi_WU^F).\end{aligned}$$

From (6), we know

$$\bar{g}(U, V) = -\phi_U\phi_V + \bar{g}(U^F, V^F), \quad (16)$$

for any $U, V \in \Gamma(T\bar{M})$; therefore we further have

$$\begin{aligned}\bar{S}(U, V)W &= S^F(U^F, V^F)W^F + \left(\frac{\varrho'}{\varrho}\right)^2(\bar{g}(V, W)U - \bar{g}(U, W)V) \\ &\quad + \frac{(\varrho')^2 - \varrho\varrho''}{\varrho^2}\{(\phi_V\phi_WU - \phi_U\phi_WV) \\ &\quad + (\bar{g}(U, W)\phi_V - \bar{g}(V, W)\phi_U)\partial_t\}.\end{aligned} \quad (17)$$

Let the statistical manifold $(F, \nabla^F = \nabla^{g_F} + \mathcal{K}^F, \nabla^{*F}, g_F)$ be of constant statistical curvature c . Then

$$S^F(U^F, V^F)W^F = c\{g_F(V^F, W^F)U^F - g_F(U^F, W^F)V^F\}. \quad (18)$$

Using (5) and (16) in (18), we obtain

$$\begin{aligned}S^F(U^F, V^F)W^F &= \frac{c}{\varrho^2}\{(\bar{g}(V, W)U - \bar{g}(U, W)V) + (\phi_V\phi_WU - \phi_U\phi_WV) \\ &\quad + (\bar{g}(U, W)\phi_V - \bar{g}(V, W)\phi_U)\partial_t\}.\end{aligned} \quad (19)$$

Hence, from (17) and (19), we derive the following result for later uses.

Theorem 3.7. *Let $(\bar{M} = I \times_{\varrho} F, \bar{g}, \bar{\nabla} = \bar{\nabla}^{\bar{g}} + \bar{\mathcal{K}}, \bar{\nabla}^*)$ be a GRW spacetime endowed with a statistical structure. If the statistical manifold $(F, g_F, \nabla^F = \nabla^{g_F} + \mathcal{K}^F, \nabla^{*F})$ is of constant statistical curvature c , then*

$$\begin{aligned}\bar{S}(U, V)W &= \frac{c + (\varrho')^2}{\varrho^2}(\bar{g}(V, W)U - \bar{g}(U, W)V) + \frac{(\varrho')^2 - \varrho\varrho'' + c}{\varrho^2}\{(\phi_V\phi_WU - \phi_U\phi_WV) \\ &\quad + (\bar{g}(U, W)\phi_V - \bar{g}(V, W)\phi_U)\partial_t\},\end{aligned} \quad (20)$$

for any $U, V, W \in \Gamma(T\bar{M})$.

Let $\overline{Ric}^{\overline{S}}$ denote the statistical Ricci tensor of a GRW spacetime $(\overline{M} = I \times_{\varrho} F, \overline{g}, \overline{\nabla} = \overline{\nabla}^{\overline{g}} + \overline{\mathcal{K}}, \overline{\nabla}^*)$ endowed with a statistical structure, defined by the statistical curvature tensor field $\overline{S}$. Consider an orthonormal frame of fields $\{e_1, \dots, e_{n+1}, \partial_t\}$ on $\overline{M}$. Then for any $U, V \in \Gamma(T\overline{M})$, we have

$$\overline{Ric}^{\overline{S}}(U, V) = \sum_{i=1}^{n+1} \overline{g}(\overline{S}(e_i, U)V, e_i) - \overline{g}(\overline{S}(\partial_t, U)V, \partial_t). \quad (21)$$

Since $\{e_1, \dots, e_{n+1}, \partial_t\}$ is an orthonormal frame of fields on $\overline{M}$, therefore from (5), $\{\varrho e_1, \dots, \varrho e_{n+1}\}$ is an orthonormal frame of fields on the fiber F . Hence by straightforward calculations, utilizing (17) and (21), we obtain the following result.

Theorem 3.8. Let $(\overline{M} = I \times_{\varrho} F, \overline{g}, \overline{\nabla} = \overline{\nabla}^{\overline{g}} + \overline{\mathcal{K}}, \overline{\nabla}^*)$ be a GRW spacetime endowed with a statistical structure. Then

$$\overline{Ric}^{\overline{S}}(U, V) = Ric^F(U^F, V^F) + n\left(\frac{\varrho'}{\varrho}\right)^2(\overline{g}(U, V) + \phi_U\phi_V) + \frac{\varrho''}{\varrho}(\overline{g}(U, V) - n\phi_U\phi_V), \quad (22)$$

for any $U, V \in \Gamma(T\overline{M})$, where $Ric^F(U^F, V^F)$ denotes the statistical Ricci tensor of the fiber F .

Moreover, if the statistical manifold $(F, g_F, \nabla^F = \nabla^{g_F} + \mathcal{K}^F, \nabla^{*F})$ is of constant statistical curvature c , then

$$\overline{Ric}^{\overline{S}}(U, V) = n\left(\frac{c + \left(\frac{\varrho'}{\varrho}\right)^2}{\varrho^2}\right)(\overline{g}(U, V) + \phi_U\phi_V) + \frac{\varrho''}{\varrho}(\overline{g}(U, V) - n\phi_U\phi_V), \quad (23)$$

for any $U, V \in \Gamma(T\overline{M})$.

Let $\Pi = \text{span}\{U, V\}$ be a non-degenerate plane spanned by orthonormal vector fields $U, V \in \Gamma(T\overline{M})$. Then using (17), the statistical sectional curvature $K_{\Pi}^{\overline{S}}(U, V)$ of the plane Π is given by

$$\begin{aligned} K_{\Pi}^{\overline{S}}(U, V) &= \overline{g}(S^F(U^F, V^F)V^F, U^F) + \left(\frac{\varrho'}{\varrho}\right)^2(\overline{g}(U, U)\overline{g}(V, V) - \overline{g}(U, V)^2) \\ &\quad + \frac{(\varrho')^2 - \varrho\varrho''}{\varrho^2}\{(\phi_U^2\overline{g}(U, U) - \phi_U\phi_V\overline{g}(U, V)) - (\overline{g}(U, V)\phi_V - \overline{g}(V, V)\phi_U)\phi_U\}. \end{aligned}$$

This further implies

$$K_{\Pi}^{\overline{S}}(U, V) = \overline{g}(S^F(U^F, V^F)V^F, U^F) + \left(\frac{\varrho'}{\varrho}\right)^2 + \frac{(\varrho')^2 - \varrho\varrho''}{\varrho^2}(\phi_U^2 + \phi_V^2). \quad (24)$$

On the other hand, using (5) and (16), we derive

$$\begin{aligned} K_{\Pi}^F(U^F, V^F) &= \frac{g_F(S^F(U^F, V^F)V^F, U^F)}{g_F(U^F, U^F)g_F(V^F, V^F) - g_F(U^F, V^F)^2} \\ &= \varrho^2 \frac{\overline{g}(S^F(U^F, V^F)V^F, U^F)}{(\overline{g}(U, U) + \phi_U^2)(\overline{g}(V, V) + \phi_V^2) - (\overline{g}(U, V) + \phi_U\phi_V)^2} \\ &= \varrho^2 \frac{\overline{g}(S^F(U^F, V^F)V^F, U^F)}{1 + \phi_U^2 + \phi_V^2}, \end{aligned} \quad (25)$$

for any orthonormal vector fields U, V . Hence, substituting (25) in (24), we obtain

$$\begin{aligned} K_{\Pi}^{\overline{S}}(U, V) &= \frac{(1 + \phi_U^2 + \phi_V^2)}{\varrho^2} K_{\Pi}^F(U^F, V^F) + \left(\frac{\varrho'}{\varrho}\right)^2(1 + \phi_U^2 + \phi_V^2) \\ &\quad - \frac{\varrho''}{\varrho}(\phi_U^2 + \phi_V^2). \end{aligned}$$

Consequently from the previous expression, we have the following result.

Theorem 3.9. Let $(\bar{M} = I \times_{\varrho} F, \bar{g}, \bar{\nabla} = \bar{\nabla}^{\bar{g}} + \bar{\mathcal{K}}, \bar{\nabla}^*)$ be a GRW spacetime endowed with a statistical structure. Let the statistical manifold $(F, g_F, \nabla^F = \nabla^{g_F} + \mathcal{K}^F, \nabla^{*F})$ be of constant statistical curvature c . Then

$$K_{\Pi}^{\bar{S}}(U, V) = \alpha + \beta(\phi_U^2 + \phi_V^2),$$

where

$$\alpha = \frac{c + (\varrho')^2}{\varrho^2} \text{ and } \beta = \frac{c + (\varrho')^2 - \varrho\varrho''}{\varrho^2}. \quad (26)$$

Let M be an $(n+1)$ -dimensional manifold. An immersion $\iota : M \rightarrow \bar{M}$, where $\bar{M}$ is the GRW spacetime, is said to be *spacelike* if the Lorentzian metric $\bar{g}$ given in (5) induces, via ι , a Riemannian metric g on M . In this case, M is called a *spacelike hypersurface* of $\bar{M}$. Since for the GRW spacetime $\bar{M}$, the coordinate vector field ∂_t is globally defined and timelike, then $\bar{M}$ is time-orientable. Therefore, for any spacelike hypersurface M of $\bar{M}$, there exists a unique unitary timelike normal vector field N globally defined on M with the same time-orientation as ∂_t , i.e., $\bar{g}(N, \partial_t) < 0$. Utilizing the reverse Cauchy-Schwarz inequality [19], we have $\bar{g}(N, \partial_t) = -\cosh \varphi \leq -1$, and the equality holds at a point $p \in M$ if and only if $N = \partial_t$, where φ represents the hyperbolic angle between N and ∂_t .

In the GRW spacetime there exists a distinguished family of spacelike hypersurfaces, known as its *spacelike slices*. A spacelike slice in the GRW spacetime $\bar{M} = I \times_{\varrho} F$ corresponds to a spacelike hypersurface defined by a constant value of t . Consequently, a spacelike slice is given by $\pi_I^{-1}(t_0) = \{t_0\} \times_{\varrho(t_0)} F$, for $t_0 \in I$. A spacelike hypersurface in $\bar{M}$ is a spacelike slice if and only if the hyperbolic angle φ is identically zero.

Assume that (M, g) is a spacelike hypersurface of a GRW spacetime $(\bar{M} = I \times_{\varrho} F, \bar{g}, \bar{\nabla} = \bar{\nabla}^{\bar{g}} + \bar{\mathcal{K}}, \bar{\nabla}^*)$ endowed with a statistical structure. Then, the local Gauss-Weingarten formulas on TM are

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \quad \bar{\nabla}_X N = -A_N X + \tau(X)N, \quad (27)$$

$$\bar{\nabla}_X^* Y = \nabla_X^* Y + B^*(X, Y)N, \quad \bar{\nabla}_X^* N = -A_N^* X + \tau^*(X)N, \quad (28)$$

for any $X, Y \in \Gamma(TM)$. Here, ∇ and ∇^* denote the induced connections of $\bar{\nabla}$ and $\bar{\nabla}^*$ on the spacelike hypersurface (M, g) , respectively. Obviously, (M, g, ∇, ∇^*) forms a statistical manifold, with ∇^* being the dual connection of ∇ with respect to the induced metric g from $\bar{g}$. The shape operators A_N and A_N^* on TM are associated with the normal vector field N , while τ and τ^* are 1-forms defined on TM .

The normal vector field N is a unitary timelike vector field and then the local second fundamental forms B and B^* on TM are expressed as $B(X, Y) = -\bar{g}(\bar{\nabla}_X Y, N)$ and $B^*(X, Y) = -\bar{g}(\bar{\nabla}_X^* Y, N)$, respectively. By differentiating $\bar{g}(N, N)$ along X , we have $X\bar{g}(N, N) = \bar{g}(\bar{\nabla}_X N, N) + \bar{g}(N, \bar{\nabla}_X^* N)$. Since N is a unitary timelike vector field, by applying equations (27) and (28), it yields

$$\tau = -\tau^*. \quad (29)$$

In a similar manner, applying (2), we obtain

$$B(X, Y) = -g(A_N^* X, Y), \quad B^*(X, Y) = -g(A_N X, Y), \quad (30)$$

for any $X, Y \in \Gamma(TM)$.

Let (M, g, ∇, ∇^*) be a statistical spacelike hypersurface of a GRW spacetime $(\bar{M} = I \times_{\varrho} F, \bar{g}, \bar{\nabla} = \bar{\nabla}^{\bar{g}} + \bar{\mathcal{K}}, \bar{\nabla}^*)$ endowed with a statistical structure. The spacelike hypersurface M is said to be *totally umbilical* with respect to $\bar{\nabla}$ (or $\bar{\nabla}^*$) if the condition $B = \mathbb{H} \otimes g$ (or $B^* = \mathbb{H}^* \otimes g$) is satisfied. Additionally, M is described as *totally geodesic* with respect to $\bar{\nabla}$ (or $\bar{\nabla}^*$) if its local second fundamental form B (or B^*) associated with $\bar{\nabla}$ (or $\bar{\nabla}^*$)

vanishes identically. The vector fields $\mathbb{H}$ and $\mathbb{H}^*$ are referred to as the *mean curvature vector fields* on M and are defined respectively by

$$\mathbb{H} = \frac{1}{n+1} \sum_{i=1}^{n+1} B(e_i, e_i) \text{ and } \mathbb{H}^* = \frac{1}{n+1} \sum_{i=1}^{n+1} B^*(e_i, e_i), \quad (31)$$

where $\{e_1, \dots, e_{n+1}\}$ is an orthonormal frame on M .

Let (M, g, ∇, ∇^*) be a statistical spacelike hypersurface of a GRW spacetime $(\bar{M} = I \times_{\varrho} F, \bar{g}, \bar{\nabla} = \bar{\nabla}^{\bar{g}} + \bar{\mathcal{K}}, \bar{\nabla}^*)$ endowed with a statistical structure. Let R and R^* denote the curvature tensors of the induced connections ∇ and ∇^* on M , respectively. By applying equations (27), (28), and (4), and performing straightforward calculations, we obtain the following significant expression

$$\begin{aligned} 2\bar{g}(\bar{S}(X, Y)Z, W) &= 2g(S(X, Y)Z, W) + B(Y, Z)B^*(X, W) - B(X, Z)B^*(Y, W) \\ &\quad + B^*(Y, Z)B(X, W) - B^*(X, Z)B(Y, W), \end{aligned} \quad (32)$$

for any $X, Y, Z, W \in \Gamma(TM)$, where S denotes the statistical curvature tensor field on the statistical spacelike hypersurface M . Let Ric^S denote the induced statistical Ricci tensor of the statistical spacelike hypersurface M of the GRW spacetime $(\bar{M} = I \times_{\varrho} F, \bar{g}, \bar{\nabla} = \bar{\nabla}^{\bar{g}} + \bar{\mathcal{K}}, \bar{\nabla}^*)$ endowed with a statistical structure. Let $\{e_1, \dots, e_{n+1}\}$ be an orthonormal frame on M . Then

$$Ric^S(X, Y) = \sum_{i=1}^{n+1} g(S(e_i, X)Y, e_i),$$

for any $X, Y \in \Gamma(TM)$. Since $\mathcal{N}$ is a globally defined, unique unitary timelike normal vector field on M , by applying (32), we obtain

$$\begin{aligned} Ric^S(X, Y) &= \left(\sum_{i=1}^{n+1} \bar{g}(\bar{S}(e_i, X)Y, e_i) - \bar{g}(\bar{S}(\mathcal{N}, X)Y, \mathcal{N}) \right) + \bar{g}(\bar{S}(\mathcal{N}, X)Y, \mathcal{N}) \\ &\quad - \frac{1}{2} \sum_{i=1}^{n+1} \left(B(X, Y)B^*(e_i, e_i) - B(e_i, Y)B^*(X, e_i) + B^*(X, Y)B(e_i, e_i) \right. \\ &\quad \left. - B^*(e_i, Y)B(X, e_i) \right). \end{aligned}$$

Further, utilizing (30) and (31), it leads to

$$\begin{aligned} Ric^S(X, Y) &= \bar{Ric}^{\bar{S}}(X, Y) + \bar{g}(\bar{S}(\mathcal{N}, X)Y, \mathcal{N}) + \frac{1}{2} \left(g(A_{\mathcal{N}}X, A_{\mathcal{N}}^*Y) + g(A_{\mathcal{N}}Y, A_{\mathcal{N}}^*X) \right. \\ &\quad \left. - (n+1)(\mathbb{H}B^*(X, Y) + \mathbb{H}^*B(X, Y)) \right), \end{aligned} \quad (33)$$

for any $X, Y \in \Gamma(TM)$.

Let $\mathcal{Q}^S$ and $\bar{\mathcal{Q}}^{\bar{S}}$ denote self-adjoint operators on M and $\bar{M}$, respectively. They are defined by $Ric^S(X, Y) = g(\mathcal{Q}^S X, Y)$ for any $X, Y \in \Gamma(TM)$, and $\bar{Ric}^{\bar{S}}(U, V) = \bar{g}(\bar{\mathcal{Q}}^{\bar{S}} U, V)$ for any $U, V \in \Gamma(T\bar{M})$. In particular, we denote $\bar{Ric}^{\bar{S}}(X, Y) = g(\bar{\mathcal{Q}}^{\bar{S}}|_M X, Y)$. Then using (30) in (33), we obtain the following result.

Theorem 3.10. *Let (M, g, ∇, ∇^*) be a statistical spacelike hypersurface of a GRW spacetime $(\bar{M} = I \times_{\varrho} F, \bar{g}, \bar{\nabla} = \bar{\nabla}^{\bar{g}} + \bar{\mathcal{K}}, \bar{\nabla}^*)$ endowed with a statistical structure. Then*

$$\begin{aligned} \mathcal{Q}^S X &= \bar{\mathcal{Q}}^{\bar{S}}|_M X - \bar{S}(\mathcal{N}, X)\mathcal{N} + \frac{1}{2} \left((A_{\mathcal{N}} \circ A_{\mathcal{N}}^*)X + (A_{\mathcal{N}}^* \circ A_{\mathcal{N}})X \right) \\ &\quad + \frac{1}{2}(n+1)(\mathbb{H}A_{\mathcal{N}}X + \mathbb{H}^*A_{\mathcal{N}}^*X), \end{aligned} \quad (34)$$

for any vector field $X \in \Gamma(TM)$, where $A_{\mathcal{N}}$ and $A_{\mathcal{N}}^*$ are the corresponding shape operators.

Let $\overline{\Omega}^{\overline{S}}$ and Ω^S denote the statistical scalar curvatures of the GRW spacetime $(\overline{M} = I \times_{\varrho} F, \overline{g}, \overline{\nabla} = \overline{\nabla}^{\overline{g}} + \overline{\mathcal{K}}, \overline{\nabla}^*)$ endowed with a statistical structure and its statistical spacelike hypersurface (M, g, ∇, ∇^*) , respectively. Therefore

$$\Omega^S = \sum_{i=1}^{n+1} Ric^S(e_i, e_i),$$

and by using (33), we get

$$\begin{aligned} \Omega^S &= \sum_{i=1}^{n+1} \overline{Ric}^{\overline{S}}(e_i, e_i) - \overline{Ric}^{\overline{S}}(\mathcal{N}, \mathcal{N}) + \overline{Ric}^{\overline{S}}(\mathcal{N}, \mathcal{N}) + \sum_{i=1}^{n+1} \overline{g}(\overline{S}(\mathcal{N}, e_i)e_i, \mathcal{N}) \\ &\quad + \frac{1}{2} \sum_{i=1}^{n+1} \left(2g(A_{\mathcal{N}}e_i, A_{\mathcal{N}}^*e_i) - (n+1)(\mathbb{H}B^*(e_i, e_i) + \mathbb{H}^*B(e_i, e_i)) \right). \end{aligned}$$

This further gives

$$\Omega^S = \overline{\Omega}^{\overline{S}} + 2\overline{Ric}^{\overline{S}}(\mathcal{N}, \mathcal{N}) + \sum_{i=1}^{n+1} g(A_{\mathcal{N}}e_i, A_{\mathcal{N}}^*e_i) - (n+1)^2\mathbb{H}\mathbb{H}^*.$$

Using (30), it is straightforward to verify that $g(A_{\mathcal{N}}e_i, A_{\mathcal{N}}^*e_i) = g(A_{\mathcal{N}}^*A_{\mathcal{N}}e_i, e_i) = g(A_{\mathcal{N}}A_{\mathcal{N}}^*e_i, e_i)$. Consequently, the above expression simplifies to

$$\Omega^S = \overline{\Omega}^{\overline{S}} + 2\overline{Ric}^{\overline{S}}(\mathcal{N}, \mathcal{N}) + \|A_{\mathcal{N}} \circ A_{\mathcal{N}}^*\| - (n+1)^2\mathbb{H}\mathbb{H}^*,$$

where $\|A_{\mathcal{N}} \circ A_{\mathcal{N}}^*\| = \sum_{i=1}^{n+1} g(A_{\mathcal{N}}A_{\mathcal{N}}^*e_i, e_i)$.

4. Ricci Solitons on Spacelike Hypersurfaces

A natural generalization of an Einstein metric is a Ricci soliton [15]. On a Riemannian manifold $(\overline{M}, \overline{g})$, a *Ricci soliton* [9] is a stationary solution of the Ricci flow equation $\frac{\partial}{\partial t}\overline{g}(t) = -2\overline{Ric}(t)$, given by $\overline{g}(t) = \chi(t)\pi_t^*$ with $\overline{g}(0) = \overline{g}$, where $\overline{Ric}(t)$ is the Ricci tensor of the evolving metric $\overline{g}(t)$, π_t represents diffeomorphisms of $\overline{M}$, and $\chi(t)$ is the scaling function. Perelman used the concept of Ricci solitons to solve the long-awaited Poincaré conjecture, which was posed in 1904. Since then, the notion of Ricci solitons has become a topic of interest for mathematicians and physicists. A complete Riemannian manifold $(\overline{M}, \overline{g})$ is considered a Ricci soliton if there exists a vector field $\xi \in \Gamma(T\overline{M})$, referred to as the potential vector field, satisfying the following relation

$$\overline{Ric} + \frac{1}{2}\mathcal{L}_{\xi}\overline{g} = \lambda\overline{g},$$

where $\mathcal{L}_{\xi}\overline{g}$ denotes the Lie derivative of $\overline{g}$ along ξ , $\overline{Ric}$ is the Ricci tensor of $\overline{M}$ and λ is constant. The Ricci soliton is denoted by $(\overline{M}, \overline{g}, \xi, \lambda)$. A Ricci soliton is classified as shrinking, steady, or expanding if λ is positive, zero, or negative, respectively. A Ricci soliton is *trivial* if ξ is zero or Killing, which implies that the metric is Einstein. The concept of a Ricci soliton is further generalized to an almost Ricci soliton [22], where the condition of λ being constant is relaxed.

Lemma 4.1. Let $(\overline{M} = I \times_{\varrho} F, \overline{g}, \overline{\nabla} = \overline{\nabla}^{\overline{g}} + \overline{\mathcal{K}}, \overline{\nabla}^*)$ be a GRW spacetime endowed with a statistical structure. Then

$$\begin{aligned} (\mathcal{E}_W\overline{g})(U, V) &= \varrho^2(\mathcal{E}_{W^F}g_F)(U^F, V^F) - (U(\phi_W))\phi_V - (V(\phi_W))\phi_U \\ &\quad + 2\varrho\varrho'\phi_Wg_F(U^F, V^F), \end{aligned} \tag{35}$$

for any vector fields $U, V, W \in \Gamma(T\overline{M})$.

Proof. It is known that $(\mathcal{E}_W \bar{g})(U, V) = \bar{g}(\bar{\nabla}_U^{\bar{g}} W, V) + \bar{g}(U, \bar{\nabla}_V^{\bar{g}} W)$, for any vector fields $U, V, W \in \Gamma(T\bar{M})$. Since $\bar{\nabla}^{\bar{g}} = \frac{1}{2}(\bar{\nabla} + \bar{\nabla}^*)$ and using the Codazzi equation, it implies that

$$\begin{aligned} (\mathcal{E}_W \bar{g})(U, V) &= \frac{1}{2}(\bar{g}(\bar{\nabla}_U W, V) + \bar{g}(U, \bar{\nabla}_V W)) + \frac{1}{2}(\bar{g}(\bar{\nabla}_U^* W, V) + \bar{g}(U, \bar{\nabla}_V^* W)) \\ &= \bar{g}(\bar{\nabla}_U W, V) + \bar{g}(U, \bar{\nabla}_V W). \end{aligned} \quad (36)$$

Utilizing (12) and (13), we obtain the required relation (35). $\square$

Chen [7] provided a useful characterization theorem, stating that a Lorentzian manifold $\bar{M}$ of dimension $n \geq 3$ is a local GRW spacetime $\bar{M} = I \times_{\varrho} F$ if and only if it possesses a timelike concircular vector field. Chen demonstrated that the vector field $\zeta = \varrho \partial_t$ satisfies this requirement.

Let $(\bar{M} = I \times_{\varrho} F, \bar{g}, \bar{\nabla} = \bar{\nabla}^{\bar{g}} + \bar{\mathcal{K}}, \bar{\nabla}^*)$ be a GRW spacetime endowed with a statistical structure. Utilizing equations (6), (14), and (15), we have

$$\bar{\nabla}_U \zeta = \varrho' U + \nu \phi_U \zeta \text{ and } \bar{\nabla}_U^* \zeta = \varrho' U - \nu \phi_U \zeta, \quad (37)$$

for any vector field $U \in \Gamma(T\bar{M})$.

Let (M, g, ∇, ∇^*) be a statistical spacelike hypersurface of a GRW spacetime $(\bar{M} = I \times_{\varrho} F, \bar{g}, \bar{\nabla} = \bar{\nabla}^{\bar{g}} + \bar{\mathcal{K}}, \bar{\nabla}^*)$ endowed with a statistical structure. Then, the closed conformal timelike vector field ζ can be expressed as $\zeta = \zeta^T + \theta \mathcal{N}$, where $\theta = -\bar{g}(\zeta, \mathcal{N}) < 0$ is the support function on M and ζ^T is the tangential component of ζ . In this context, θ is given by $\theta = \varrho \cosh \varphi$. Similarly, the vector field ∂t can also be represented as $\partial t = (\partial t)^T + \cosh \varphi \mathcal{N}$, where $(\partial t)^T$ denotes the tangential component of ∂t and $\mathcal{N} = \mathcal{N}^F + \cosh \varphi \partial t$. One has $\bar{g}(\mathcal{N}, \mathcal{N}) = \bar{g}(\partial t, \partial t) = -1$, $\bar{g}(\mathcal{N}^F, \mathcal{N}^F) = \bar{g}((\partial t)^T, (\partial t)^T) = \sinh^2 \varphi$.

Remark 4.2. Utilizing (35), we have

$$(\mathcal{E}_{\zeta} \bar{g})(U, V) = -(U(\phi_{\zeta}))\phi_V - (V(\phi_{\zeta}))\phi_U + 2\varrho\varrho'\phi_{\zeta}g_F(U^F, V^F),$$

for any vector fields $U, V \in \Gamma(T\bar{M})$. It is known that $\phi_{\zeta} = -\bar{g}(\zeta, \partial t) = \varrho$, therefore using (16), it follows that

$$(\mathcal{E}_{\zeta} \bar{g})(U, V) = 2\varrho'\bar{g}(U, V). \quad (38)$$

Hence, ζ is a conformal vector field on $\bar{M}$ with conformal factor ϱ' .

Next, from (27), we have

$$\bar{\nabla}_X \zeta = \nabla_X \zeta^T + B(X, \zeta^T)\mathcal{N} + (X\theta)\mathcal{N} + \theta(-A_{\mathcal{N}}X + \tau(X)\mathcal{N}),$$

for any vector field $X \in \Gamma(TM)$.

By comparing the tangential and normal components of the expression and using equations (30) and (37), we derive

$$\nabla_X \zeta^T - \theta A_{\mathcal{N}}X - \nu \phi_X \zeta^T = \varrho' X,$$

and

$$g(A_{\mathcal{N}}^* \zeta^T, X) - (X\theta) - \theta \tau(X) + \nu \theta \phi_X = 0, \quad (39)$$

respectively. Similarly

$$\nabla_X^* \zeta^T - \theta A_{\mathcal{N}}^* X + \nu \phi_X \zeta^T = \varrho' X$$

and

$$g(A_{\mathcal{N}} \zeta^T, X) - (X\theta) - \theta \tau^*(X) - \nu \theta \phi_X = 0. \quad (40)$$

By adding equations (39) and (40), and using (29), we obtain $2(X\theta) = \bar{g}(A_{\mathcal{N}} \zeta^T, X) + \bar{g}(A_{\mathcal{N}}^* \zeta^T, X)$. Consequently, the following result arises immediately.

Proposition 4.3. Let (M, g, ∇, ∇^*) be a statistical spacelike hypersurface of a GRW spacetime $(\bar{M} = I \times_{\varrho} F, \bar{g}, \bar{\nabla} = \bar{\nabla}^{\bar{g}} + \bar{\mathcal{K}}, \bar{\nabla}^*)$ endowed with a statistical structure. Then, the support function θ on M is constant function if and only if $A_N \zeta^T = -A_N^* \zeta^T$.

Using (2), we have $(\mathcal{E}_{\zeta} \bar{g})(X, Y) = \bar{g}(\bar{\nabla}_X \zeta, Y) + \bar{g}(X, \bar{\nabla}_Y^* \zeta)$ for any $X, Y \in \Gamma(TM)$. Since $\zeta = \zeta^T + \theta N$, and applying equations (27) and (28), we also obtain

$$\begin{aligned} (\mathcal{E}_{\zeta} \bar{g})(X, Y) &= \bar{g}(\nabla_X \zeta^T, Y) - \theta \bar{g}(A_N X, Y) + \bar{g}(X, \nabla_Y^* \zeta^T) - \theta \bar{g}(X, A_N^* Y) \\ &= (\mathcal{E}_{\zeta^T} \bar{g})(X, Y) + \theta(B(X, Y) + B^*(X, Y)) \\ &= (\mathcal{E}_{\zeta^T} \bar{g})(X, Y) + 2\theta B^{\bar{g}}(X, Y), \end{aligned}$$

where $B^{\bar{g}}(X, Y)$ denotes the local second fundamental form on TM with respect to the Levi-Civita connection $\bar{\nabla}^{\bar{g}}$ on $\bar{M}$. Hence using (38), we further obtain

$$(\mathcal{E}_{\zeta^T} \bar{g})(X, Y) = 2(\varrho' g(X, Y) - \theta B^{\bar{g}}(X, Y)), \quad (41)$$

for any vector fields $X, Y \in \Gamma(TM)$. Consequently, we have the following result immediately.

Proposition 4.4. Let (M, g, ∇, ∇^*) be a statistical spacelike hypersurface of a GRW spacetime $(\bar{M} = I \times_{\varrho} F, \bar{g}, \bar{\nabla} = \bar{\nabla}^{\bar{g}} + \bar{\mathcal{K}}, \bar{\nabla}^*)$ endowed with a statistical structure. Then:

- (i) The vector field ζ^T on M is conformal if and only if M is totally umbilical with respect to the Levi-Civita connection $\bar{\nabla}^{\bar{g}}$.
- (ii) Let the vector field ζ^T on M be either conformal or Killing. Then M is totally umbilical with respect to $\bar{\nabla}$ if and only if it is totally umbilical with respect to $\bar{\nabla}^*$.

Theorem 4.5. Let (M, g, ∇, ∇^*) be a statistical spacelike hypersurface of a GRW spacetime $(\bar{M} = I \times_{\varrho} F, \bar{g}, \bar{\nabla} = \bar{\nabla}^{\bar{g}} + \bar{\mathcal{K}}, \bar{\nabla}^*)$ endowed with a statistical structure. Then, the quadruple (M, g, ζ^T, λ) is a statistical Ricci soliton if and only if

$$\mathcal{Q}^S = \frac{1}{n+1} \Omega^S - \frac{1}{2} \theta (\mathbb{H} + \mathbb{H}^* + A_N + A_N^*),$$

with $\mathcal{Q}^S, \Omega^S, \mathbb{H}, \mathbb{H}^*, A_N$ and A_N^* , defined before.

Proof. Assume that (M, g, ζ^T, λ) is a statistical Ricci soliton, satisfying the equation

$$\text{Ric}^S(X, Y) + \frac{1}{2} (\mathcal{E}_{\zeta^T} g)(X, Y) = \lambda g(X, Y), \quad (42)$$

for any vector fields $X, Y \in \Gamma(TM)$. Utilizing (41), we derive the expression

$$\text{Ric}^S(X, Y) = (\lambda - \varrho') g(X, Y) + \theta B^{\bar{g}}(X, Y).$$

This further implies

$$\mathcal{Q}^S X = (\lambda - \varrho') X - \frac{1}{2} \theta (A_N X + A_N^* X), \quad (43)$$

where $\mathcal{Q}^S$ is the self-adjoint Ricci operator on M which is defined by $\text{Ric}^S(X, Y) = g(\mathcal{Q}^S X, Y)$.

Let $\{e_1, \dots, e_{n+1}\}$ be an orthonormal frame on M . By contracting equation (42) with this frame, it follows that

$$\Omega^S + \text{div}(\zeta^T) = (n+1)\lambda. \quad (44)$$

On the other hand, by contracting equation (41) with this frame, we obtain

$$\operatorname{div}(\zeta^T) = (n+1)\left(\varrho' - \frac{1}{2}\theta(\mathbb{H} + \mathbb{H}^*)\right). \quad (45)$$

From (44) and (45), it follows that

$$\lambda - \varrho' = \frac{1}{n+1}\left(\Omega^S - \frac{1}{2}(n+1)\theta(\mathbb{H} + \mathbb{H}^*)\right). \quad (46)$$

Hence, from (43) and (46), the proof is completed. $\square$

Theorem 4.6. Let (M, g, ∇, ∇^*) be a statistical spacelike hypersurface of a GRW spacetime $(\overline{M} = I \times_{\varrho} F, \overline{g}, \overline{\nabla} = \overline{\nabla}^g + \overline{\mathcal{K}}, \overline{\nabla}^*)$ endowed with a statistical structure. Let the statistical manifold $(F, g_F, \nabla^F = \nabla^{g_F} + \mathcal{K}^F, \nabla^{*F})$ be of constant statistical curvature c . If the quadruple (M, g, ζ^T, λ) is a statistical Ricci soliton, then the statistical scalar curvature Ω^S of the statistical spacelike hypersurface M is of the form

$$\Omega^S = n(n+1)\alpha + \beta\left((n-1)\sinh^2 \varphi + (n+1)\left(\frac{\theta^2}{\varrho^2} - 1\right)\right) + \|A_{\mathcal{N}} \circ A_{\mathcal{N}}^*\| - (n+1)^2 \mathbb{H} \mathbb{H}^*, \quad (47)$$

where α and β are given in equation (26).

Proof. Let the quadruple (M, g, ζ^T, λ) be a statistical Ricci soliton. Then, utilizing (34) and (43), the Ricci operator $\overline{\mathcal{Q}}^{\overline{S}}$ on $\overline{M}$ is given by

$$\begin{aligned} \overline{\mathcal{Q}}^{\overline{S}}|_M X &= \overline{S}(\mathcal{N}, X)\mathcal{N} - \frac{1}{2}\left((A_{\mathcal{N}} \circ A_{\mathcal{N}}^*)X + (A_{\mathcal{N}}^* \circ A_{\mathcal{N}})X\right) + (\lambda - \varrho')X \\ &\quad - \frac{1}{2}(n+1)\left(\mathbb{H}A_{\mathcal{N}}X + \mathbb{H}^*A_{\mathcal{N}}^*X\right) - \frac{1}{2}\theta(A_{\mathcal{N}}X + A_{\mathcal{N}}^*X), \end{aligned}$$

for any vector field $X \in \Gamma(TM)$. This further implies that

$$\begin{aligned} \overline{Ric}^{\overline{S}}(X, Y) &= \overline{g}(\overline{S}(\mathcal{N}, X)\mathcal{N}, Y) - \frac{1}{2}\left(g((A_{\mathcal{N}} \circ A_{\mathcal{N}}^*)X, Y) + g((A_{\mathcal{N}}^* \circ A_{\mathcal{N}})X, Y)\right) \\ &\quad - \frac{1}{2}(n+1)\left(\mathbb{H}g(A_{\mathcal{N}}X, Y) + \mathbb{H}^*g(A_{\mathcal{N}}^*X, Y)\right) + (\lambda - \varrho')g(X, Y) \\ &\quad - \frac{1}{2}\theta\left(g(A_{\mathcal{N}}X, Y) + g(A_{\mathcal{N}}^*X, Y)\right), \end{aligned} \quad (48)$$

for any vector field $X, Y \in \Gamma(TM)$. Since the statistical manifold $(F, g_F, \nabla^F = \nabla^{g_F} + \mathcal{K}^F, \nabla^{*F})$ is of constant statistical curvature c , using (20) and (26), we have

$$\overline{S}(\mathcal{N}, X)\mathcal{N} = \alpha X + \beta\left(\frac{\theta}{\varrho}\phi_X\mathcal{N} - \left(\frac{\theta}{\varrho}\right)^2 X - \phi_X \partial t\right). \quad (49)$$

Hence, using (23) and (49) in (48), we obtain

$$\begin{aligned} (n\alpha - \beta)g(X, Y) + n\beta\phi_X\phi_Y &= \beta\left(-\frac{\theta^2}{\varrho^2}g(X, Y) + \phi_X\phi_Y\right) + (\lambda - \varrho')g(X, Y) \\ &\quad - \frac{1}{2}\left(g((A_{\mathcal{N}} \circ A_{\mathcal{N}}^*)X, Y) + g((A_{\mathcal{N}}^* \circ A_{\mathcal{N}})X, Y)\right) \\ &\quad - \frac{1}{2}(n+1)\left(\mathbb{H}g(A_{\mathcal{N}}X, Y) + \mathbb{H}^*g(A_{\mathcal{N}}^*X, Y)\right) \\ &\quad - \frac{1}{2}\theta\left(g(A_{\mathcal{N}}X, Y) + g(A_{\mathcal{N}}^*X, Y)\right). \end{aligned}$$

Let $\{e_1, \dots, e_{n+1}\}$ be an orthonormal frame on M . On contracting the above expression with this frame, we derive

$$\begin{aligned} (n\alpha - \beta)(n+1) + n\beta \sinh^2 \varphi &= \beta \left(-\frac{\theta^2}{\varrho^2}(n+1) + \sinh^2 \varphi \right) + (\lambda - \varrho')(n+1) \\ &\quad - \|A_N \circ A_N^*\| + (n+1)^2 \mathbb{H} \mathbb{H}^* + \frac{1}{2}(n+1)\theta(\mathbb{H} + \mathbb{H}^*), \end{aligned}$$

where $\sum_{i=1}^{n+1} \phi_{e_i}^2 = g((\partial t)^T, (\partial t)^T) = \sinh^2 \varphi$. Finally, using (46) in the last expression, we get the desired result. $\square$

Remark 4.7. In the context of Theorem 4.6, consider that the statistical spacelike hypersurface (M, g, ∇, ∇^*) of the GRW spacetime $(\bar{M} = I \times_{\varrho} F, \bar{g}, \bar{\nabla} = \bar{\nabla}^{\bar{g}} + \bar{\mathcal{K}}, \bar{\nabla}^*)$ endowed with a statistical structure is a spacelike slice. In this case, the hyperbolic angle φ is identically zero, which implies $\partial t = N$ and $\theta = \varrho \cosh \varphi = \varrho$. Additionally, applying assertion (ii) of Lemma 3.1 and noting that $\bar{\mathcal{K}}(X^F, \partial t) = 0$, we obtain $\bar{\nabla}_{X^F} N = \frac{\varrho'}{\varrho} X^F$. Further, using the Gauss formula, it follows that $A_N X^F = -\frac{\varrho'}{\varrho} X^F$. Similarly, $A_N^* X^F = -\frac{\varrho'}{\varrho} X^F$ and, consequently, we have

$$\|A_N \circ A_N^*\| = \sum_{i=1}^{n+1} g(A_N e_i, A_N^* e_i) = \left(\frac{\varrho'}{\varrho}\right)^2 (n+1).$$

Additionally, we also have $\mathbb{H} = \mathbb{H}^* = \frac{\varrho'}{\varrho}$. Let Ω^{slice} denote the statistical scalar curvature of the spacelike slice. Then, by applying all these results in (47), we obtain

$$\Omega^{\text{slice}} = n(n+1) \frac{c}{\varrho^2}.$$

Theorem 4.8. Let $(\bar{M} = I \times_{\varrho} F, \bar{g}, \bar{\nabla} = \bar{\nabla}^{\bar{g}} + \bar{\mathcal{K}}, \bar{\nabla}^*)$ be a GRW spacetime endowed with a statistical structure. Let the quadruple (F, g_F, W^F, λ^F) be a statistical Ricci soliton. Then, $(\bar{M}, \bar{g}, W, \lambda)$ is a statistical almost Ricci soliton if and only if

$$(\mathcal{E}_W \bar{g})(U, V) = \frac{2\varrho^2}{1-\varrho^2} \left(\bar{\lambda} - (n+1) \frac{\varrho''}{\varrho} \right) \phi_U \phi_V - \frac{1}{1-\varrho^2} \left((U(\phi_W)) \phi_V + (V(\phi_W)) \phi_U \right),$$

for any $U, V, W \in \Gamma(T\bar{M})$.

Proof. Let the quadruple (F, g_F, W^F, λ^F) be a statistical Ricci soliton. Then, for any $U, V, W \in \Gamma(T\bar{M})$, we have

$$\text{Ric}^F(U^F, V^F) + \frac{1}{2}(\mathcal{E}_{W^F} g_F)(U^F, V^F) = \lambda^F g_F(U^F, V^F). \quad (50)$$

By using (22) and (35) in (50), we derive

$$\begin{aligned} \bar{\text{Ric}}^{\bar{g}}(U, V) + \frac{1}{2}(\mathcal{E}_W \bar{g})(U, V) &= n \left(\frac{\varrho'}{\varrho} \right)^2 (\bar{g}(U, V) + \phi_U \phi_V) + \frac{\varrho''}{\varrho} (\bar{g}(U, V) - n \phi_U \phi_V) \\ &\quad + \frac{1}{2} \left(1 - \frac{1}{\varrho^2} \right) (\mathcal{E}_W \bar{g})(U, V) + \frac{\varrho'}{\varrho} \phi_W g_F(U^F, V^F) \\ &\quad - \frac{1}{2\varrho^2} \left((U(\phi_W)) \phi_V + (V(\phi_W)) \phi_U \right) \\ &\quad + \lambda^F g_F(U^F, V^F). \end{aligned} \quad (51)$$

Further, utilizing (5) and (16) in (51), we obtain

$$\begin{aligned} \overline{Ric}^{\bar{S}}(U, V) + \frac{1}{2}(\mathcal{E}_W \bar{g})(U, V) &= \left(\frac{\lambda^F}{\varrho^2} + n \left(\frac{\varrho'}{\varrho} \right)^2 + \frac{\varrho''}{\varrho} + \frac{\varrho'}{\varrho^3} \phi_W \right) \bar{g}(U, V) \\ &\quad + \left(\frac{\lambda^F}{\varrho^2} + n \left(\frac{\varrho'}{\varrho} \right)^2 - n \frac{\varrho''}{\varrho} + \frac{\varrho'}{\varrho^3} \phi_W \right) \phi_U \phi_V \\ &\quad - \frac{1}{2\varrho^2} \left((U(\phi_W)) \phi_V + (V(\phi_W)) \phi_U \right) \\ &\quad + \frac{(\varrho^2 - 1)}{2\varrho^2} (\mathcal{E}_W \bar{g})(U, V). \end{aligned}$$

This can be written as

$$\begin{aligned} \overline{Ric}^{\bar{S}}(U, V) + \frac{1}{2}(\mathcal{E}_W \bar{g})(U, V) &= \bar{\lambda} \bar{g}(U, V) + \left(\bar{\lambda} - (n+1) \frac{\varrho''}{\varrho} \right) \phi_U \phi_V \\ &\quad - \frac{1}{2\varrho^2} \left((U(\phi_W)) \phi_V + (V(\phi_W)) \phi_U \right) \\ &\quad + \frac{(\varrho^2 - 1)}{2\varrho^2} (\mathcal{E}_W \bar{g})(U, V), \end{aligned} \quad (52)$$

where $\bar{\lambda} = \frac{\lambda^F}{\varrho^2} + n \left(\frac{\varrho'}{\varrho} \right)^2 + \frac{\varrho''}{\varrho} + \frac{\varrho'}{\varrho^3} \phi_W$. Thus, the proof follows from (52). $\square$

A Riemannian manifold $(\bar{M}, \bar{g})$ is said to be a *quasi-Yamabe soliton* [8] if it admits a vector field ξ such that

$$\frac{1}{2} \mathcal{L}_\xi \bar{g} = (\bar{\Omega} - \bar{\lambda}) \bar{g} + \mu \psi \otimes \psi,$$

for some constant $\bar{\lambda}$ and some function μ , where ψ is the dual 1-form of ξ and $\bar{\Omega}$ is the scalar curvature of $\bar{M}$.

Since $\zeta = \varrho \partial t$, it follows that $\zeta^F = 0$. Assume that the quadruple $(F, g_F, \zeta^F, \lambda^F)$ is a statistical trivial Ricci soliton, it implies that $Ric^F(U^F, V^F) = \lambda^F g_F(U^F, V^F)$. Substituting this into (22), we obtain

$$\overline{Ric}^{\bar{S}}(U, V) = \left(\frac{\lambda^F}{\varrho^2} + n \left(\frac{\varrho'}{\varrho} \right)^2 + \frac{\varrho''}{\varrho} \right) \bar{g}(U, V) + \left(\frac{\lambda^F}{\varrho^2} + n \left(\frac{\varrho'}{\varrho} \right)^2 - n \frac{\varrho''}{\varrho} \right) \phi_U \phi_V,$$

for any $U, V \in \Gamma(T\bar{M})$. Further, using the previous expression with (38), we obtain

$$\overline{Ric}^{\bar{S}}(U, V) + \frac{1}{2}(\mathcal{E}_\zeta \bar{g})(U, V) = \widehat{\lambda} \bar{g}(U, V) + \frac{1}{\varrho^2} \left(\widehat{\lambda} - \varrho' - (n+1) \frac{\varrho''}{\varrho} \right) \omega(U) \omega(V), \quad (53)$$

where

$$\widehat{\lambda} = \frac{\lambda^F}{\varrho^2} + \varrho' + n \left(\frac{\varrho'}{\varrho} \right)^2 + \frac{\varrho''}{\varrho} \quad (54)$$

and ω is the dual 1-form of ζ . Hence, we obtain the following result.

Theorem 4.9. Let $(\bar{M} = I \times_\varrho F, \bar{g}, \bar{\nabla} = \bar{\nabla}^g + \bar{\mathcal{K}}, \bar{\nabla}^*)$ be a GRW spacetime endowed with a statistical structure. Let the quadruple $(F, g_F, \zeta^F, \lambda^F)$ be a statistical trivial Ricci soliton. Then, $(\bar{M}, \bar{g}, \zeta, \widehat{\lambda})$ is a statistical quasi almost Ricci soliton.

The following example supports the claim made in the previous theorem.

Example 4.10. Let $F = \{(x, y, z, u) \in \mathbb{R}^4 : (x, y, z, u) \neq (0, 0, 0, 0)\}$ be a Riemannian manifold equipped with a Riemannian metric

$$g_F = \frac{1}{z^2}(dx^2 + dz^2) + \frac{1}{u^2}(dy^2 + du^2).$$

Let ∇^{g_F} denote the Levi-Civita connection of (F, g_F) with respect to the metric g_F . Then, by straightforward calculations, we obtain

$$\begin{aligned}\nabla_{\partial_x}^{g_F} \partial x &= \frac{1}{z} \partial z, & \nabla_{\partial_x}^{g_F} \partial y &= 0, & \nabla_{\partial_x}^{g_F} \partial z &= -\frac{1}{z} \partial x, & \nabla_{\partial_x}^{g_F} \partial u &= 0, \\ \nabla_{\partial_y}^{g_F} \partial x &= 0, & \nabla_{\partial_y}^{g_F} \partial y &= \frac{1}{u} \partial u, & \nabla_{\partial_y}^{g_F} \partial z &= 0, & \nabla_{\partial_y}^{g_F} \partial u &= -\frac{1}{u} \partial y, \\ \nabla_{\partial_z}^{g_F} \partial x &= -\frac{1}{z} \partial x, & \nabla_{\partial_z}^{g_F} \partial y &= 0, & \nabla_{\partial_z}^{g_F} \partial z &= -\frac{1}{z} \partial z, & \nabla_{\partial_z}^{g_F} \partial u &= 0, \\ \nabla_{\partial_u}^{g_F} \partial x &= 0, & \nabla_{\partial_u}^{g_F} \partial y &= -\frac{1}{u} \partial y, & \nabla_{\partial_u}^{g_F} \partial z &= 0, & \nabla_{\partial_u}^{g_F} \partial u &= -\frac{1}{u} \partial u.\end{aligned}$$

For the Riemannian metric g_F of (F, g_F) , we define the difference tensor field $\mathcal{K}^F \in \Gamma(TF)$ on F by

$$\begin{aligned}\mathcal{K}^F(\partial x, \partial x) &= -\frac{1}{z} \partial z, & \mathcal{K}^F(\partial x, \partial y) &= 0, & \mathcal{K}^F(\partial x, \partial z) &= -\frac{1}{z} \partial x, & \mathcal{K}^F(\partial x, \partial u) &= 0, \\ \mathcal{K}^F(\partial y, \partial x) &= 0, & \mathcal{K}^F(\partial y, \partial y) &= -\frac{1}{u} \partial u, & \mathcal{K}^F(\partial y, \partial z) &= 0, & \mathcal{K}^F(\partial y, \partial u) &= -\frac{1}{u} \partial y, \\ \mathcal{K}^F(\partial z, \partial x) &= -\frac{1}{z} \partial x, & \mathcal{K}^F(\partial z, \partial y) &= 0, & \mathcal{K}^F(\partial z, \partial z) &= \frac{1}{z} \partial z, & \mathcal{K}^F(\partial z, \partial u) &= 0, \\ \mathcal{K}^F(\partial u, \partial x) &= 0, & \mathcal{K}^F(\partial u, \partial y) &= -\frac{1}{u} \partial y, & \mathcal{K}^F(\partial u, \partial z) &= 0, & \mathcal{K}^F(\partial u, \partial u) &= \frac{1}{u} \partial u.\end{aligned}$$

It is then easy to verify that the difference tensor field $\mathcal{K}^F$ satisfies the identities in (3). Thus, $(\nabla^F = \nabla^{g_F} + \mathcal{K}^F, g_F)$ is a statistical structure on (F, g_F) , and given by

$$\begin{aligned}\nabla_{\partial_x}^F \partial x &= 0, & \nabla_{\partial_x}^F \partial y &= 0, & \nabla_{\partial_x}^F \partial z &= -\frac{2}{z} \partial x, & \nabla_{\partial_x}^F \partial u &= 0, \\ \nabla_{\partial_y}^F \partial x &= 0, & \nabla_{\partial_y}^F \partial y &= 0, & \nabla_{\partial_y}^F \partial z &= 0, & \nabla_{\partial_y}^F \partial u &= -\frac{2}{u} \partial y, \\ \nabla_{\partial_z}^F \partial x &= -\frac{2}{z} \partial x, & \nabla_{\partial_z}^F \partial y &= 0, & \nabla_{\partial_z}^F \partial z &= 0, & \nabla_{\partial_z}^F \partial u &= 0, \\ \nabla_{\partial_u}^F \partial x &= 0, & \nabla_{\partial_u}^F \partial y &= -\frac{2}{u} \partial y, & \nabla_{\partial_u}^F \partial z &= 0, & \nabla_{\partial_u}^F \partial u &= 0.\end{aligned}\tag{55}$$

Utilizing $\nabla^{*F} = \nabla^{g_F} - \mathcal{K}^F$, the dual statistical connection ∇^{*F} of ∇^F with respect to g_F is given by

$$\begin{aligned}\nabla_{\partial_x}^{*F} \partial x &= \frac{2}{z} \partial z, & \nabla_{\partial_x}^{*F} \partial y &= 0, & \nabla_{\partial_x}^{*F} \partial z &= 0, & \nabla_{\partial_x}^{*F} \partial u &= 0, \\ \nabla_{\partial_y}^{*F} \partial x &= 0, & \nabla_{\partial_y}^{*F} \partial y &= \frac{2}{u} \partial u, & \nabla_{\partial_y}^{*F} \partial z &= 0, & \nabla_{\partial_y}^{*F} \partial u &= 0, \\ \nabla_{\partial_z}^{*F} \partial x &= 0, & \nabla_{\partial_z}^{*F} \partial y &= 0, & \nabla_{\partial_z}^{*F} \partial z &= -\frac{2}{z} \partial z, & \nabla_{\partial_z}^{*F} \partial u &= 0, \\ \nabla_{\partial_u}^{*F} \partial x &= 0, & \nabla_{\partial_u}^{*F} \partial y &= 0, & \nabla_{\partial_u}^{*F} \partial z &= 0, & \nabla_{\partial_u}^{*F} \partial u &= -\frac{2}{u} \partial u.\end{aligned}\tag{56}$$

Thus, $(F, g_F, \nabla^F = \nabla^{g_F} + \mathcal{K}^F, \nabla^{*F})$ is a statistical manifold.

Let $\bar{M} = I \times_{\exp t} F$ be a warped product manifold, where the base I is an open interval of the real line $\mathbb{R}$, the fiber (F, g_F) is a Riemannian manifold, and the warping function $\exp t$ is defined on I . Let the warped product manifold $\bar{M} = I \times_{\exp t} F$ be equipped with metric tensor $\bar{g}$ given by

$$\bar{g} = -dt^2 + (\exp t)^2 g_F,$$

where t serves as a natural parameter on the base I . Let $\bar{\nabla}^{\bar{g}}$ denote the Levi-Civita connection of the warped product manifold $(\bar{M} = I \times_{\exp t} F, \bar{g})$ with respect to the metric $\bar{g}$. Then, by straightforward calculations, we obtain

$$\begin{aligned}\bar{\nabla}_{\partial t}^{\bar{g}} \partial t &= 0, & \bar{\nabla}_{\partial t}^{\bar{g}} \partial x &= \partial x, & \bar{\nabla}_{\partial t}^{\bar{g}} \partial y &= \partial y, & \bar{\nabla}_{\partial t}^{\bar{g}} \partial z &= \partial z, & \bar{\nabla}_{\partial t}^{\bar{g}} \partial u &= \partial u, \\ \bar{\nabla}_{\partial x}^{\bar{g}} \partial t &= \partial x, & \bar{\nabla}_{\partial x}^{\bar{g}} \partial x &= \frac{1}{z} \partial z + \frac{\exp 2t}{z^2} \partial t, & \bar{\nabla}_{\partial x}^{\bar{g}} \partial y &= 0, & \bar{\nabla}_{\partial x}^{\bar{g}} \partial z &= -\frac{1}{z} \partial x, & \bar{\nabla}_{\partial x}^{\bar{g}} \partial u &= 0, \\ \bar{\nabla}_{\partial y}^{\bar{g}} \partial t &= \partial y, & \bar{\nabla}_{\partial y}^{\bar{g}} \partial x &= 0, & \bar{\nabla}_{\partial y}^{\bar{g}} \partial y &= \frac{1}{u} \partial u + \frac{\exp 2t}{u^2} \partial t, & \bar{\nabla}_{\partial y}^{\bar{g}} \partial z &= 0, & \bar{\nabla}_{\partial y}^{\bar{g}} \partial u &= -\frac{1}{u} \partial y, \\ \bar{\nabla}_{\partial z}^{\bar{g}} \partial t &= \partial z, & \bar{\nabla}_{\partial z}^{\bar{g}} \partial x &= -\frac{1}{z} \partial x, & \bar{\nabla}_{\partial z}^{\bar{g}} \partial y &= 0, & \bar{\nabla}_{\partial z}^{\bar{g}} \partial z &= -\frac{1}{z} \partial z + \frac{\exp 2t}{z^2} \partial t, & \bar{\nabla}_{\partial z}^{\bar{g}} \partial u &= 0, \\ \bar{\nabla}_{\partial u}^{\bar{g}} \partial t &= \partial u, & \bar{\nabla}_{\partial u}^{\bar{g}} \partial x &= 0, & \bar{\nabla}_{\partial u}^{\bar{g}} \partial y &= -\frac{1}{u} \partial y, & \bar{\nabla}_{\partial u}^{\bar{g}} \partial z &= 0, & \bar{\nabla}_{\partial u}^{\bar{g}} \partial u &= -\frac{1}{u} \partial u + \frac{\exp 2t}{u^2} \partial t.\end{aligned}$$

For the Lorentzian metric $\bar{g}$ of the warped product manifold $(\bar{M} = I \times_{\exp t} F, \bar{g})$, we define the difference tensor field $\bar{\mathcal{K}} \in \Gamma(T\bar{M})$ on $\bar{M}$ by

$$\bar{\mathcal{K}}(U^F, V^F) = \mathcal{K}^F(U^F, V^F), \quad \bar{\mathcal{K}}(\partial_t, U^F) = \bar{\mathcal{K}}(U^F, \partial_t) = 0, \quad \bar{\mathcal{K}}(\partial_t, \partial_t) = v \partial_t,$$

where $U^F, V^F \in \Gamma(TF)$, v is a smooth function defined on I . This gives

$$\bar{g}(\bar{\mathcal{K}}(U, V), W) = (\exp t)^2 g_F(\mathcal{K}^F(U^F, V^F), W^F) - v \phi_U \phi_V \phi_W,$$

for any $U, V, W \in \Gamma(T\bar{M})$. Since $\mathcal{K}^F$ satisfies the identities in (3), $\bar{\mathcal{K}}$ also satisfies the identities in (3), consequently $(\bar{\nabla} = \bar{\nabla}^{\bar{g}} + \bar{\mathcal{K}}, \bar{g})$ is a statistical structure on $(\bar{M}, \bar{g})$. Hence

$$\begin{aligned}\bar{\nabla}_{\partial t} \partial t &= v \partial t, & \bar{\nabla}_{\partial t} \partial x &= \partial x, & \bar{\nabla}_{\partial t} \partial y &= \partial y, & \bar{\nabla}_{\partial t} \partial z &= \partial z, & \bar{\nabla}_{\partial t} \partial u &= \partial u, \\ \bar{\nabla}_{\partial x} \partial t &= \partial x, & \bar{\nabla}_{\partial x} \partial x &= \frac{\exp 2t}{z^2} \partial t, & \bar{\nabla}_{\partial x} \partial y &= 0, & \bar{\nabla}_{\partial x} \partial z &= -\frac{2}{z} \partial x, & \bar{\nabla}_{\partial x} \partial u &= 0, \\ \bar{\nabla}_{\partial y} \partial t &= \partial y, & \bar{\nabla}_{\partial y} \partial x &= 0, & \bar{\nabla}_{\partial y} \partial y &= \frac{\exp 2t}{u^2} \partial t, & \bar{\nabla}_{\partial y} \partial z &= 0, & \bar{\nabla}_{\partial y} \partial u &= -\frac{2}{u} \partial y, \\ \bar{\nabla}_{\partial z} \partial t &= \partial z, & \bar{\nabla}_{\partial z} \partial x &= -\frac{2}{z} \partial x, & \bar{\nabla}_{\partial z} \partial y &= 0, & \bar{\nabla}_{\partial z} \partial z &= \frac{\exp 2t}{z^2} \partial t, & \bar{\nabla}_{\partial z} \partial u &= 0, \\ \bar{\nabla}_{\partial u} \partial t &= \partial u, & \bar{\nabla}_{\partial u} \partial x &= 0, & \bar{\nabla}_{\partial u} \partial y &= -\frac{2}{u} \partial y, & \bar{\nabla}_{\partial u} \partial z &= 0, & \bar{\nabla}_{\partial u} \partial u &= \frac{\exp 2t}{u^2} \partial t.\end{aligned}$$

The dual statistical connection $\bar{\nabla}^*$ of $\bar{\nabla}$ with respect to $\bar{g}$ is given by

$$\begin{aligned}\bar{\nabla}_{\partial t}^* \partial t &= -v \partial t, & \bar{\nabla}_{\partial t}^* \partial x &= \partial x, & \bar{\nabla}_{\partial t}^* \partial y &= \partial y, & \bar{\nabla}_{\partial t}^* \partial z &= \partial z, & \bar{\nabla}_{\partial t}^* \partial u &= \partial u, \\ \bar{\nabla}_{\partial x}^* \partial t &= \partial x, & \bar{\nabla}_{\partial x}^* \partial x &= \frac{2}{z} \partial z + \frac{\exp 2t}{z^2} \partial t, & \bar{\nabla}_{\partial x}^* \partial y &= 0, & \bar{\nabla}_{\partial x}^* \partial z &= 0, & \bar{\nabla}_{\partial x}^* \partial u &= 0, \\ \bar{\nabla}_{\partial y}^* \partial t &= \partial y, & \bar{\nabla}_{\partial y}^* \partial x &= 0, & \bar{\nabla}_{\partial y}^* \partial y &= \frac{2}{u} \partial u + \frac{\exp 2t}{u^2} \partial t, & \bar{\nabla}_{\partial y}^* \partial z &= 0, & \bar{\nabla}_{\partial y}^* \partial u &= 0, \\ \bar{\nabla}_{\partial z}^* \partial t &= \partial z, & \bar{\nabla}_{\partial z}^* \partial x &= 0, & \bar{\nabla}_{\partial z}^* \partial y &= 0, & \bar{\nabla}_{\partial z}^* \partial z &= -\frac{2}{z} \partial z + \frac{\exp 2t}{z^2} \partial t, & \bar{\nabla}_{\partial z}^* \partial u &= 0, \\ \bar{\nabla}_{\partial u}^* \partial t &= \partial u, & \bar{\nabla}_{\partial u}^* \partial x &= 0, & \bar{\nabla}_{\partial u}^* \partial y &= 0, & \bar{\nabla}_{\partial u}^* \partial z &= 0, & \bar{\nabla}_{\partial u}^* \partial u &= -\frac{2}{u} \partial u + \frac{\exp 2t}{u^2} \partial t.\end{aligned}$$

Thus, $(\overline{M} = I \times_{\exp t} F, \overline{g}, \overline{\nabla} = \overline{\nabla}^{\overline{g}} + \overline{\mathcal{K}}, \overline{\nabla}^*)$ is an indefinite statistical manifold.

Next, for the statistical manifold $(F, g_F, \nabla^F = \nabla^{g_F} + \mathcal{K}^F, \nabla^{*F})$, denote the curvature tensor fields of ∇^F and ∇^{*F} by R^F and R^{*F} , respectively. Utilizing (55) and (56), we obtain

$$R^F(\partial x, \partial z)\partial z = -\frac{6}{z^2}\partial x, \quad R^F(\partial y, \partial u)\partial u = -\frac{6}{u^2}\partial y$$

and

$$R^{*F}(\partial x, \partial z)\partial x = \frac{6}{z^2}\partial z, \quad R^{*F}(\partial y, \partial u)\partial y = \frac{6}{u^2}\partial u,$$

and all other components of the curvature tensor fields R^F and R^{*F} vanish identically. We denote by S^F the statistical curvature tensor field of $(F, g_F, \nabla^F = \nabla^{g_F} + \mathcal{K}^F, \nabla^{*F})$. Consider $\{e_1 = z\partial x, e_2 = u\partial y, e_3 = z\partial z, e_4 = u\partial u\}$ an orthonormal frame on the statistical manifold $(F, g_F, \nabla^F = \nabla^{g_F} + \mathcal{K}^F, \nabla^{*F})$. Using the previous expressions, we get

$$S^F(e_1, e_3)e_3 = -3e_1, \quad S^F(e_2, e_4)e_4 = -3e_2,$$

$$S^F(e_1, e_3)e_1 = 3e_3, \quad S^F(e_2, e_4)e_2 = 3e_4,$$

and all other components of the statistical curvature tensor field S^F vanish identically. Let $\text{Ric}^F(U^F, V^F)$ denote the statistical Ricci tensor of the statistical manifold $(F, g_F, \nabla^F = \nabla^{g_F} + \mathcal{K}^F, \nabla^{*F})$. Then, we derive

$$\text{Ric}^F(e_1, e_1) = \text{Ric}^F(e_2, e_2) = \text{Ric}^F(e_3, e_3) = \text{Ric}^F(e_4, e_4) = -3.$$

Since $\zeta = \rho\partial t$, it follows that $\zeta^F = 0$. Thus, the fiber $(F, g_F, \zeta^F, \lambda^F)$ is a statistical trivial Ricci soliton with $\lambda^F = -3$.

Since $\{e_1, e_2, e_3, e_4\}$ is an orthonormal frame of fields on the fiber $(F, g_F, \nabla^F = \nabla^{g_F} + \mathcal{K}^F, \nabla^{*F})$, it implies that $\{\partial t, E_1 = \frac{1}{\rho}e_1, E_2 = \frac{1}{\rho}e_2, E_3 = \frac{1}{\rho}e_3, E_4 = \frac{1}{\rho}e_4\}$ is an orthonormal frame of fields on the warped product manifold $(\overline{M} = I \times_{\exp t} F, \overline{g}, \overline{\nabla} = \overline{\nabla}^{\overline{g}} + \overline{\mathcal{K}}, \overline{\nabla}^*)$. It is known that $(\mathcal{E}_W\overline{g})(U, V) = \overline{g}(\overline{\nabla}_U^{\overline{g}}W, V) + \overline{g}(U, \overline{\nabla}_V^{\overline{g}}W)$, for any vector fields $U, V, W \in \Gamma(T\overline{M})$. By straightforward calculations, we obtain

$$(\mathcal{E}_{\zeta}\overline{g})(\partial t, \partial t) = -2\exp t, \quad (\mathcal{E}_{\zeta}\overline{g})(E_i, E_i) = 2\exp t, i = 1, \dots, 4.$$

Let $\overline{\text{Ric}}^{\overline{S}}$ denote the statistical Ricci tensor of the warped product manifold $(\overline{M} = I \times_{\exp t} F, \overline{g}, \overline{\nabla} = \overline{\nabla}^{\overline{g}} + \overline{\mathcal{K}}, \overline{\nabla}^*)$. By applying (22), we get

$$\overline{\text{Ric}}^{\overline{S}}(\partial t, \partial t) = -4, \quad \overline{\text{Ric}}^{\overline{S}}(E_i, E_i) = -\frac{3}{\exp 2t} + 4, \quad i = 1, \dots, 4.$$

Hence

$$\overline{\text{Ric}}^{\overline{S}}(\partial t, \partial t) + \frac{1}{2}(\mathcal{E}_{\zeta}\overline{g})(\partial t, \partial t) = -4 - \exp t \quad (57)$$

and

$$\overline{\text{Ric}}^{\overline{S}}(E_i, E_i) + \frac{1}{2}(\mathcal{E}_{\zeta}\overline{g})(E_i, E_i) = -\frac{3}{\exp 2t} + 4 + \exp t, \quad (58)$$

for $i = 1, \dots, 4$. By utilizing (54), we have $\widehat{\lambda} = 4 + \exp t - \frac{3}{\exp 2t}$. Moreover, it is easy to verify that

$$\widehat{\lambda}\overline{g}(\partial t, \partial t) + \frac{1}{\rho^2}\left(\widehat{\lambda} - \rho' - (n+1)\frac{\rho''}{\rho}\right)\omega(\partial t)\omega(\partial t) = -4 - \exp t \quad (59)$$

and

$$\widehat{\lambda}\overline{g}(E_i, E_i) + \frac{1}{\rho^2}\left(\widehat{\lambda} - \rho' - (n+1)\frac{\rho''}{\rho}\right)\omega(E_i)\omega(E_i) = 4 + \exp t - \frac{3}{\exp 2t}. \quad (60)$$

Thus, from (57), (58), (59) and (60), it follows that the orthonormal frame of fields $\{\partial t, E_1 = \frac{1}{\varrho}e_1, E_2 = \frac{1}{\varrho}e_2, E_3 = \frac{1}{\varrho}e_3, E_4 = \frac{1}{\varrho}e_4\}$ on $\overline{M}$ satisfy the expression in (53). Consequently, we find that if the fiber $(F, g_F, \zeta^F, \lambda^F)$ is a statistically trivial Ricci soliton, then $(\overline{M}, \overline{g}, \zeta, \widehat{\lambda})$ is a statistical quasi almost Ricci soliton.

Lemma 4.11. Let $(\overline{M} = I \times_{\varrho} F, \overline{g}, \overline{\nabla} = \overline{\nabla}^{\overline{g}} + \overline{\mathcal{K}}, \overline{\nabla}^*)$ be a GRW spacetime endowed with a statistical structure. Then

$$\begin{aligned} (\mathcal{E}_W \overline{g})(U, V) &= (\mathcal{E}_{W^T} g)(U^T, V^T) - B(U^T, W^T)\eta(V) - B^*(V^T, W^T)\eta(U) \\ &\quad - \left((U(\eta(W)))\eta(V) + (V(\eta(W)))\eta(U) \right) + \eta(U)\overline{g}(\overline{\nabla}_N W, V) \\ &\quad + \eta(W)(B(U^T, V^T) + B^*(U^T, V^T)) + \eta(V)\overline{g}(\overline{\nabla}_N^* W, U) \\ &\quad - \tau(U^T)\eta(W)\eta(V) + \tau(V^T)\eta(W)\eta(U) \\ &\quad + 2(\mathcal{N} \cdot \eta(W))\eta(U)\eta(V), \end{aligned} \quad (61)$$

for any vector fields $U, V, W \in \Gamma(T\overline{M})$. In particular,

$$\begin{aligned} (\mathcal{E}_{\zeta} \overline{g})(U, V) &= (\mathcal{E}_{\zeta^T} g)(U^T, V^T) + \varrho \cosh \varphi (B(U^T, V^T) + B^*(U^T, V^T)) \\ &\quad - 2\varrho' \eta(U)\eta(V). \end{aligned} \quad (62)$$

Proof. Since M is a spacelike hypersurface of $\overline{M}$ and $\mathcal{N}$ is a unique unitary timelike normal vector field globally defined on M , for any $U \in \Gamma(T\overline{M})$, we can write $U = U^T + \eta(U)\mathcal{N}$, where U^T is the tangential component of U and $\eta(U) = -\overline{g}(U, \mathcal{N})$. Then, utilizing the local Gauss-Weingarten formulas on TM , we have

$$\begin{aligned} \overline{\nabla}_U W &= \nabla_{U^T} W^T + B(U^T, W^T)\mathcal{N} + (U(\eta(W)))\mathcal{N} - \eta(W)A_N U^T + \eta(W)\tau(U^T)\mathcal{N} \\ &\quad + \eta(U)(\overline{\nabla}_N W^T + \eta(W)\overline{\nabla}_N \mathcal{N}) \end{aligned}$$

and

$$\begin{aligned} \overline{\nabla}_V^* W &= \nabla_{V^T}^* W^T + B^*(V^T, W^T)\mathcal{N} + (V(\eta(W)))\mathcal{N} - \eta(W)A_N^* V^T + \eta(W)\tau^*(V^T)\mathcal{N} \\ &\quad + \eta(V)(\overline{\nabla}_N^* W^T + \eta(W)\overline{\nabla}_N^* \mathcal{N}). \end{aligned}$$

Using above expressions in (36), we derive

$$\begin{aligned} (\mathcal{E}_W \overline{g})(U, V) &= (\mathcal{E}_{W^T} g)(U^T, V^T) - B(U^T, W^T)\eta(V) - B^*(V^T, W^T)\eta(U) \\ &\quad - \left((U(\eta(W)))\eta(V) + (V(\eta(W)))\eta(U) \right) \\ &\quad + \eta(W)(B(U^T, V^T) + B^*(U^T, V^T)) \\ &\quad + \eta(U)(\mathcal{N} \cdot \overline{g}(W^T, V) - \overline{g}(W^T, \overline{\nabla}_N^* V)) \\ &\quad + \eta(V)(\mathcal{N} \cdot \overline{g}(U, W^T) - \overline{g}(\overline{\nabla}_N U, W^T)) \\ &\quad - \tau(U^T)\eta(W)\eta(V) + \eta(U)\eta(W)(\mathcal{N} \cdot \overline{g}(\mathcal{N}, V) - \overline{g}(\overline{\nabla}_N^* V, \mathcal{N})) \\ &\quad + \tau(V^T)\eta(W)\eta(U) + \eta(V)\eta(W)(\mathcal{N} \cdot \overline{g}(U, \mathcal{N}) - \overline{g}(\overline{\nabla}_N U, \mathcal{N})). \end{aligned}$$

This expression leads to the final result. $\square$

Since for any $U \in \Gamma(\overline{TM})$, we can write $U = U^T + \eta(U)\mathcal{N}$, then by applying the local Gauss-Weingarten formulas on TM , we derive

$$\begin{aligned}\overline{R}(e_i, U^T)\mathcal{N} &= -(\nabla_{e_i}A_{\mathcal{N}})(U^T) + (\nabla_{U^T}A_{\mathcal{N}})(e_i) - \tau(U^T)A_{\mathcal{N}}e_i + \tau(e_i)A_{\mathcal{N}}U^T \\ &\quad - (B(e_i, A_{\mathcal{N}}U^T) - B(U^T, A_{\mathcal{N}}e_i) - 2d\tau(e_i, U^T))\mathcal{N},\end{aligned}$$

where $\{e_1, \dots, e_{n+1}\}$ is an orthonormal frame of fields on M and

$$(\nabla_{e_i}A_{\mathcal{N}})(U^T) = \nabla_{e_i}(A_{\mathcal{N}}U^T) - A_{\mathcal{N}}(\nabla_{e_i}U^T). \quad (63)$$

The preceding expression yields

$$\begin{aligned}\overline{g}(\overline{R}(e_i, U^T)\mathcal{N}, e_i) &= -g((\nabla_{e_i}A_{\mathcal{N}})(U^T), e_i) + g((\nabla_{U^T}A_{\mathcal{N}})(e_i), e_i) \\ &\quad + \tau(U^T)B^*(e_i, e_i) - \tau(e_i)B^*(U^T, e_i).\end{aligned}$$

Furthermore, by applying (4), we obtain

$$\begin{aligned}2\overline{g}(\overline{S}(e_i, U^T)\mathcal{N}, e_i) &= -g((\nabla_{e_i}A_{\mathcal{N}})(U^T), e_i) + g((\nabla_{U^T}A_{\mathcal{N}})(e_i), e_i) \\ &\quad - g((\nabla_{e_i}^*A_{\mathcal{N}}^*)(U^T), e_i) + g((\nabla_{U^T}^*A_{\mathcal{N}}^*)(e_i), e_i) \\ &\quad + \tau(U^T)(B^*(e_i, e_i) - B(e_i, e_i)) \\ &\quad - \tau(e_i)(B^*(U^T, e_i) - B(U^T, e_i)).\end{aligned} \quad (64)$$

As noted in [26], the covariant derivative of the second fundamental form is equivalently defined as $(\nabla_X A)(\mathcal{N}, Y) = \nabla_X(A_{\mathcal{N}}Y) - A_{\nabla_X^{\perp}\mathcal{N}}Y - A_{\mathcal{N}}(\nabla_X Y)$ for any vector fields X and Y tangent to M and any vector field $\mathcal{N}$ normal to M . We denote

$$\begin{aligned}\text{trace}(A_{\mathcal{N}}U^T) &= \sum_{i=1}^{n+1} g((\nabla_{e_i}A)(\mathcal{N}, U^T), e_i) \\ &= \sum_{i=1}^{n+1} g(\nabla_{e_i}(A_{\mathcal{N}}U^T) - A_{\nabla_{e_i}^{\perp}\mathcal{N}}U^T - A_{\mathcal{N}}(\nabla_{e_i}U^T), e_i).\end{aligned}$$

Further, using (30) and (63), we obtain

$$\text{trace}(A_{\mathcal{N}}U^T) = \sum_{i=1}^{n+1} \{g((\nabla_{e_i}A_{\mathcal{N}})(U^T), e_i) + \tau(e_i)B^*(U^T, e_i)\}. \quad (65)$$

Similarly, by using (29), we obtain

$$\text{trace}(A_{\mathcal{N}}^*U^T) = \sum_{i=1}^{n+1} \{g((\nabla_{e_i}^*A_{\mathcal{N}}^*)(U^T), e_i) - \tau(e_i)B(U^T, e_i)\}. \quad (66)$$

We also denote

$$\text{trace}(\nabla_{U^T}A_{\mathcal{N}}) = \sum_{i=1}^{n+1} g((\nabla_{U^T}A)(\mathcal{N}, e_i), e_i),$$

and then by using (30) and (63), we obtain

$$\text{trace}(\nabla_{U^T}A_{\mathcal{N}}) = \sum_{i=1}^{n+1} \{g((\nabla_{U^T}A_{\mathcal{N}})(e_i), e_i) + \tau(U^T)B^*(e_i, e_i)\}. \quad (67)$$

Similarly

$$\text{trace}(\nabla_{U^T}^* A_N^*) = \sum_{i=1}^{n+1} \left\{ g((\nabla_{U^T}^* A_N^*)(e_i), e_i) - \tau(U^T)B(e_i, e_i) \right\}. \quad (68)$$

We know that

$$\begin{aligned} \overline{\text{Ric}}^{\bar{S}}(U^T, \mathcal{N}) &= \sum_{i=1}^{n+1} \bar{g}(\bar{S}(e_i, U^T)\mathcal{N}, e_i) - \bar{g}(\bar{S}(\mathcal{N}, U^T)\mathcal{N}, \mathcal{N}) \\ &= \sum_{i=1}^{n+1} \bar{g}(\bar{S}(e_i, U^T)\mathcal{N}, e_i). \end{aligned} \quad (69)$$

Thus, by applying the expressions from equations (64)–(68) in (69), we arrive at the following result.

Lemma 4.12. *Let $(\bar{M} = I \times_{\varrho} F, \bar{g}, \bar{\nabla} = \bar{\nabla}^{\bar{g}} + \bar{\mathcal{K}}, \bar{\nabla}^*)$ be a GRW spacetime endowed with a statistical structure. Then*

$$\begin{aligned} \overline{\text{Ric}}^{\bar{S}}(U^T, \mathcal{N}) &= \frac{1}{2} \left(\text{trace}(\nabla_{U^T} A_N) + \text{trace}(\nabla_{U^T}^* A_N^*) - \text{trace}(A_N U^T) \right. \\ &\quad \left. - \text{trace}(A_N^* U^T) \right), \end{aligned} \quad (70)$$

for any vector field U tangent to $\bar{M}$.

Let (M, g, ∇, ∇^*) be a statistical spacelike hypersurface of a GRW spacetime $(\bar{M} = I \times_{\varrho} F, \bar{g}, \bar{\nabla} = \bar{\nabla}^{\bar{g}} + \bar{\mathcal{K}}, \bar{\nabla}^*)$ endowed with a statistical structure. Then, for any vector fields U and V tangent to $\bar{M}$, we have

$$\begin{aligned} \overline{\text{Ric}}^{\bar{S}}(U, V) &= \overline{\text{Ric}}^{\bar{S}}(U^T, V^T) + \eta(V)\overline{\text{Ric}}^{\bar{S}}(U^T, \mathcal{N}) + \eta(U)\overline{\text{Ric}}^{\bar{S}}(V^T, \mathcal{N}) \\ &\quad + \eta(U)\eta(V)\overline{\text{Ric}}^{\bar{S}}(\mathcal{N}, \mathcal{N}). \end{aligned} \quad (71)$$

Using (33) and (70) in (71), we derive

$$\begin{aligned} \overline{\text{Ric}}^{\bar{S}}(U, V) &= \text{Ric}^S(U^T, V^T) - \bar{g}(\bar{S}(\mathcal{N}, U^T)V^T, \mathcal{N}) - \frac{1}{2} \left(g((A_N^* \circ A_N)U^T, V^T) \right. \\ &\quad \left. + g((A_N \circ A_N^*)U^T, V^T) + (n+1)(\mathbb{H}B^*(U^T, V^T) + \mathbb{H}^*B(U^T, V^T)) \right) \\ &\quad + \frac{1}{2} \eta(V) \left(\text{trace}(\nabla_{U^T} A_N) + \text{trace}(\nabla_{U^T}^* A_N^*) - \text{trace}(A_N U^T) \right. \\ &\quad \left. - \text{trace}(A_N^* U^T) \right) + \frac{1}{2} \eta(U) \left(\text{trace}(\nabla_{V^T} A_N) + \text{trace}(\nabla_{V^T}^* A_N^*) \right. \\ &\quad \left. - \text{trace}(A_N V^T) - \text{trace}(A_N^* V^T) \right) + \eta(U)\eta(V)\overline{\text{Ric}}^{\bar{S}}(\mathcal{N}, \mathcal{N}). \end{aligned} \quad (72)$$

Hence, by applying (61) and (72), we derive the following result.

Theorem 4.13. *Let (M, g, ∇, ∇^*) be a statistical spacelike hypersurface of a GRW spacetime $(\bar{M} = I \times_{\varrho} F, \bar{g}, \bar{\nabla} = \bar{\nabla}^{\bar{g}} + \bar{\mathcal{K}}, \bar{\nabla}^*)$ endowed with a statistical structure. Let the quadruple $(\bar{M}, \bar{g}, W, \tilde{\lambda})$ be a statistical Ricci soliton. Then,*

the quadruple $(M, g, W^T, \tilde{\lambda})$ is a statistical Ricci soliton if and only if

$$\begin{aligned} & (\tilde{\lambda} + (\mathcal{N} \cdot \eta(W)) + 2\overline{Ric}^{\tilde{S}}(\mathcal{N}, \mathcal{N}))\eta(U)\eta(V) = \bar{g}(\bar{S}(\mathcal{N}, U^T)V^T, \mathcal{N}) \\ & + \frac{1}{2}(g((A_{\mathcal{N}}^* \circ A_{\mathcal{N}})U^T, V^T) + g((A_{\mathcal{N}} \circ A_{\mathcal{N}}^*)U^T, V^T)) \\ & + \frac{1}{2}((n+1)\mathbb{H} - \eta(W))B^*(U^T, V^T) + \frac{1}{2}((n+1)\mathbb{H}^* - \eta(W))B(U^T, V^T) \\ & - \frac{1}{2}\eta(V)(\text{trace}(\nabla_{U^T}A_{\mathcal{N}}) + \text{trace}(\nabla_{U^T}^*A_{\mathcal{N}}^*) - \text{trace}(A_{\mathcal{N}}U^T) - \text{trace}(A_{\mathcal{N}}^*U^T)) \\ & - \frac{1}{2}\eta(U)(\text{trace}(\nabla_{V^T}A_{\mathcal{N}}) + \text{trace}(\nabla_{V^T}^*A_{\mathcal{N}}^*) - \text{trace}(A_{\mathcal{N}}V^T) - \text{trace}(A_{\mathcal{N}}^*V^T)) \\ & + \frac{1}{2}(B^*(V^T, W^T) + V(\eta(W)) - \bar{g}(\bar{\nabla}_{\mathcal{N}}W, V) - \tau(V^T)\eta(W))\eta(U) \\ & + \frac{1}{2}(B(U^T, W^T) + U(\eta(W)) - \bar{g}(\bar{\nabla}_{\mathcal{N}}^*W, U) + \tau(U^T)\eta(W))\eta(V), \end{aligned}$$

for any vector fields U, V, W tangent to $\bar{M}$.

By applying the expressions from (61) and (72) to (52), we obtain the following result.

Theorem 4.14. Let (M, g, ∇, ∇^*) be a statistical spacelike hypersurface of a GRW spacetime $(\bar{M} = I \times_{\varrho} F, \bar{g}, \bar{\nabla} = \bar{\nabla}^{\bar{g}} + \bar{\mathcal{K}}, \bar{\nabla}^*)$ endowed with a statistical structure. Let the quadruple (F, g_F, W^F, λ^F) be a statistical Ricci soliton. Then, the quadruple $(M, g, W^T, \bar{\lambda})$ is a statistical Ricci soliton if and only if

$$\begin{aligned} (\mathcal{E}_{W^T}g)(U^T, V^T) &= \frac{2\varrho^2}{1-\varrho^2}(\bar{\lambda} - (n+1)\frac{\varrho''}{\varrho})\phi_U\phi_V + \frac{1}{\varrho^2-1}(U(\phi_W)\phi_V + V(\phi_W)\phi_U) \\ &+ \frac{1}{1-\varrho^2}((n+1)\varrho^2\mathbb{H} - \eta(W))B^*(U^T, V^T) + \frac{1}{1-\varrho^2}((n+1)\varrho^2\mathbb{H}^* - \eta(W))B(U^T, V^T) \\ &+ \frac{\varrho^2}{1-\varrho^2}(2\bar{g}(\bar{S}(\mathcal{N}, U^T)V^T, \mathcal{N}) + g((A_{\mathcal{N}}^* \circ A_{\mathcal{N}})U^T, V^T) + g((A_{\mathcal{N}} \circ A_{\mathcal{N}}^*)U^T, V^T)) \\ &+ \frac{1}{1-\varrho^2}(B^*(V^T, W^T) + V(\eta(W)) - \bar{g}(\bar{\nabla}_{\mathcal{N}}W, V) - \tau(V^T)\eta(W))\eta(U) \\ &+ \frac{\varrho^2}{\varrho^2-1}(\text{trace}(\nabla_{V^T}A_{\mathcal{N}}) + \text{trace}(\nabla_{V^T}^*A_{\mathcal{N}}^*) - \text{trace}(A_{\mathcal{N}}V^T) - \text{trace}(A_{\mathcal{N}}^*V^T))\eta(U) \\ &+ \frac{1}{1-\varrho^2}(B(U^T, W^T) + U(\eta(W)) - \bar{g}(\bar{\nabla}_{\mathcal{N}}^*W, U) + \tau(U^T)\eta(W))\eta(V) \\ &+ \frac{\varrho^2}{\varrho^2-1}(\text{trace}(\nabla_{U^T}A_{\mathcal{N}}) + \text{trace}(\nabla_{U^T}^*A_{\mathcal{N}}^*) - \text{trace}(A_{\mathcal{N}}U^T) - \text{trace}(A_{\mathcal{N}}^*U^T))\eta(V) \\ &+ \frac{2\varrho^2}{\varrho^2-1}(\bar{\lambda} + \overline{Ric}^{\bar{S}}(\mathcal{N}, \mathcal{N}) + \frac{1}{\varrho^2}(\mathcal{N} \cdot \eta(W)))\eta(U)\eta(V), \end{aligned}$$

for any vector fields U, V, W tangent to $\bar{M}$.

Let the quadruple $(F, g_F, \zeta^F, \lambda^F)$ be a statistical trivial Ricci soliton. Then, $Ric^F(U^F, V^F) = \lambda^F g_F(U^F, V^F)$ for any vector fields U and V tangent to $\bar{M}$. Using (22) and (38), we obtain

$$\begin{aligned} \overline{Ric}^{\bar{S}}(U, V) + \frac{1}{2}(\mathcal{E}_{\zeta}\bar{g})(U, V) &= \lambda^*(g(U^T, V^T) - \eta(U)\eta(V)) \\ &+ (\lambda^* - (n+1)\frac{\varrho''}{\varrho})\phi_U\phi_V, \end{aligned}$$

where $\lambda^* = \frac{\lambda^F}{\varrho^2} + n\left(\frac{\varrho'}{\varrho}\right)^2 + \frac{\varrho''}{\varrho} + \varrho'$. Thus, using (62) and (72) in the previous expression, we arrive at the following result.

Theorem 4.15. Let $(\overline{M} = I \times_{\varrho} F, \overline{g}, \overline{\nabla} = \overline{\nabla}^F + \overline{\mathcal{K}}, \overline{\nabla}^*)$ be a GRW spacetime endowed with a statistical structure. Let the quadruple $(F, g_F, \zeta^F, \lambda^F)$ be a statistical trivial Ricci soliton. Then, $(M, g, \zeta^T, \lambda^*)$ is a statistical almost Ricci soliton if and only if

$$\begin{aligned} & \overline{g}(\overline{S}(\mathcal{N}, U^T)V^T, \mathcal{N}) + \frac{1}{2}\left(g((A_N^* \circ A_N)U^T, V^T) + g((A_N \circ A_N^*)U^T, V^T)\right) \\ & + \frac{1}{2}\left((n+1)\mathbb{H}^* - \varrho \cosh \varphi\right)B(U^T, V^T) + \frac{1}{2}\left((n+1)\mathbb{H} - \varrho \cosh \varphi\right)B^*(U^T, V^T) \\ & - \frac{1}{2}\eta(V)\left(\text{trace}(\nabla_{U^T} A_N) + \text{trace}(\nabla_{U^T}^* A_N^*) - \text{trace}(A_N U^T) - \text{trace}(A_N^* U^T)\right) \\ & - \frac{1}{2}\eta(U)\left(\text{trace}(\nabla_{V^T} A_N) + \text{trace}(\nabla_{V^T}^* A_N^*) - \text{trace}(A_N V^T) - \text{trace}(A_N^* V^T)\right) \\ & - \left(\overline{\text{Ric}}^{\overline{S}}(\mathcal{N}, \mathcal{N}) - \varrho' + \lambda^*\right)\eta(U)\eta(V) + \left(\lambda^* - (n+1)\frac{\varrho''}{\varrho}\right)\phi_U\phi_V = 0, \end{aligned}$$

for any vector fields U and V tangent to $\overline{M}$.

References

- [1] L. Alias, A. Romero, M. Sanchez, *Uniqueness of complete space-like hypersurfaces of constant mean curvature in generalized Robertson-Walker spacetimes*, Gen. Relativ. Gravit. **27** (1995), 71–84.
- [2] S. Amari, *Information geometry of the EM and EM algorithms for neural networks*, Neural Networks **8** (1995), 1379–1408.
- [3] S. Amari, H. Nagaoka, *Methods of information geometry*, Amer. Math. Soc., Providence, 2000.
- [4] S. Amari, *Differential geometric methods in statistics*, Lecture Notes in Statistics, Springer, New York, 1985.
- [5] A. Barros, J. N. Gomes, E. Ribeiro, *Immersion of almost Ricci solitons into a Riemannian manifold*, Matematica Contemp. **40** (2011), 91–102.
- [6] A. M. Blaga, B. Y. Chen, *Gradient solitons on statistical manifolds*, J. Geom. Phys. **164** (2021), 1–10.
- [7] B. Y. Chen, *A simple characterization of generalized Robertson-Walker spacetimes*, Gen. Relativ. Gravit. **46** (2014), 1–5.
- [8] B. Y. Chen, S. Deshmukh, *Yamabe and quasi-Yamabe solitons on Euclidean submanifolds*, Mediterr. J. Math. **15** (2018), 1–9.
- [9] B. Chow, P. Lu, L. Ni, *Hamilton's Ricci flow*, Graduate Studies in Mathematics, Amer. Mat. Soc., Providence, 2010.
- [10] K. De, M.N.I. Khan, U.C. De, *Characterizations of generalized Robertson-Walker spacetimes concerning gradient solitons*, Heliyon **10** (2024), Art. No. e25702.
- [11] U.C. De, A. Gezer, *A short note on generalized Robertson Walker spacetimes*, Turk. J. Math. **48** (2024), 955–964.
- [12] U.C. De, F. Mofarreh, *A note on generalized Robertson–Walker spacetimes and $f(R^*)$ -gravity*, Int. J. Geom. Meth. Mod. Phys. **22** (2025), Art. No. 2450262.
- [13] H. Furuhashi, I. Hasegawa, *Submanifold theory in holomorphic statistical manifolds*, In: S. Dragomir, M.H. Shahid, F.R. Al-Solamy (eds) Geometry of Cauchy-Riemann Submanifolds, Springer, Singapore, 2016.
- [14] H. Furuhashi, *Hypersurfaces in statistical manifolds*, Diff. Geom. Appl. **27** (2009), 420–429.
- [15] R. S. Hamilton, *The Ricci flow on surfaces, mathematics and general relativity*, Contemp. Math. **71** (1998), 237–262.
- [16] V. Jain, A. P. Singh, R. Kumar, *On the geometry of lightlike submanifolds of indefinite statistical manifolds*, Int. J. Geom. Methods Mod. Phys. **17**(7) (2020), 1–17.
- [17] C.A. Mantica, L.G. Molinari, U.C. De, *A note on generalized Robertson–Walker spacetimes*, Int. J. Geom. Meth. Mod. Phys. **13** (2016), Art. No. 1650079.
- [18] C.A. Mantica, L.G. Molinari, U.C. De, *A condition for a perfect-fluid space-time to be a generalized Robertson-Walker space-time*, J. Math. Phys. **57** (2016), Art. No. 022508.
- [19] B. O'Neill, *Semi-Riemannian geometry with applications to relativity*, Academic Press, New York, 1982.
- [20] B. Opozda, *Curvature bounded conjugate symmetric statistical structures with complete metric*, Ann. Global Anal. Geom. **55** (2019), 687–702.
- [21] B. Opozda, *Completeness in affine and statistical geometry*, Ann. Global Anal. Geom. **59** (2021), 367–383.
- [22] S. Pigola, M. Rigoli, M. Rimoldi, A. G. Setti, *Ricci almost solitons*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **10** (2011), 757–799.
- [23] R. K. Sachs, L. H. Wu, *General relativity for mathematicians*, Springer, New York, 1977.
- [24] A. Sardar, U.C. De, *Characterizations of GRW spacetimes admitting Ricci–Yamabe solitons*, Int. J. Geom. Meth. Mod. Phys. **21** (2024), Art. No. 2440002.
- [25] A. P. Singh, C. Atindogbe, R. Kumar, V. Jain, *Rigged scalar curvature on the null hypersurfaces of GRW spacetimes*, Publicationes Math. Debrecen **104** (2024), 137–157.
- [26] K. Yano, M. Kon, *Structures on manifolds*, World Scientific Pub., Singapore, 1984.