Filomat 39:27 (2025), 9403–9413
https://doi.org/10.2298/FIL2527403G

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A new approach to integral characterizations in dichotomy theory

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Abstract. This paper introduces a new approach to integral characterizations in the study of dynamical systems, utilizing the connection between uniform dichotomy with differentiable growth rates and uniform exponential dichotomy. Within this framework, we establish integral conditions for these dichotomies, considering both invariant projection-valued functions and projection-valued functions compatible with a skew-evolution cocycle. The method relies on incorporating differentiable growth rates, which extend existing results and provide a broader perspective on dichotomy theory. We hope that these results may contribute to the dichotomy theory of nonautonomous dynamical systems.

1. Introduction

Widely applied in science, engineering, and economics, differential equations are a strong tool in the theory of dynamical systems and it is important to study the behavior of their solutions.

For instance, the notion of exponential dichotomy introduced by Perron in [31], is a key concept in the asymptotic theory of dynamical systems. (We refer the reader to the seminal works of Daleckii and Krein [9] and Massera and Schäffer [21].) This theory was consolidated and extended by Coppel [8], who offered a comprehensive synthesis and enhancement of the existing results up to 1978. For results in infinite-dimensional spaces, we refer to the work of Chicone and Latushkin [6]. In [36], Sacker and Sell introduced the notion of exponential dichotomy for skew-product semiflows, imposing the condition that the unstable subspace be finite-dimensional. Later, Chow and Leiva [7] generalized this notion, presenting a weaker form of exponential dichotomy for linear skew-product semiflows, which relaxes the finite-dimensionality assumption of the unstable subspace.

A large variety of methods has been introduced, beginning with different characterizations of exponential dichotomies and their applications, and progressing to detailed analyses of the properties of systems admitting exponential dichotomies, as established in [1–3, 14, 19, 20, 24, 28, 33, 37].

To the best of our knowledge, Pinto investigated for the first time the stability properties of dynamical systems under perturbations in the presence of a growth rate [32]. The notions of uniform and nonuniform dichotomy with growth rates or differentiable growth rates have been the subject of extensive research,

2020 *Mathematics Subject Classification*. Primary 34D05, 34D09; Secondary 40C10.
Keywords. skew-evolution cocycles, uniform strong h-dichotomy, uniform exponential dichotomy, projection-valued function.
Received: 21 March 2025; Revised: 02 August 2025; Accepted: 07 August 2025
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leading to significant results obtained through various approaches. The results in [4, 13, 15, 16, 27, 38, 41] and the references therein provide significant contributions to this topic.

One of the fundamental results in the stability theory was established by Datko in 1970 [10] for the case of C_0-semigroups. He proved that a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$, on a complex Hilbert space X, is uniformly exponentially stable if and only if, for every $x \in X$, the mapping $t \mapsto \|T(t)x\|$ belongs to $L^2(\mathbb{R}_+)$. This criterion was later extended by Pazy in 1983 to the more general setting of C_0-semigroups on Banach spaces, replacing $L^2(\mathbb{R}_+)$ with $L^p(\mathbb{R}_+)$, for all $p \geq 1$ [30].

In 1972, Datko extended the previous results to a more general setting [11]. Precisely, he proved that an evolution family $\{U(t,s)\}_{t\geq s\geq 0}$ on a Banach space X which has uniform exponential growth is uniformly exponentially stable if and only if there exists $p \geq 1$ such that

$$\sup_{s\geq 0} \int_s^\infty \|U(\tau,s)x\|^p d\tau < \infty, x \in X.$$

The result mentioned above was extended to dichotomy by Preda and Megan in 1985 [34]. Later, in 1986, Rolewicz [35] provided a further generalization considering a continuous, non-decreasing function $N : \mathbb{R}_+ \to \mathbb{R}_+$ with $N(0) = 0$ and $N(t) > 0$, for all $t > 0$. He established that if an evolution family $\{U(t,s)\}_{t\geq s\geq 0}$ has exponential growth and satisfies the condition

$$\sup_{s\geq 0} \int_s^\infty N(\|U(\tau,s)x\|)\, d\tau < \infty, x \in X,$$

then $\{U(t,s)\}_{t\geq s\geq 0}$ is uniformly exponentially stable.

Van Neerven [29] provides a unified treatment of the theorems of Datko–Pazy and Rolewicz. In [23], Megan, Sasu and Sasu established necessary and sufficient conditions for the uniform exponential stability of evolution equations on Banach spaces, highlighting techniques from skew-product semiflows. Their results extend and generalize previous results of Datko, Rolewicz and Van Neerven.

Let A be a bounded linear operator on the Banach space W and let φ be an evolution semiflow on the metric space Y. The solution of the equation

$$\dot{w}(t) = A(\varphi(t,s,y))w(t), \quad t \geq s \geq 0,$$

where $y \in Y$, defines a *skew-evolution cocycle*. This is given by the pair $C = (\Phi, \varphi)$, where Φ is a skew-evolution semiflow and φ is an evolution semiflow. The concept of a skew-evolution cocycle was introduced by Megan and Stoica in [26] as a generalization of classical notions of C_0-semigroups, evolution operators and skew-product semiflows, which have been studied by Chow and Leiva [7], Elaydi and Hajek [14], Latushkin and Schnaubelt [18], Megan, Sasu, and Sasu [25], Wang and Wang [40], along with other references in the literature. Further work on the asymptotic behavior of dynamical systems described by skew-evolution cocycles can be found in the studies of Hai [17], Megan, Găină, and Boruga (Toma) [22], Mihiţ and Megan [27], Stoica and Megan [39], Yue, Song and Li [42], etc.

The idea behind this paper originates from [4], where differentiable growth rates were introduced to analyze the asymptotic behavior of dynamical systems. Within this framework, we introduce the concept of uniform strong h-dichotomy, also referred to as uniform dichotomy with differentiable growth rates. Using this notion, we develop a new method that exploits the connection between uniform strong h-dichotomy and uniform exponential dichotomy to establish integral characterizations.

Motivated by a recent work of Dragičević [12], we intend to generalize a classical result of Datko for uniform strong h-dichotomy. This leads to new integral characterizations for both invariant projection-valued function and projection-valued function compatible with a skew-evolution cocycle. Moreover, incorporating differentiable growth rates into this framework, we extend existing results and provides a new perspective on integral characterizations in the study of dynamical systems.

2. Preliminaries

Let Y be a metric space and W be a Banach space. Denote by $\mathcal{B}(W)$ the Banach algebra of all bounded linear operators on W, and by I the identity operator on W. The norms on W and $\mathcal{B}(W)$ are both denoted by $\|\cdot\|$. We also consider the following sets:

$$\Delta = \{(t,s) \in \mathbb{R}_+^2 : t \geq s\} \text{ and } T = \{(t,s,t_0) \in \mathbb{R}_+^3 : t \geq s \geq t_0\}.$$

We recall that a mapping $\varphi : \Delta \times Y \to Y$, is an *evolution semiflow* on Y, if it satisfies the following properties:

(es_1) $\varphi(s,s,y) = y$, for all $(s,y) \in \mathbb{R}_+ \times Y$;
(es_2) $\varphi(t,s,\varphi(s,t_0,y_0)) = \varphi(t,t_0,y_0)$, for all $(t,s,t_0,y_0) \in T \times Y$.

Similarly, a mapping $\Phi : \Delta \times Y \to \mathcal{B}(W)$ is called a *skew-evolution semiflow* on $Y \times W$ over the evolution semiflow φ if it satisfies:

(ses_1) $\Phi(s,s,y) = I$, for all $(s,y) \in \mathbb{R}_+ \times Y$;
(ses_2) $\Phi(t,s,\varphi(s,t_0,y_0))\Phi(s,t_0,y_0) = \Phi(t,t_0,y_0)$, for all $(t,s,t_0,y_0) \in T \times Y$.

When φ is an evolution semiflow and Φ is a skew-evolution semiflow over φ, the pair $C = (\Phi,\varphi)$ is called a *skew-evolution cocycle*.

The skew-evolution cocycle C is considered *strongly measurable* if, for any $(s,y,w) \in \mathbb{R}_+ \times Y \times W$, the function $t \mapsto \|\Phi(t,s,y)w\|$ is measurable on $[s,\infty)$.

Definition 2.1. A mapping $P : \mathbb{R}_+ \times Y \to \mathcal{B}(W)$ is a *projection-valued function* if $s \mapsto P(s,y)w$ is continuous for all $(y,w) \in Y \times W$ and the following property takes place

$$P^2(s,y) = P(s,y), \text{ for all } (s,y) \in \mathbb{R}_+ \times Y.$$

If $P : \mathbb{R}_+ \times Y \to \mathcal{B}(W)$ is a projection-valued function, then $Q : \mathbb{R}_+ \times Y \to \mathcal{B}(W)$ defined by $Q(s,y) = I - P(s,y)$, is also a projection-valued function, called *the complementary projection of P.*

Definition 2.2. A projection-valued function $P : \mathbb{R}_+ \times Y \to \mathcal{B}(W)$ is said to be *compatible* with the skew-evolution cocycle C if the following conditions hold:

(i) it is *invariant* for the skew-evolution cocycle C, precisely

$$\Phi(t,s,y)P(s,y) = P(t,\varphi(t,s,y))\Phi(t,s,y), \text{ for all } (t,s,y) \in \Delta \times Y;$$

(ii) the mapping $\Phi(t,s,y) : Range\ Q(s,y) \to Range\ Q(t,\varphi(t,s,y))$ is invertible, for all $(t,s,y) \in \Delta \times Y$.

In what follows, we consider

$$\Phi_P : \Delta \times Y \to \mathcal{B}(W), \Phi_P(t,s,y) = \Phi(t,s,y)P(s,y) \text{ and } \Phi_Q : \Delta \times Y \to \mathcal{B}(W), \Phi_Q(t,s,y) = \Phi(t,s,y)Q(s,y).$$

Moreover, for any $(t,s,t_0,y_0) \in T \times Y$, the following identity holds:

$$\Phi_P(t,t_0,y_0) = \Phi_P(t,s,\varphi(s,t_0,y_0))\Phi_P(s,t_0,y_0).$$

Remark 2.3. If the projection-valued function $P : \mathbb{R}_+ \times Y \to \mathcal{B}(W)$ is compatible with the skew-evolution cocycle C, then there exists $\Psi : \Delta \times Y \to \mathcal{B}(W)$ such that:

(i) $\Psi(t,s,y)$ is an isomorphism from $Range\ Q(t,\varphi(t,s,y))$ to $Range\ Q(s,y)$,
(ii) $\Psi(t,s,y)\Phi(t,s,y)Q(s,y) = Q(s,y)$,
(iii) $\Phi(t,s,y)\Psi(t,s,y)Q(t,\varphi(t,s,y)) = Q(t,\varphi(t,s,y))$,

for all $(t,s,y) \in \Delta \times Y$.

We set

$$\Psi_Q(t,s,y) = \Psi(t,s,y)Q(t,\varphi(t,s,y)), (t,s,y) \in \Delta \times Y.$$

We recall that a function $h : \mathbb{R}_+ \to [1,\infty)$ is called a *growth rate* if it is strictly increasing and continuous with $h(0) = 1$ and $\lim\limits_{t\to\infty} h(t) = \infty$.

In particular, a growth rate of class C^1 will be called a differentiable growth rate.

Definition 2.4. Assume that h is a growth rate. The pair (C,P) is called *uniformly h-dichotomic* if there exist two constants $N \geq 1$ and $\nu > 0$ satisfying:

(uhd_1) $h(t)^\nu \|\Phi_P(t,s,y)w\| \leq Nh(s)^\nu \|P(s,y)w\|$;
(uhd_2) $h(t)^\nu \|Q(s,y)w\| \leq Nh(s)^\nu \|\Phi_Q(t,s,y)w\|$,

for all $(t,s,y,w) \in \Delta \times Y \times W$.

Definition 2.5. If in particular, h is a differentiable growth rate, the pair (C,P) is said to be *uniformly strongly h-dichotomic* if there are two constants $N \geq 1$ and $\nu > 0$ such that:

$(ushd_1)$ $h(t)^\nu \|\Phi_P(t,s,y)w\| \leq N\frac{h'(s)}{h(s)} h(s)^\nu \|P(s,y)w\|$;
$(ushd_2)$ $h(t)^\nu \|Q(s,y)w\| \leq N\frac{h'(t)}{h(t)} h(s)^\nu \|\Phi_Q(t,s,y)w\|$,

for all $(t,s,y,w) \in \Delta \times Y \times W$.

Setting $h(t) = e^t$ we encounter the classical notion of *uniform exponential dichotomy.*

Remark 2.6. The pair (C,P) is uniformly strongly h-dichotomic if and only if there are $N \geq 1, \nu > 0$ such that:

$(ushd_1')$ $h(t)^\nu \|\Phi_P(t,s,y)w\| \leq N\frac{h'(s)}{h(s)} h(s)^\nu \|P(s,y)w\|$;
$(ushd_2')$ $h(t)^\nu \|\Psi_Q(t,s,y)w\| \leq N\frac{h'(t)}{h(t)} h(s)^\nu \|Q(t,\varphi(t,s,y))w\|$,

for all $(t,s,y,w) \in \Delta \times Y \times W$.

Proof. Necessity. From Definition 2.5 it suffices to prove the inequality in $(ushd_2')$. We have

$$\begin{aligned} h(t)^\nu \|\Psi_Q(t,s,y)w\| &= h(t)^\nu \|Q(s,y)\Psi_Q(t,s,y)w\| \\ &\leq N\frac{h'(t)}{h(t)} h(s)^\nu \|\Phi_Q(t,s,y)\Psi_Q(t,s,y)w\| \\ &= N\frac{h'(t)}{h(t)} h(s)^\nu \|Q(t,\varphi(t,s,y))w\|, \end{aligned}$$

for all $(t,s,y,w) \in \Delta \times Y \times W$.

Sufficiency. Similarly, we have

$$h(t)^\nu \|Q(s,y)w\| = h(t)^\nu \|\Psi_Q(t,s,y)\Phi_Q(t,s,y)w\| \leq N\frac{h'(t)}{h(t)} h(s)^\nu \|\Phi_Q(t,s,y)w\|,$$

for all $(t,s,y,w) \in \Delta \times Y \times W$. □

In what follows we denote by $\mathcal{H}$ and $\mathcal{H}_1$ the sets of functions that provide a comprehensive framework for analyzing differentiable growth rates in the context of our study, precisely:

• $\mathcal{H}$ is the set of all differentiable growth rates $h : \mathbb{R}_+ \to [1,\infty)$ with the property that there exists $H > 1$ such that $h'(t) \leq Hh(t)$, for all $t \geq 0$;

• $\mathcal{H}_1$ is the set of all differentiable growth rates $h : \mathbb{R}_+ \to [1,\infty)$ with the property that there exists $m > 0$ such that $h'(t) \geq mh(t)$, for all $t \geq 0$.

Remark 2.7. If $h \in \mathcal{H}$, then the concept of uniform strong h-dichotomy implies the concept of uniform h-dichotomy. The converse implication is not necessarily valid.

Example 2.8. For any fixed differentiable growth rate $h : \mathbb{R}_+ \to [1, \infty)$, let

$$\Phi(t,s,y) = \frac{h(s)}{h(t)}P(s,y) + \frac{h(t)}{h(s)}Q(s,y),$$

where $P : \mathbb{R}_+ \times Y \to \mathcal{B}(W)$ is a projection-valued function satisfying the property

$$P(t, \varphi(t,t_0,y_0)) = P(t_0,y_0), \text{ for all } (t,t_0,y_0) \in \Delta \times Y.$$

One can easily check that $C = (\Phi, \varphi)$ is a skew-evolution cocycle for some evolution semiflow $\varphi : \Delta \times Y \to Y$. Furthermore, the pair (C, P) is uniformly h-dichotomic.

Case 1: Let us consider the function $h(t) = e^{t^2+t}$. We observe that the pair (C, P) is uniformly strongly h-dichotomic. Indeed, since $\left(\frac{h(s)}{h(t)}\right)^{1-\nu} \leq 1$, for any fixed $\nu \in (0, 1)$ and for all $(t, s) \in \Delta$, we have

$$\begin{aligned} \|\Phi_P(t,s,y)w\| &= \frac{h(s)}{h(t)}\|P(s,y)w\| = \left(\frac{h(s)}{h(t)}\right)^{\nu}\left(\frac{h(s)}{h(t)}\right)^{1-\nu}\|P(s,y)w\| \\ &\leq \left(\frac{h(s)}{h(t)}\right)^{\nu}\|P(s,y)w\| \leq (2s+1)\left(\frac{h(t)}{h(s)}\right)^{-\nu}\|P(s,y)w\| \\ &= \frac{h'(s)}{h(s)}\left(\frac{h(t)}{h(s)}\right)^{-\nu}\|P(s,y)w\|, \end{aligned}$$

for all $(t, s, y, w) \in \Delta \times Y \times W$. Similarly, one may check the inequality $(ushd_2)$.

From the above computations, we remark that if $h \in \mathcal{H}_1$, then (C, P) is uniformly strongly h-dichotomic.

Case 2: For $h(t) = t + 1$, $h \in \mathcal{H}$ and the pair (C, P) is not uniformly strongly h-dichotomic.

Indeed, assuming that there exist $N \geq 1$ and $\nu > 0$ such that

$$(t+1)^{\nu}\|Q(s,y)w\| \leq N(s+1)^{\nu-1}\|Q(s,y)w\|, \text{for all } (t,s,y,w) \in \Delta \times Y \times W$$

and setting $s = 0$, we obtain

$$(t+1)^{\nu} \leq N, \text{ for all } t \geq 0,$$

which is false for $t \to \infty$. Thus, the pair (C, P) is not uniformly strongly h-dichotomic.

Definition 2.9. Assume that h is a differentiable growth rate. The pair (C, P) has *uniform strong h-growth* if there are $M \geq 1, \omega > 0$ such that:

$(ushg_1)$ $h(s)^{\omega}\|\Phi_P(t,s,y)w\| \leq M\frac{h'(s)}{h(s)}h(t)^{\omega}\|P(s,y)w\|$;

$(ushg_2)$ $h(s)^{\omega}\|Q(s,y)w\| \leq M\frac{h'(t)}{h(t)}h(t)^{\omega}\|\Phi_Q(t,s,y)w\|$,

for all $(t, s, y, w) \in \Delta \times Y \times W$.

Letting $h(t) = e^t$ we encounter the classical concept of *uniform exponential growth*.

For any fixed growth rate h, we consider the skew-evolution cocycle $C^h = (\Phi^h, \varphi^h)$, where

$$\varphi^h : \Delta \times Y \to Y, \varphi^h(t,s,y) = \varphi(h^{-1}(e^t), h^{-1}(e^s), y)$$

and

$$\Phi^h : \Delta \times Y \to \mathcal{B}(W), \Phi^h(t,s,y) = \Phi(h^{-1}(e^t), h^{-1}(e^s), y).$$

Let also

$$P^h : \mathbb{R}_+ \times Y \to \mathcal{B}(W), P^h(t,y) = P(h^{-1}(e^t), y).$$

The following result plays a crucial role in this work, establishing the connections between uniform strong h-dichotomy and uniform exponential dichotomy.

Lemma 2.10. *Let $h \in \mathcal{H} \cap \mathcal{H}_1$. The pair (C, P) is uniformly strongly h-dichotomic if and only if the pair (C^h, P^h) is uniformly exponentially dichotomic.*

Proof. *Indeed, assuming that (C, P) is uniformly strongly h-dichotomic, as $h \in \mathcal{H}$, we have*

$$\begin{aligned} \|\Phi^h_{P^h}(t,s,y)w\| &= \|\Phi_P(h^{-1}(e^t), h^{-1}(e^s), y)w\| \\ &\leq N\frac{h'(h^{-1}(e^s))}{h(h^{-1}(e^s))}\Big(\frac{h(h^{-1}(e^t))}{h(h^{-1}(e^s))}\Big)^{-\nu}\|P(h^{-1}(e^s), y)w\| \\ &\leq NHe^{-\nu(t-s)}\|P^h(s,y)w\|, \end{aligned}$$

for all $(t, s, y, w) \in \Delta \times Y \times W$.

Similarly, we get

$$\begin{aligned} \|Q^h(s,y)w\| &= \|Q(h^{-1}(e^s), y)w\| \\ &\leq N\frac{h'(h^{-1}(e^t))}{h(h^{-1}(e^t))}\Big(\frac{h(h^{-1}(e^t))}{h(h^{-1}(e^s))}\Big)^{-\nu}\|\Phi_Q(h^{-1}(e^t), h^{-1}(e^s), y)w\| \\ &\leq NHe^{-\nu(t-s)}\|\Phi^h_{Q^h}(t,s,y)w\|, \end{aligned}$$

for all $(t, s, y, w) \in \Delta \times Y \times W$.

The above inequalities show that that the pair (C^h, P^h) is uniformly exponentially dichotomic.

Conversely, if we suppose that the pair (C^h, P^h) is uniformly exponentially dichotomic, we obtain

$$\|\Phi^h_{P^h}(t,s,y)w\| \leq Ne^{-\nu(t-s)}\|P^h(s,y)w\|,$$

for all $(t, s, y, w) \in \Delta \times Y \times W$, or equivalently,

$$\|\Phi_P(h^{-1}(e^t), h^{-1}(e^s), y)w\| \leq Ne^{-\nu(t-s)}\|P(h^{-1}(e^s), y)w\|.$$

Let $u_0 = h^{-1}(e^t)$ and $v_0 = h^{-1}(e^s)$. Then

$$\|\Phi_P(u_0, v_0, y)w\| \leq \frac{N}{m}\frac{h'(v_0)}{h(v_0)}\Big(\frac{h(u_0)}{h(v_0)}\Big)^{-\nu}\|P(v_0, y)w\|.$$

Similarly, we get

$$\|Q(v_0, y)w\| \leq \frac{N}{m}\frac{h'(u_0)}{h(u_0)}\Big(\frac{h(u_0)}{h(v_0)}\Big)^{-\nu}\|\Phi_Q(u_0, v_0, y)w\|.$$

Therefore, the pair (C, P) is uniformly strongly h-dichotomic. □

Let us remark that the function $h(t) = e^{3t}(t+1)^2, t \geq 0$ belongs to $\mathcal{H} \cap \mathcal{H}_1$. This emphasizes the significance of our results.

3. Characterizing integrals through invariant projections

In this section, we present a new approach to Datko-type integral characterizations using invariant projection-valued functions. A key element of our method is the connection between uniform strong h-dichotomy and uniform exponential dichotomy, which allows us to establish integral conditions for dichotomy.

In the following, $C = (\Phi, \varphi)$ is a strongly measurable skew-evolution cocycle, $P : \mathbb{R}_+ \times Y \to \mathcal{B}(W)$ is an invariant projection-valued function and $h : \mathbb{R}_+ \to [1, \infty)$ is a differentiable growth rate.

Using the same arguments as in [5], we obtain the following two characterizations of uniform exponential dichotomy for skew-evolution cocycles.

Proposition 3.1. *Assume that (C, P) has uniform exponential growth. Then the pair (C, P) is uniformly exponentially dichotomic if and only if there exist $D \geq 1$ and $d \in (0, 1)$ satisfying:*

(ueD_1) $\int_s^\infty e^{d(\tau-s)}\|\Phi_P(\tau, s, y)w\|d\tau \leq D\|P(s, y)w\|$, *for all* $(s, y, w) \in \mathbb{R}_+ \times Y \times W$;

(ueD_2) $\int_s^\infty \frac{e^{d(\tau-s)}}{\|\Phi_Q(\tau,s,y)w\|}d\tau \leq \frac{D}{\|Q(s,y)w\|}$, *for all* $(s, y, w) \in \mathbb{R}_+ \times Y \times W$ *with* $Q(s, y)w \neq 0$.

Proposition 3.2. *Assume that (C, P) has uniform exponential growth. Then (C, P) is uniformly exponentially dichotomic if and only if there exist $D \geq 1, d \in (0, 1)$ such that:*

(ueD_1') $\int_s^t \frac{e^{d(t-\tau)}}{\|\Phi_P(\tau,s,y)w\|}d\tau \leq \frac{D}{\|\Phi_P(t,s,y)w\|}$, *for all* $(t, s, y, w) \in \Delta \times Y \times W$ *with* $\Phi_P(t, s, y)w \neq 0$;

(ueD_2') $\int_s^t e^{d(t-\tau)}\|\Phi_Q(\tau, s, y)w\|d\tau \leq D\|\Phi_Q(t, s, y)w\|$, *for all* $(t, s, y, w) \in \Delta \times Y \times W$.

The following theorem presents a Datko-type characterization of uniform strong h-dichotomy.

Theorem 3.3. *Assume that the pair (C, P) has uniform strong h-growth and $h \in \mathcal{H} \cap \mathcal{H}_1$. Then (C, P) is uniformly strongly h-dichotomic if and only if there exist two constants $D \geq 1$ and $d \in (0, 1)$ such that*

$(ushD_1)$ $\int_s^\infty \frac{h'(\tau)}{h(\tau)}h(\tau)^d\|\Phi_P(\tau, s, y)w\|d\tau \leq Dh(s)^d\frac{h'(s)}{h(s)}\|P(s, y)w\|$, *for all* $(s, y, w) \in \mathbb{R}_+ \times Y \times W$;

$(ushD_2)$ $\int_s^\infty \frac{h'(\tau)}{h(\tau)}\frac{h(\tau)^d}{\|\Phi_Q(\tau,s,y)w\|}d\tau \leq \frac{h'(s)}{h(s)}\frac{Dh(s)^d}{\|Q(s,y)w\|}$, *for all* $(s, y, w) \in \mathbb{R}_+ \times Y \times W$ *with* $Q(s, y)w \neq 0$.

Proof. Necessity. Suppose that (C, P) is uniformly strongly h-dichotomic. According to Lemma 2.10, this implies that the pair (C^h, P^h) is uniformly exponentially dichotomic. Furthermore, by applying Proposition 3.1, we obtain the existence of two constants $D \geq 1$ and $d \in (0, 1)$ such that

(a) $\int_s^\infty e^{d(\tau-s)}\|\Phi^h_{P^h}(\tau, s, y)w\|d\tau \leq D\|P^h(s, y)w\|$, for all $(s, y, w) \in \mathbb{R}_+ \times Y \times W$;

(b) $\int_s^\infty \frac{e^{d(\tau-s)}}{\|\Phi^h_{Q^h}(\tau,s,y)w\|}d\tau \leq \frac{D}{\|Q^h(s,y)w\|}$, for all $(s, y, w) \in \mathbb{R}_+ \times Y \times W$ with $Q^h(s, y)w \neq 0$.

Then
$$\int_s^\infty e^{d(\tau-s)}\|\Phi_P(h^{-1}(e^\tau), h^{-1}(e^s), y)w\|d\tau \leq D\|P(h^{-1}(e^s), y)w\|.$$

Changing the variables $h^{-1}(e^\tau) = u_0$ and setting $h^{-1}(e^s) = v_0$, we get

$$\begin{aligned}\int_{v_0}^\infty \frac{h'(u_0)}{h(u_0)}\Big(\frac{h(u_0)}{h(v_0)}\Big)^d\|\Phi_P(u_0, v_0, y)w\|du_0 \quad &\leq \quad D\|P(v_0, y)w\| \\ &\leq \quad \frac{D}{m}\frac{h'(v_0)}{h(v_0)}\|P(v_0, y)w\|.\end{aligned}$$

Similarly,
$$\int_{v_0}^\infty \frac{h'(u_0)}{h(u_0)}\Big(\frac{h(u_0)}{h(v_0)}\Big)^d\frac{1}{\|\Phi_Q(u_0, v_0, y)w\|}du_0 \leq \frac{D}{m}\frac{h'(v_0)}{h(v_0)}\frac{1}{\|Q(v_0, y)w\|}.$$

Sufficiency. Assume that there exist $D \geq 1$ and $d \in (0, 1)$ satisfying $(ushD_1)$ and $(ushD_2)$. By Lemma 2.10 it suffices to prove that the pair (C^h, P^h) is uniformly exponentially dichotomic. So, we obtain

$$\int_s^\infty e^{d(\tau-s)}\|\Phi^h_{P^h}(\tau, s, y)w\|d\tau = \int_s^\infty e^{d(\tau-s)}\|\Phi_P(h^{-1}(e^\tau), h^{-1}(e^s), y)w\|d\tau.$$

Setting $h^{-1}(e^\tau) = u_0$, we have

$$\int_{v_0}^\infty \frac{h'(u_0)}{h(u_0)}\Big(\frac{h(u_0)}{h(v_0)}\Big)^d\|\Phi_P(u_0, v_0, y)w\|du_0 \leq D\frac{h'(v_0)}{h(v_0)}\|P(v_0, y)w\| \leq DH\|P^h(s, y)w\|,$$

where $h^{-1}(e^s) = v_0$.

Similarly,

$$\int_s^\infty \frac{e^{d(\tau-s)}}{\|\Phi^h_{Q^h}(\tau,s,y)w\|}d\tau = \int_{v_0}^\infty \frac{h'(u_0)}{h(u_0)}\left(\frac{h(u_0)}{h(v_0)}\right)^d \frac{1}{\|\Phi_Q(u_0,v_0,y)w\|}du_0 \leq \frac{DH}{\|Q^h(s,y)w\|}.$$

□

Using the same type of arguments, one can easily deduce another Datko-type characterization of uniform strong h-dichotomy.

Theorem 3.4. *Assume that (C,P) has uniform strong h-growth for some $h \in \mathcal{H} \cap \mathcal{H}_1$. The pair (C,P) is uniformly strongly h-dichotomic if and only if there exist $D \geq 1$ and $d \in (0,1)$ with*

$(ushD_1')$ $\int_s^t \frac{h'(\tau)}{h(\tau)} \frac{h(\tau)^{-d}}{\|\Phi_P(\tau,s,y)w\|}d\tau \leq \frac{h'(t)}{h(t)} \frac{Dh(t)^{-d}}{\|\Phi_P(t,s,y)w\|}$*, for all* $(t,s,y,w) \in \Delta \times Y \times W$ *with* $\Phi_P(t,s,y)w \neq 0$;

$(ushD_2')$ $\int_s^t \frac{h'(\tau)}{h(\tau)} h(\tau)^{-d}\|\Phi_Q(\tau,s,y)w\|d\tau \leq D\frac{h'(t)}{h(t)}h(t)^{-d}\|\Phi_Q(t,s,y)w\|$*, for all* $(t,s,y,w) \in \Delta \times Y \times W$.

4. Projections compatible with the skew-evolution cocycle C in integral characterizations

This section is dedicated to integral characterizations that involve projection-valued functions compatible with C and the operator Ψ_Q, introduced in *Remark 2.3*. Our results will emphasize the role of a projection-valued function compatible with a skew-evolution cocycle in dichotomy framework.

Theorem 4.1. *Assume that (C,P) has uniform exponential growth. Then the pair (C,P) is uniformly exponentially dichotomic if and only if there exist two constants $D \geq 1, d \in (0,1)$ with the following properties:*

(ueD_1'') $\int_s^\infty e^{d(\tau-s)}\|\Phi_P(\tau,s,y)w\|d\tau \leq D\|P(s,y)w\|$*, for all* $(s,y,w) \in \mathbb{R}_+ \times Y \times W$;

(ueD_2'') $\int_s^\infty \frac{e^{d(\tau-s)}}{\|\Psi_Q(t,\tau,\varphi(\tau,s,y))w\|}d\tau \leq \frac{D}{\|\Psi_Q(t,s,y)w\|}$*, for all* $(t,s,y,w) \in \Delta \times Y \times W$ *with* $\Psi_Q(t,s,y)w \neq 0$.

Proof. Necessity. Let $d \in (0,\nu)$. The inequality (ueD_1'') is explicitly mentioned in Proposition 3.1 and for the second inequality we have:

$$\int_s^\infty \frac{e^{d(\tau-s)}}{\|\Psi_Q(t,\tau,\varphi(\tau,s,y))w\|}d\tau \leq N\int_s^\infty e^{-\nu(\tau-s)}e^{d(\tau-s)}\frac{1}{\|\Psi_Q(t,s,y)w\|}d\tau = \frac{N}{\nu-d}\frac{1}{\|\Psi_Q(t,s,y)w\|},$$

for all $(t,s,y,w) \in \Delta \times Y \times W$ with $\Psi_Q(t,s,y)w \neq 0$.

Sufficiency. Proposition 3.1 establishes the first inequality. In order to prove the second one we establish two cases:

In the first case we suppose that $t \geq s+1$ and $\Psi_Q(t,s,y)w \neq 0$. Then

$$\begin{aligned}
\frac{1}{\|Q(t,\varphi(t,s,y))w\|} &= \int_{t-1}^{t} \frac{1}{\|\Phi_Q(t,s,y)\Psi_Q(t,s,y)w\|}d\tau \\
&\leq \int_{t-1}^{t} Me^{\omega(t-\tau)}\frac{1}{\|\Phi_Q(\tau,s,y)\Psi_Q(t,s,y)w\|}d\tau \\
&= M\int_{t-1}^{t} e^{-d(t-s)}e^{(t-\tau)(\omega+d)}e^{d(\tau-s)}\frac{1}{\|\Phi_Q(\tau,s,y)\Psi_Q(t,s,y)w\|}d\tau \\
&\leq Me^{\omega+d}e^{-d(t-s)}\int_s^\infty \frac{e^{d(\tau-s)}}{\|\Psi_Q(t,\tau,\varphi(\tau,s,y))w\|}d\tau \\
&\leq Me^{\omega+d}e^{-d(t-s)}\frac{D}{\|\Psi_Q(t,s,y)w\|},
\end{aligned}$$

for all $(t,s,y,w) \in \Delta \times Y \times W$ with $\Psi_Q(t,s,y)w \neq 0$.

In the second case we consider $t \in [s, s+1)$.

$$\|\Psi_Q(t,s,y)w\| \leq Me^{\omega(t-s)}\|Q(t,\varphi(t,s,y))w\| \leq Me^{\omega+d}e^{-d(t-s)}\|Q(t,\varphi(t,s,y))w\|,$$

for all $(t,s,y,w) \in \Delta \times Y \times W$.

Therefore (C,P) is uniformly exponentially dichotomic. □

Taking into account that

$$\|\Psi^h_{Q^h}(t,t_0,y_0)w_0\| \leq Ne^{-\nu(s-t_0)}\|\Psi^h_{Q^h}(t,s,\varphi^h(s,t_0,y_0))w_0\|,$$

for all $(t,s,t_0,y_0,w_0) \in T \times Y \times W$, an analog result for uniform strong h-dichotomy is presented below.

Theorem 4.2. *Assume that the pair (C,P) has uniform strong h-growth, for some $h \in \mathcal{H} \cap \mathcal{H}_1$. Then (C,P) is uniformly strongly h-dichotomic if and only if there exist $D \geq 1, d \in (0,1)$ such that:*

$(ushD_1'')$ $\int_s^\infty \frac{h'(\tau)}{h(\tau)}h(\tau)^d\|\Phi_P(\tau,s,y)w\|d\tau \leq Dh(s)^d\frac{h'(s)}{h(s)}\|P(s,y)w\|$, *for all* $(s,y,w) \in \mathbb{R}_+ \times Y \times W$;

$(ushD_2'')$ $\int_s^\infty \frac{h'(\tau)}{h(\tau)}\frac{h(\tau)^d}{\|\Psi_Q(t,\tau,\varphi(\tau,s,y))w\|}d\tau \leq \frac{h'(s)}{h(s)}\frac{Dh(s)^d}{\|\Psi_Q(t,s,y)w\|}$, *for all* $(t,s,y,w) \in \Delta \times Y \times W$ *with* $\Psi_Q(t,s,y)w \neq 0$.

Proof. Necessity. From Theorem 3.3 we obtain $(ushD_1'')$. In order to obtain $(ushD_2'')$ we use Theorem 4.1 and the connection between the uniform strong h-dichotomy and the uniform exponential dichotomy. Indeed, we have

$$\int_s^\infty \frac{e^{d(\tau-s)}}{\|\Psi^h_{Q^h}(t,\tau,\varphi^h(\tau,s,y))w\|}d\tau \leq \frac{D}{\|\Psi^h_{Q^h}(t,s,y)w\|},$$

for all $(t,s,y,w) \in \Delta \times Y \times W$ with $\Psi^h_{Q^h}(t,s,y)w \neq 0$, which is equivalent to

$$\int_s^\infty \frac{e^{d(\tau-s)}}{\|\Psi_Q(h^{-1}(e^t),h^{-1}(e^\tau),\varphi(h^{-1}(e^\tau),h^{-1}(e^s),y))w\|}d\tau \leq \frac{D}{\|\Psi_Q(h^{-1}(e^t),h^{-1}(e^s),y)w\|}.$$

By applying the change of variables $h^{-1}(e^\tau) = u_0$ and introducing the notations $h^{-1}(e^s) = v_0$ and $h^{-1}(e^t) = u$, we get:

$$\int_{v_0}^\infty \frac{h'(u_0)}{h(u_0)}\left(\frac{h(u_0)}{h(v_0)}\right)^d\frac{1}{\|\Psi_Q(u,u_0,\varphi(u_0,v_0,y))w\|}du_0 \leq \frac{D}{m}\frac{h'(v_0)}{h(v_0)}\frac{1}{\|\Psi_Q(u,v_0,y)w\|}.$$

Sufficiency. The first part of the argument follows directly from Theorem 3.3.

For the second part, we suppose the existence of $D \geq 1$ and $d \in (0,1)$ satisfying $(ushD_2'')$, and we use the connection between uniform strong h-dichotomy and uniform exponential dichotomy. Then, we have

$$\int_s^\infty \frac{e^{d(\tau-s)}}{\|\Psi^h_{Q^h}(t,\tau,\varphi^h(\tau,s,y))w\|}d\tau = \int_s^\infty \frac{e^{d(\tau-s)}}{\|\Psi_Q(h^{-1}(e^t),h^{-1}(e^\tau),\varphi(h^{-1}(e^\tau),h^{-1}(e^s),y))w\|}d\tau.$$

Using the fact that $h \in \mathcal{H}$ and setting $h^{-1}(e^\tau) = u_0$, along with the notations $h^{-1}(e^s) = v_0$ and $h^{-1}(e^t) = u$, we get

$$\int_{v_0}^\infty \frac{h'(u_0)}{h(u_0)}\Big(\frac{h(u_0)}{h(v_0)}\Big)^d \frac{1}{\|\Psi_Q(u,u_0,\varphi(u_0,v_0,y))w\|}du_0 \leq \frac{DH}{\|\Psi^h_{Q^h}(t,s,y)w\|},$$

if $\Psi^h_{Q^h}(t,s,y)w \neq 0$. $\square$

Adapting the arguments from [27], which consider the case of exponential splitting, we establish the following characterization of uniform exponential dichotomy for skew-evolution cocycles.

Theorem 4.3. Assume that (C,P) has uniform exponential growth. Then the pair (C,P) is uniformly exponentially dichotomic if and only if there exist $D \geq 1$ and $d \in (0,1)$ with:

(ueD_1''') $\int_s^t \frac{e^{d(t-\tau)}}{\|\Phi_P(\tau,s,y)w\|}d\tau \leq \frac{D}{\|\Phi_P(t,s,y)w\|}$, for all $(t,s,y,w) \in \Delta \times Y \times W$ with $\Phi_P(t,s,y)w \neq 0$;
(ueD_2''') $\int_s^t e^{d(t-\tau)}\|\Psi_Q(t,\tau,\varphi(\tau,s,y))w\|d\tau \leq D\|Q(t,\varphi(t,s,y))w\|$, for all $(t,s,y,w) \in \Delta \times Y \times W$.

Following the same approach as in the proof of Theorem 4.2, we deduce a new characterization of the uniform strong h-dichotomy of skew-evolution cocycles through an integral approach.

Theorem 4.4. Let (C,P) be a pair with uniform strong h-growth, where $h \in \mathcal{H} \cap \mathcal{H}_1$. Then (C,P) is uniformly strongly h-dichotomic if and only if there exist $D \geq 1$ and $d \in (0,1)$ such that:

$(ushD_1''')$ $\int_s^t \frac{h'(\tau)}{h(\tau)}\frac{h(\tau)^{-d}}{\|\Phi_P(\tau,s,y)w\|}d\tau \leq \frac{h'(t)}{h(t)}\frac{Dh(t)^{-d}}{\|\Phi_P(t,s,y)w\|}$, for all $(t,s,y,w) \in \Delta \times Y \times W$ with $\Phi_P(t,s,y)w \neq 0$;
$(ushD_2''')$ $\int_s^t h(\tau)^{-d}\frac{h'(\tau)}{h(\tau)}\|\Psi_Q(t,\tau,\varphi(\tau,s,y))w\|d\tau \leq Dh(t)^{-d}\frac{h'(t)}{h(t)}\|Q(t,\varphi(t,s,y))w\|$, for all $(t,s,y,w) \in \Delta \times Y \times W$.

Acknowledgements

The author would like to thank Professor Emeritus Mihail Megan for his support and valuable suggestions that greatly contributed to the finalization of this paper. Sincere thanks are also extended to the referees for their insightful comments, which have significantly improved the quality of the paper.

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