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Overlapping subset neighborhoods and applications via generalized approximation spaces

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Abstract. In rough set theory, numerous studies have employed various neighborhood systems to develop novel rough approximation models, aiming to enhance accuracy and preserve as many characteristics as possible of the reference approximation space established by Pawlak. To contribute to this field, we introduce new rough neighborhoods, referred to as overlapping subset rough neighborhoods. They are defined under any arbitrary binary relation using subset relations between right and left neighborhoods, as well as between minimal right and minimal left neighborhoods. We examine their fundamental properties and identify the types of binary relations (i.e., serial, inverse serial, transitive, and quasi-order relations) under which they satisfy specific interrelations between them, as well as their relationships to previously established neighborhoods. We also successfully derive indicators based on the proposed neighborhoods to determine whether a relation is symmetric or both symmetric and transitive. Furthermore, we explore the relationships between the proposed neighborhoods as they transition between two generalized approximation spaces, where the smaller relation is serial (inverse serial) and transitive. Next, we put forward novel rough set models derived from the suggested neighborhoods and examine their core properties. We demonstrate that these models retain most of the characterizations of Pawlak approximation space and identify the binary relations under which they outperform previous models in minimizing uncertainty and satisfy the monotonicity property. Additionally, we design an algorithm to compute the boundary area and accuracy measures and provide a practical example related to book authorship to highlight the effectiveness of the proposed technique in extracting information. Finally, we conduct a comparative analysis that showcases the main advantages of our method while also referring to its limitations in comparison to various prior approaches.

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1. Introduction

1.1. An overview of advances in classification with rough neighborhood systems

Uncertainty and vagueness are inherent aspects of the real world, often resulting in problems that lack precise definitions. Conventional approaches frequently struggle to address these ambiguities effectively. In response to this challenge, Pawlak [42, 43] founded the rough set theory as a robust mathematical framework for handling imprecise information and facilitating knowledge extraction. This theory is built upon the operators of upper and lower approximations, which distinguish between possible and confirmed knowledge derived from data. Ten fundamental properties of each one of these operators have been established, serving as a standard for comparing later models and as an indication of their reliability. These operators are used to define positive, negative, and boundary regions and accuracy measures, which offer valuable insights for decision-makers into the structure and completeness of the extracted information.

Over time, the reliance on equivalence relations has posed significant limitations in the practical application of rough set theory, particularly as new challenges have emerged. To overcome these constraints, researchers have sought to generalize and extend the framework, enhancing its applicability to a broader range of scenarios. A breakthrough in this endeavor was achieved in 1996 by Yao [55], who introduced an alternative approach to obtaining granules of computation corresponding to equivalence classes, known as right neighborhoods and left neighborhoods. In this approach, these new types of neighborhoods serve as fundamental building blocks for representing knowledge rather than equivalence classes. Unlike the traditional rough set model of Pawlak, Yao's method did not impose any conditions on the underlying relations, paving the way for the development of rough neighborhood systems.

Despite these advancements, Yao identified that the approximation operators derived from these neighborhoods failed to preserve fully some key properties of upper and lower approximation operators defined within the Pawlak rough set model. As a result, some researchers introduced additional constraints on binary relations to ensure consistency with Pawlak's framework. To develop Yao's contributions, further rough set models have been proposed by incorporating diverse types of binary relations, including quasi-order [44, 56], similarity [24], tolerance [51], and binary relations [29].

Later, various rough neighborhoods were proposed based on relationships between Yao's neighborhoods, such as intersection, union, containment, and equality. These rough neighborhoods were utilized to develop multiple generalized rough set models, which were then applied to address practical issues across various fields. Among them, minimal right and left neighborhoods were introduced by Abo-Tabl [1] as new blocks for defining upper and lower approximations. Additionally, union and intersection neighborhoods were proposed by Allam et al. [5, 6], which they subsequently used to explore rough approximation operators. Equality neighborhoods and their corresponding rough set models were introduced and studied by Mareay [37], and later discussed by Atef et al. [19]. The concept of maximal right neighborhoods was defined by Dai et al. [24] under a similarity relation, after which several types of maximal neighborhoods were extensively investigated by Al-shami [9], who applied them to categorize patients infected with COVID-19.

Al-shami and colleagues investigated novel paradigms derived from new types of neighborhoods they introduced, such as containment neighborhoods [8], subset neighborhoods [10], intersection neighborhoods [12], and cardinality neighborhoods [14, 15]. Their study identified key characteristics and interrelationships among these paradigms and demonstrated their potential applications in preventing the spread of COVID-19 and alleviating pressure on healthcare facilities. In 2024, Demiralp [25] investigated novel insights into rough set theory by studying Al-shami et al.'s neighborhoods [12] under the name of transitive neighborhoods. Her approach involves using different types of intersection neighborhoods rather than relying on the same types. Right afterwards, Al-shami and Mhemdi [16] adopted this approach to provide a new version of containment neighborhoods, namely overlapping containment neighborhoods, and initiated their generalized approximation spaces. They developed Demiralp's framework by enhancing accuracy measures and ensuring the retention of Pawlak approximation properties. Following this line of research, Al-shami [7] refined the approach proposed in [16] by employing an equality relation instead of an inclusion relation to achieve greater accuracy.

Abu-Donia [3] and Qian et al. [45, 46] proposed novel techniques for studying rough neighborhoods and their corresponding rough set models by varying the number of given binary relations. Abu-Donia explored new types of rough neighborhoods by defining Yao's neighborhoods over the intersection of a finite number of relations rather than a single relation. He applied this approach to introduce rough set models and highlight their key characteristics. Qian et al. developed multi-granulation rough set paradigms by utilizing multiple relations to compute their corresponding upper and lower approximations. The aforementioned studies reflect an ongoing effort to advance rough set models and enhance their usage in practical applications.

Since the rough set theory has drawn significant interest from researchers across various fields of mathematics and computer science, it has led to the integration of abstract mathematical structures with rough set theory, as well as the study of approximation operators and regions of uncertainty within these frameworks. For instance, the close relationship between the closure operator in topological spaces and upper rough approximation, as well as between the interior operator in topological spaces and lower rough approximation, has been leveraged to develop new rough set models and offer new frameworks to deal with incomplete knowleadage, as illustrated in [2, 11, 17, 20, 21, 26, 31, 36, 39, 41, 47, 52]. Additionally, some researchers have combined rough neighborhoods with ideals, filters, and primal structures, as discussed in [27, 30–33, 35].

1.2. Motivations of study

Although rough set models based on neighborhood systems have evolved rapidly and are well-documented in the literature, several types of rough neighborhoods remain unexplored. Therefore, we write this manuscript to introduce a novel framework of classifications induced from new rough neighborhoods that does not rely on equivalence relations and is constructed using subset relations between right and left neighborhoods, as well as between minimal right and minimal left neighborhoods. This approach is independent of the subset neighborhood-based method [10, 41] and provides a robust framework for addressing challenges that require the integration of left and right neighborhoods. The following three key advantages of the proposed rough set paradigms:

- It is highly reliable, as it preserves most properties of the reference model introduced by Pawlak, and in certain cases of $\jmath$, it retains all properties when specific conditions are imposed on the given relation.
- It satisfies the monotonicity property when the smaller relation is serial (inverse serial) and transitive, and the larger relation is an equivalence relation.
- It increases the amount of corroborated information derived from rough subsets while reducing uncertainty in boundary regions, thereby enhancing the accuracy of information extracted from data sets.

1.3. Manuscript organization

The structure of this manuscript is as follows. After this introduction, Section 2 provides essential definitions and properties of Pawlak approximation space along with recent generalizations of approximation spaces necessary for understanding this work. In Section 3, we create a novel class of neighborhoods, referred to as overlapping subset rough neighborhoods, which are formulated using the subset relation between left and right neighborhoods as well as minimal left and minimal right neighborhoods. We explore their fundamental characteristics, derive conditions on binary relations to establish explicit formulas for their computation, and identify key equivalences among them. Section 4 is dedicated to developing new rough set models inspired by the proposed rough neighborhoods and examining their core properties. In Section 5, we present a practical example to demonstrate the superiority of the proposed approach compared to previous models introduced in [9, 12, 24, 25, 54, 55] under specific types of binary relations. Section 6 provides a comparative analysis to highlight the effectiveness and validity of the proposed method, including a discussion of its strengths and limitations. Finally, in Section 7, we summarize the key findings and suggest directions for future research.

2. Background on some types of rough neighborhoods and their generalized approximation spaces

This section provides a concise overview of the fundamental rough neighborhood types and their associated rough set paradigms, and mentions the essential concepts related to them.

2.1. Pawlak approximation space

The first introduction of rough approximation spaces was proposed by Pawlak [42, 43], so we refer to it as the Pawlak approximation space throughout this work. We start by recalling some types of binary relations that we shall use to prove some results herein.

Definition 2.1. (see [8]) For a nonempty universal set $\mathbf{U}$, we say Ω is a binary relation on $\mathbf{U}$ if $\Omega \subseteq \mathbf{U} \times \mathbf{U}$. We consider (b, f) as an element of Ω if b and f are associated together with respect to Ω; i.e., $(\mathsf{b}, \mathsf{f}) \in \Omega$. Consider the following conditions imposed on a relations Ω on $\mathbf{U}$:

(i) for each $\mathsf{b} \in \Omega$ there is $\mathsf{f} \in \Omega$ such that $\mathsf{b}\Omega\mathsf{f}$.

(ii) for each $\mathsf{b} \in \Omega$ there is $\mathsf{f} \in \Omega$ such that $\mathsf{f}\Omega\mathsf{b}$.

(iii) for each $\mathsf{b} \in \mathbf{U}$ we have $(\mathsf{b}, \mathsf{b}) \in \Omega$.

(iv) $\mathsf{b}\Omega\mathsf{f} \Longleftrightarrow \mathsf{f}\Omega\mathsf{b}$.

(v) $\mathsf{b}\Omega\mathsf{f}$ and $\mathsf{f}\Omega\mathsf{b} \Longrightarrow \mathsf{b} = \mathsf{f}$.

(vi) $\mathsf{b}\Omega\mathsf{f}$ whenever $\mathsf{b}\Omega\mathsf{k}$ and $\mathsf{k}\Omega\mathsf{f}$.

(vii) Ω reflexive and transitive.

(viii) Ω is symmetric and quasi-order.

(ix) Ω is antisymmetric and quasi-order.

If a condition (i) (resp., (ii), (iii), (iv), (v), (vi), (vii), (viii), (ix)) is satisfied, then Ω is said to be serial (resp., inverse serial, reflexive, symmetric, antisymmetric, transitive, quasi-order, equivalence, partial order).

The next definition shows how one can approximate a subset using other two sets defined by an equivalence relation.

Definition 2.2. ([42, 43]) Let Ω be an equivalence relation on $\mathbf{U}$ and $\mathbf{U}/\Omega$ be an equivalence classes of $\mathbf{U}$ generated by Ω. Then, the lower approximation and upper approximation of a subset Y of $\mathbf{U}$ are given by:

$$\underline{\Omega}(Y) = \bigcup\{G \in \mathbf{U}/\Omega : G \subseteq Y\}.$$

$$\overline{\Omega}(Y) = \bigcup\{G \in \mathbf{U}/\Omega : G \bigcap Y \neq \emptyset\}, \text{ respectively.}$$

Remark 2.3. From now on, we call the pair $(\mathbf{U}, \Omega)$ Pawlak approximation space ($\mathcal{P}$-approximation space, in short) or Pawlak rough set model ($\mathcal{P}$-rough set model, in short) if Ω is an equivalence relation on $\mathbf{U}$.

The next proposition exhibits the essential properties of $\mathcal{P}$-rough set model.

Proposition 2.4. ([42, 43]) *Let $(\boldsymbol{U}, \Omega)$ be $\mathcal{P}$-approximation space, $Y, Z \subseteq \boldsymbol{U}$, and $\boldsymbol{U}/\Omega$ be a equivalence classes of $\boldsymbol{U}$. The subsequent characterizations are fulfilled.*

$(\mathcal{LP}1)$ $\underline{\Omega}(Y) \subseteq Y$	$(\mathcal{UP}1)$ $Y \subseteq \overline{\Omega}(Y)$
$(\mathcal{LP}2)$ $\underline{\Omega}(\emptyset) = \emptyset$	$(\mathcal{UP}2)$ $\overline{\Omega}(\emptyset) = \emptyset$
$(\mathcal{LP}3)$ $\underline{\Omega}(\boldsymbol{U}) = \boldsymbol{U}$	$(\mathcal{UP}3)$ $\overline{\Omega}(\boldsymbol{U}) = \boldsymbol{U}$
$(\mathcal{LP}4)$ $If\, Y \subseteq Z,\ then\ \underline{\Omega}(Y) \subseteq \underline{\Omega}(Z)$	$(\mathcal{UP}4)$ $If\, Y \subseteq Z,\ then\ \overline{\Omega}(Y) \subseteq \overline{\Omega}(Z)$
$(\mathcal{LP}5)$ $\underline{\Omega}(Y \cap Z) = \underline{\Omega}(Y) \cap \underline{\Omega}(Z)$	$(\mathcal{UP}5)$ $\overline{\Omega}(Y \cap Z) \subseteq \overline{\Omega}(Y) \cap \overline{\Omega}(Z)$
$(\mathcal{LP}6)$ $\underline{\Omega}(Y) \cup \underline{\Omega}(Z) \subseteq \underline{\Omega}(Y \cup Z)$	$(\mathcal{UP}6)$ $\overline{\Omega}(Y \cup Z) = \overline{\Omega}(Y) \cup \overline{\Omega}(Z)$
$(\mathcal{LP}7)$ $\underline{\Omega}(Y^c) = (\overline{\Omega}(Y))^c$	$(\mathcal{UP}7)$ $\overline{\Omega}(Y^c) = (\underline{\Omega}(Y))^c$
$(\mathcal{LP}8)$ $\underline{\Omega}(\underline{\Omega}(Y)) = \underline{\Omega}(Y)$	$(\mathcal{UP}8)$ $\overline{\Omega}(\overline{\Omega}(Y)) = \overline{\Omega}(Y)$
$(\mathcal{LP}9)$ $\underline{\Omega}((\underline{\Omega}(Y))^c) = (\overline{\Omega}(Y))^c$	$(\mathcal{UP}9)$ $\overline{\Omega}((\overline{\Omega}(Y))^c) = (\underline{\Omega}(Y))^c$
$(\mathcal{LP}10)$ $\underline{\Omega}(Y) = Y, \forall Y \in \boldsymbol{U}/\Omega$	$(\mathcal{UP}10)$ $\overline{\Omega}(Y) = Y, \forall Y \in \boldsymbol{U}/\Omega$

Definition 2.5. A subset Y of $\mathcal{P}$-approximation space $(\mathbf{U}, \Omega)$ is named a definable (or exact) subset if its upper approximation and lower approximation are equal; otherwise, it is called a rough set.

Numerically, rough sets can be characterized using the following measures derived from the lower and upper approximations.

Definition 2.6. ([42, 43]) Let Y be a subset of $\mathcal{P}$-approximation space $(\mathbf{U}, \Omega)$. The ξ-accuracy and $\hat{\xi}$-roughness measures of Y are respectively calculated by

$$\xi(Y) = \frac{|\underline{\Omega}(Y)|}{|\overline{\Omega}(Y)|}, \mid \overline{\Omega}(Y) \mid \neq 0.$$

$$\hat{\xi}(Y) = 1 - \xi(Y).$$

Easily one can remark that a subset Y is a definable (respectively, rough) set iff

$\xi(Y) = 1 \quad$ (respectively, $\xi(Y) < 1$).

2.2. Rough neighborhoods inspired by a binary relation and their rough set models

In this subsection, we present some generalizations of $\mathcal{P}$-approximation space that are constructed by replacing the equivalence relation with an arbitrary binary relation. On the one hand, this approach expands the scope of rough set theory since many practical scenarios cannot be modeled by an equivalence relation. On the other hand, rough set models initiated by this approach fail to fulfill some properties of $\mathcal{P}$-approximation space (display in Proposition 2.4). Thus, the ability of the proposed rough set models induced by this approach to retain as many of these properties as possible is seen as an advantage of these models.

Remark 2.7. From now on, we call the pair $(\mathbf{U}, \Omega)$ generalized approximation space ($\mathcal{G}$-approximation space, in short) whenever Ω is not an equivalence relation on $\mathbf{U}$.

We begin by presenting the Yao neighborhoods, which serve as the cornerstone for constructing all other rough neighborhood systems and their corresponding approximation spaces.

Definition 2.8. ([54, 55]) Let $(\mathbf{U}, \Omega)$ be a $\mathcal{G}$-approximation space and $\jmath \in \{r, l, i, u\}$. For each $\mathsf{b} \in \mathbf{U}$, the $\mathcal{A}_\jmath$-neighborhoods are given by the following formulas:

(i) $\mathcal{A}_r(\mathsf{b}) = \{\mathsf{f} \in \mathbf{U} : \mathsf{b}\,\Omega\,\mathsf{f}\}$.

(ii) $\mathcal{A}_l(\mathsf{b}) = \{\mathsf{f} \in \mathbf{U} : \mathsf{f}\,\Omega\,\mathsf{b}\}$.

(iii) $\mathcal{A}_i(\mathsf{b}) = \mathcal{A}_r(\mathsf{b}) \bigcap \mathcal{A}_l(\mathsf{b})$.

(iv) $\mathcal{A}_u(\mathsf{b}) = \mathcal{A}_r(\mathsf{b}) \bigcup \mathcal{A}_l(\mathsf{b})$.

Then, some authors set up new formulas to generate rough neighborhoods as follows:

Definition 2.9. ([1, 5, 6]) Let $(\mathbf{U}, \Omega)$ be a $\mathcal{G}$-approximation space and $\jmath \in \{\langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$. For each $\mathsf{b} \in \mathbf{U}$, the $\mathcal{A}_\jmath$-neighborhoods are given by the following formulas:

(i)

$$\mathcal{A}_{\langle r\rangle}(\mathsf{b}) = \begin{cases} \bigcap\limits_{\mathsf{b}\in\mathcal{A}_r(\mathsf{f})} \mathcal{A}_r(\mathsf{f}) & : \ \exists\, \mathcal{A}_r(\mathsf{f}) \text{ involving } \mathsf{b} \\ \emptyset & : \ \textit{Otherwise} \end{cases}$$

(ii)

$$\mathcal{A}_{\langle l\rangle}(\mathsf{b}) = \begin{cases} \bigcap\limits_{\mathsf{b}\in\mathcal{A}_l(\mathsf{f})} \mathcal{A}_l(\mathsf{f}) & : \ \exists\, \mathcal{A}_l(\mathsf{f}) \text{ involving } \mathsf{b} \\ \emptyset & : \ \textit{Otherwise} \end{cases}$$

(iii) $\mathcal{A}_{\langle i\rangle}(\mathsf{b}) = \mathcal{A}_{\langle r\rangle}(\mathsf{b}) \bigcap \mathcal{A}_{\langle l\rangle}(\mathsf{b})$.

(iv) $\mathcal{A}_{\langle u\rangle}(\mathsf{b}) = \mathcal{A}_{\langle r\rangle}(\mathsf{b}) \bigcup \mathcal{A}_{\langle l\rangle}(\mathsf{b})$.

Henceforth, we consider $\jmath$ to belong to the set $\{r, l, i, u, \langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$, unless otherwise specified.

The next proposition demonstrates the necessary conditions for the correspondence between $\mathcal{A}_\jmath$-neighborhoods introduced in Definition 2.8 and their analogous displayed in Definition 2.9.

Proposition 2.10. *Let $(\boldsymbol{U}, \Omega)$ be a $\mathcal{G}$-approximation space such that Ω is a quasi-order. Then $\mathcal{A}_\jmath = \mathcal{A}_{\langle\jmath\rangle}$ for $\jmath \in \{r, l, i, u\}$*

Lemma 2.11. ([16]) *In G-approximation spaces $(\boldsymbol{U}, \Omega_1)$ and $(\boldsymbol{U}, \Omega_2)$, if $\Omega_1 \subseteq \Omega_2$, then the next relationships are fulfilled.*

(i) $\mathcal{A}_{1\xi}(\mathsf{b}) \subseteq \mathcal{A}_{2\xi}(\mathsf{b})$ *for each $\jmath \in \{r, l, i, u\}$.*

(ii) $\mathcal{A}_{1\xi}(\mathsf{b}) \subseteq \mathcal{A}_{2\xi}(\mathsf{b})$ *for each $\jmath \in \{\langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$ providing that Ω_1 is reflexive and Ω_2 is transitive.*

Yao [54, 55] and other researchers and scholars [1, 5, 6] employed the aforesaid sorts of neighborhoods to establish various $\mathcal{G}$-approximation spaces and compared them with $\mathcal{P}$-approximation space in terms of approximation properties and accuracy values.

Definition 2.12. ([1, 5, 6, 54, 55]) In a $\mathcal{G}$-approximation space $(\mathbf{U}, \Omega)$, the upper and lower approximations of a subset Y are calculated with respect to $\mathcal{A}_\jmath$-neighborhoods using the next formulas:

$$\mathcal{H}^{\mathcal{A}_\jmath}(Y) = \{\mathsf{b} \in \mathbf{U} : \mathcal{A}_\xi(\mathsf{b}) \cap Y \neq \emptyset\}.$$

$$\mathcal{H}_{\mathcal{A}_\jmath}(Y) = \{\mathsf{b} \in \mathbf{U} : \mathcal{A}_\xi(\mathsf{b}) \subseteq Y\},$$

Definition 2.13. ([1, 5, 6, 54, 55]) In a $\mathcal{G}$-approximation space $(\mathbf{U}, \Omega)$, the $\xi_{\mathcal{A}_\jmath}$-accuracy and $\hat{\xi}_{\mathcal{A}_\jmath}$-roughness measures of a subset $Y \neq \emptyset$ are calculated with respect to $\mathcal{A}_\jmath$-neighborhoods using the next formulas:

$$\xi_{\mathcal{A}_\jmath}(Y) = \frac{|\mathcal{H}_{\mathcal{A}_\jmath}(Y) \cap Y|}{|\mathcal{H}^{\mathcal{A}_\jmath}(Y) \cup Y|}, \text{ and}$$

$$\hat{\xi}_{\mathcal{A}_j}(Y) = 1 - \xi_{\mathcal{A}_j}(Y).$$

Definition 2.14. (see [24]) If $\Omega_1 \subseteq \Omega_2$ where Ω_1 and Ω_2 are relations on $\mathbf{U}$, we say that the approximations, defined by $\mathcal{A}_j$-neighborhoods, have the property of monotonicity in accuracy (respectively, roughness) if $\xi_{\mathcal{A}_{1\xi}}(Y) \geq \xi_{\mathcal{A}_{2\xi}}(Y)$ (respectively, $\hat{\xi}_{\mathcal{A}_{1\xi}}(Y) \leq \hat{\xi}_{\mathcal{A}_{2\xi}}(Y)$) for each subset Y.

Recently, several types of neighborhood systems and their $\mathcal{G}$-approximation spaces have been introduced. Some of them are recalled in the following.

Definition 2.15. ([9, 24]) Let $(\mathbf{U}, \Omega)$ be a $\mathcal{G}$-approximation space. For each $\mathsf{b} \in \mathbf{U}$, the $\mathbb{M}_j$-neighborhoods are given by the following formulas:

(i) $\mathbb{M}_r(\mathsf{b}) = \bigcup_{\mathsf{b}\in\mathcal{A}_r(\mathsf{f})} \mathcal{A}_r(\mathsf{f})$.

(ii) $\mathbb{M}_l(\mathsf{b}) = \bigcup_{\mathsf{b}\in\mathcal{A}_l(\mathsf{f})} \mathcal{A}_l(\mathsf{f})$.

(iii) $\mathbb{M}_i(\mathsf{b}) = \mathbb{M}_r(\mathsf{b}) \cap \mathbb{M}_l(\mathsf{b})$.

(iv) $\mathbb{M}_u(\mathsf{b}) = \mathbb{M}_r(\mathsf{b}) \cup \mathbb{M}_l(\mathsf{b})$.

(v) $\mathbb{M}_{\langle r\rangle}(\mathsf{b}) = \bigcup_{\mathsf{b}\in\mathcal{A}_{\langle r\rangle}(\mathsf{f})} \mathcal{A}_{\langle r\rangle}(\mathsf{f})$.

(vi) $\mathbb{M}_{\langle l\rangle}(\mathsf{b}) = \bigcup_{\mathsf{b}\in\mathcal{A}_{\langle l\rangle}(\mathsf{f})} \mathcal{A}_{\langle l\rangle}(\mathsf{f})$.

(vii) $\mathbb{M}_{\langle i\rangle}(\mathsf{b}) = \mathbb{M}_{\langle r\rangle}(\mathsf{b}) \cap \mathbb{M}_{\langle l\rangle}(\mathsf{b})$.

(viii) $\mathbb{M}_{\langle u\rangle}(\mathsf{b}) = \mathbb{M}_{\langle r\rangle}(\mathsf{b}) \cup \mathbb{M}_{\langle l\rangle}(\mathsf{b})$.

Definition 2.16. ([8]) Let $(\mathbf{U}, \Omega)$ be a $\mathcal{G}$-approximation space. For each $\mathsf{b} \in \mathbf{U}$, the $\mathbb{C}_j$-neighborhoods are given by the following formulas:

(i) $\mathbb{C}_r(\mathsf{b}) = \{\mathsf{f} \in \mathbf{U} : \mathcal{A}_r(\mathsf{f}) \subseteq \mathcal{A}_r(\mathsf{b})\}$.

(ii) $\mathbb{C}_l(\mathsf{b}) = \{\mathsf{f} \in \mathbf{U} : \mathcal{A}_l(\mathsf{f}) \subseteq \mathcal{A}_l(\mathsf{b})\}$.

(iii) $\mathbb{C}_i(\mathsf{b}) = \mathbb{C}_r(\mathsf{b}) \cap \mathbb{C}_l(\mathsf{b})$.

(iv) $\mathbb{C}_u(\mathsf{b}) = \mathbb{C}_r(\mathsf{b}) \cup \mathbb{C}_l(\mathsf{b})$.

(v) $\mathbb{C}_{\langle r\rangle}(\mathsf{b}) = \{\mathsf{f} \in \mathbf{U} : \mathcal{A}_{\langle r\rangle}(\mathsf{f}) \subseteq \mathcal{A}_{\langle r\rangle}(\mathsf{b})\}$.

(vi) $\mathbb{C}_{\langle l\rangle}(\mathsf{b}) = \{\mathsf{f} \in \mathbf{U} : \mathcal{A}_{\langle l\rangle}(\mathsf{f}) \subseteq \mathcal{A}_{\langle l\rangle}(\mathsf{b})\}$.

(vii) $\mathbb{C}_{\langle i\rangle}(\mathsf{b}) = \mathbb{C}_{\langle r\rangle}(\mathsf{b}) \cap \mathbb{C}_{\langle l\rangle}(\mathsf{b})$.

(viii) $\mathbb{C}_{\langle u\rangle}(\mathsf{b}) = \mathbb{C}_{\langle r\rangle}(\mathsf{b}) \cup \mathbb{C}_{\langle l\rangle}(\mathsf{b})$.

Definition 2.17. ([10]) Let $(\mathbf{U}, \Omega)$ be a $\mathcal{G}$-approximation space. For each $\mathsf{b} \in \mathbf{U}$, the $\mathbb{S}_j$-neighborhoods are given by the following formulas:

(i) $\mathbb{S}_r(\mathsf{b}) = \{\mathsf{f} \in \mathbf{U} : \mathcal{A}_r(\mathsf{b}) \subseteq \mathcal{A}_r(\mathsf{f})\}$.

(ii) $\mathbb{S}_l(\mathsf{b}) = \{\mathsf{f} \in \mathbf{U} : \mathcal{A}_l(\mathsf{b}) \subseteq \mathcal{A}_l(\mathsf{f})\}$.

(iii) $\mathbb{S}_i(\mathsf{b}) = \mathbb{S}_r(\mathsf{b}) \cap \mathbb{S}_l(\mathsf{b})$.

(iv) $\mathbb{S}_u(\mathsf{b}) = \mathbb{S}_r(\mathsf{b}) \cup \mathbb{S}_l(\mathsf{b})$.

(v) $\mathbb{S}_{\langle r\rangle}(\mathsf{b}) = \{\mathsf{f} \in \mathbf{U} : \mathcal{A}_{\langle r\rangle}(\mathsf{b}) \subseteq \mathcal{A}_{\langle r\rangle}(\mathsf{f})\}$.

(vi) $\mathbb{S}_{\langle l\rangle}(\mathsf{b}) = \{\mathsf{f} \in \mathbf{U} : \mathcal{A}_{\langle l\rangle}(\mathsf{b}) \subseteq \mathcal{A}_{\langle l\rangle}(\mathsf{f})\}$.

(vii) $\mathbb{S}_{\langle i\rangle}(\mathsf{b}) = \mathbb{S}_{\langle r\rangle}(\mathsf{b}) \cap \mathbb{S}_{\langle l\rangle}(\mathsf{b})$.

(viii) $\mathbb{S}_{\langle u\rangle}(\mathsf{b}) = \mathbb{S}_{\langle r\rangle}(\mathsf{b}) \cup \mathbb{S}_{\langle l\rangle}(\mathsf{b})$.

Definition 2.18. ([12]) Let $(\mathbf{U}, \Omega)$ be a $\mathcal{G}$-approximation space. For each $\mathsf{b} \in \mathbf{U}$, the $\mathbb{E}_J$-neighborhoods are given by the following formulas:

(i) $\mathbb{E}_r(\mathsf{b}) = \{\mathsf{f} \in \mathbf{U} : \mathcal{A}_r(\mathsf{f}) \cap \mathcal{A}_r(\mathsf{b}) \neq \emptyset\}$.

(ii) $\mathbb{E}_l(\mathsf{b}) = \{\mathsf{f} \in \mathbf{U} : \mathcal{A}_l(\mathsf{f}) \cap \mathcal{A}_l(\mathsf{b}) \neq \emptyset\}$.

(iii) $\mathbb{E}_i(\mathsf{b}) = \mathbb{E}_r(\mathsf{b}) \cap \mathbb{E}_l(\mathsf{b})$.

(iv) $\mathbb{E}_u(\mathsf{b}) = \mathbb{E}_r(\mathsf{b}) \cup \mathbb{E}_l(\mathsf{b})$.

(v) $\mathbb{E}_{\langle r\rangle}(\mathsf{b}) = \{\mathsf{f} \in \mathbf{U} : \mathcal{A}_{\langle r\rangle}(\mathsf{f}) \cap \mathcal{A}_{\langle r\rangle}(\mathsf{b}) \neq \emptyset\}$.

(vi) $\mathbb{E}_{\langle l\rangle}(\mathsf{b}) = \{\mathsf{f} \in \mathbf{U} : \mathcal{A}_{\langle l\rangle}(\mathsf{f}) \cap \mathcal{A}_{\langle l\rangle}(\mathsf{b}) \neq \emptyset\}$.

(vii) $\mathbb{E}_{\langle i\rangle}(\mathsf{b}) = \mathbb{E}_{\langle r\rangle}(\mathsf{b}) \cap \mathbb{E}_{\langle l\rangle}(\mathsf{b})$.

(viii) $\mathbb{E}_{\langle u\rangle}(\mathsf{b}) = \mathbb{E}_{\langle r\rangle}(\mathsf{b}) \cup \mathbb{E}_{\langle l\rangle}(\mathsf{b})$.

Definition 2.19. ([8–10, 12, 24]) In a $\mathcal{G}$-approximation space $(\mathbf{U}, \Omega)$, the lower and upper approximations of a subset Y are calculated with respect to $\mathbb{M}_J$-neighborhoods, $\mathbb{C}_J$-neighborhoods, $\mathbb{S}_J$-neighborhoods, and $\mathbb{E}_J$-neighborhoods using the following formulas:

$$\begin{aligned}
&\mathcal{H}_{\mathbb{M}_J}(Y) = \{\mathsf{b} \in \mathbf{U} : \mathbb{M}_\xi(\mathsf{b}) \subseteq Y\} && \mathcal{H}^{\mathbb{M}_J}(Y) = \{\mathsf{b} \in \mathbf{U} : \mathbb{M}_\xi(\mathsf{b}) \cap Y \neq \emptyset\}\\
&\mathcal{H}_{\mathbb{C}_J}(Y) = \{\mathsf{b} \in \mathbf{U} : \mathbb{C}_\xi(\mathsf{b}) \subseteq Y\} && \mathcal{H}^{\mathbb{C}_J}(Y) = \{\mathsf{b} \in \mathbf{U} : \mathbb{C}_\xi(\mathsf{b}) \cap Y \neq \emptyset\}\\
&\mathcal{H}_{\mathbb{S}_J}(Y) = \{\mathsf{b} \in \mathbf{U} : \mathbb{S}_\xi(\mathsf{b}) \subseteq Y\} && \mathcal{H}^{\mathbb{S}_J}(Y) = \{\mathsf{b} \in \mathbf{U} : \mathbb{S}_\xi(\mathsf{b}) \cap Y \neq \emptyset\}\\
&\mathcal{H}_{\mathbb{E}_J}(Y) = \{\mathsf{b} \in \mathbf{U} : \mathbb{E}_\xi(\mathsf{b}) \subseteq Y\} && \mathcal{H}^{\mathbb{E}_J}(Y) = \{\mathsf{b} \in \mathbf{U} : \mathbb{E}_\xi(\mathsf{b}) \cap Y \neq \emptyset\}
\end{aligned}$$

Definition 2.20. ([8–10, 12, 24]) In a $\mathcal{G}$-approximation space $(\mathbf{U}, \Omega)$, the ξ-accuracy and $\hat{\xi}$-roughness of a subset $Y \neq \emptyset$ are calculated with respect to $\mathbb{M}_J$-neighborhoods, $\mathbb{C}_J$-neighborhoods, $\mathbb{S}_J$-neighborhoods, and $\mathbb{E}_J$-neighborhoods using the following formulas:

$$\begin{aligned}
&\xi_{\mathbb{M}_J}(Y) = \frac{|\,\mathcal{H}_{\mathbb{M}_J}(Y) \cap Y\,|}{|\,\mathcal{H}^{\mathbb{M}_J}(Y) \cup Y\,|} && \hat{\xi}_{\mathbb{M}_J}(Y) = 1 - \xi_{\mathbb{M}_J}(Y)\\
&\xi_{\mathbb{C}_J}(Y) = \frac{|\,\mathcal{H}_{\mathbb{C}_J}(Y)\,|}{|\,\mathcal{H}^{\mathbb{C}_J}(Y)\,|} && \hat{\xi}_{\mathbb{C}_J}(Y) = 1 - \xi_{\mathbb{C}_J}(Y)\\
&\xi_{\mathbb{S}_J}(Y) = \frac{|\,\mathcal{H}_{\mathbb{S}_J}(Y)\,|}{|\,\mathcal{H}^{\mathbb{S}_J}(Y)\,|} && \hat{\xi}_{\mathbb{S}_J}(Y) = 1 - \xi_{\mathbb{S}_J}(Y)\\
&\xi_{\mathbb{E}_J}(Y) = \frac{|\,\mathcal{H}_{\mathbb{E}_J}(Y) \cap Y\,|}{|\,\mathcal{H}^{\mathbb{E}_J}(Y) \cup Y\,|} && \hat{\xi}_{\mathbb{E}_J}(Y) = 1 - \xi_{\mathbb{E}_J}(Y)
\end{aligned}$$

Definition 2.21. ([25]) Let $(\mathbf{U}, \Omega)$ be a $\mathcal{G}$-approximation space. For each $\mathsf{b} \in \mathbf{U}$, the $\widetilde{\mathbb{E}}_J$-neighborhoods are given by the following formulas:

(i) $\widetilde{\mathbb{E}}_r(\mathsf{b}) = \{\mathsf{f} \in \mathbf{U} : \mathcal{A}_l(\mathsf{f}) \cap \mathcal{A}_r(\mathsf{b}) \neq \emptyset\}$.

(ii) $\widetilde{\mathbb{E}}_l(\mathsf{b}) = \{\mathsf{f} \in \mathbf{U} : \mathcal{A}_r(\mathsf{f}) \cap \mathcal{A}_l(\mathsf{b}) \neq \emptyset\}$.

(iii) $\widetilde{\mathbb{E}}_i(\mathsf{b}) = \widetilde{\mathbb{E}}_r(\mathsf{b}) \cap \widetilde{\mathbb{E}}_l(\mathsf{b})$.

(iv) $\widetilde{\mathbb{E}}_u(\mathsf{b}) = \widetilde{\mathbb{E}}_r(\mathsf{b}) \cup \widetilde{\mathbb{E}}_l(\mathsf{b})$.

(v) $\widetilde{\mathbb{E}}_{\langle r\rangle}(\mathsf{b}) = \{\mathsf{f} \in \mathbf{U} : \mathcal{A}_{\langle l\rangle}(\mathsf{f}) \cap \mathcal{A}_{\langle r\rangle}(\mathsf{b}) \neq \emptyset\}$.

(vi) $\widetilde{\mathbb{E}}_{\langle l\rangle}(\mathsf{b}) = \{\mathsf{f} \in \mathbf{U} : \mathcal{A}_{\langle r\rangle}(\mathsf{f}) \cap \mathcal{A}_{\langle l\rangle}(\mathsf{b}) \neq \emptyset\}$.

(vii) $\widetilde{\mathbb{E}}_{\langle i\rangle}(\mathsf{b}) = \widetilde{\mathbb{E}}_{\langle r\rangle}(\mathsf{b}) \cap \widetilde{\mathbb{E}}_{\langle l\rangle}(\mathsf{b})$.

(viii) $\widetilde{\mathbb{E}}_{\langle u\rangle}(\mathsf{b}) = \widetilde{\mathbb{E}}_{\langle r\rangle}(\mathsf{b}) \cup \widetilde{\mathbb{E}}_{\langle l\rangle}(\mathsf{b})$.

Definition 2.22. ([16]) Let $(\mathbf{U}, \Omega)$ be a $\mathcal{G}$-approximation space. For each $\mathsf{b} \in \mathbf{U}$, the $\widetilde{\mathbb{C}}_J$-neighborhoods are given by the following formulas:

(i) $\widetilde{\mathbb{C}}_r(\mathsf{b}) = \{\mathsf{f} \in \mathbf{U} : \mathcal{A}_l(\mathsf{f}) \subseteq \mathcal{A}_r(\mathsf{b})\}$.

(ii) $\widetilde{\mathbb{C}}_l(\mathsf{b}) = \{\mathsf{f} \in \mathbf{U} : \mathcal{A}_r(\mathsf{f}) \subseteq \mathcal{A}_l(\mathsf{b})\}$.

(iii) $\widetilde{\mathbb{C}}_i(\mathsf{b}) = \widetilde{\mathbb{C}}_r(\mathsf{b}) \cap \widetilde{\mathbb{C}}_l(\mathsf{b})$.

(iv) $\widetilde{\mathbb{C}}_u(\mathsf{b}) = \widetilde{\mathbb{C}}_r(\mathsf{b}) \cup \widetilde{\mathbb{C}}_l(\mathsf{b})$.

(v) $\widetilde{\mathbb{C}}_{\langle r\rangle}(\mathsf{b}) = \{\mathsf{f} \in \mathbf{U} : \mathcal{A}_{\langle l\rangle}(\mathsf{f}) \subseteq \mathcal{A}_{\langle r\rangle}(\mathsf{b})\}$.

(vi) $\widetilde{\mathbb{C}}_{\langle l\rangle}(\mathsf{b}) = \{\mathsf{f} \in \mathbf{U} : \mathcal{A}_{\langle r\rangle}(\mathsf{f}) \subseteq \mathcal{A}_{\langle l\rangle}(\mathsf{b})\}$.

(vii) $\widetilde{\mathbb{C}}_{\langle i\rangle}(\mathsf{b}) = \widetilde{\mathbb{C}}_{\langle r\rangle}(\mathsf{b}) \cap \widetilde{\mathbb{C}}_{\langle l\rangle}(\mathsf{b})$.

(viii) $\widetilde{\mathbb{C}}_{\langle u\rangle}(\mathsf{b}) = \widetilde{\mathbb{C}}_{\langle r\rangle}(\mathsf{b}) \cup \widetilde{\mathbb{C}}_{\langle l\rangle}(\mathsf{b})$.

Definition 2.23. ([16, 25]) In a $\mathcal{G}$-approximation space $(\mathbf{U}, \Omega)$, the lower and upper approximations of a subset Y are calculated with respect to $\widetilde{\mathbb{E}}_J$-neighborhoods and $\widetilde{\mathbb{C}}_J$-neighborhoods using the following formulas:

$$\mathcal{H}_{\widetilde{\mathbb{E}}_J}(Y) = \left(\bigcup_{\widetilde{\mathbb{E}}_J(\mathsf{b})\subseteq Y} \widetilde{\mathbb{E}}_\sigma(\mathsf{b})\right) \qquad \mathcal{H}^{\widetilde{\mathbb{E}}_J}(Y) = \left(\bigcap_{\widetilde{\mathbb{E}}_J(\mathsf{b})\cap Y\neq\emptyset} \widetilde{\mathbb{E}}_J(\mathsf{b})\right) \cup Y$$

$$\mathcal{H}_{\widetilde{\mathbb{C}}_J} = Y \cap \{\mathsf{b} \in \mathbf{U} : \widetilde{\mathbb{C}}_J(\mathsf{b}) \subseteq Y\} \qquad \mathcal{H}^{\widetilde{\mathbb{C}}_J}(Y) = Y \cup \{\mathsf{b} \in \mathbf{U} : \widetilde{\mathbb{C}}_J(\mathsf{b}) \cap Y \neq \emptyset\}$$

Definition 2.24. ([16, 25]) In a $\mathcal{G}$-approximation space $(\mathbf{U}, \Omega)$, the ξ-accuracy and $\hat{\xi}$-roughness of a subset $Y \neq \emptyset$ are calculated with respect to $\widetilde{\mathbb{E}}_J$-neighborhoods and $\widetilde{\mathbb{C}}_J$-neighborhoods using the following formulas:

$$\xi_{\widetilde{\mathbb{E}}_J}(Y) = \frac{|\ \mathcal{H}_{\widetilde{\mathbb{E}}_J}(Y)\ |}{|\ \mathcal{H}^{\widetilde{\mathbb{E}}_J}(Y)\ |} \qquad \hat{\xi}_{\widetilde{\mathbb{E}}_J}(Y) = 1 - \xi_{\widetilde{\mathbb{E}}_J}(Y)$$

$$\xi_{\widetilde{\mathbb{C}}_J}(Y) = \frac{|\ \mathcal{H}_{\widetilde{\mathbb{C}}_J}(Y)\ |}{|\ \mathcal{H}^{\widetilde{\mathbb{C}}_J}(Y)\ |} \qquad \hat{\xi}_{\widetilde{\mathbb{C}}_J}(Y) = 1 - \xi_{\widetilde{\mathbb{C}}_J}(Y)$$

To maintain the systematic formulas used to describe the approximations and accuracy while facilitating comparisons among the rough set models, we refine the definitions of lower and upper approximations, as well as accuracy measures, established using $\widetilde{\mathbb{E}}_\jmath$-neighborhoods, as follows.

Definition 2.25. In a $\mathcal{G}$-approximation space $(\mathbf{U}, \Omega)$, the lower approximation, upper approximation, and ξ-accuracy of a subset $Y \neq \emptyset$ are calculated with respect to $\widetilde{\mathbb{E}}_\jmath$-neighborhoods using the following formulas:

$$\mathcal{H}_{\widetilde{\mathbb{E}}_\jmath}(Y) = \{\mathsf{b} \in \mathbf{U} : \widetilde{\mathbb{E}}_\jmath(\mathsf{b}) \subseteq Y\},$$

$$\mathcal{H}^{\widetilde{\mathbb{E}}_\jmath}(Y) = \{\mathsf{b} \in \mathbf{U} : \widetilde{\mathbb{E}}_\jmath(\mathsf{b}) \cap Y \neq \emptyset\}, \text{ and}$$

$$\xi_{\widetilde{\mathbb{E}}_\jmath}(Y) = \frac{|\underline{O}_{\widetilde{\mathbb{E}}_\jmath}(Y) \cap Y|}{|\overline{O}_{\widetilde{\mathbb{E}}_\jmath}(Y) \cup Y|}.$$

Proposition 2.26. ([16]) *In a G-approximation space $(\boldsymbol{U}, \Omega)$, If Ω is inverse serial (resp., serial), then for each* $\mathsf{b} \in \boldsymbol{U}$ *we have* $\widetilde{\mathbb{C}}_\jmath(\mathsf{b}) \subseteq \widetilde{\mathbb{E}}_\jmath(\mathsf{b})$ *for each $\jmath \in \{r, \langle l\rangle\}$ (resp.,* $\widetilde{\mathbb{C}}_\jmath(\mathsf{b}) \subseteq \widetilde{\mathbb{E}}_\jmath(\mathsf{b})$ *for each $\jmath \in \{l, \langle r\rangle\}$).*

3. Overlapping subset rough neighborhood systems

In this segment, we introduce a novel system of neighborhoods, called overlapping subset rough neighborhoods, based on any binary relation. We explore their key features and provide illustrative examples to clarify the derived implementations. We also extensively discuss the interrelations between them and existing types, and identify the conditions under which certain equalities hold.

Definition 3.1. Let $(\mathbf{U}, \Omega)$ be a $\mathcal{G}$-approximation space. For each $\mathsf{b} \in \mathbf{U}$, the $\widetilde{\mathbb{S}}_\jmath$-neighborhoods are given by the following formulas:

(i) $\widetilde{\mathbb{S}}_r(\mathsf{b}) = \{\mathsf{f} \in \mathbf{U} : \mathcal{A}_r(\mathsf{b}) \subseteq \mathcal{A}_l(\mathsf{f})\}$.

(ii) $\widetilde{\mathbb{S}}_l(\mathsf{b}) = \{\mathsf{f} \in \mathbf{U} : \mathcal{A}_l(\mathsf{b}) \subseteq \mathcal{A}_r(\mathsf{f})\}$.

(iii) $\widetilde{\mathbb{S}}_i(\mathsf{b}) = \widetilde{\mathbb{S}}_r(\mathsf{b}) \cap \widetilde{\mathbb{S}}_l(\mathsf{b})$.

(iv) $\widetilde{\mathbb{S}}_u(\mathsf{b}) = \widetilde{\mathbb{S}}_r(\mathsf{b}) \cup \widetilde{\mathbb{S}}_l(\mathsf{b})$.

(v) $\widetilde{\mathbb{S}}_{\langle r\rangle}(\mathsf{b}) = \{\mathsf{f} \in \mathbf{U} : \mathcal{A}_{\langle r\rangle}(\mathsf{b}) \subseteq \mathcal{A}_{\langle l\rangle}(\mathsf{f})\}$.

(vi) $\widetilde{\mathbb{S}}_{\langle l\rangle}(\mathsf{b}) = \{\mathsf{f} \in \mathbf{U} : \mathcal{A}_{\langle l\rangle}(\mathsf{b}) \subseteq \mathcal{A}_{\langle r\rangle}(\mathsf{f})\}$.

(vii) $\widetilde{\mathbb{S}}_{\langle i\rangle}(\mathsf{b}) = \widetilde{\mathbb{S}}_{\langle r\rangle}(\mathsf{b}) \cap \widetilde{\mathbb{S}}_{\langle l\rangle}(\mathsf{b})$.

(viii) $\widetilde{\mathbb{S}}_{\langle u\rangle}(\mathsf{b}) = \widetilde{\mathbb{S}}_{\langle r\rangle}(\mathsf{b}) \cup \widetilde{\mathbb{S}}_{\langle l\rangle}(\mathsf{b})$.

The notions of $\widetilde{\mathbb{S}}_\jmath$-neighborhoods and $\mathbb{S}_\jmath$-neighborhoods are unrelated in the general case. To demonstrate this, we will compute both the $\widetilde{\mathbb{S}}_\jmath$-neighborhoods and $\mathbb{S}_\jmath$-neighborhoods for each $\mathsf{b} \in \mathbf{U}$ in the subsequent instance.

Example 3.2. Take Ω a binary relation on $\mathbf{U} = \{\mathsf{b}, \mathsf{f}, \mathsf{h}, \mathsf{k}\}$ defined as follows:

$$\Omega = \{(\mathsf{b}, \mathsf{b}), (\mathsf{b}, \mathsf{f}), (\mathsf{b}, \mathsf{h}), (\mathsf{b}, \mathsf{k}), (\mathsf{f}, \mathsf{b}), (\mathsf{h}, \mathsf{f}), (\mathsf{h}, \mathsf{k})\}.$$

For each element of $\mathbf{U}$, we compute $\widetilde{\mathbb{S}}_\jmath$-neighborhoods and $\mathbb{C}_\jmath$-neighborhoods in Table 1 and Table 2, respectively.

	b	f	h	k
$\mathcal{A}_r$	$\mathbf{U}$	{b}	{f, k}	$\emptyset$
$\mathcal{A}_l$	{b, f}	{b, h}	{b}	{b, h}
$\mathcal{A}_{\langle r\rangle}$	{b}	{f, k}	$\mathbf{U}$	{f, k}
$\mathcal{A}_{\langle l\rangle}$	{b}	{b, f}	{b, h}	$\emptyset$
$\widetilde{\mathbb{S}}_r$	$\emptyset$	$\mathbf{U}$	$\emptyset$	$\mathbf{U}$
$\widetilde{\mathbb{S}}_l$	{b}	{b}	{b, f}	{b}
$\widetilde{\mathbb{S}}_i$	$\emptyset$	{b}	$\emptyset$	{b}
$\widetilde{\mathbb{S}}_u$	{b}	$\mathbf{U}$	{b, f}	$\mathbf{U}$
$\widetilde{\mathbb{S}}_{\langle r\rangle}$	{b, f, h}	$\emptyset$	$\emptyset$	$\emptyset$
$\widetilde{\mathbb{S}}_{\langle l\rangle}$	{b, h}	{h}	{h}	$\mathbf{U}$
$\widetilde{\mathbb{S}}_{\langle i\rangle}$	{b, h}	$\emptyset$	$\emptyset$	$\emptyset$
$\widetilde{\mathbb{S}}_{\langle u\rangle}$	{b, f, h}	{h}	{h}	$\mathbf{U}$

Table 1: The $\mathcal{A}_\jmath$-neighborhoods and $\widetilde{\mathbb{S}}_\jmath$-neighborhoods of all elements in $\mathbf{U}$

	b	f	h	k
$\mathbb{S}_r$	{b}	{b, f}	{b, h}	$\mathbf{U}$
$\mathbb{S}_l$	{b}	{f, k}	$\mathbf{U}$	{f, k}
$\mathbb{S}_i$	{b}	{f}	{b, h}	{f, k}
$\mathbb{S}_u$	{b}	{b, f, k}	$\mathbf{U}$	$\mathbf{U}$
$\mathbb{S}_{\langle r\rangle}$	{b, h}	{f, h, k}	{h}	{f, h, k}
$\mathbb{S}_{\langle l\rangle}$	{b, f, h}	{f}	{h}	$\mathbf{U}$
$\mathbb{S}_{\langle i\rangle}$	{b, h}	{f}	{h}	{f, h, k}
$\mathbb{S}_{\langle u\rangle}$	{b, f, h}	{f, h, k}	{h}	$\mathbf{U}$

Table 2: The $\mathbb{S}_\jmath$-neighborhoods of all elements in $\mathbf{U}$

We now investigate the connections between $\widetilde{\mathbb{S}}_\jmath$-neighborhoods based on an arbitrary relation and examine certain equalities that arise between them under particular types of binary relations.

Proposition 3.3. *In a G-approximation space $(\boldsymbol{U}, \Omega)$, we have the following inclusions relations for each* $\mathsf{b} \in \boldsymbol{U}$.

(i) $\widetilde{\mathbb{S}}_i(\mathsf{b}) \subseteq \widetilde{\mathbb{S}}_r(\mathsf{b}) \subseteq \widetilde{\mathbb{S}}_u(\mathsf{b})$ *and* $\widetilde{\mathbb{S}}_i(\mathsf{b}) \subseteq \widetilde{\mathbb{S}}_l(\mathsf{b}) \subseteq \widetilde{\mathbb{S}}_u(\mathsf{b})$.

(ii) $\widetilde{\mathbb{S}}_{\langle i\rangle}(\mathsf{b}) \subseteq \widetilde{\mathbb{S}}_{\langle r\rangle}(\mathsf{b}) \subseteq \widetilde{\mathbb{S}}_{\langle u\rangle}(\mathsf{b})$ *and* $\widetilde{\mathbb{S}}_{\langle i\rangle}(\mathsf{b}) \subseteq \widetilde{\mathbb{S}}_{\langle l\rangle}(\mathsf{b}) \subseteq \widetilde{\mathbb{S}}_{\langle u\rangle}(\mathsf{b})$.

(iii) $\widetilde{\mathbb{S}}_i(\mathsf{b}) \subseteq \{\mathsf{f} \in \boldsymbol{U} : \mathcal{A}_i(\mathsf{b}) \subseteq \mathcal{A}_i(\mathsf{f})\}$.

(iv) $\widetilde{\mathbb{S}}_{\langle i\rangle}(\mathsf{b}) \subseteq \{\mathsf{f} \in \boldsymbol{U} : \mathcal{A}_{\langle i\rangle}(\mathsf{b}) \subseteq \mathcal{A}_{\langle i\rangle}(\mathsf{f})\}$.

(v) *If* $\mathcal{A}_\jmath(\mathsf{b}) = \mathcal{A}_\jmath(\mathsf{f})$, *then* $\widetilde{\mathbb{S}}_\jmath(\mathsf{b}) = \widetilde{\mathbb{S}}_\jmath(\mathsf{f})$ *for each* $\jmath \in \{r, \langle r\rangle, l, \langle l\rangle\}$.

(vi) *If* $\mathcal{A}_\jmath(\mathsf{b}) = \emptyset$, *then* $\widetilde{\mathbb{S}}_\jmath(\mathsf{b}) = \boldsymbol{U}$ *for each* $\jmath \in \{r, \langle r\rangle, l, \langle l\rangle\}$.

Proof. The first two properties directly follow by Definition 3.1. To show (iii), assume that $\mathsf{f} \in \widetilde{\mathbb{S}}_i(\mathsf{b})$. Then, $\mathsf{f} \in \widetilde{\mathbb{S}}_r(\mathsf{b})$ and $\mathsf{f} \in \widetilde{\mathbb{S}}_l(\mathsf{b})$, so $\mathcal{A}_r(\mathsf{b}) \subseteq \mathcal{A}_l(\mathsf{f})$ and $\mathcal{A}_l(\mathsf{b}) \subseteq \mathcal{A}_r(\mathsf{f})$. Accordingly, we obtain $\mathcal{A}_r(\mathsf{b}) \cap \mathcal{A}_l(\mathsf{b}) \subseteq \mathcal{A}_l(\mathsf{f}) \cap \mathcal{A}_r(\mathsf{f})$, so $\mathcal{A}_i(\mathsf{b}) \subseteq \mathcal{A}_i(\mathsf{f})$. Hence, $\mathsf{f} \in \{\mathsf{f} \in \mathbf{U} : \mathcal{A}_i(\mathsf{b}) \subseteq \mathcal{A}_i(\mathsf{f})\}$.
As we do above, one can achieve (iv).
The proofs of (v) and (vi) are straightforward. □

By Table 1, we see that the inclusion relations of Proposition 3.3 cannot be replaced by equality relations.

Proposition 3.4. *In a G-approximation space $(\boldsymbol{U}, \Omega)$, if Ω is symmetric, then for each $\mathsf{b} \in \boldsymbol{U}$ we have:*

(i) $\mathsf{b} \in \widetilde{\mathbb{S}}_{\jmath}(\mathsf{b})$ *for every* $\jmath$.

(ii) $\widetilde{\mathbb{S}}_r(\mathsf{b}) = \widetilde{\mathbb{S}}_l(\mathsf{b}) = \widetilde{\mathbb{S}}_i(\mathsf{b}) = \widetilde{\mathbb{S}}_u(\mathsf{b})$ *and* $\widetilde{\mathbb{S}}_{\langle r\rangle}(\mathsf{b}) = \widetilde{\mathbb{S}}_{\langle l\rangle}(\mathsf{b}) = \widetilde{\mathbb{S}}_{\langle i\rangle}(\mathsf{b}) = \widetilde{\mathbb{S}}_{\langle u\rangle}(\mathsf{b})$.

Proof. We have $\mathcal{A}_l(\mathsf{b}) = \mathcal{A}_r(\mathsf{b})$ and $\mathcal{A}_{\langle l\rangle}(\mathsf{b}) = \mathcal{A}_{\langle r\rangle}(\mathsf{b})$ for each $\mathsf{b} \in \mathbf{U}$ whenever Ω is symmetric. So that, $\mathsf{b} \in \widetilde{\mathbb{S}}_{\jmath}(\mathsf{b})$ for each $\jmath$. Furthermore, we obtain the two equalities stated in (ii). □

By Table 1, we see that a symmetric relation condition imposed by Proposition 3.4 is indispensable.

Proposition 3.5. *In a G-approximation space $(\boldsymbol{U}, \Omega)$ we have $\widetilde{\mathbb{S}}_{\jmath}(\mathsf{b}) = \widetilde{\mathbb{S}}_{\langle\jmath\rangle}(\mathsf{b})$ for each $\mathsf{b} \in \boldsymbol{U}$ and $\jmath \in \{r, l, i, u\}$ whenever Ω is a quasi-order relation, i.e., reflexive and transitive.*

Proof. It is a direct result of Proposition 2.10. □

Proposition 3.6. *In a G-approximation space $(\boldsymbol{U}, \Omega)$, if Ω is symmetric and transitive. Then, for each $\jmath$ and $\mathsf{b}, \mathsf{f} \in \boldsymbol{U}$ we have*

$$\widetilde{\mathbb{S}}_{\jmath}(\mathsf{b}) = \widetilde{\mathbb{S}}_{\jmath}(\mathsf{f}) \text{ or } \widetilde{\mathbb{S}}_{\jmath}(\mathsf{b}) \cap \widetilde{\mathbb{S}}_{\jmath}(\mathsf{f}) \subseteq \{\mathsf{h} \in \boldsymbol{U} : (\mathsf{h}, \mathsf{h}) \in \Omega\}.$$

Proof. Take $\jmath = r$. Suppose that $\widetilde{\mathbb{S}}_r(\mathsf{b}) \neq \widetilde{\mathbb{S}}_r(\mathsf{f})$. Without loss of generality, let us consider that there exists $\mathsf{k} \in \mathbf{U}$ such that $\mathsf{k} \in \widetilde{\mathbb{S}}_r(\mathsf{b})$ and $\mathsf{k} \notin \widetilde{\mathbb{S}}_r(\mathsf{f})$. Then, $\mathcal{A}_r(\mathsf{b}) \subseteq \mathcal{A}_l(\mathsf{k})$ and $\mathcal{A}_r(\mathsf{f}) \nsubseteq \mathcal{A}_l(\mathsf{k})$. This adheres that

$$\mathcal{A}_r(\mathsf{f}) \neq \emptyset. \tag{1}$$

Now, we prove that

$$\widetilde{\mathbb{S}}_r(\mathsf{b}) \cap \widetilde{\mathbb{S}}_r(\mathsf{f}) \subseteq \{\mathsf{h} \in \mathbf{U} : (\mathsf{h}, \mathsf{h}) \in \Omega\}.$$

if $\mathsf{x} \in \widetilde{\mathbb{S}}_r(\mathsf{b}) \cap \widetilde{\mathbb{S}}_r(\mathsf{f})$, then $\mathcal{A}_r(\mathsf{b}) \subseteq \mathcal{A}_l(\mathsf{x})$ and $\mathcal{A}_r(\mathsf{f}) \subseteq \mathcal{A}_l(\mathsf{x})$. By (1), it follows that $\mathcal{A}_l(\mathsf{x}) \neq \emptyset$. This yields that there is k such that $(\mathsf{k}, \mathsf{x}) \in \Omega$. By symmetric and transitive of Ω, we get $(\mathsf{x}, \mathsf{x}) \in \Omega$. Thus, $\widetilde{\mathbb{S}}_r(\mathsf{b}) \cap \widetilde{\mathbb{S}}_r(\mathsf{f}) \subseteq \{\mathsf{h} \in \mathbf{U} : (\mathsf{h}, \mathsf{h}) \in \Omega\}$, which completes the proof. □

The following example elucidates that the inclusion relation stated in Proposition 3.6 is proper.

Example 3.7. Let $\Omega = \{(\mathsf{b}, \mathsf{b}), (\mathsf{f}, \mathsf{f}), (\mathsf{k}, \mathsf{k}), (\mathsf{f}, \mathsf{k}), (\mathsf{k}, \mathsf{f})\}$ be a binary relation on $\mathbf{U} = \{\mathsf{b}, \mathsf{f}, \mathsf{h}, \mathsf{k}\}$. In Table 3, we calculate $\mathcal{A}_{\jmath}$- and $\widetilde{\mathbb{S}}_{\jmath}$-neighborhoods for all elements in $\mathbf{U}$.

	b	f	h	k
$\mathcal{A}_r = \mathcal{A}_l = \mathcal{A}_i = \mathcal{A}_u$	$\{\mathsf{b}\}$	$\{\mathsf{f}, \mathsf{k}\}$	$\emptyset$	$\{\mathsf{f}, \mathsf{k}\}$
$\widetilde{\mathbb{S}}_r = \widetilde{\mathbb{S}}_l = \widetilde{\mathbb{S}}_i = \widetilde{\mathbb{S}}_u$	$\{\mathsf{b}\}$	$\{\mathsf{f}, \mathsf{k}\}$	$\mathbf{U}$	$\{\mathsf{f}, \mathsf{k}\}$

Table 3: The $\mathcal{A}_{\jmath}$- and $\widetilde{\mathbb{S}}_{\jmath}$-neighborhoods of all elements in $\mathbf{U}$

It is clear that Ω is symmetric and transitive. However, we note that $\widetilde{\mathbb{S}}_{\jmath}(\mathsf{b}) \neq \widetilde{\mathbb{S}}_{\jmath}(\mathsf{h})$ and $\widetilde{\mathbb{S}}_{\jmath}(\mathsf{b}) \cap \widetilde{\mathbb{S}}_{\jmath}(\mathsf{h}) = \{\mathsf{b}\}$ is a proper subset of $\{\mathsf{b}, \mathsf{f}, \mathsf{k}\}$.

The next two propositions explore some criteria for determining a type of the given binary relation. It is worth noting that these results do not hold for rough neighborhoods defined over the same kinds of Yao neighborhoods.

Proposition 3.8. *Let $(\boldsymbol{U}, \Omega)$ be a G-approximation space. Then, for each $\jmath \in \{r, l, i\}$ we have*

$$\mathsf{b} \in \widetilde{\mathbb{S}}_{\jmath}(\mathsf{b}) \text{ for each } \mathsf{b} \in \boldsymbol{U} \Longleftrightarrow \Omega \text{ is a symmetric relation.}$$

Proof. When $\jmath = r$. Let the necessary condition be satisfied. Presume Ω in not symmetric. This implies that there exist $\mathsf{h}, \mathsf{k} \in \mathbf{U}$ such that $(\mathsf{h}, \mathsf{k}) \in \Omega$ and $(\mathsf{k}, \mathsf{h}) \notin \Omega$. Accordingly, we have $\mathsf{k} \in \mathcal{A}_r(\mathsf{h})$ and $\mathsf{k} \notin \mathcal{A}_l(\mathsf{h})$. Consequently, $\mathcal{A}_r(\mathsf{h}) \nsubseteq \mathcal{A}_l(\mathsf{h})$, so $\mathsf{h} \notin \widetilde{\mathbb{S}}_r(\mathsf{h})$. This is a contradiction with the necessary condition. Thus, Ω is a symmetric relation. Conversely, item (i) of Proposition 3.4 proves the sufficient condition. □

The next lemma will assist in clarifying the upcoming proposition and corollary.

Lemma 3.9. *[16] In a G-approximation space $(\boldsymbol{U}, \Omega)$, the next property is satisfied for each $\jmath$ and each $\mathsf{b}, \mathsf{f} \in \boldsymbol{U}$ whenever Ω is symmetric and transitive.*

$$\mathcal{A}_\jmath(\mathsf{b}) \cap \mathcal{A}_\jmath(\mathsf{f}) = \emptyset \text{ or } \mathcal{A}_\jmath(\mathsf{b}) = \mathcal{A}_\jmath(\mathsf{f}).$$

Proposition 3.10. *In a G-approximation space $(\boldsymbol{U}, \Omega)$, the next property is satisfied for each $\jmath$:*

$$\Omega \text{ is symmetric and transitive if and only if } \mathcal{A}_\jmath(\mathsf{b}) \subseteq \widetilde{\mathbb{S}}_\jmath(\mathsf{b}) \text{ for each } \mathsf{b} \in \boldsymbol{U}.$$

Proof. We prove the proposition when $\jmath = r$.
Necessity: Let $\mathsf{f} \in \mathcal{A}_r(\mathsf{b})$. Then, $(\mathsf{b}, \mathsf{f}), (\mathsf{f}, \mathsf{b}), (\mathsf{b}, \mathsf{b}), (\mathsf{f}, \mathsf{f}) \in \Omega$. By Lemma 3.9, we find that $\mathcal{A}_r(\mathsf{b}) = \mathcal{A}_l(\mathsf{f})$. Thus, $\mathsf{f} \in \widetilde{\mathbb{S}}_r(\mathsf{b})$. Hence, $\mathcal{A}_\jmath(\mathsf{b}) \subseteq \widetilde{\mathbb{S}}_\jmath(\mathsf{b})$, as required.
Sufficiency: First, suppose that $(\mathsf{b}, \mathsf{f}) \in \Omega$. Now, $\mathcal{A}_r(\mathsf{b}) \subseteq \widetilde{\mathbb{S}}_r(\mathsf{b})$. By our assumption, $\mathsf{f} \in \widetilde{\mathbb{S}}_r(\mathsf{b})$, which means that $\mathcal{A}_r(\mathsf{b}) \subseteq \mathcal{A}_l(\mathsf{f})$. This leads to that $(\mathsf{f}, \mathsf{f}) \in \Omega$. We also have by the given condition that $\mathcal{A}_l(\mathsf{f}) \subseteq \widetilde{\mathbb{S}}_l(\mathsf{f})$. Therefore, $\mathcal{A}_l(\mathsf{f}) \subseteq \mathcal{A}_r(\mathsf{f})$. Again by our assumption, $\mathsf{b} \in \mathcal{A}_r(\mathsf{f})$, so $(\mathsf{f}, \mathsf{b}) \in \Omega$. Thus, Ω is symmetric. Second, suppose that $(\mathsf{b}, \mathsf{f}), (\mathsf{f}, \mathsf{h}) \in \Omega$. By the given conditions, we have $\mathcal{A}_l(\mathsf{f}) \subseteq \widetilde{\mathbb{S}}_l(\mathsf{f})$. Then, $\mathsf{b} \in \widetilde{\mathbb{S}}_l(\mathsf{f})$. This automatically yields to $\mathcal{A}_l(\mathsf{f}) \subseteq \mathcal{A}_r(\mathsf{b})$. As we prove that Ω is symmetric, then we obtain $\mathcal{A}_r(\mathsf{f}) \subseteq \mathcal{A}_r(\mathsf{b})$. According to our assumption, $\mathsf{h} \in \mathcal{A}_r(\mathsf{b})$, which yields that $(\mathsf{b}, \mathsf{h}) \in \Omega$. Thus, Ω is transitive. Hence, we complete the proof. □

Corollary 3.11. *In a G-approximation space $(\boldsymbol{U}, \Omega)$, let $\mathcal{A}_\jmath(\mathsf{b}) \subseteq \widetilde{\mathbb{S}}_\jmath(\mathsf{b})$ hold for each $\mathsf{b} \in \boldsymbol{U}$ and $\jmath$. Then:*

(i) *if Ω is serial (or inverse serial), then it is an equivalence relation.*

(ii) *if Ω is antisymmetric, then it is an diagonal relation, i.e. $\Omega = \{(\mathsf{b}, \mathsf{b}) : \mathsf{b} \in \boldsymbol{U}\}$.*

Lemma 3.12. *In a $\mathcal{G}$-approximation space $(\boldsymbol{U}, \Omega)$, if Ω is symmetric and transitive, then all $\mathcal{A}_\jmath$ are identical.*

Proof. Since Ω is symmetric, we have $\mathcal{A}_r = \mathcal{A}_l = \mathcal{A}_i = \mathcal{A}_u$ and $\mathcal{A}_{\langle r\rangle} = \mathcal{A}_{\langle l\rangle} = \mathcal{A}_{\langle i\rangle} = \mathcal{A}_{\langle u\rangle}$. So we need to prove that $\mathcal{A}_r = \mathcal{A}_{\langle r\rangle}$. To do suppose $\mathsf{f} \in \mathcal{A}_r(\mathsf{b})$. Now, for each $\mathsf{h} \in \mathbf{U}$ such that $\mathsf{b} \in \mathcal{A}_r(\mathsf{h})$ we have $(\mathsf{h}, \mathsf{b}) \in \Omega$. Then, by assumption and transitive we obtain $(\mathsf{h}, \mathsf{f}) \in \Omega$, which directly leads to that $\mathsf{f} \in \mathcal{A}_r(\mathsf{h})$. Thus, $\mathsf{f} \in \mathcal{A}_{\langle r\rangle}(\mathsf{b})$, which proves that $\mathcal{A}_r(\mathsf{b}) \subseteq \mathcal{A}_{\langle r\rangle}(\mathsf{b})$. Conversely, suppose that $\mathsf{f} \in \mathcal{A}_{\langle r\rangle}(\mathsf{b})$. Then, there exists $\mathsf{h} \in \mathbf{U}$ such that $\mathsf{b}, \mathsf{f} \in \mathcal{A}_r(\mathsf{h})$, so $(\mathsf{h}, \mathsf{b}), (\mathsf{h}, \mathsf{f}) \in \Omega$. It yields by the given conditions that $(\mathsf{f}, \mathsf{b}) \in \Omega$, which proves that $\mathcal{A}_{\langle r\rangle}(\mathsf{b}) \subseteq \mathcal{A}_r(\mathsf{b})$, as required. □

Corollary 3.13. *In a $\mathcal{G}$-approximation space $(\boldsymbol{U}, \Omega)$, if Ω is symmetric and transitive, then all $\widetilde{\mathbb{S}}_\jmath$ are identical. Moreover, all types of rough neighbourhoods mentioned in Section 2 are identical.*

In the subsequent discussions, we uncover the connections between the current types of neighborhoods and the previous ones. Additionally, we elucidate some relationships that hold under certain types of binary relations.

Proposition 3.14. *In a $\mathcal{G}$-approximation space $(\boldsymbol{U}, \Omega)$, if Ω is symmetric and transitive, then for each $\jmath$ we have*

(i) $\widetilde{\mathbb{S}}_\jmath(\mathsf{b}) \subseteq \widetilde{\mathbb{C}}_\jmath(\mathsf{b})$ *providing that* $(\mathsf{b}, \mathsf{b}) \in \Omega$*, and*

(ii) $\widetilde{\mathbb{C}}_\jmath(\mathsf{b}) \subseteq \widetilde{\mathbb{S}}_\jmath(\mathsf{b})$ *providing that* $(\mathsf{b}, \mathsf{b}) \notin \Omega$*.*

Proof. First of all, it follows from Corollary 3.13 that all $\widetilde{\mathbb{S}}_\jmath$ are identical.
To prove (i), let $f \in \widetilde{\mathbb{S}}_\jmath(b)$ such that $(b, b) \in \Omega$. Then, $\mathcal{A}_\jmath(b) \subseteq \mathcal{A}_\jmath(f)$ such that $\mathcal{A}_\jmath(b) \neq \emptyset$. According to Lemma 3.9, it follows that $\mathcal{A}_\jmath(b) = \mathcal{A}_\jmath(f)$; thus, $f \in \widetilde{\mathbb{C}}_\jmath(b)$.
To prove (ii), let $f \in \widetilde{\mathbb{C}}_\jmath(b)$ such that $(b, b) \notin \Omega$. Then, $\mathcal{A}_\jmath(f) \subseteq \mathcal{A}_\jmath(b)$. Since Ω is symmetric and transitive, we have $\mathcal{A}_\jmath(b) = \emptyset$, so $\mathcal{A}_\jmath(f) = \emptyset$ as well. Thus, $f \in \widetilde{\mathbb{S}}_\jmath(b)$. □

We show by Example 3.15 that the converse of Proposition 3.14 fails.

Example 3.15. Let $\Omega = \{(b, b), (f, f), (b, f), (f, b)\}$ be a binary relation on $\mathbf{U} = \{b, f, h\}$. In Table 4, we calculate $\mathcal{A}_\jmath$-, $\widetilde{\mathbb{S}}_\jmath$-, and $\widetilde{\mathbb{C}}_\jmath$-neighborhoods for all elements in $\mathbf{U}$.

	b	f	h
$\mathcal{A}_\jmath$	$\{b, f\}$	$\{b, f\}$	$\emptyset$
$\widetilde{\mathbb{C}}_\jmath$	$\mathbf{U}$	$\mathbf{U}$	$\{h\}$
$\widetilde{\mathbb{S}}_\jmath$	$\{b, f\}$	$\{b, f\}$	$\mathbf{U}$

Table 4: The $\mathcal{A}_\jmath$-, $\widetilde{\mathbb{S}}_\jmath$-, and $\widetilde{\mathbb{C}}_\jmath$-neighborhoods of all elements in $\mathbf{U}$

Obviously, Ω is symmetric and transitive. It can be seen that $\widetilde{\mathbb{S}}_\jmath(b)$ and $\widetilde{\mathbb{S}}_\jmath(f)$ are proper subsets of $\mathbb{C}_\jmath(b)$ and $\widetilde{\mathbb{S}}_\jmath(b)$, respectively, where $(b, b), (f, f) \in \Omega$. On the other hand, $\widetilde{\mathbb{C}}_\jmath(h)$ is a proper subset of $\mathbb{S}_\jmath(h)$, where $(h, h) \notin \Omega$.

Proposition 3.16. *In a G-approximation space $(\boldsymbol{U}, \Omega)$, the following relationships are fulfilled for each $b \in \boldsymbol{U}$.*

(i) *If Ω is symmetric, then $\widetilde{\mathbb{S}}_\jmath(b) = \mathbb{S}_\jmath(b)$ for each $\jmath$.*

(ii) *If Ω is reflexive, then $\widetilde{\mathbb{S}}_\jmath(b) \subseteq \mathcal{A}_\jmath(b)$ for each $\jmath \in \{r, l, i, u\}$.*

(iii) *If Ω is serial (or inverse serial) and symmetric, then $\widetilde{\mathbb{S}}_\jmath(b) \subseteq \mathcal{A}_\jmath(b)$ for each $\jmath$.*

(iv) *If Ω is serial and transitive, then $\widetilde{\mathbb{S}}_r(b) \subseteq \mathcal{A}_r(b)$.*

(v) *If Ω is inverse serial and transitive, then $\widetilde{\mathbb{S}}_l(b) \subseteq \mathcal{A}_l(b)$.*

Proof. (i): Obvious.
(ii): When $\jmath = r$. Let $f \in \widetilde{\mathbb{S}}_r(b)$. Then, $\mathcal{A}_r(b) \subseteq \mathcal{A}_l(f)$. Since Ω is reflexive, we obtain $b \in \mathcal{A}_l(f)$, which directly means that $f \in \mathcal{A}_r(b)$. Hence, we prove the desired result.
(iii): When $\jmath = \langle r \rangle$. Let $f \in \widetilde{\mathbb{S}}_{\langle r \rangle}(b)$. Then, $\mathcal{A}_{\langle r \rangle}(b) \subseteq \mathcal{A}_{\langle l \rangle}(f)$. By symmetric of Ω, we have $\mathcal{A}_{\langle l \rangle}(b) \subseteq \mathcal{A}_{\langle r \rangle}(f)$. Since Ω is serial, we obtain $b \in \mathcal{A}_{\langle r \rangle}(f)$, as required.
(iv): Let $f \in \widetilde{\mathbb{S}}_r(b)$. Then, $\mathcal{A}_r(b) \subseteq \mathcal{A}_l(f)$. Since Ω is serial, there exists $h \in \mathcal{A}_r(b)$. Accordingly, we have $(b, h), (h, f) \in \Omega$. Since Ω is transitive, $(b, f) \in \Omega$. Therefore, $f \in \mathcal{A}_r(b)$. Hence, $\widetilde{\mathbb{S}}_r(b) \subseteq \mathcal{A}_r(b)$, as required.
(v): Following a similar technique of (iv), we obtain the proof. □

Corollary 3.17. *In a G-approximation space $(\boldsymbol{U}, \Omega)$, the following relationships are fulfilled for each $b \in \boldsymbol{U}$.*

(i) *If Ω is serial, inverse serial, and transitive, then $\widetilde{\mathbb{S}}_\jmath(b) \subseteq \mathcal{A}_\jmath(b)$ for each $\jmath \in \{r, l, i, u\}$.*

(ii) *If Ω is a quasi-order, then $\widetilde{\mathbb{S}}_\jmath(b) \subseteq \mathcal{A}_\jmath(b)$ for each $\jmath$.*

Proof. (i): Warranted by (iv) and (v) of Proposition 3.16.
(ii): By Proposition 2.10 and items (iv) and (v) of Proposition 3.16, we obtain $\widetilde{\mathbb{S}}_\jmath(\mathsf{b}) \subseteq \mathcal{A}_\jmath(\mathsf{b})$ for each $\jmath$. □

Proposition 3.18. *In a G-approximation space $(\boldsymbol{U}, \Omega)$, the following relationships are fulfilled for each $\mathsf{b} \in \boldsymbol{U}$.*

(i) *If Ω is serial, then $\widetilde{\mathbb{S}}_\jmath(\mathsf{b}) \subseteq \widetilde{\mathbb{E}}_\jmath(\mathsf{b})$ for each $\jmath \in \{r, \langle l\rangle\}$.*

(ii) *If Ω is inverse serial, then $\widetilde{\mathbb{S}}_\jmath(\mathsf{b}) \subseteq \widetilde{\mathbb{E}}_\jmath(\mathsf{b})$ for each $\jmath \in \{l, \langle r\rangle\}$.*

Proof. We show the first property, and one can prove the second one similarly.
When $\jmath = r$: Assume $\mathsf{f} \in \widetilde{\mathbb{S}}_r(\mathsf{b})$. Then, $\mathcal{A}_r(\mathsf{b}) \subseteq \mathcal{A}_l(\mathsf{f})$, so, by the serial condition, we find $\mathcal{A}_r(\mathsf{b}) \neq \emptyset$. Consequentially, the intersection between $\mathcal{A}_r(\mathsf{b})$ and $\mathcal{A}_l(\mathsf{f})$ is nonempty. Hence, $\mathsf{f} \in \widetilde{\mathbb{E}}_r(\mathsf{b})$, which proves that $\widetilde{\mathbb{S}}_r(\mathsf{b}) \subseteq \widetilde{\mathbb{E}}_r(\mathsf{b})$.
When $\jmath = \langle l\rangle$. Assume $\mathsf{f} \in \widetilde{\mathbb{S}}_{\langle l\rangle}(\mathsf{b})$. Then, $\mathcal{A}_{\langle l\rangle}(\mathsf{b}) \subseteq \mathcal{A}_{\langle r\rangle}(\mathsf{f})$. By the serial condition, there is an element h belongs to $\mathbf{U}$ such that $\mathsf{b} \in \mathcal{A}_l(\mathsf{h})$, which means that $\mathcal{A}_{\langle l\rangle}(\mathsf{b}) \neq \emptyset$. Thus, the intersection between $\mathcal{A}_{\langle l\rangle}(\mathsf{b})$ and $\mathcal{A}_{\langle r\rangle}(\mathsf{f})$ is nonempty. Hence, $\mathsf{f} \in \widetilde{\mathbb{E}}_{\langle l\rangle}(\mathsf{b})$, which proves that $\widetilde{\mathbb{S}}_l(\mathsf{b}) \subseteq \widetilde{\mathbb{E}}_l(\mathsf{b})$. □

Corollary 3.19. *In a G-approximation space $(\boldsymbol{U}, \Omega)$, if Ω is inverse serial and serial (or reflexive), then $\widetilde{\mathbb{S}}_\jmath(\mathsf{b}) \cup \widetilde{\mathbb{C}}_\jmath(\mathsf{b}) \subseteq \widetilde{\mathbb{E}}_\jmath(\mathsf{b})$ for each $\mathsf{b} \in \boldsymbol{U}$ and $\jmath$.*

Proof. For each $\mathsf{b} \in \mathbf{U}$ and $\jmath$, it follows from Proposition 3.18 that $\widetilde{\mathbb{S}}_\jmath(\mathsf{b}) \subseteq \widetilde{\mathbb{E}}_\jmath(\mathsf{b})$ and it follows from Proposition 2.26 that $\widetilde{\mathbb{C}}_\jmath(\mathsf{b}) \subseteq \widetilde{\mathbb{E}}_\jmath(\mathsf{b})$. Hence, the proof is complete. □

Corollary 3.20. *In a G-approximation space $(\boldsymbol{U}, \Omega)$, if Ω is serial (or inverse serial) and symmetric, then $\widetilde{\mathbb{S}}_\jmath(\mathsf{b}) \cup \widetilde{\mathbb{C}}_\jmath(\mathsf{b}) \subseteq \mathbb{E}_\jmath(\mathsf{b})$ for each $\mathsf{b} \in \boldsymbol{U}$ and $\jmath$.*

Proposition 3.21. *In a G-approximation space $(\boldsymbol{U}, \Omega)$, the following relationships are satisfied.*

(i) *If f belongs to $\widetilde{\mathbb{S}}_r(\mathsf{b})$, then $\widetilde{\mathbb{S}}_l(\mathsf{f})$ is a subset of $\mathbb{S}_r(\mathsf{b})$.*

(ii) *If f belongs to $\widetilde{\mathbb{S}}_{\langle r\rangle}(\mathsf{b})$, then $\widetilde{\mathbb{S}}_{\langle l\rangle}(\mathsf{f})$ is a subset of $\mathbb{S}_{\langle r\rangle}(\mathsf{b})$.*

(iii) *If f belongs to $\widetilde{\mathbb{S}}_l(\mathsf{b})$, then $\widetilde{\mathbb{S}}_r(\mathsf{f})$ is a subset of $\mathbb{S}_l(\mathsf{b})$.*

(iv) *If f belongs to $\widetilde{\mathbb{S}}_{\langle l\rangle}(\mathsf{b})$, then $\widetilde{\mathbb{S}}_{\langle r\rangle}(\mathsf{f})$ is a subset of $\mathbb{S}_{\langle l\rangle}(\mathsf{b})$.*

Proof. We provide the proof for case (i), while the remaining cases can be established in a similar fashion. Suppose that $\mathsf{f} \in \widetilde{\mathbb{S}}_r(\mathsf{b})$. Then, $\mathcal{A}_r(\mathsf{b}) \subseteq \mathcal{A}_l(\mathsf{f})$. Now, let $\mathsf{h} \in \widetilde{\mathbb{S}}_l(\mathsf{f})$. Then, $\mathcal{A}_l(\mathsf{f}) \subseteq \mathcal{A}_r(\mathsf{h})$. By these two inclusions, we obtain $\mathcal{A}_r(\mathsf{b}) \subseteq \mathcal{A}_r(\mathsf{h})$, which directly gives rise to that $\mathsf{h} \in \mathbb{S}_r(\mathsf{b})$. Thus, we prove that $\widetilde{\mathbb{S}}_l(\mathsf{f})$ is a subset of $\mathbb{S}_r(\mathsf{b})$. □

Corollary 3.22. *In a G-approximation space $(\boldsymbol{U}, \Omega)$, if $\mathsf{f} \in \widetilde{\mathbb{S}}_\jmath(\mathsf{b})$, then for each $\jmath \in \{i, \langle i\rangle\}$ we have $\widetilde{\mathbb{S}}_\jmath(\mathsf{f}) \subseteq \mathbb{S}_\jmath(\mathsf{b})$.*

The following four results can be readily proven, so we omit their proofs.

Proposition 3.23. *In a G-approximation space $(\boldsymbol{U}, \Omega)$, the following relationships hold true.*

(i) $\mathsf{b} \in \widetilde{\mathbb{S}}_r(\mathsf{h}) \cap \mathbb{C}_r(\mathsf{h}) \Longleftrightarrow \mathcal{A}_r(\mathsf{b}) \subseteq \mathcal{A}_r(\mathsf{h}) \subseteq \mathcal{A}_l(\mathsf{b})$.

(ii) $\mathsf{b} \in \widetilde{\mathbb{S}}_l(\mathsf{h}) \cap \mathbb{C}_l(\mathsf{h}) \Longleftrightarrow \mathcal{A}_l(\mathsf{b}) \subseteq \mathcal{A}_l(\mathsf{h}) \subseteq \mathcal{A}_r(\mathsf{b})$.

(iii) $\mathsf{b} \in \widetilde{\mathbb{S}}_{\langle r\rangle}(\mathsf{h}) \cap \mathbb{C}_{\langle r\rangle}(\mathsf{h}) \Longleftrightarrow \mathcal{A}_{\langle r\rangle}(\mathsf{b}) \subseteq \mathcal{A}_{\langle r\rangle}(\mathsf{h}) \subseteq \mathcal{A}_{\langle l\rangle}(\mathsf{b})$.

(iv) $b \in \widetilde{\mathbb{S}}_{\langle l \rangle}(h) \cap \mathbb{C}_{\langle l \rangle}(h) \Longleftrightarrow \mathcal{A}_{\langle l \rangle}(b) \subseteq \mathcal{A}_{\langle l \rangle}(h) \subseteq \mathcal{A}_{\langle r \rangle}(b)$.

Corollary 3.24. *In a G-approximation space $(\boldsymbol{U}, \Omega)$, the following relationships hold true.*

(i) $b \in \widetilde{\mathbb{S}}_i(h) \cap \mathbb{C}_i(h) \Longrightarrow \mathcal{A}_i(b) = \mathcal{A}_i(h)$ *and* $\mathcal{A}_u(b) = \mathcal{A}_u(h)$.

(ii) $b \in \widetilde{\mathbb{S}}_{\langle i \rangle}(h) \cap \mathbb{C}_{\langle i \rangle}(h) \Longrightarrow \mathcal{A}_{\langle i \rangle}(b) = \mathcal{A}_{\langle i \rangle}(h)$ *and* $\mathcal{A}_{\langle u \rangle}(b) = \mathcal{A}_{\langle u \rangle}(h)$.

Proposition 3.25. *In a G-approximation space $(\boldsymbol{U}, \Omega)$, the following relationships hold true.*

(i) $b \in \widetilde{\mathbb{S}}_r(h) \cap \mathbb{S}_r(h) \Longleftrightarrow \mathcal{A}_r(h) \subseteq \mathcal{A}_i(b)$.

(ii) $b \in \widetilde{\mathbb{S}}_l(h) \cap \mathbb{S}_l(h) \Longleftrightarrow \mathcal{A}_l(h) \subseteq \mathcal{A}_i(b)$.

(iii) $b \in \widetilde{\mathbb{S}}_{\langle r \rangle}(h) \cap \mathbb{S}_{\langle r \rangle}(h) \Longleftrightarrow \mathcal{A}_{\langle r \rangle}(h) \subseteq \mathcal{A}_{\langle i \rangle}(b)$.

(iv) $b \in \widetilde{\mathbb{S}}_{\langle l \rangle}(h) \cap \mathbb{S}_{\langle l \rangle}(h) \Longleftrightarrow \mathcal{A}_{\langle l \rangle}(h) \subseteq \mathcal{A}_{\langle i \rangle}(b)$.

Corollary 3.26. *In a G-approximation space $(\boldsymbol{U}, \Omega)$, the following relationships hold true.*

(i) $b \in \widetilde{\mathbb{S}}_i(h) \cap \mathbb{S}_i(h) \Longrightarrow \mathcal{A}_u(h) \subseteq \mathcal{A}_i(b)$.

(ii) $b \in \widetilde{\mathbb{S}}_{\langle i \rangle}(h) \cap \mathbb{S}_{\langle i \rangle}(h) \Longrightarrow \mathcal{A}_{\langle u \rangle}(h) \subseteq \mathcal{A}_{\langle i \rangle}(b)$.

Remark 3.27. We highlight that the previously discussed types of neighborhoods, namely maximal neighborhoods [9], containment and subset neighborhoods [8, 10], intersection neighborhoods [12], and equality neighborhoods [19, 37], are equivalent to their corresponding Yao's neighborhoods when the binary relation is an equivalence relation. This property also applies to the neighborhoods introduced in this work. However, it does not hold for cardinality rough neighborhoods [14].

In what follows, we compare the proposed rough neighborhoods when induced by two binary relations. This comparison will help us determine how the rough set models, introduced in the next section, behave with respect to the monotonicity property.

Proposition 3.28. *In G-approximation spaces $(\boldsymbol{U}, \Omega_1)$ and $(\boldsymbol{U}, \Omega_2)$, if $\Omega_1 \subseteq \Omega_2$ such that Ω_1 is serial and transitive, and Ω_2 is symmetric and transitive, then* $\widetilde{\mathbb{S}}_{1r}(b) \subseteq \widetilde{\mathbb{S}}_{2r}(b)$.

Proof. Since Ω_1 is serial and transitive, it follows from (iv) of Proposition 3.16 that

$$\widetilde{\mathbb{S}}_{1r}(b) \subseteq \mathcal{A}_{1r}(b) \tag{2}$$

It follows from Lemma 2.11 that

$$\mathcal{A}_{1r}(b) \subseteq \mathcal{A}_{2r}(b) \tag{3}$$

Since Ω_2 is symmetric and transitive, it follows from Proposition 3.10 that

$$\mathcal{A}_{2r}(b) \subseteq \widetilde{\mathbb{S}}_{2r}(b) \tag{4}$$

From (2) and (3), we get

$$\widetilde{\mathbb{S}}_{1r}(b) \subseteq \mathcal{A}_{2r}(b) \tag{5}$$

And from (4) and (5), we obtain $\widetilde{\mathbb{S}}_{1r}(b) \subseteq \widetilde{\mathbb{S}}_{2r}(b)$, as required. □

Following a similar approach of the proof of Proposition 3.28 and using (v) of Proposition 3.16 and Corollary 3.17, one can prove the subsequent propositions.

Proposition 3.29. *In G-approximation spaces $(\boldsymbol{U}, \Omega_1)$ and $(\boldsymbol{U}, \Omega_2)$, if $\Omega_1 \subseteq \Omega_2$ such that Ω_1 is inverse serial and transitive, and Ω_2 is symmetric and transitive, then $\widetilde{\mathbb{S}}_{1l}(\mathsf{b}) \subseteq \widetilde{\mathbb{S}}_{2l}(\mathsf{b})$.*

Proposition 3.30. *In G-approximation spaces $(\boldsymbol{U}, \Omega_1)$ and $(\boldsymbol{U}, \Omega_2)$, we have the following properties.*

(i) *If $\Omega_1 \subseteq \Omega_2$, Ω_1 is serial, inverse serial and transitive, and Ω_2 is symmetric and transitive, then $\widetilde{\mathbb{S}}_{1\jmath}(\mathsf{b}) \subseteq \widetilde{\mathbb{S}}_{2\jmath}(\mathsf{b})$ for each $\jmath \in \{r, l, \langle r\rangle, \langle l\rangle\}$.*

(ii) *If $\Omega_1 \subseteq \Omega_2$, Ω_1 is a quasi-order, and Ω_2 is symmetric and transitive, then $\widetilde{\mathbb{S}}_{1\jmath}(\mathsf{b}) \subseteq \widetilde{\mathbb{S}}_{2\jmath}(\mathsf{b})$ for each $\jmath$.*

We close this section by displaying some characterizations of overlapping containment rough neighborhoods [16, Proposition 3.31 and Proposition 3.32] that are missing for overlapping subset rough neighborhoods, as illustrated by Example 3.33 and Example 3.34.

Proposition 3.31. ([16]) *In a G-approximation space $(\boldsymbol{U}, \Omega)$, if Ω is a partial order relation, then the next properties hold.*

(i) $\widetilde{\mathbb{C}}_r(\mathsf{b}) \subseteq \{\mathsf{b}\}$ *or* $\widetilde{\mathbb{C}}_l(\mathsf{b}) \subseteq \{\mathsf{b}\}$.

(ii) $\widetilde{\mathbb{C}}_i(\mathsf{b}) \subseteq \{\mathsf{b}\}$.

(iii) $\widetilde{\mathbb{C}}_u(\mathsf{b}) = \widetilde{\mathbb{C}}_r(\mathsf{b})$ *or* $\widetilde{\mathbb{C}}_u(\mathsf{b}) = \widetilde{\mathbb{C}}_l(\mathsf{b})$.

Proposition 3.32. ([16]) *In a G-approximation space $(\boldsymbol{U}, \Omega)$, if Ω is symmetric and transitive. Then, for each $\jmath$ and $\mathsf{b}, \mathsf{f} \in \boldsymbol{U}$ we have*

$$\widetilde{\mathbb{C}}_\jmath(\mathsf{b}) \cap \widetilde{\mathbb{C}}_\jmath(\mathsf{f}) = \{\mathsf{h} \in \boldsymbol{U} : (\mathsf{h}, \mathsf{h}) \notin \Omega\} \text{ or } \widetilde{\mathbb{C}}_\jmath(\mathsf{b}) = \widetilde{\mathbb{C}}_\jmath(\mathsf{f}).$$

Example 3.33. Let $\Omega = \{(\mathsf{b}, \mathsf{b}), (\mathsf{f}, \mathsf{f}), (\mathsf{h}, \mathsf{h}), (\mathsf{b}, \mathsf{f}), (\mathsf{f}, \mathsf{h}), (\mathsf{b}, \mathsf{h})\}$ be a partial order relation on $\mathbf{U} = \{\mathsf{b}, \mathsf{f}, \mathsf{h}\}$. In Table 5, we calculate $\mathcal{A}_\jmath$-, $\widetilde{\mathbb{S}}_\jmath$-, and $\widetilde{\mathbb{C}}_\jmath$-neighborhoods for all elements in $\mathbf{U}$.

	b	f	h
$\mathcal{A}_r = \mathcal{A}_{\langle r\rangle}$	$\mathbf{U}$	$\{\mathsf{f}, \mathsf{h}\}$	$\{\mathsf{h}\}$
$\mathcal{A}_l = \mathcal{A}_{\langle l\rangle}$	$\{\mathsf{b}\}$	$\{\mathsf{b}, \mathsf{f}\}$	$\mathbf{U}$
$\widetilde{\mathbb{S}}_r = \widetilde{\mathbb{S}}_{\langle r\rangle}$	$\{\mathsf{h}\}$	$\{\mathsf{h}\}$	$\{\mathsf{h}\}$
$\widetilde{\mathbb{S}}_l = \widetilde{\mathbb{S}}_{\langle l\rangle}$	$\{\mathsf{b}\}$	$\{\mathsf{b}\}$	$\{\mathsf{b}\}$
$\widetilde{\mathbb{S}}_i = \widetilde{\mathbb{S}}_{\langle i\rangle}$	$\emptyset$	$\emptyset$	$\emptyset$
$\widetilde{\mathbb{S}}_u = \widetilde{\mathbb{S}}_{\langle u\rangle}$	$\{\mathsf{b}, \mathsf{h}\}$	$\{\mathsf{b}, \mathsf{h}\}$	$\{\mathsf{b}, \mathsf{h}\}$
$\widetilde{\mathbb{C}}_r = \widetilde{\mathbb{C}}_{\langle r\rangle}$	$\mathbf{U}$	$\emptyset$	$\emptyset$
$\widetilde{\mathbb{C}}_l = \widetilde{\mathbb{C}}_{\langle l\rangle}$	$\emptyset$	$\emptyset$	$\mathbf{U}$
$\widetilde{\mathbb{C}}_i = \widetilde{\mathbb{C}}_{\langle i\rangle}$	$\emptyset$	$\emptyset$	$\emptyset$
$\widetilde{\mathbb{C}}_u = \widetilde{\mathbb{C}}_{\langle u\rangle}$	$\mathbf{U}$	$\emptyset$	$\mathbf{U}$

Table 5: The $\mathcal{A}_\jmath$-, $\widetilde{\mathbb{S}}_\jmath$-, and $\widetilde{\mathbb{C}}_\jmath$-neighborhoods of all elements in $\mathbf{U}$

Example 3.34. By Example 3.15, Ω is symmetric and transitive. However, we note that $\widetilde{\mathbb{S}}_\jmath(\mathsf{f}) \neq \widetilde{\mathbb{S}}_\jmath(\mathsf{h})$ and $\widetilde{\mathbb{S}}_\jmath(\mathsf{f}) \cap \widetilde{\mathbb{S}}_\jmath(\mathsf{h}) = \{\mathsf{b}, \mathsf{f}\}$, where $(\mathsf{b}, \mathsf{b}), (\mathsf{f}, \mathsf{f}) \in \Omega$.

4. Innovative rough set models based on $\widetilde{\mathbb{S}}_\jmath$-neighborhoods

In this section, we utilize rough neighborhoods, defined in the previous section, to introduce and analyze novel rough approximations, referred to as $\underline{O}_{\widetilde{\mathbb{S}}_\jmath}$-lower and $\overline{O}_{\widetilde{\mathbb{S}}_\jmath}$-upper approximations. These approximations are then used to define new set regions and establish accuracy measures. We demonstrate that the most precise approximations and accuracy levels for a subset occur when $\jmath \in \{i, \langle i\rangle\}$. Additionally, we leverage the equivalences proven in the previous section to clarify the conditions under which the proposed models outperform existing ones. Finally, we structure an algorithm to verify whether a subset is $\widetilde{\mathbb{S}}$-definable.

Definition 4.1. In a G-approximation space $(\mathbf{U}, \Omega)$, the upper and lower approximations of a subset Y, denoted by $\overline{O}_{\widetilde{\mathbb{S}}_\jmath}(Y)$ and $\underline{O}_{\widetilde{\mathbb{S}}_\jmath}(Y)$, are calculated with respect to $\widetilde{\mathbb{S}}_\jmath$-neighborhoods using the following formulas:

$$\overline{O}_{\widetilde{\mathbb{S}}_\jmath}(Y) = Y \cup \{\mathsf{b} \in \mathbf{U} : \widetilde{\mathbb{S}}_\jmath(\mathsf{b}) \cap Y \neq \emptyset\}, \text{ and}$$

$$\underline{O}_{\widetilde{\mathbb{S}}_\jmath}(Y) = Y \cap \{\mathsf{b} \in \mathbf{U} : \widetilde{\mathbb{S}}_\jmath(\mathsf{b}) \subseteq Y\}, \text{ respectively.}$$

Now, we introduce the next definitions to classify regions and measures for any subset Y of a G-approximation space $(\mathbf{U}, \Omega)$.

Definition 4.2. The next regions are defined for a subset Y of a G-approximation space $(\mathbf{U}, \Omega)$ with respect to $\widetilde{\mathbb{S}}_\jmath$-neighborhoods:

$$\widetilde{\mathbb{S}}_\jmath\text{-boundary region: } \delta_{\widetilde{\mathbb{S}}_\jmath}(Y) = \overline{O}_{\widetilde{\mathbb{S}}_\jmath}(Y) \setminus \underline{O}_{\widetilde{\mathbb{S}}_\jmath}(Y),$$

$$\widetilde{\mathbb{S}}_\jmath\text{-positive region: } \mathcal{P}_{\widetilde{\mathbb{S}}_\jmath}(Y) = \underline{O}_{\widetilde{\mathbb{S}}_\jmath}(Y), \text{ and}$$

$$\widetilde{\mathbb{S}}_\jmath\text{-negative region: } \mathcal{N}_{\widetilde{\mathbb{S}}_\jmath}(Y) = \mathbf{U} \setminus \overline{O}_{\widetilde{\mathbb{S}}_\jmath}(Y).$$

Definition 4.3. The next measures are defined for a subset Y of a G-approximation space $(\mathbf{U}, \Omega)$ with respect to $\widetilde{\mathbb{S}}_\jmath$-neighborhoods:

$$\widetilde{\mathbb{S}}_\jmath\text{-accuracy measure: } \xi_{\widetilde{\mathbb{S}}_\jmath}(Y) = \frac{|\underline{O}_{\widetilde{\mathbb{S}}_\jmath}(Y)|}{|\overline{O}_{\widetilde{\mathbb{S}}_\jmath}(Y)|}, \ |\overline{O}_{\widetilde{\mathbb{S}}_\jmath}(Y)| \neq 0.$$

$$\widetilde{\mathbb{S}}_\jmath\text{-roughness measure: } \hat{\xi}_{\widetilde{\mathbb{S}}_\jmath}(Y) = 1 - \xi_{\widetilde{\mathbb{S}}_\jmath}(Y).$$

Proposition 4.4. *In a G-approximation space $(\boldsymbol{U}, \Omega)$, if Ω is symmetric, then for each $\jmath \in \{r, l, i\}$ we have:*

(i) $\overline{O}_{\widetilde{\mathbb{S}}_\jmath}(Y) = \{\mathsf{b} \in \boldsymbol{U} : \widetilde{\mathbb{S}}_\jmath(\mathsf{b}) \cap Y \neq \emptyset\}$, *and*

(ii) $\underline{O}_{\widetilde{\mathbb{S}}_\jmath}(Y) = \{\mathsf{b} \in \boldsymbol{U} : \widetilde{\mathbb{S}}_\jmath(\mathsf{b}) \subseteq Y\}$.

Proof. According to Proposition 3.8, we have $\mathsf{b} \in \widetilde{\mathbb{S}}_\jmath(\mathsf{b})$ for each $\mathsf{b} \in \mathbf{U}$ and $\jmath \in \{r, l, i\}$, so if $\mathsf{b} \in Y$, then $\widetilde{\mathbb{S}}_\jmath(\mathsf{b}) \cap Y \neq \emptyset$. This means that $Y \subseteq \{\mathsf{b} \in \mathbf{U} : \widetilde{\mathbb{S}}_\jmath(\mathsf{b}) \cap Y \neq \emptyset\}$. Hence, $\overline{O}_{\widetilde{\mathbb{S}}_\jmath}(Y) = Y \cup \{\mathsf{b} \in \mathbf{U} : \widetilde{\mathbb{S}}_\jmath(\mathsf{b}) \cap Y \neq \emptyset\} = \{\mathsf{b} \in \mathbf{U} : \widetilde{\mathbb{S}}_\jmath(\mathsf{b}) \cap Y \neq \emptyset\}$. In a similar way, it can be proved (ii). □

In what follows, we explore the key characterizations of $\overline{O}_{\widetilde{\mathbb{S}}_\jmath}$ and $\underline{O}_{\widetilde{\mathbb{S}}_\jmath}$ approximations. We identify which of Pawlak's properties they preserve.

Theorem 4.5. *In a G-approximation space $(\boldsymbol{U}, \Omega)$, let $Y, Z \subseteq \boldsymbol{U}$. Then, the next results are fulfilled for each $\jmath$.*

(i) $\underline{O}_{\widetilde{\mathbb{S}}_j}(Y) \subseteq Y$.

(ii) $\underline{O}_{\widetilde{\mathbb{S}}_j}(\emptyset) = \emptyset$.

(iii) $\underline{O}_{\widetilde{\mathbb{S}}_j}(\boldsymbol{U}) = \boldsymbol{U}$.

(iv) *If* $Y \subseteq Z$, *then* $\underline{O}_{\widetilde{\mathbb{S}}_j}(Y) \subseteq \underline{O}_{\widetilde{\mathbb{S}}_j}(Z)$.

(v) $\underline{O}_{\widetilde{\mathbb{S}}_j}(Y \cap Z) = \underline{O}_{\widetilde{\mathbb{S}}_j}(Y) \cap \underline{O}_{\widetilde{\mathbb{S}}_j}(Z)$.

(vi) $\underline{O}_{\widetilde{\mathbb{S}}_j}(Y^c) = (\overline{O}_{\widetilde{\mathbb{S}}_j}(Y))^c$.

Proof. The proofs of (i) and (ii) follows directly by noting that $\underline{O}_{\widetilde{\mathbb{S}}_j}(Y) = Y \cap \{\mathsf{b} \in \mathbf{U} : \widetilde{\mathbb{S}}_j(\mathsf{b}) \subseteq Y\} \subseteq Y$.
(iii): For every $\mathsf{b} \in \mathbf{U}$ we have $\widetilde{\mathbb{S}}_j(\mathsf{b}) \subseteq \mathbf{U}$, so we conclude that $\underline{O}_{\widetilde{\mathbb{S}}_j}(\mathbf{U}) = \mathbf{U}$.
(iv): If $Y \subseteq Z$, then $\underline{O}_{\widetilde{\mathbb{S}}_j}(Y) = Y \cap \{\mathsf{b} \in \mathbf{U} : \widetilde{\mathbb{S}}_j(\mathsf{b}) \subseteq Y\} \subseteq Z \cap \{\mathsf{b} \in \mathbf{U} : \widetilde{\mathbb{S}}_j(\mathsf{b}) \subseteq Z\} = \underline{O}_{\widetilde{\mathbb{S}}_j}(Z)$.
(v): Follows from (iv) that $\underline{O}_{\widetilde{\mathbb{S}}_j}(Y \cap Z) \subseteq \underline{O}_{\widetilde{\mathbb{S}}_j}(Y) \cap \underline{O}_{\widetilde{\mathbb{S}}_j}(Z)$. Conversely, assume that $\mathsf{b} \in \underline{O}_{\widetilde{\mathbb{S}}_j}(Y) \cap \underline{O}_{\widetilde{\mathbb{S}}_j}(Z)$. Accordingly, it follows $\mathsf{b} \in \underline{O}_{\widetilde{\mathbb{S}}_j}(Y)$ and $\mathsf{b} \in \underline{O}_{\widetilde{\mathbb{S}}_j}(Z)$. Now, $\mathsf{b} \in Y$, $\widetilde{\mathbb{S}}_j(\mathsf{b}) \subseteq Y$ and $\mathsf{b} \in Z$, $\widetilde{\mathbb{S}}_j(\mathsf{b}) \subseteq Z$. Thus, $\mathsf{b} \in Y \cap Z$ and $\widetilde{\mathbb{S}}_j(\mathsf{b}) \subseteq Y \cap Z$, which leads to that $\mathsf{b} \in \underline{O}_{\widetilde{\mathbb{S}}_j}(Y \cap Z)$. This proves that $\underline{O}_{\widetilde{\mathbb{S}}_j}(Y) \cap \underline{O}_{\widetilde{\mathbb{S}}_j}(Z) \subseteq \underline{O}_{\widetilde{\mathbb{S}}_j}(Y \cap G)$.
(vi):
$$\begin{aligned} \mathsf{b} \in \underline{O}_{\widetilde{\mathbb{S}}_j}(Y^c) &\Longleftrightarrow \mathsf{b} \in Y^c \text{ and } \widetilde{\mathbb{S}}_j(\mathsf{b}) \subseteq Y^c \\ &\Longleftrightarrow \mathsf{b} \notin Y \text{ and } \widetilde{\mathbb{S}}_j(\mathsf{b}) \cap Y = \emptyset \\ &\Longleftrightarrow \mathsf{b} \notin \overline{O}_{\widetilde{\mathbb{S}}_j}(Y) \\ &\Longleftrightarrow \mathsf{b} \in (\overline{O}_{\widetilde{\mathbb{S}}_j}(Y))^c. \end{aligned}$$ □

Corollary 4.6. *In a G-approximation space* $(\boldsymbol{U}, \Omega)$, *let* $Y, Z \subseteq \boldsymbol{U}$. *Then, the next results are fulfilled for each* j.

(i) $\underline{O}_{\widetilde{\mathbb{S}}_j}(Y) \cup \underline{O}_{\widetilde{\mathbb{S}}_j}(Z) \subseteq \underline{O}_{\widetilde{\mathbb{S}}_j}(Y \cup Z)$

(ii) $\underline{O}_{\widetilde{\mathbb{S}}_j}(\underline{O}_{\widetilde{\mathbb{S}}_j}(Y)) \subseteq \underline{O}_{\widetilde{\mathbb{S}}_j}(Y)$.

Theorem 4.7. *In a G-approximation space* $(\boldsymbol{U}, \Omega)$, *let* $Y, Z \subseteq \boldsymbol{U}$. *Then, the next results are fulfilled for each* j.

(i) $Y \subseteq \overline{O}_{\widetilde{\mathbb{S}}_j}(Y)$.

(ii) $\overline{O}_{\widetilde{\mathbb{S}}_j}(\emptyset) = \emptyset$.

(iii) $\overline{O}_{\widetilde{\mathbb{S}}_j}(\boldsymbol{U}) = \boldsymbol{U}$.

(iv) *If* $Y \subseteq Z$, *then* $\overline{O}_{\widetilde{\mathbb{S}}_j}(Y) \subseteq \overline{O}_{\widetilde{\mathbb{S}}_j}(Z)$.

(v) $\overline{O}_{\widetilde{\mathbb{S}}_j}(Y \cup Z) = \overline{O}_{\widetilde{\mathbb{S}}_j}(Y) \cup \overline{O}_{\widetilde{\mathbb{S}}_j}(Z)$.

(vi) $\overline{O}_{\widetilde{\mathbb{S}}_j}(Y^c) = (\underline{O}_{\widetilde{\mathbb{S}}_j}(Y))^c$.

Proof. We leave the proof for the reader as it closely resembles the one used in Theorem 4.5. □

Corollary 4.8. *In a G-approximation space* $(\boldsymbol{U}, \Omega)$, *let* $Y, Z \subseteq \boldsymbol{U}$. *Then, the next results are fulfilled for each* j.

(i) $\overline{O}_{\widetilde{\mathbb{S}}_j}(Y \cap Z) \subseteq \overline{O}_{\widetilde{\mathbb{S}}_j}(Y) \cap \overline{O}_{\widetilde{\mathbb{S}}_j}(Z)$.

(ii) $\overline{O}_{\widetilde{\mathbb{S}}_\jmath}(Y) \subseteq \overline{O}_{\widetilde{\mathbb{S}}_\jmath}(\overline{O}_{\widetilde{\mathbb{S}}_\jmath}(Y))$.

The example that follows clarifies that the converses of items (i) and (iv) in Theorems 4.5 and 4.7, as well as Corollaries 4.6 and 4.8, do not hold.

Example 4.9. In a G-approximation space $(\mathbf{U}, \Omega)$ displayed in Example 3.2, let $Y = \{\mathsf{b}\}$ and $Z = \{\mathsf{f}\}$. Then,

(i) $\underline{O}_{\widetilde{\mathbb{S}}_l}(Z) = \emptyset \subset Z$ and $Y \subset \overline{O}_{\widetilde{\mathbb{S}}_l}(Y) = \mathbf{U}$.

(ii) $\underline{O}_{\widetilde{\mathbb{S}}_l}(Y) = Y$ and $\underline{O}_{\widetilde{\mathbb{S}}_l}(Z) = \emptyset$. Now, $\underline{O}_{\widetilde{\mathbb{S}}_l}(Y \cup Z) = \{\mathsf{b}, \mathsf{f}\}$ but $\underline{O}_{\widetilde{\mathbb{S}}_l}(Y) \cup \underline{O}_{\widetilde{\mathbb{S}}_l}(Z) = \{\mathsf{b}\}$.

(iii) $\overline{O}_{\widetilde{\mathbb{S}}_l}(Y) = \mathbf{U}$ and $\overline{O}_{\widetilde{\mathbb{S}}_l}(Z) = \{\mathsf{f}, \mathsf{h}\}$. Now, $\overline{O}_{\widetilde{\mathbb{S}}_l}(Y \cap Z) = \emptyset$ but $\overline{O}_{\widetilde{\mathbb{S}}_l}(Y) \cap \overline{O}_{\widetilde{\mathbb{S}}_l}(Z) = \{\mathsf{f}, \mathsf{h}\}$. Also, $\overline{O}_{\widetilde{\mathbb{S}}_l}(Z) \subseteq \overline{O}_{\widetilde{\mathbb{S}}_l}(Y)$ in spite that Z is not a subset of Y.

Proposition 4.10. *In a G-approximation space $(\boldsymbol{U}, \Omega)$, the next results are fulfilled for every subset Y of $\boldsymbol{U}$ and each $\jmath$.*

(i) $\underline{O}_{\widetilde{\mathbb{S}}_u}(Y) \subseteq \underline{O}_{\widetilde{\mathbb{S}}_r}(Y) \cap \underline{O}_{\widetilde{\mathbb{S}}_l}(Y) \subseteq \underline{O}_{\widetilde{\mathbb{S}}_r}(Y) \cup \underline{O}_{\widetilde{\mathbb{S}}_l}(Y) \subseteq \underline{O}_{\widetilde{\mathbb{S}}_i}(Y)$.

(ii) $\overline{O}_{\widetilde{\mathbb{S}}_i}(Y) \subseteq \overline{O}_{\widetilde{\mathbb{S}}_r}(Y) \cap \overline{O}_{\widetilde{\mathbb{S}}_l}(Y) \subseteq \overline{O}_{\widetilde{\mathbb{S}}_r}(Y) \cup \overline{O}_{\widetilde{\mathbb{S}}_l}(Y) \subseteq \overline{O}_{\widetilde{\mathbb{S}}_u}(Y)$.

(iii) $\underline{O}_{\widetilde{\mathbb{S}}_{\langle u\rangle}}(Y) \subseteq \underline{O}_{\widetilde{\mathbb{S}}_{\langle r\rangle}}(Y) \cap \underline{O}_{\widetilde{\mathbb{S}}_{\langle l\rangle}}(Y) \subseteq \underline{O}_{\widetilde{\mathbb{S}}_{\langle r\rangle}}(Y) \cup \underline{O}_{\widetilde{\mathbb{S}}_{\langle l\rangle}}(Y) \subseteq \underline{O}_{\widetilde{\mathbb{S}}_{\langle i\rangle}}(Y)$.

(iv) $\overline{O}_{\widetilde{\mathbb{S}}_{\langle i\rangle}}(Y) \subseteq \overline{O}_{\widetilde{\mathbb{S}}_{\langle r\rangle}}(Y) \cap \overline{O}_{\widetilde{\mathbb{S}}_{\langle l\rangle}}(Y) \subseteq \overline{O}_{\widetilde{\mathbb{S}}_{\langle r\rangle}}(Y) \cup \overline{O}_{\widetilde{\mathbb{S}}_{\langle l\rangle}}(Y) \subseteq \overline{O}_{\widetilde{\mathbb{S}}_{\langle u\rangle}}(Y)$.

Proof. According to Definition 4.1 and by (i) and (ii) of Proposition 3.3, the proof follows. □

Corollary 4.11. *In a G-approximation space $(\boldsymbol{U}, \Omega)$, the next results are fulfilled for every subset Y of $\boldsymbol{U}$ and each $\jmath$.*

(i) $\xi_{\widetilde{\mathbb{S}}_u}(Y) \leq \xi_{\widetilde{\mathbb{S}}_r}(Y) \leq \xi_{\widetilde{\mathbb{S}}_i}(Y)$.

(ii) $\xi_{\widetilde{\mathbb{S}}_u}(Y) \leq \xi_{\widetilde{\mathbb{S}}_l}(Y) \leq \xi_{\widetilde{\mathbb{S}}_i}(Y)$.

(iii) $\xi_{\widetilde{\mathbb{S}}_{\langle u\rangle}}(Y) \leq \xi_{\widetilde{\mathbb{S}}_{\langle r\rangle}}(Y) \leq \xi_{\widetilde{\mathbb{S}}_{\langle i\rangle}}(Y)$.

(iv) $\xi_{\widetilde{\mathbb{S}}_{\langle u\rangle}}(Y) \leq \xi_{\widetilde{\mathbb{S}}_{\langle l\rangle}}(Y) \leq \xi_{\widetilde{\mathbb{S}}_{\langle i\rangle}}(Y)$.

Proof. (i): Given that $\underline{O}_{\widetilde{\mathbb{S}}_u}(Y) \subseteq \underline{O}_{\widetilde{\mathbb{S}}_r}(Y) \subseteq \underline{O}_{\widetilde{\mathbb{S}}_i}(Y)$, we get

$$|\, \underline{O}_{\widetilde{\mathbb{S}}_u}(Y) \,|\leq|\, \underline{O}_{\widetilde{\mathbb{S}}_r}(Y) \,|\leq|\, \underline{O}_{\widetilde{\mathbb{S}}_i}(Y) \,| \tag{6}$$

Considering $\overline{O}_{\widetilde{\mathbb{S}}_i}(Y) \subseteq \overline{O}_{\widetilde{\mathbb{S}}_r}(Y) \subseteq \overline{O}_{\widetilde{\mathbb{S}}_u}(Y)$, $|\, \overline{O}_{\widetilde{\mathbb{S}}_i}(Y) \,|\leq|\, \overline{O}_{\widetilde{\mathbb{S}}_r}(Y) \,|\leq|\, \overline{O}_{\widetilde{\mathbb{S}}_u}(Y) \,|$. Given the assumption that Y is nonempty, it follows that $|\, \overline{O}_{\widetilde{\mathbb{S}}_\jmath}(Y) \,|> 0$ for all $\jmath$. Therefore,

$$\frac{1}{|\, \overline{O}_{\widetilde{\mathbb{S}}_u}(Y) \,|} \leq \frac{1}{|\, \overline{O}_{\widetilde{\mathbb{S}}_r}(Y) \,|} \leq \frac{1}{|\, \overline{O}_{\widetilde{\mathbb{S}}_i}(Y) \,|}. \tag{7}$$

Based on (6) and (7), we obtain that

$$\frac{|\, \underline{O}_{\widetilde{\mathbb{S}}_u}(Y) \,|}{|\, \overline{O}_{\widetilde{\mathbb{S}}_u}(Y) \,|} \leq \frac{|\, \underline{O}_{\widetilde{\mathbb{S}}_r}(Y) \,|}{|\, \overline{O}_{\widetilde{\mathbb{S}}_r}(Y) \,|} \leq \frac{|\, \underline{O}_{\widetilde{\mathbb{S}}_i}(Y) \,|}{|\, \overline{O}_{\widetilde{\mathbb{S}}_i}(Y) \,|}$$

Hence, we finish the proof.
The remaining items can be proven by utilizing a similar approach. □

Corollary 4.12. *In a G-approximation space $(\boldsymbol{U}, \Omega)$, the next results are fulfilled for every subset Y of $\boldsymbol{U}$ and each $\jmath$.*

(i) $\delta_{\widetilde{\mathbb{S}}_u}(Y) \subseteq \delta_{\widetilde{\mathbb{S}}_r}(Y) \subseteq \delta_{\widetilde{\mathbb{S}}_i}(Y)$.

(ii) $\delta_{\widetilde{\mathbb{S}}_u}(Y) \subseteq \delta_{\widetilde{\mathbb{S}}_l}(Y) \subseteq \delta_{\widetilde{\mathbb{S}}_i}(Y)$.

(iii) $\delta_{\widetilde{\mathbb{S}}_{\langle u\rangle}}(Y) \subseteq \delta_{\widetilde{\mathbb{S}}_{\langle r\rangle}}(Y) \subseteq \delta_{\widetilde{\mathbb{S}}_{\langle i\rangle}}(Y)$.

(iv) $\delta_{\widetilde{\mathbb{S}}_{\langle u\rangle}}(Y) \subseteq \delta_{\widetilde{\mathbb{S}}_{\langle l\rangle}}(Y) \subseteq \delta_{\widetilde{\mathbb{S}}_{\langle i\rangle}}(Y)$.

To illustrate the invalidity of the converse statements in Proposition 4.10 and Corollary 10, we consider Example 3.2 and calculate the approximation operators and accuracy measures for all sets, as exhibited in Table 6 and Table 7.

Proposition 4.13. *In a $\mathcal{G}$-approximation space $(\boldsymbol{U}, \Omega)$, if Ω is symmetric and transitive, then all types of $\overline{O}_{\widetilde{\mathbb{S}}_\jmath}$-approximations (resp., $\underline{O}_{\widetilde{\mathbb{S}}_\jmath}$-approximations, $\xi_{\widetilde{\mathbb{S}}_\jmath}$-accuracy) are equal.*

Proof. Warranted by Corollary 3.13. □

Proposition 4.14. *In a G-approximation space $(\boldsymbol{U}, \Omega)$, let $Y \subseteq \boldsymbol{U}$. Then, $\xi_{\widetilde{\mathbb{S}}_\jmath}(\boldsymbol{U}) = 1$ and $0 \leq \xi_{\widetilde{\mathbb{S}}_\jmath}(Y) \leq 1$ for each $\jmath$.*

Proof. Straightforward. □

Definition 4.15. In a G-approximation space $(\mathbf{U}, \Omega)$, a set Y is called $\widetilde{\mathbb{S}}_\jmath$-rough if $\xi_{\widetilde{\mathbb{S}}_\jmath}(Y) < 1$. If $\xi_{\widetilde{\mathbb{S}}_\jmath}(Y) = 1$, then we name Y an $\widetilde{\mathbb{S}}_\jmath$-definable set.

We now design Algorithm 1 to demonstrate the process of determining whether a given set is $\widetilde{\mathbb{S}}_\jmath$-rough or $\widetilde{\mathbb{S}}_\jmath$-definable in the case of $\jmath = r$.

Ultimately, we examine the conditions necessary for the present rough set paradigms to satisfy the monotonicity property.

Proposition 4.16. *In G-approximation spaces $(\boldsymbol{U}, \Omega_1)$ and $(\boldsymbol{U}, \Omega_2)$, if $\Omega_1 \subseteq \Omega_2$ such that Ω_1 is serial (resp., inverse serial) and transitive, and Ω_2 is symmetric and transitive, then for each subset Y of $\boldsymbol{U}$ we have*

(i) $\xi_{\widetilde{\mathbb{S}}_{2r}}(Y) \leq \xi_{\widetilde{\mathbb{S}}_{1r}}(Y)$ *(resp.,* $\xi_{\widetilde{\mathbb{S}}_{2l}}(Y) \leq \xi_{\widetilde{\mathbb{S}}_{1l}}(Y)$*).*

(ii) $\hat{\xi}_{\widetilde{\mathbb{S}}_{1r}}(Y) \leq \hat{\xi}_{\widetilde{\mathbb{S}}_{2r}}(Y)$ *(resp.,* $\hat{\xi}_{\widetilde{\mathbb{S}}_{1r}}(Y) \leq \hat{\xi}_{\widetilde{\mathbb{S}}_{2r}}(Y)$*).*

Proof. According to Proposition 3.28 we obtain that $\widetilde{\mathbb{S}}_{1r}(\mathsf{b}) \subseteq \widetilde{\mathbb{S}}_{2r}(\mathsf{b})$. This yields that, for a set Y, we get $\overline{O}_{\widetilde{\mathbb{S}}_{1r}}(Y) \subseteq \overline{O}_{\widetilde{\mathbb{S}}_{2r}}(Y)$ and $\underline{O}_{\widetilde{\mathbb{S}}_{2r}}(Y) \subseteq \underline{O}_{\widetilde{\mathbb{S}}_{1r}}(Y)$. Thus, $\xi_{\widetilde{\mathbb{S}}_{2r}}(Y) = \frac{|\underline{O}_{\widetilde{\mathbb{S}}_{2r}}(Y)|}{|\overline{O}_{\widetilde{\mathbb{S}}_{2r}}(Y)|} \leq \frac{|\underline{O}_{\widetilde{\mathbb{S}}_{1r}}(Y)|}{|\overline{O}_{\widetilde{\mathbb{S}}_{1r}}(Y)|} = \xi_{\widetilde{\mathbb{S}}_{1\jmath}}(Y)$, which confirms the validity of (i). By Proposition 3.29, one can following a similar approach to verify the result in the case of an inverse serial relation.

Since $\xi_{\widetilde{\mathbb{S}}_{2\jmath}}(Y) \leq \xi_{\widetilde{\mathbb{S}}_{1\jmath}}(Y) \Longleftrightarrow 1 - \xi_{\widetilde{\mathbb{S}}_{1\jmath}}(Y) \leq 1 - \xi_{\widetilde{\mathbb{S}}_{2\jmath}}(Y)$, it follows from (i) above the proof of (ii). □

Corollary 4.17. *Under the conditions stated in Proposition 4.16, the rough set models suggested in this work satisfy the monotonicity property in the cases of $\jmath \in \{r, l\}$.*

Proposition 4.18. *In G-approximation spaces $(\boldsymbol{U}, \Omega_1)$ and $(\boldsymbol{U}, \Omega_2)$, if Ω_1 is serial, inverse serial and transitive, and Ω_2 is symmetric and transitive, then for each subset Y of $\boldsymbol{U}$ we have*

(i) $\xi_{\widetilde{\mathbb{S}}_{2\jmath}}(Y) \leq \xi_{\widetilde{\mathbb{S}}_{1\jmath}}(Y)$ *for each $\jmath \in \{r, l, \langle r\rangle, \langle l\rangle\}$.*

(ii) $\hat{\xi}_{\widetilde{\mathbb{S}}_{1\jmath}}(Y) \leq \hat{\xi}_{\widetilde{\mathbb{S}}_{2\jmath}}(Y)$ *for each $\jmath \in \{r, l, \langle r\rangle, \langle l\rangle\}$.*

Y	$\underline{O}_{\widetilde{\mathbb{S}}_r}(Y)$	$\overline{O}_{\widetilde{\mathbb{S}}_r}(Y)$	$\xi_{\widetilde{\mathbb{S}}_r}(Y)$	$\underline{O}_{\widetilde{\mathbb{S}}_l}(Y)$	$\overline{O}_{\widetilde{\mathbb{S}}_l}(Y)$	$\xi_{\widetilde{\mathbb{S}}_l}(Y)$	$\underline{O}_{\widetilde{\mathbb{S}}_i}(Y)$	$\overline{O}_{\widetilde{\mathbb{S}}_i}(Y)$	$\xi_{\widetilde{\mathbb{S}}_i}(Y)$	$\underline{O}_{\widetilde{\mathbb{S}}_u}(Y)$	$\overline{O}_{\widetilde{\mathbb{S}}_u}(Y)$	$\xi_{\widetilde{\mathbb{S}}_u}(Y)$
{b}	{b}	{b, f, k}	1/3	{b}	**U**	1/4	{b}	{b, f, k}	1/3	{b}	**U**	1/4
{f}	∅	{f, k}	0	∅	{f, h}	0	∅	{f}	0	∅	{f, h, k}	0
{h}	{h}	{f, h, k}	1/3	∅	{h}	0	{h}	{h}	1	∅	{f, h, k}	0
{k}	∅	{f, k}	0	∅	{k}	0	∅	{k}	0	∅	{k}	0
{b, f}	{b}	{b, f, k}	1/3	{b, f}	**U**	1/2	{b, f}	{b, f, k}	2/3	{b}	**U**	1/4
{b, h}	{b, h}	**U**	1/2	{b}	**U**	1/4	{b, h}	**U**	1/2	{b}	**U**	1/4
{b, k}	{b}	{b, f, k}	1/3	{b, k}	**U**	1/2	{b, k}	{b, f, k}	2/3	{b}	**U**	1/4
{f, h}	{h}	{f, h, k}	1/3	∅	{f, h}	0	{h}	{f, h}	1/2	∅	{f, h, k}	0
{f, k}	∅	{f, k}	0	∅	{f, h, k}	0	∅	{f, k}	0	∅	{f, h, k}	0
{h, k}	{h}	{f, h, k}	1/3	∅	{h, k}	0	{h}	{h, k}	1/2	∅	{f, h, k}	0
{b, f, h}	{b, h}	**U**	1/2	{b, f, h}	**U**	3/4	{b, f, h}	**U**	3/4	{b, h}	**U**	1/2
{b, f, k}	{b}	{b, f, k}	1/3	{b, f, k}	**U**	3/4	{b, f, k}	{b, f, k}	1	{b}	**U**	1/4
{b, h, k}	{b, h}	**U**	1/2	{b, k}	**U**	1/2	{b, h, k}	**U**	3/4	{b}	**U**	1/4
{f, h, k}	{h}	{f, h, k}	1/3	∅	{f, h, k}	0	{h}	{f, h, k}	1/3	∅	{f, h, k}	0
U	**U**	**U**	1	**U**	**U**	1	**U**	**U**	1	**U**	**U**	1

Table 6: $\underline{O}_{\widetilde{\mathbb{S}}_\jmath}, \overline{O}_{\widetilde{\mathbb{S}}_\jmath}$, and $\xi_{\widetilde{\mathbb{S}}_\jmath}$ for each subsets and $\jmath \in \{r, l, i, u\}$

Y	$\underline{O}_{\widetilde{\mathbb{S}}_{\langle r\rangle}}(Y)$	$\overline{O}_{\widetilde{\mathbb{S}}_{\langle r\rangle}}(Y)$	$\xi_{\widetilde{\mathbb{S}}_{\langle r\rangle}}(Y)$	$\underline{O}_{\widetilde{\mathbb{S}}_{\langle l\rangle}}(Y)$	$\overline{O}_{\widetilde{\mathbb{S}}_{\langle l\rangle}}(Y)$	$\xi_{\widetilde{\mathbb{S}}_{\langle l\rangle}}(Y)$	$\underline{O}_{\widetilde{\mathbb{S}}_{\langle i\rangle}}(Y)$	$\overline{O}_{\widetilde{\mathbb{S}}_{\langle i\rangle}}(Y)$	$\xi_{\widetilde{\mathbb{S}}_{\langle i\rangle}}(Y)$	$\underline{O}_{\widetilde{\mathbb{S}}_{\langle u\rangle}}(Y)$	$\overline{O}_{\widetilde{\mathbb{S}}_{\langle u\rangle}}(Y)$	$\xi_{\widetilde{\mathbb{S}}_{\langle u\rangle}}(Y)$
{b}	∅	{b}	0	∅	{b, k}	0	∅	{b}	0	∅	{b, k}	0
{f}	{f}	{b, f}	1/2	∅	{f, k}	0	{f}	{f}	1	∅	{b, f, k}	0
{h}	{h}	{b, h}	1/2	{h}	**U**	1/4	{h}	{b, h}	1/2	{h}	**U**	1/4
{k}	{k}	{k}	1	∅	{k}	0	{k}	{k}	1	∅	{k}	0
{b, f}	{f}	{b, f}	1/2	∅	{b, f, k}	0	{f}	{b, f}	1/2	∅	{b, f, k}	0
{b, h}	{h}	{b, h}	1/2	{b, h}	**U**	1/2	{b, h}	{b, h}	1	{h}	**U**	1/4
{b, k}	{k}	{b, k}	1/2	∅	{b, k}	0	{k}	{b, k}	1/2	∅	{b, k}	0
{f, h}	{f, h}	{b, f, h}	2/3	{f, h}	**U**	1/2	{f, h}	{b, f, h}	2/3	{f, h}	**U**	1/2
{f, k}	{f, k}	{b, f, k}	2/3	∅	{f, k}	0	{f, k}	{f, k}	1	∅	{b, f, k}	0
{h, k}	{h, k}	{b, h, k}	2/3	{h}	**U**	1/4	{h, k}	{b, h, k}	2/3	{h}	**U**	1/4
{b, f, h}	{b, f, h}	{b, f, h}	1	{b, f, h}	**U**	3/4	{b, f, h}	{b, f, h}	1	{b, f, h}	**U**	3/4
{b, f, k}	{f, k}	{b, f, k}	2/3	∅	{b, f, k}	0	{f, k}	{b, f, k}	2/3	∅	{b, f, k}	0
{b, h, k}	{h, k}	{b, h, k}	2/3	{b, h}	**U**	1/2	{b, h, k}	{b, h, k}	1	∅	**U**	0
{f, h, k}	{f, h, k}	**U**	3/4	{f, h}	**U**	1/2	{f, h, k}	**U**	3/4	{f, h}	**U**	1/2
U	**U**	**U**	1	**U**	**U**	1	**U**	**U**	1	**U**	**U**	1

Table 7: $\underline{O}_{\widetilde{\mathbb{S}}_\jmath}, \overline{O}_{\widetilde{\mathbb{S}}_\jmath}$, and $\xi_{\widetilde{\mathbb{S}}_\jmath}$ for each subsets and $\jmath \in \{\langle r\rangle, \langle l\rangle, \langle i\rangle, \langle u\rangle\}$

Input : The entire set **U** represents the sample we want to study.
Output: Determine the classification of each set: $\widetilde{\mathbb{S}}_r$-rough or $\widetilde{\mathbb{S}}_r$-definable.

Define a binary relation Ω describes the relationship between the elements of **U**;
for *every* $\mathsf{b} \in \boldsymbol{U}$ **do**
  Calculate $\mathcal{A}_r(\mathsf{b}) = \{\mathsf{f} \in \mathbf{U} : (\mathsf{b}, \mathsf{f}) \in \Omega\}$;
  Calculate $\mathcal{A}_l(\mathsf{b}) = \{\mathsf{f} \in \mathbf{U} : (\mathsf{f}, \mathsf{b}) \in \Omega$
end
for *every* $\mathsf{b} \in \boldsymbol{U}$ **do**
  Calculate $\widetilde{\mathbb{S}}_r(\mathsf{b}) = \{\mathsf{f} \in \mathbf{U} : \mathcal{A}_r(\mathsf{b}) \subseteq \mathcal{A}_l(\mathsf{f})\}$
end
for *every proper set* $Y \neq \emptyset$ *in* $\boldsymbol{U}$ **do**
  Calculate $\widetilde{\mathbb{S}}_r$-lower approximation: $\underline{O}_{\widetilde{\mathbb{S}}_r}(Y) = Y \cap \{\mathsf{b} \in \mathbf{U} : \widetilde{\mathbb{S}}_\jmath(\mathsf{b}) \subseteq Y\}$;
  Calculate $\widetilde{\mathbb{S}}_r$-upper approximation: $\overline{O}_{\widetilde{\mathbb{S}}_r}(Y) = Y \cup \{\mathsf{b} \in \mathbf{U} : \widetilde{\mathbb{S}}_\jmath(\mathsf{b}) \cap Y \neq \emptyset\}$;
  Calculate $\widetilde{\mathbb{S}}_r$-accuracy: $\xi_{\widetilde{\mathbb{S}}_r}(Y) = \frac{|\underline{O}_{\widetilde{\mathbb{S}}_\jmath}(Y)|}{|\overline{O}_{\widetilde{\mathbb{S}}_\jmath}(Y)|}$;
  if $\xi_{\widetilde{\mathbb{S}}_r}(Y) = 1$ **then**
    return a set Y is $\widetilde{\mathbb{S}}_r$-definable
  else
    a set Y is $\widetilde{\mathbb{S}}_r$-rough
  end
end

Algorithm 1: Identifying whether a set is $\widetilde{\mathbb{S}}_r$-rough or $\widetilde{\mathbb{S}}_r$-definable

Proof. According to (i) of Proposition 3.30, this proposition can be proved using a technique similar to that of Proposition 4.16. □

Corollary 4.19. *Under the conditions stated in Proposition 4.18, the rough set models suggested in this work satisfy the monotonicity property in the cases of* $\jmath \in \{r, l, \langle r\rangle, \langle l\rangle\}$.

Proposition 4.20. *In G-approximation spaces* $(\boldsymbol{U}, \Omega_1)$ *and* $(\boldsymbol{U}, \Omega_2)$*, if* Ω_1 *is quasi-order and* Ω_2 *is symmetric and transitive, then for each* $\jmath$ *and each subset* Y *of* $\boldsymbol{U}$ *we have*

(i) $\xi_{\widetilde{\mathbb{S}}_{2\jmath}}(Y) \leq \xi_{\widetilde{\mathbb{S}}_{1\jmath}}(Y)$.

(ii) $\hat{\xi}_{\widetilde{\mathbb{S}}_{1\jmath}}(Y) \leq \hat{\xi}_{\widetilde{\mathbb{S}}_{2\jmath}}(Y)$.

Proof. According to (ii) of Proposition 3.30, this proposition can be proved using a technique similar to that of Proposition 4.16. □

Corollary 4.21. *Under the conditions stated in Proposition 4.20, the rough set models suggested in this work satisfy the monotonicity property for each* $\jmath$.

Remark 4.22. We draw the reader's attention to the fact that $\widetilde{\mathbb{S}}_{1\jmath}(\mathsf{b})$ and $\widetilde{\mathbb{S}}_{2\jmath}(\mathsf{b})$, derived from the G-approximation spaces $(\mathbf{U}, \Omega_1)$ and $(\mathbf{U}, \Omega_2)$, are generally independent for all (or some) $\mathsf{b} \in \mathbf{U}$ and each $\jmath$. In other words, overlapping subset neighborhoods do not preserve the property of monotonicity. However, they satisfy the monotonicity property under specific conditions imposed on the binary relations Ω_1 and Ω_2, as established in Proposition 4.16, Proposition 4.18, and Proposition 4.20.

5. Computing the accuracy and uncertainty of knowledge derived from information systems

5.1. Problem statement and algorithm

Let us assume that $\mathbf{U}$ is a set of objects under study, each element characterized by some criteria. We systematically arrange these objects and their corresponding criteria in an information table. The analyst or decision-maker responsible for extracting insights from this table will calculate the confirmed and possible knowledge for the desired data subsets using both the current and some previously established types of neighborhoods.

Following this approach, we calculate the approximation operators and accuracy measures for subsets using the steps outlined below:

Step 1: Structure the information system by organizing objects along with their associated attributes (criteria).
Step 2: Define the relation that links objects based on their criteria values.
Step 3: Determine whether the defined relation satisfies specific properties such as reflexivity, serial, inverse serial, symmetry, or transitivity. Identifying the given relation helps streamline certain complex calculations in the subsequent steps.
Step 4: Calculate the $\mathcal{A}_J$-neighborhoods, representing the fundamental granular computing framework.
Step 5: Calculate the $\widetilde{\mathbb{S}}_J$-neighborhoods, along with other types of neighborhoods if different rough set models are to be examined.
Step 6: For selected subsets of objects, calculate the $\widetilde{\mathbb{S}}_J$-lower and $\widetilde{\mathbb{S}}_J$-upper approximation operators, along with their counterparts derived from the rough set models under investigation.
Step 7: Find the $\widetilde{\mathbb{S}}_J$-boundary regions and the $\widetilde{\mathbb{S}}_J$-accuracy for these subsets.

5.2. Numerical example

We adopt an empirical example to apply the proposed rough neighborhoods in describing a practical scenario and demonstrating how extracted information can be classified. To this end, suppose there is a set $\mathbf{U}$ of published books written by a set of authors, denoted by $\mathbf{Y}$, and categorized into a set of topics, denoted by $\mathbf{Z}$. Let us consider these sets are:
$\mathbf{U} = \{b_1, b_2, b_3, b_4, b_5, b_6, b_7\}$,
$\mathbf{Y} = \{\text{Tareq} = x_1, \text{Yang} = x_2, \text{Muneerah} = x_3, \text{Lashin} = x_4, \text{Ibrahim} = x_5, \text{Faiz} = x_6, \text{Rizwan} = x_7, \text{Skowron} = x_8, \text{Liang} = x_9\}$, and
$\mathbf{Z} = \{y_1, y_2, y_3, y_4, y_5, y_6\}$, where these topics are: fuzzy sets, artificial intelligence, rough sets, multi-criteria decision-making, soft sets, and graph theory.
Consider the mappings $F_1 : \mathbf{U} \to 2^{\mathbf{Y}}$ and $F_2 : \mathbf{U} \to 2^{\mathbf{Z}}$, where $F_1(b)$ represents the authors who participated in writing the book b, and $F_2(b)$ represents the topic(s) of the book b. The representation of these mappings is displayed in Table 8.

Books	b_1	b_2	b_3	b_4	b_5	b_6	b_7
Authors	$\{x_3, x_5\}$	$\{x_1, x_7, x_8, x_9\}$	$\{x_3, x_4, x_5\}$	$\{x_1, x_9\}$	$\{x_2, x_3, x_7\}$	$\{x_2, x_7\}$	$\{x_3, x_5\}$
Topics	$\{y_1, y_4, y_6\}$	$\{y_5\}$	$\{y_1, y_2, y_4\}$	$\{y_2, y_4\}$	$\{y_3\}$	$\{y_3, y_5\}$	$\{y_4, y_6\}$

Table 8: Information system of the proposed example

Suppose the system administrators suggested the following binary relations:

$$(b_r, b_s) \in \Omega \Longleftrightarrow F_1(b_r) \subseteq F_1(b_s) \text{ or } |F_2(b_r) \cap F_2(b_s)| \geq 2$$

Accordingly, we have $\Omega = \{(b_1, b_1), (b_2, b_2), (b_3, b_3), (b_4, b_4), (b_5, b_5), (b_6, b_6), (b_7, b_7), (b_1, b_3), (b_3, b_1), (b_1, b_7), (b_7, b_1), (b_3, b_4), (b_4, b_3), (b_4, b_2), (b_6, b_5), (b_7, b_3)\}$. To conduct the comparisons demonstrating the superiority of the proposed rough set models, we first calculate the granule blocks (which include some existing rough neighborhoods and the proposed one) of the models being compared, as given in Table 9.

	b_1	b_2	b_3	b_4	b_5	b_6	b_7
$\mathcal{A}_r$	$\{b_1, b_3, b_7\}$	$\{b_2\}$	$\{b_1, b_3, b_4\}$	$\{b_2, b_3, b_4\}$	$\{b_5\}$	$\{b_5, b_6\}$	$\{b_1, b_3, b_7\}$
$\mathcal{A}_l$	$\{b_1, b_3, b_7\}$	$\{b_2, b_4\}$	$\{b_1, b_3, b_4, b_7\}$	$\{b_3, b_4\}$	$\{b_5, b_6\}$	$\{b_6\}$	$\{b_1, b_7\}$
$\mathcal{A}_{\langle r\rangle}$	$\{b_1, b_3\}$	$\{b_2\}$	$\{b_3\}$	$\{b_3, b_4\}$	$\{b_5\}$	$\{b_5, b_6\}$	$\{b_1, b_3, b_7\}$
$\mathcal{A}_{\langle l\rangle}$	$\{b_1, b_7\}$	$\{b_2, b_4\}$	$\{b_3\}$	$\{b_4\}$	$\{b_5, b_6\}$	$\{b_6\}$	$\{b_1, b_7\}$
$\widetilde{\mathbb{S}}_r$	$\{b_1, b_3\}$	$\{b_2\}$	$\{b_3\}$	$\emptyset$	$\{b_5\}$	$\{b_5\}$	$\{b_1, b_3\}$
$\widetilde{\mathbb{S}}_l$	$\{b_1, b_7\}$	$\{b_4\}$	$\emptyset$	$\{b_3, b_4\}$	$\{b_6\}$	$\{b_6\}$	$\{b_1, b_7\}$
$\widetilde{\mathbb{S}}_{\langle r\rangle}$	$\emptyset$	$\{b_2\}$	$\{b_3\}$	$\{b_3, b_4\}$	$\{b_5\}$	$\{b_5\}$	$\emptyset$
$\widetilde{\mathbb{S}}_{\langle l\rangle}$	$\{b_7\}$	$\emptyset$	$\{b_1, b_3, b_4, b_7\}$	$\{b_4\}$	$\{b_6\}$	$\{b_6\}$	$\{b_7\}$
$\widetilde{\mathbb{C}}_r$	$\{b_1, b_7\}$	$\emptyset$	$\{b_4\}$	$\{b_2, b_4\}$	$\emptyset$	$\{b_5, b_6\}$	$\{b_1, b_7\}$
$\widetilde{\mathbb{C}}_l$	$\{b_1, b_7\}$	$\{b_2\}$	$\{b_1, b_3, b_7\}$	$\emptyset$	$\{b_5, b_6\}$	$\emptyset$	$\emptyset$
$\widetilde{\mathbb{C}}_{\langle r\rangle}$	$\{b_3\}$	$\emptyset$	$\{b_3\}$	$\{b_3, b_4\}$	$\emptyset$	$\{b_5, b_6\}$	$\{b_1, b_3, b_7\}$
$\widetilde{\mathbb{C}}_{\langle l\rangle}$	$\emptyset$	$\{b_2\}$	$\{b_3\}$	$\emptyset$	$\{b_5, b_6\}$	$\emptyset$	$\emptyset$
$\widetilde{\mathbb{E}}_l$	$\{b_1, b_3, b_4, b_7\}$	$\{b_2, b_3, b_4\}$	$\mathbf{U} \setminus \{b_2, b_5, b_6\}$	$\{b_1, b_3, b_4, b_7\}$	$\{b_5, b_6\}$	$\{b_5, b_6\}$	$\{b_1, b_3, b_7\}$
$\widetilde{\mathbb{E}}_r$	$\{b_1, b_3, b_4, b_7\}$	$\{b_2\}$	$\mathbf{U} \setminus \{b_5, b_6\}$	$\{b_1, b_2, b_3, b_4\}$	$\{b_5\}$	$\{b_5, b_6\}$	$\{b_1, b_3, b_4, b_7\}$
$\widetilde{\mathbb{E}}_{\langle l\rangle}$	$\{b_1, b_7\}$	$\{b_2, b_4\}$	$\{b_1, b_3, b_4, b_7\}$	$\{b_4\}$	$\{b_5, b_6\}$	$\{b_6\}$	$\{b_1, b_7\}$

Table 9: $\mathcal{A}_\jmath$-, $\widetilde{\mathbb{S}}_\jmath$-, $\widetilde{\mathbb{C}}_\jmath$-, $\mathbb{E}_\jmath$-, and $\widetilde{\mathbb{E}}_\jmath$-neighborhoods of all members in $\mathbf{U}$

Now, we calculate the approximation operators and accuracy measures for some subsets via the proposed rough set models and some of the previous ones. To do this, we choose three subsets of $\mathbf{U}$ as following: $M = \{b_1, b_2, b_3, b_6\}$, $T = \{b_2, b_3, b_4, b_5\}$, and $Q = \{b_1, b_2, b_3\}$. The information that can be extracted from these subsets is presented as follows:

(i) By using the proposed rough set model in the case of $\jmath = r$ and its counterpart generated by $\mathcal{A}_r$-neighborhood:

$$\mathcal{H}_{\mathcal{N}_r}(M) = \{b_2\} \qquad \mathcal{H}^{\mathcal{N}_r}(M) = \mathbf{U} \setminus \{b_5\} \qquad \delta_{\mathcal{N}_r}(M) = \mathbf{U} \setminus \{b_2, b_5\}, \text{ and } \xi_{\mathcal{N}_r}(M) = 1/6$$

$$\underline{O}_{\widetilde{\mathbb{S}}_r}(M) = \{b_1, b_2, b_3\} \qquad \overline{O}_{\widetilde{\mathbb{S}}_r}(M) = \mathbf{U} \setminus \{b_4, b_5\} \qquad \delta_{\widetilde{\mathbb{S}}_r}(M) = \{b_6, b_7\}, \text{ and } \xi_{\widetilde{\mathbb{S}}_r}(M) = 3/5$$

(ii) By using the proposed rough set model in the case of $\jmath = l$ and its counterpart generated by $\mathbb{E}_l$-neighborhood:

$$\mathcal{H}_{\widetilde{\mathbb{E}}_l}(T) = \{b_2\} \qquad \mathcal{H}^{\widetilde{\mathbb{E}}_l}(T) = \mathbf{U} \qquad \delta_{\widetilde{\mathbb{E}}_l}(T) = \mathbf{U} \setminus \{b_2\}, \text{ and } \xi_{\widetilde{\mathbb{E}}_l}(T) = 1/7$$

$$\underline{O}_{\widetilde{\mathbb{S}}_l}(T) = \{b_2, b_3, b_5\} \qquad \overline{O}_{\widetilde{\mathbb{S}}_l}(T) = \{b_2, b_3, b_4, b_5\} \qquad \delta_{\widetilde{\mathbb{S}}_l}(T) = \{b_4\}, \text{ and } \xi_{\widetilde{\mathbb{S}}_l}(T) = 3/4$$

(iii) By using the proposed rough set model in the case of $\jmath = \langle l\rangle$ and its counterpart generated by $\widetilde{\mathbb{C}}_{\langle l\rangle}$-neighborhood:

$$\mathcal{H}_{\widetilde{\mathbb{C}}_{\langle l\rangle}}(M) = M \qquad \mathcal{H}^{\widetilde{\mathbb{C}}_{\langle l\rangle}}(M) = \mathbf{U} \setminus \{b_4, b_7\} \qquad \delta_{\widetilde{\mathbb{C}}_{\langle l\rangle}}(M) = \{b_5\}, \text{ and } \xi_{\widetilde{\mathbb{C}}_{\langle l\rangle}}(M) = 4/5$$

$$\underline{O}_{\widetilde{\mathbb{S}}_{\langle l\rangle}}(M) = \{b_2, b_6\} \qquad \overline{O}_{\widetilde{\mathbb{S}}_{\langle l\rangle}}(M) = \mathbf{U} \setminus \{b_4, b_7\} \qquad \delta_{\widetilde{\mathbb{S}}_{\langle l\rangle}}(M) = \{b_1, b_3, b_5\}, \text{ and } \xi_{\widetilde{\mathbb{S}}_{\langle l\rangle}}(M) = 2/5.$$

(iv) By using the proposed rough set model in the case of $\jmath = r$ and its counterpart generated by $\widetilde{\mathbb{C}}_r$-neighborhood:

$$\mathcal{H}_{\widetilde{\mathbb{C}}_r}(Q) = \{b_2\} \qquad \mathcal{H}^{\widetilde{\mathbb{C}}_r}(Q) = \{b_1, b_2, b_3, b_4, b_7\} \qquad \delta_{\widetilde{\mathbb{C}}_r}(Q) = \{b_1, b_3, b_4, b_7\}, \text{ and } \xi_{\widetilde{\mathbb{C}}_r}(Q) = 1/5$$

$$\underline{O}_{\widetilde{\mathbb{S}}_r}(Q) = Q \qquad \overline{O}_{\widetilde{\mathbb{S}}_r}(Q) = \{b_1, b_2, b_3, b_7\} \qquad \delta_{\widetilde{\mathbb{S}}_r}(Q) = \{b_7\}, \text{ and } \xi_{\widetilde{\mathbb{S}}_r}(Q) = 3/4$$

Based on the above computations, we conclude the following facts. First, the traditional rough set model fails to represent this information system since the proposed relation Ω is neither symmetric nor transitive. Second, it follows from (i) and (ii) that the proposed rough set models enhance the accuracy measures compared to the rough set models induced by $\mathcal{A}_\jmath$-neighborhoods and $\widetilde{\mathbb{E}}_\jmath$-neighborhoods. This improvement occurs since our models expand the lower approximation and reduce the upper approximation, directly leading to a smaller boundary region. These outcomes support the validity of the results obtained in (ii) of Proposition 3.16 and Proposition 3.18. Finally, one can observe from (iii) and (iv) the independence of our models from their counterparts induced by $\widetilde{\mathbb{C}}_\jmath$-neighborhoods, even when the given binary relation is reflexive.

6. Comparative analysis and discussion

We allocate this part to perform a comparative analysis incorporating both quantitative and qualitative factors to highlight the effectiveness and superiority of the proposed approach over existing ones. Additionally, we provide a detailed investigation of its advantages and limitations. The step-by-step comparison process is presented below.

6.1. Quantitative comparison

In this subsection, we will demonstrate some results that confirm the superiority of the proposed approach in this manuscript over some existing techniques under some conditions imposed on the binary relations. We will base our demonstration of these comparisons on the results we proved in Section 3 and our method of defining the current rough set models introduced in Section 4.

We illustrate under which types of binary relations the present rough set models outperform their counterparts established by $\mathcal{A}_\jmath$-neighborhoods, $\widetilde{\mathbb{C}}_\jmath$-neighborhoods, $\widetilde{\mathbb{E}}_\jmath$-neighborhoods.

Proposition 6.1. *In a G-approximation space $(\boldsymbol{U}, \Omega)$, the following relationships are warranted when Ω is a serial and symmetric relation, or an inverse serial and symmetric relation.*

(i) $\mathcal{H}_{\mathcal{A}_\jmath}(Y) \subseteq \underline{O}_{\widetilde{\mathbb{S}}_\jmath}(Y)$ *for each $Y \subseteq \boldsymbol{U}$ and each $\jmath$.*

(ii) $\overline{O}_{\widetilde{\mathbb{S}}_\jmath}(Y) \subseteq \mathcal{H}^{\mathcal{A}_\jmath}(Y)$ *for each $Y \subseteq \boldsymbol{U}$ and each $\jmath$.*

Proof. In all given cases Ω is symmetric, so it follows from Proposition 4.4 that $\overline{O}_{\widetilde{\mathbb{S}}_\jmath}(Y) = \{b \in \mathbf{U} : \widetilde{\mathbb{S}}_\jmath(b) \cap Y \neq \emptyset\}$ and $\underline{O}_{\widetilde{\mathbb{S}}_\jmath}(Y) = \{b \in \mathbf{U} : \widetilde{\mathbb{S}}_\jmath(b) \subseteq Y\}$. Now, it follows from (iii) of Proposition 3.16 that $\widetilde{\mathbb{S}}_\jmath(b) \subseteq \mathcal{A}_\jmath(b)$ for each $b \in \mathbf{U}$ and each $\jmath$ when Ω is a serial and symmetric relation or an inverse serial and symmetric relation. Accordingly, we derive that $\{b \in \mathbf{U} : \mathcal{A}_\jmath(b) \subseteq Y\} \subseteq \{b \in \mathbf{U} : \widetilde{\mathbb{S}}_\jmath(b) \subseteq Y\}$ and $\{b \in \mathbf{U} : \widetilde{\mathbb{S}}_\jmath(b) \cap Y \neq \emptyset\} \subseteq \{b \in \mathbf{U} : \mathcal{A}_\jmath(b) \cap Y \neq \emptyset\}$; hence, we demonstrate the desired relationships. □

Corollary 6.2. *In a G-approximation space $(\boldsymbol{U}, \Omega)$, if Ω is a serial and symmetric relation, or an inverse serial and symmetric relation, then $\xi_{\widetilde{\mathbb{S}}_\jmath}(Y) \geq \xi_{\mathcal{A}_\jmath}(Y)$ for each $Y \subseteq \boldsymbol{U}$ and each $\jmath$.*

Proposition 6.3. *In a G-approximation space $(\boldsymbol{U}, \Omega)$, if Ω is a reflexive relation, or a serial and transitive relation, or an inverse serial and transitive relation, or a quasi-order relation, then $\xi_{\widetilde{\mathbb{S}}_\jmath}(Y) \geq \xi_{\mathcal{A}_\jmath}(Y)$ for each $Y \subseteq \boldsymbol{U}$ and each $\jmath$.*

Proof. For each $\mathsf{b} \in \mathbf{U}$ we have the following cases:

(i) if Ω is a reflexive relation, then by (ii) of Proposition 3.16 we obtain $\widetilde{\mathbb{S}}_j(\mathsf{b}) \subseteq \mathcal{A}_j(\mathsf{b})$ for each j.

(ii) if Ω is a serial (or inverse serial) and transitive relation, then by (iv) and (v) of Proposition 3.16 we obtain $\widetilde{\mathbb{S}}_j(\mathsf{b}) \subseteq \mathcal{A}_j(\mathsf{b})$ for each j.

(iii) if Ω is a quasi-order relation, then by (ii) of Corollary 3.17 we obtain $\widetilde{\mathbb{S}}_j(\mathsf{b}) \subseteq \mathcal{A}_j(\mathsf{b})$ for each j.

According to the above cases, for any $Y \subseteq \mathbf{U}$ and each j we derive that $\mathcal{H}_{\mathcal{A}_j}(Y) \cap Y \subseteq \underline{O}_{\widetilde{\mathbb{S}}_j}(Y)$ and $\overline{O}_{\widetilde{\mathbb{S}}_j}(Y) \subseteq \mathcal{H}^{\mathcal{A}_j}(Y) \cup Y$. Consequentially, $\xi_{\widetilde{\mathbb{S}}_j}(Y) = \frac{|\underline{O}_{\widetilde{\mathbb{S}}_j}(Y)|}{|\overline{O}_{\widetilde{\mathbb{S}}_j}(Y)|} \geq \frac{|\mathcal{H}_{\mathcal{A}_j}(Y) \cap Y|}{|\mathcal{H}^{\mathcal{A}_j}(Y) \cup (Y)|} = \xi_{\mathcal{A}_j}(Y)$, as required. □

Proposition 6.4. *In a G-approximation space $(\boldsymbol{U}, \Omega)$, the following relationships are satisfied for each $Y \subseteq \boldsymbol{U}$.*

(i) *if Ω is a serial and symmetric relation, then $\mathcal{H}_{\widetilde{\mathbb{E}}_j}(Y) \subseteq \underline{O}_{\widetilde{\mathbb{S}}_j}(Y)$ and $\overline{O}_{\widetilde{\mathbb{S}}_j}(Y) \subseteq \mathcal{H}^{\widetilde{\mathbb{E}}_j}(Y)$ for each $j \in \{r, \langle l \rangle\}$.*

(ii) *if Ω is an inverse serial and symmetric relation, then $\mathcal{H}_{\widetilde{\mathbb{E}}_j}(Y) \subseteq \underline{O}_{\widetilde{\mathbb{S}}_j}(Y)$ and $\overline{O}_{\widetilde{\mathbb{S}}_j}(Y) \subseteq \mathcal{H}^{\widetilde{\mathbb{E}}_j}(Y)$ for each $j \in \{l, \langle r \rangle\}$.*

Proof. The proof is achieved by using the relationships proved in Proposition 3.18 and following a similar technique in the proof of Proposition 6.1. □

Corollary 6.5. *In a G-approximation space $(\boldsymbol{U}, \Omega)$, the following relationships are satisfied for each $Y \subseteq \boldsymbol{U}$.*

(i) *if Ω is a serial relation, then $\xi_{\widetilde{\mathbb{S}}_j}(Y) \geq \xi_{\widetilde{\mathbb{E}}_j}(Y)$ for each $j \in \{r, \langle l \rangle\}$.*

(ii) *if Ω is an inverse serial relation, then $\xi_{\widetilde{\mathbb{S}}_j}(Y) \geq \xi_{\widetilde{\mathbb{E}}_j}(Y)$ for each $j \in \{l, \langle r \rangle\}$.*

The calculations of upper and lower approximations, along with accuracy measures for sets in the practical example of Subsection 5.2 and the previous examples in Section 3, indicate that the converse of the aforementioned results does not always hold. This highlights that the proposed approach broadens the scope of confirmed knowledge while minimizing uncertainty, ultimately leading to higher accuracy values and more precise decision-making. In general, our approach enriches the area of generalized rough set models and offers more efficient measures compared to certain previous methods under specific types of binary relations.

6.2. *Qualitative comparison*

In this subsection, we present a comprehensive comparative analysis of our approach against several existing methods, focusing on their adherence to the properties of $\mathcal{P}$-approximation space. This comparison encompasses techniques based on $\mathcal{A}_j$-neighborhoods [54, 55], $\mathbb{E}_j$-neighborhoods [12], $\mathbb{C}_j$-neighborhoods [8], maximal neighborhoods [9, 24], and $\mathbb{S}_j$-neighborhoods [10].

In Table 10, we reveal how the upper and lower approximations behave with respect to the properties of $\mathcal{P}$-approximation space; we use the symbols ♦ and ◊ to indicate whether a property is satisfied or not, respectively. As shown in Table 10, the proposed rough set models retain the most properties of $\mathcal{P}$-approximation space. Furthermore, they outperform rough set models based on $\mathcal{A}_j$-neighborhoods, $\mathbb{E}_j$-neighborhoods, and maximal neighborhoods in terms of preserving these properties, while demonstrating similar behavior to rough set models derived from $\mathbb{C}_j$-neighborhoods and $\mathbb{S}_j$-neighborhoods.

Pawlak's properties	approach inspired by $\mathcal{A}_j$-neighborhood [54, 55]	approach inspired by $\mathbb{C}_j$-neighborhood [8]	approach inspired by $\mathbb{S}_j$-neighborhood [10]	approach inspired by maximal neighborhood [9, 24]	approach inspired by $\mathbb{E}_j$-neighborhood [12]	Our approach
($\mathcal{LP}$1)	◊	♦	♦	◊	◊	♦
($\mathcal{LP}$2)	◊	♦	♦	◊	◊	♦
($\mathcal{LP}$3)	♦	♦	♦	♦	♦	♦
($\mathcal{LP}$4)	♦	♦	♦	♦	♦	♦
($\mathcal{LP}$5)	♦	♦	♦	♦	♦	♦
($\mathcal{LP}$6)	♦	♦	♦	♦	♦	♦
($\mathcal{LP}$7)	♦	♦	♦	♦	♦	♦
($\mathcal{LP}$8)	◊	◊	◊	◊	◊	◊
($\mathcal{LP}$9)	◊	◊	◊	◊	◊	◊
($\mathcal{UP}$1)	◊	♦	♦	◊	◊	♦
($\mathcal{UP}$2)	♦	♦	♦	♦	♦	♦
($\mathcal{UP}$3)	◊	♦	♦	◊	◊	♦
($\mathcal{UP}$4)	♦	♦	♦	♦	♦	♦
($\mathcal{UP}$5)	♦	♦	♦	♦	♦	♦
($\mathcal{UP}$6)	♦	♦	♦	♦	♦	♦
($\mathcal{UP}$7)	♦	♦	♦	♦	♦	♦
($\mathcal{UP}$8)	◊	◊	◊	◊	◊	◊
($\mathcal{UP}$9)	◊	◊	◊	◊	◊	◊

Table 10: Comparison of various methods regarding the properties of $\mathcal{P}$-approximation space

Table 10 displays a comparative analysis between our proposed method and several existing approaches in the literature, focusing on their ability to preserve the properties of lower and upper approximations inspired by $\mathcal{P}$-approximation space, as outlined in Proposition 2.4. The results we proved in this work indicate that the proposed approach successfully maintains the first seven properties of lower and upper approximations within the $\mathcal{P}$-approximation space.

Moreover, it demonstrates superior performance compared to methods based on maximal neighborhoods [9, 24], $\mathcal{A}_j$-neighborhoods [54, 55], and $\mathbb{E}_j$-neighborhoods [12] in retaining the first three properties of lower and upper approximations of $\mathcal{P}$-approximation space. These properties are expressed by the following relation:

$$\underline{O}_{\widetilde{\mathbb{S}}_j}(Y) \subseteq Y \subseteq \overline{O}_{\widetilde{\mathbb{S}}_j}(Y) \text{ for each } j.$$

To show that these properties are missing in the framework generated by $\mathcal{A}_j$-neighborhoods, we consider a G-approximation space presented in Example 3.2. If $Y = \{\mathsf{b}, \mathsf{k}\}$, then $\mathcal{H}_{\mathcal{A}_r}(Y) = \{\mathsf{f}, \mathsf{k}\}$ and $\mathcal{H}^{\mathcal{A}_r}(Y) = \{\mathsf{b}, \mathsf{f}, \mathsf{h}\}$. Hence, $\mathcal{H}_{\mathcal{A}_r}(Y)$ is not a subset of Y and Y is not a subset of $\mathcal{H}^{\mathcal{A}_r}(Y)$.

Thus, based on the aforesaid discussion, the results obtained using our method are highly reliable.

In addition, the properties $\mathcal{LP}8$ and $\mathcal{UP}8$ of $\mathcal{P}$-approximation space are also considered as criteria for comparing our models with the introduced rough set models. On the one hand, we demonstrated in Section 4 that $\underline{O}_{\widetilde{\mathbb{S}}_j}(\underline{O}_{\widetilde{\mathbb{S}}_j}(Y))$ is a proper subset of $\underline{O}_{\widetilde{\mathbb{S}}_j}(Y)$ and $\overline{O}_{\widetilde{\mathbb{S}}_j}(Y)$ is a proper subset of $\overline{O}_{\widetilde{\mathbb{S}}_j}(\overline{O}_{\widetilde{\mathbb{S}}_j}(Y))$, which indicates that the proposed rough set models meet one direction of the $\mathcal{LP}8$ and $\mathcal{UP}8$ properties of $\mathcal{P}$-approximation space.

On the other hand, the behavior of rough set models derived from $\mathcal{A}_j$-neighborhoods, maximal neighborhoods, $\mathbb{E}_j$-neighborhoods, and $\mathbb{C}_j$-neighborhoods with respect to these properties is as follows:

(i) These properties are completely missing within the frameworks derived from $\mathcal{A}_j$-neighborhoods, maximal neighborhoods $\mathbb{E}_j$-neighborhoods. To show that they are incomparable in the case of $\mathcal{A}_j$-neighborhoods, we consider a G-approximation space presented in Example 3.2. If $Y = \{\mathsf{b}, \mathsf{h}\}$ and $Z = \{\mathsf{h}, \mathsf{k}\}$, then

$$\mathcal{H}_{\mathcal{A}_r}(Y) = \{\mathsf{f}, \mathsf{k}\} \text{ and } \mathcal{H}_{\mathcal{A}_r}(\mathcal{H}_{\mathcal{A}_r}(Y)) = \{\mathsf{h}, \mathsf{k}\} \text{ and}$$

$$\mathcal{H}^{\mathcal{A}_r}(Z) = \{\mathsf{b}, \mathsf{h}\} \text{ and } \mathcal{H}^{\mathcal{A}_r}(\mathcal{H}^{\mathcal{A}_r}(Z)) = \{\mathsf{b}, \mathsf{f}\} \text{ and .}$$

Accordingly, $\mathcal{H}_{\mathcal{A}_r}(Y)$ is not a subset of $\mathcal{H}_{\mathcal{A}_r}(\mathcal{H}_{\mathcal{A}_r}(Y))$ and $\mathcal{H}_{\mathcal{A}_l}(\mathcal{H}_{\mathcal{A}_l}(Y))$ is not a subset of $\mathcal{H}_{\mathcal{A}_r}(Y)$, as well as $\mathcal{H}^{\mathcal{A}_r}(Z)$ is not a subset of $\mathcal{H}^{\mathcal{A}_r}(\mathcal{H}^{\mathcal{A}_r}(Z))$ and $\mathcal{H}^{\mathcal{A}_l}(\mathcal{H}^{\mathcal{A}_l}(Z))$ is not a subset of $\mathcal{H}^{\mathcal{A}_r}(Z)$.

(ii) These properties are satisfied within the frameworks derived from $\mathbb{S}_\jmath$-neighborhoods and $\mathbb{C}_\jmath$-neighborhoods satisfy these properties when $\jmath \in \{r, l, i, \langle r\rangle, \langle l\rangle, \langle i\rangle\}$. That is, for each $\jmath \in \{r, l, i, \langle r\rangle, \langle l\rangle, \langle i\rangle\}$ we have

$$\mathcal{H}_{\mathbb{C}_\jmath}(\mathcal{H}_{\mathbb{C}_\jmath}(Y)) = \mathcal{H}_{\mathbb{C}_\jmath}(Y) \text{ and } \mathcal{H}^{\mathbb{C}_\jmath}(\mathcal{H}^{\mathbb{C}_\jmath}(Y)) = \mathcal{H}^{\mathbb{C}_\jmath}(Y)$$

$$\mathcal{H}_{\mathbb{S}_\jmath}(\mathcal{H}_{\mathbb{S}_\jmath}(Y)) = \mathcal{H}_{\mathbb{S}_\jmath}(Y) \text{ and } \mathcal{H}^{\mathbb{S}_\jmath}(\mathcal{H}^{\mathbb{S}_\jmath}(Y)) = \mathcal{H}^{\mathbb{S}_\jmath}(Y),$$

which represents an advantage of these rough set models in the determined cases of $\jmath$.

6.3. Advantages

The key benefits of the proposed approach are highlighted as follows:

(i) The neighborhood systems and rough set paradigms introduced in this work provide an innovative framework for modeling empirical scenarios that necessitate two types of $\mathcal{A}_\jmath$-neighborhoods related by a subset relation. This framework holds potential for applications by integrating with graph theory to specify the major symptoms in diseases, analyze electrical models, and classify individuals on social media.

(ii) In contrast to many existing models in the literature, which often rely on predefined relations such as equivalence [42, 43] and similarity [1, 24, 50] to define rough set frameworks, the approach presented here does not impose such constraints. Instead, the proposed rough set paradigms are introduced without requiring any specific conditions on the binary relations used to construct $\widetilde{\mathbb{S}}_\jmath$-neighborhoods and their approximation operators. This flexibility allows for their application across a broad spectrum of real-world scenarios.

(iii) As illustrated, the first seven properties of lower and upper approximations within the $\mathcal{P}$-approximation space hold in the proposed rough set models, whereas the eighth and ninth properties are only partially satisfied. Thus, most of the lower and upper approximation properties within the $\mathcal{P}$-approximation space, which are partially or entirely lost in certain earlier rough set models, such as those discussed in [9, 12, 24, 54, 55], are preserved in the proposed models, as detailed in Subsection 6.2.

(iv) In comparison to previous models, such as those in [9, 24, 25, 54, 55], our approach reduces uncertainty and enhances the accuracy of validated knowledge derived from the analyzed information systems. This advantage is achieved under specific types of binary relations. Through this content, we demonstrate that the proposed rough set models surpass their counterparts established by:

1) $\mathcal{A}_\jmath$-neighborhoods for each case of $\jmath$ under any one of the following binary relations:
- symmetric and transitive relations,
- serial and symmetric relations,
- inverse serial and symmetric relations, and
- quasi-order relations.

2) $\mathcal{A}_\jmath$-neighborhoods for each case of $\jmath \in \{r, l, i, u\}$ under a reflexive relation

3) $\mathcal{A}_r$-neighborhoods (resp., $\mathcal{A}_l$-neighborhoods) under a serial and transitive (resp., inverse serial and transitive) relation.

4) $\widetilde{\mathbb{C}}_\jmath$-neighborhoods for the elements that are irreflexive; that is, elements with $(\mathsf{b}, \mathsf{b}) \notin \Omega$.

5) $\widetilde{\mathbb{E}}_r$-neighborhoods and $\widetilde{\mathbb{E}}_{\langle l\rangle}$ under a serial relation, as well as $\widetilde{\mathbb{E}}_l$-neighborhoods and $\widetilde{\mathbb{E}}_{\langle r\rangle}$ under an inverse serial relation.

6) $\widetilde{\mathbb{E}}_\jmath$-neighborhoods for all cases of $\jmath$ under a serial and inverse serial relation or a reflexive relation.

(v) The proposed model upholds the monotonicity property when the smaller relation is serial and transitive, while the larger relation is symmetric and transitive. This implies that as the relation expands, the accuracy measure decreases. This characteristic, established under these specific types of relations, allows for the comparison of accuracy values between two rough set models, established in this study, based on the accuracy of one.

6.4. Limitations

(i) Some $\widetilde{\mathbb{S}}_J$-neighborhoods are empty for certain elements when the binary relation is not serial or inverse serial. This is an undesirable characteristic for approximating subsets using classical formulas, as it results in lower approximations of some sets not being contained within those sets, and some sets not being contained within their upper approximations.

(ii) Rough set models induced by $\mathbb{C}_J$-neighborhoods and $\mathbb{S}_J$-neighborhoods fully adhere to the $\mathcal{LP}8$ and $\mathcal{UP}8$ properties of $\mathcal{P}$-approximation space for each $J \in \{r, l, i, \langle r\rangle, \langle l\rangle, \langle i\rangle\}$, which makes them superior to the proposed models, as the latter only partially meet these properties for each J.

(iii) The execution time of Algorithm 1 is longer compared to its counterpart algorithm for classification based on subset neighborhoods. This difference arises because the counterpart algorithm restricts numerical computations to a single type of $\mathcal{A}_J$-neighborhoods instead of two. Thus, when classifying subsets using subset neighborhoods, either step 3 or step 4 of Algorithm 1 is omitted.

(iv) Rough set models described by overlapping equality rough neighborhoods [7] produce higher accuracy measures than their counterparts proposed herein.

7. Concluding remarks and future research plans

One of the most powerful techniques for extracting information from incomplete systems is the rough set theory, which has been extended in various ways, including through neighborhood systems defined over non-equivalence relations. Many existing contributions inspired by this approach enhance rough set models and broaden their applicability in practical situations.

In this work, we introduce a novel neighborhood system called overlapping subset rough neighborhoods, denoted by $\widetilde{\mathbb{S}}$-neighborhoods, and utilize it to construct G-approximation spaces. This approach fundamentally differs from all previous methods, as its definition relies on two different kinds of $\mathcal{A}_J$-neighborhoods under a subset relation. In contrast, earlier approaches relied on a single kind of $\mathcal{A}_J$-neighborhoods related to each other by some relations, such as equality [19, 37], inclusion [8], subset [10], and intersection [12]. As a result, the proposed approach fills a gap in addressing classification problems across various disciplines that require the integration of two different kinds of $\mathcal{A}_J$-neighborhoods.

Throughout this study, we have explored the fundamental properties of $\widetilde{\mathbb{S}}$-neighborhoods and examined their relationships with both each other and preceding models, using elucidative examples. In this context, we have investigated the conditions under which certain identities hold and demonstrated unique properties, such as:

- all $\widetilde{\mathbb{S}}_J$ are equivalent under a symmetric and transitive relation.
- $\widetilde{\mathbb{S}}_J(\mathsf{b}) \subseteq \mathbb{C}_J(\mathsf{b})$ for each $(\mathsf{b}, \mathsf{b}) \notin \Omega$.
- $\widetilde{\mathbb{S}}_J(\mathsf{b}) \subseteq \mathcal{A}_J(\mathsf{b})$ for each $J \in \{r, l, i, u\}$ under a reflexive relation, and this relation holds true for each J under serial (or inverse serial) and symmetric relations.
- $\widetilde{\mathbb{S}}_J(\mathsf{b}) \subseteq \widetilde{\mathbb{E}}_J(\mathsf{b})$ for each $J \in \{r, \langle l\rangle\}$ under a serial relation.
- $\widetilde{\mathbb{S}}_J(\mathsf{b}) \subseteq \widetilde{\mathbb{E}}_J(\mathsf{b})$ for each $J \in \{l, \langle r\rangle\}$ under an inverse serial relation.

Furthermore, we derived criteria for identifying specific types of binary relations, as follows:

$$\text{For each } J \in \{r, l, i\} \text{ and each } \mathsf{b} \in \mathbf{U} \text{ we have } : \mathsf{b} \in \widetilde{\mathbb{S}}_J(\mathsf{b}) \Longleftrightarrow \Omega \text{ is a symmetric relation.}$$

$$\text{For each } J \text{ and each } \mathsf{b} \in \mathbf{U} \text{ we have } : \mathcal{A}_J(\mathsf{b}) \subseteq \widetilde{\mathbb{S}}_J(\mathsf{b}) \Longleftrightarrow \Omega \text{ is a symmetric and transitive relation.}$$

We then defined upper and lower approximations by using $\widetilde{\mathbb{S}}$-neighborhoods. To eliminate illogical or undefined cases caused by previous formulas for calculating accuracy measures, we modified the classical methods by redefining these approximations through their intersection and union with the sets under consideration and proved that the proposed formulas are equivalent with the classical ones of calculating when the relation is symmetric. We have elucidated that the proposed rough set models are characterized by satisfying the monotonicity property under certain kinds of binary relations and retaining all the properties of Pawlak approximation spaces, except for the properties $\mathcal{LP}8$ and $\mathcal{UP}8$, which are only partially preserved. Furthermore, we constructed a practical example to highlight the superiority of the proposed approach over some existing rough set models like [9, 12, 24, 25, 54, 55], and presented its main advantages through a comparative analysis. Specifically, we showed that our approach operates without the strict requirement of an equivalence relation and surpasses several existing rough set models in computing accuracy measures under specific types of binary relations.

Our future research aims to explore the following key areas:

- Integrating $\widetilde{\mathbb{S}}_J$-neighborhoods with ideal, filter, and primal structures to develop a fresh framework with the following upper and lower approximations

$$\overline{O}_{\widetilde{\mathbb{S}}_J}(Y) = Y \cup \{\mathsf{b} \in \mathbf{U} : \widetilde{\mathbb{S}}_J(\mathsf{b}) \cap Y \notin \mathbb{I}\}, \text{ and}$$

$$\underline{O}_{\widetilde{\mathbb{S}}_J}(Y) = Y \cap \{\mathsf{b} \in \mathbf{U} : \widetilde{\mathbb{S}}_J(\mathsf{b}) \setminus Y \in \mathbb{I}\},$$

  respectively, where $\mathbb{I}$ refers to ideal, or filter, or primal structures defined over the universe $\mathbf{U}$.

  This integration will provides additional desirable properties, particularly in minimizing the upper approximation and maximizing the lower approximation.

- Establishing a topological framework for the proposed rough set paradigms by structuring them according to one of the following formulation:

$$\text{The collection } \{Y \subseteq \mathbf{U} : \widetilde{\mathbb{S}}_J(\mathsf{b}) \subseteq Y, \forall \mathsf{b} \in Y\} \text{ forms a topology on } \mathbf{U}.$$

$$\text{The collection } \{\widetilde{\mathbb{S}}_J(\mathsf{b}) : \forall \mathsf{b} \in Y\} \text{ forms a subbase for topology on } \mathbf{U}.$$

  These topological simulations will be particularly beneficial for researchers specializing in topological spaces and their generalizations, as well as those in algebraic structures, as they prefer utilizing topological and algebraic techniques due to the ease of deriving approximation operators from closure and interior operators.

- Extending the proposed approach to different mathematical frameworks, including soft set theory [38, 49] and fuzzy set theory [28, 53], to develop its applicability to transect with complicated situations.

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