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Binary hypersoft sets: Theory and topology

Zeravan A. Omar[a], Baravan A. Asaad[a,b,*]

[a] *Department of Mathematics, College of Science, University of Zakho, Zakho, Iraq*
[b] *Department of Computer Science, College of Science, Cihan University-Duhok, Iraq*

Abstract. This paper introduces a novel extension of hypersoft sets, termed binary hypersoft sets (BnHySSs), representing an advanced generalization of binary soft sets over two universal sets and a parameter set. It presents fundamental operations of BnHySSs such as subset, superset, equality, complement, null and absolute sets, extended union/intersection, union, intersection, difference, AND, and OR operations. We also examine foundational properties and provide a comparative analysis of BnHySSs, HySSs, and BnSSs. Building on this foundation, we introduce the notion of binary hypersoft topology and its corresponding subspace concept. Further, we explore topological notions within this framework, including limit points, neighborhoods, closure, interior, and boundary in the context of BnHySSs.

1. Introduction

In the realm of mathematical modeling and decision-making, dealing with uncertainty, vagueness, and incomplete information is a common challenge. Traditional mathematical tools like classical set theory, fuzzy set theory, and rough set theory have been employed to address these issues. However, in 1999, Molodtsov [22] introduced a new approach called Soft Set Theory, which provides a more general and flexible framework for handling uncertainty. Soft set theory is based on the concept of parameterization. Unlike classical sets that focus on object membership, soft sets associate parameters with subsets of a universe, allowing for a more nuanced representation of data. This parameterized structure makes soft sets particularly useful in fields such as decision-making, data analysis, engineering, medical diagnosis, and social sciences.

In [23], Molodtsov et al. effectively used soft sets in fields including probability, theory of measurement, Riemann integration, Perron integration, operations research, theories of games, and smoothness of functions. In 2005, Pie and Miao [30] enhanced the outcomes of Maji et al. [20]. Lately, in 2011 Shabir and Naz [38] started delving into the realm of soft topological spaces, alongside other academics, such as Aygunoglu, [11], Ahmad [4], Maji [21], Hussain [17] continued work on soft topology. Some theoretical studies on the theory of soft sets can be found in [5–8, 15, 19] in more details.

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* Corresponding author: Baravan A. Asaad

Email addresses: zeravan5101987@gmail.com (Zeravan A. Omar), baravan.asaad@uoz.edu.krd (Baravan A. Asaad)

ORCID iDs: https://orcid.org/0009-0002-7616-8275 (Zeravan A. Omar), https://orcid.org/0000-0002-2049-2044 (Baravan A. Asaad)

Building upon Molodtsov's foundational work on soft sets, researchers have proposed various extensions to enhance the theory's applicability and structural richness. One such extension is the concept of binary soft sets, introduced to address problems involving binary relations under uncertainty. In 2016, Acikğoz and TaŞ [3], presented a novel framework that integrates binary relations into the structure of soft sets called binary soft set (BnSS) theory on two initial universal sets and examined a few features, allowing for a more expressive representation of relational data. This approach not only generalizes classical soft set theory but also opens new avenues for research in relational decision-making and other applied disciplines. In 2017, Benchalli, Patil, Dodamani, and Pradeepkumar [12] introduced binary soft topological spaces and binary soft operators. Later on, Hussain [16] investigated further properties of binary soft topological spaces. Many researchers have worked on binary soft sets to explore their structure, properties, and applications such as [13, 14, 29, 37].

In the evolving landscape of mathematical tools used to handle uncertainty, ambiguity, and vagueness, Hypersoft Set Theory has emerged as a novel and promising approach. Introduced as an extension of soft set theory, hypersoft sets provide a more refined structure to model complex decision-making problems where multi-parameter and multi-subparameter relationships are involved. Unlike classical soft sets, which associate a single set of parameters with approximate values, hypersoft sets allow for a multi-layered parameterization, enabling a more detailed and flexible representation of data. This enhanced granularity makes hypersoft set theory particularly suitable for real-world applications in fields such as medical diagnosis, engineering, data analysis, and artificial intelligence, where nuanced and hierarchical data structures often arise. By accommodating sub-parameter values within the decision-making process, hypersoft sets overcome limitations of existing frameworks like fuzzy sets, rough sets, and intuitionistic fuzzy sets, offering a more adaptable and comprehensive mathematical foundation for dealing with indeterminate and imprecise information.

Smarandache [39] explored the fundamental concepts of hypersoft set (HySS) theory, reviews its algebraic structure, and discusses its advantages over traditional soft computing methods. Furthermore, potential applications and future research directions are outlined, emphasizing the theory's significance in contemporary data science and intelligent systems. This approach is better suited for decision-making problems and is more flexible than soft sets. Smarandache also introduced fuzzy HySSs, intuitionistic fuzzy HySSs, neutrosophic HySSs, and plithogenic HySSs as extensions of the HySSs. Based on the HySSs and their extension, many researchers have developed various operators, properties, and applications [1, 9, 25, 26, 34, 35]. In 2022, Musa and Asaad [24] presented the concept of bipolar HySS which is a novel extension of HySS. They explored bipolar HyS topological space [27]. Recentrly, Musa, Mohammed and Asaad [28] introduced N-hypersoft sets which is an enriched and versatile extension of HyS sets. Many researchers have worked on the development and application of hypersoft set theory to enhance its mathematical foundation and practical utility such as [2, 10, 18, 31–33, 36]

The structure of the paper is as follows: Section 2 provides a brief overview of HySS, HyST, BnSS and BnST and some relevant properties. In Section 3, we define a novel extension of hypersoft sets called the binary hypersoft sets (BnHySSs) which is an advanced generalization of binary soft sets over two universal sets and a parameter set. Later, we present some operations on binary hypersoft sets such as BnHyS subset, BnHyS superset, BnHyS equality, BnHyS complement, BnHyS null, BnHyS absolute, BnHyS extended union, BnHyS extended intersection, BnHyS union, intersection, BnHyS difference, BnHyS AND and BnHyS OR. Furthermore, we explore some of their basic properties. Also, we compare among BnHySSs, HySSs and BnSSs. Additionally, in Section 4, we introduce the concept of binary hypersoft topology and we present the concept of binary hypersoft subspace. In Section 5, we investigate binary hypersoft limit points, binary hypersoft neighborhood, binary hypersoft closure, binary hypersoft interior and binary hypersoft boundary. Finally, Section 6 concludes with a summary of findings and potential avenues for future research.

2. Preliminaries

This section explores the foundational principles and operations associated with soft sets and their extension such as hypersoft sets and binary soft sets.

2.1. Hypersoft sets and hypersoft topology

This section introduces the core concepts and results related to hypersoft sets and hypersoft topology. Hypersoft sets, an advanced generalization of soft sets, introduce enhanced parameterization for handling complex data. Key relationships such as hypersoft subset and hypersoft equal are defined to facilitate comparisons between hypersoft sets. Operations like hypersoft union, hypersoft], hypersoft complement, and hypersoft difference are presented to demonstrate their utility in managing hypersoft structures. Additionally, logical operators, including hypersoft AND and hypersoft OR, are analyzed for their applications in decision-making and problem-solving processes.

Let $\coprod$ be an initial universe set and the non empty set ₤ be an entire set of parameters. The power set of $\coprod$ can be represented as $\beta(\coprod)$, and let $\emptyset \neq \pi_i, \vartheta_i \subseteq$ ₤ with $i = 1, 2, ..., n$.

Definition 2.1. [22] Let $\pi \subseteq$ ₤. A pair (η, π) is referred to as a soft set over $\coprod$, where the mapping η is provided by $\eta : \pi \longrightarrow \beta(\coprod)$. Stated differently, a parameterized family of subsets of the universe $\coprod$ is referred to as a soft set over $\coprod$. One way to think of $\eta(\varrho)$ is as the set of e-approximate members of the soft set for a given $\varrho \in \pi$.

Definition 2.2. [39] It is possible to identify a hypersoft set (HySS) by the pair $(\eta, \pi_1 \times \pi_2 \times \ldots \times \pi_n)$, where:

$$\eta : \pi_1 \times \pi_2 \times \ldots \times \pi_n \longrightarrow \beta(\coprod).$$

In order to keep things simple, we write π for $\pi_1 \times \pi_2 \times \ldots \times \pi_n$ and ϱ for an element of the set π. We also suppose that none of the set π_i is empty for each i.

Definition 2.3. [34] It is said (η, π) is a hypersoft (HyS) subset of (μ, ϑ) if $\pi \subseteq \vartheta$, where $\vartheta = \vartheta_1 \times \vartheta_2 \times \ldots \times \vartheta_n$, and $\eta(\varrho) \subseteq \mu(\varrho)$ for each $\varrho \in \pi$. We write $(\eta, \pi) \,\widetilde{\subseteq}\, (\mu, \vartheta)$.
An HySS (η, π) is claimed to be an HyS superset of (μ, ϑ), if (μ, ϑ) is an HyS subset of (η, π). We write (η, π) $\widetilde{\supseteq}\, (\mu, \vartheta)$.

Definition 2.4. [34] Two HySSs (η, π) and (μ, ϑ) are claimed to be an HyS equal if (η, π) is an HyS subset of (μ, ϑ) and (μ, ϑ) is an HyS subset of (η, π).

Definition 2.5. [35] Let (η, π) and (μ, ϑ) be two HySSs over $\coprod$. Then the HyS extended union of (η, π) and (μ, ϑ) is symbolized by $(\mho, Ḑ) = (\eta, \pi) \widetilde{\cup}_{\mathcal{E}} (\mu, \vartheta)$ with $Ḑ = Ḑ_1 \times Ḑ_2 \times ... \times Ḑ_n$ where $Ḑ_i = \pi_i \cup \vartheta_i$ with $i = 1, 2, ..., n$, and $\mho$ can be characterized by

$$\mho(\varrho) = \begin{cases} \eta(\varrho) & if \varrho \in \pi - \vartheta \\ \mu(\varrho) & if \varrho \in \vartheta - \pi \\ \eta(\varrho) \cup \mu(\varrho) & if \varrho \in \pi \cap \vartheta \neq \emptyset \end{cases}$$

where $\varrho = (Ḑ_1, Ḑ_2, ..., Ḑ_n) \in Ḑ$.

Remark 2.6. [35] It should be noted that when two HySSs are united, the set of parameters is a Cartesian product of the sets of parameters; when two soft sets are united, the set of parameters is simply the union of the sets of parameters.

Definition 2.7. [35] Let (η, π) and (μ, ϑ) be two HySSs over $\coprod$. Then the HyS extended intersection of (η, π) and (μ, ϑ) is symbolized by $(\mho, Ḑ) = (\eta, \pi) \widetilde{\cap}_{\mathcal{E}} (\mu, \vartheta)$ with $Ḑ = Ḑ_1 \times Ḑ_2 \times ... \times Ḑ_n$ where $Ḑ_i = \pi_i \cup \vartheta_i$ with $i = 1, 2, ..., n$, and $\mho$ can be characterized by

$$\mho(\varrho) = \begin{cases} \eta(\varrho) & if \varrho \in \pi - \vartheta \\ \mu(\varrho) & if \varrho \in \vartheta - \pi \\ \eta(\varrho) \cap \mu(\varrho) & if \varrho \in \pi \cap \vartheta \neq \emptyset \end{cases}$$

where $\varrho = (Ḑ_1, Ḑ_2, ..., Ḑ_n) \in Ḑ$.

Definition 2.8. [35] Let (η, π) and (μ, ϑ) be two HySSs over $\coprod$. Then the HyS union of (η, π) and (μ, ϑ) is symbolized by $(\mho, Ḑ)=(\eta, \pi)\widetilde{\cup}(\mu, \vartheta)$ with $Ḑ = Ḑ_1 \times Ḑ_2 \times ... \times Ḑ_n$ where $Ḑ_i = \pi_i \cap \vartheta_i \neq \emptyset$ with $i = 1, 2, ..., n$, and $\mho$ can be characterized by

$$\mho(\varrho) = \eta(\pi) \cup \mu(\vartheta)$$

where $\varrho = (Ḑ_1, Ḑ_2, ..., Ḑ_n) \in Ḑ$. If, for some i, $Ḑ_i$ is an empty set, then $(\eta, \pi)\widetilde{\cup}(\mu, \vartheta)$ is defined to be a null HySS.

Definition 2.9. [35] Let (η, π) and (μ, ϑ) be two HySSs over $\coprod$. Then the HyS intersection of (η, π) and (μ, ϑ) is symbolized by $(\mho, Ḑ)=(\eta, \pi)\widetilde{\cap}(\mu, \vartheta)$ with $Ḑ = Ḑ_1 \times Ḑ_2 \times ... \times Ḑ_n$ where $Ḑ_i = \pi_i \cap \vartheta_i \neq \emptyset$ with $i = 1, 2, ..., n$, and $\mho$ can be characterized by

$$\mho(\varrho) = \eta(\pi) \cap \mu(\vartheta)$$

where $\varrho = (Ḑ_1, Ḑ_2, ..., Ḑ_n) \in Ḑ$. If, for some i, $Ḑ_i$ is an empty set, then $(\eta, \pi)\widetilde{\cap}(\mu, \vartheta)$ is defined to be a null HySS.

Definition 2.10. [35] Let (η, π) and (μ, ϑ) be two HySSs over $\coprod$. Then the HyS difference of (η, π) and (μ, ϑ) is symbolized by $(\mho, Ḑ)=(\eta, \pi)\widetilde{\setminus}(\mu, \vartheta)$ with $Ḑ = Ḑ_1 \times Ḑ_2 \times ... \times Ḑ_n$ where $Ḑ_i = \pi_i \cap \vartheta_i \neq \emptyset$ with $i = 1, 2, ..., n$, and $\mho$ can be characterized by

$$\mho(\varrho) = \eta(\pi) \setminus \mu(\vartheta)$$

where $\varrho = (Ḑ_1, Ḑ_2, ..., Ḑ_n) \in Ḑ$.

Definition 2.11. [34] The HyS complement of an HySS (η, π), symbolized by $(\eta, \pi)^c$, can be characterized by (η^c, π) where $\eta^c : \pi \longrightarrow \beta(\coprod)$ is a mapping given by $\eta^c(\varrho) = \coprod \setminus \eta(\varrho)$ for each $\varrho \in \pi$.

Definition 2.12. [35] It is said an HySS (η, π) a null HySS, symbolized by $(\widetilde{\emptyset}, \pi)$, if $\eta(\varrho) = \emptyset$ for each $\varrho \in \pi$.

Definition 2.13. [35] It is said an HySS (η, π) an absolute HySS, symbolized by $(\widetilde{\coprod}, \pi)$, if $\eta(\varrho) = \coprod$ for each $\varrho \in \pi$.

Definition 2.14. [26] Let (η, π) be an HySS over $\coprod$ and $u \in \coprod$. Then $u \in (\eta, \pi)$ if $u \in \eta(\varrho)$ for each $\varrho \in \pi$. Keep in mind that for every $u \in \coprod$, $u \notin (\eta, \pi)$, if $u \notin \eta(\varrho)$ for some $\varrho \in \pi$.

Definition 2.15. [26] Let $\tau_{\mathcal{H}}$ be the collection of HySSs over $\coprod$, then $\tau_{\mathcal{H}}$ is claimed to be a hypersoft topology (HyST) on $\coprod$ if:

1. $(\widetilde{\emptyset}, \pi)$,$(\widetilde{\coprod}, \pi)$ belongs to $\tau_{\mathcal{H}}$.
2. The HyS intersection of any two HySSs in $\tau_{\mathcal{H}}$ belongs to $\tau_{\mathcal{H}}$.
3. The HyS union of any number of HySSs in $\tau_{\mathcal{H}}$ belongs to $\tau_{\mathcal{H}}$.

Then $(\widetilde{\coprod}, \tau_{\mathcal{H}}, €̧)$ is known as hypersoft topological space (HySTS). The members of $\tau_{\mathcal{H}}$ are claimed to be HyS open sets in $\coprod$. An HySS (η, π) over $\coprod$ is claimed to be an HyS closed set in $\coprod$, if its HyS complement $(\eta, \pi)^c$ belongs to $\tau_{\mathcal{H}}$.

Definition 2.16. [26] Let $(\widetilde{\coprod}, \tau_{\mathcal{H}}, €̧)$ be an HySTS over $\coprod$, and $\mathcal{V}$ be a non empty subset of $\coprod$. Then

$$\tau_{\mathcal{H}\mathcal{V}} = \{(\eta_{\mathcal{V}}, \pi) \mid (\eta, \pi) \in \tau_{\mathcal{H}}\}$$

is claimed to be relative HyST on $\mathcal{V}$ and $(\mathcal{V}, \tau_{\mathcal{H}\mathcal{V}}, €̧)$ is known as an HyS subspace of $(\widetilde{\coprod}, \tau_{\mathcal{H}}, €̧)$.
It is simple to confirm that $\tau_{\mathcal{H}\mathcal{V}}$ is an HyST on $\mathcal{V}$.

Definition 2.17. [26] Let $(\widetilde{\coprod}, \tau_{\mathcal{H}}, ₤)$ be an HySTS over $\coprod$ and let (η, π) be an HySS over $\coprod$. A point $u \in \coprod$ is called a hypersoft limit point of (η, π) if $(\eta, \pi) \widetilde{\cap} (\mu, \vartheta) \setminus \{u\} \neq \widetilde{\emptyset}$ for every HyS open set (μ, ϑ) containing u. The set of all hypersoft limit points of (η, π) is called the hypersoft derived set of (η, π) and is denoted by $(\eta, \pi)^d$.

Proposition 2.18. *[26] Let $(\widetilde{\coprod}, \tau_{\mathcal{H}}, ₤)$ be an HySTS over $\coprod$ and let (η, π) and (μ, ϑ) be two HySSs over $\coprod$. Then*

1. $(\eta, \pi) \widetilde{\subseteq} (\mu, \vartheta)$ *implies* $(\eta, \pi)^d \widetilde{\subseteq} (\mu, \vartheta)^d$.
2. $((\eta, \pi) \widetilde{\cap} (\mu, \vartheta))^d \widetilde{\subseteq} (\eta, \pi)^d \widetilde{\cap} (\mu, \vartheta)^d$.
3. $((\eta, \pi) \widetilde{\cup} (\mu, \vartheta))^d = (\eta, \pi)^d \widetilde{\cup} (\mu, \vartheta)^d$.

Definition 2.19. [26] Let $(\widetilde{\coprod}, \tau_{\mathcal{H}}, ₤)$ be an HySTS over $\coprod$ and $u \in \coprod$. Then an HySS (η, π) over $\coprod$ is claimed to be an HyS neighborhood of u if there exists an HyS open set (μ, π) such that $u \in (\mu, \pi) \subseteq (\eta, \pi)$

Definition 2.20. [26] Let $(\widetilde{\coprod}, \tau_{\mathcal{H}}, ₤)$ be an HySTS and (η, π) be an HySS over $\coprod$. The HyS intersection of all HyS closed supersets of (η, π) is known as the HyS closure of (η, π) and is symbolized by $Cl(\eta, \pi)$.
In other words, $Cl(\eta, \pi) = \widetilde{\cap}\{(\mu, \pi) \mid (\mu, \pi)^c \in \tau_{\mathcal{H}}, (\eta, \pi) \subseteq (\mu, \pi)\}$.
That is, $Cl(\eta, \pi)$ is the smallest HyS closed set containing (η, π).

Definition 2.21. [26] Let $(\coprod, \tau_{\mathcal{H}}, ₤)$ be an HySTS and (η, π) be an HySS over $\coprod$. Then HyS interior of HySS (η, π) over $\coprod$ is symbolized by $Int(\eta, \pi)$ and is described as the HyS union of all HyS open sets contained in (η, π).
In other words, $Int(\eta, \pi) = \widetilde{\cup}\{(\mu, \pi)\} \mid (\mu, \pi) \in \tau_{\mathcal{H}}, (\mu, \pi)\widetilde{\subseteq}(\eta, \pi)\}$.
That is, $Int(\eta, \pi)$ is the largest HyS open set contained in (η, π).

Definition 2.22. [26] Let $(\widetilde{\coprod}, \tau_{\mathcal{H}}, ₤)$ be an HySTS over $\coprod$, then HyS boundary of HySS (η, π) over $\coprod$ is symbolized by $b(\eta, \pi)$ and is described as

$$b(\eta, \pi) = Cl(\eta, \pi) \widetilde{\cap} Cl(\eta, \pi)^c.$$

2.2. Binary soft sets and binary soft topology

Let $\coprod_1, \coprod_2$ be two initial universe sets and the non empty set ₤ be an entire set of parameters. Let $\beta(\coprod_1), \beta(\coprod_2)$ indicate the power set of $\coprod_1, \coprod_2$, respectively. Also, let $\emptyset \neq \pi, \vartheta \subseteq ₤$.

Definition 2.23. [3] A pair (η, π) is claimed to be a binary soft set (BnSS) over $\coprod_1, \coprod_2$, where η is described as below:

$$\eta : \pi \longrightarrow \beta(\coprod_1) \times \beta(\coprod_2),$$

where $\eta(\varrho) = (\mathcal{X}, \mathcal{Y})$ for each $\varrho \in \pi$ such that $\mathcal{X} \subseteq \coprod_1, \mathcal{Y} \subseteq \coprod_2$.

Definition 2.24. [3] Let (η, π) and (μ, ϑ) be two BnSSs over the common $\coprod_1, \coprod_2$. (η, π) is known as a binary soft (BnS) subset of (μ, ϑ) if

1. $\pi \subseteq \vartheta$.
2. $\mathcal{X}_1 \subseteq \mathcal{X}_2 \subseteq \coprod_1$ and $\mathcal{Y}_1 \subseteq \mathcal{Y}_2 \subseteq \coprod_2$ such that $\eta(\varrho) = (\mathcal{X}_1, \mathcal{Y}_1), \mu(\varrho) = (\mathcal{X}_2, \mathcal{Y}_2)$ for each $\varrho \in \pi$.

We indicate it $(\eta, \pi) \widetilde{\subseteq} (\mu, \pi)$, briefly.
(η, π) is known as a BnS super set of (μ, ϑ) if (μ, ϑ) is a BnS subset of (η, π). We write $(\eta, \pi) \widetilde{\supseteq}(\mu, \vartheta)$.

Definition 2.25. [3] Let $(\eta, \pi), (\mu, \vartheta)$ be two BnSSs over $\coprod_1, \coprod_2$. (η, π) is known as a BnS equal of (μ, ϑ) if (η, π) is a BnS subset of (μ, ϑ) and (μ, ϑ) is a BnS subset of (η, π). We indicate it (η, π)=(μ, ϑ).

Definition 2.26. [3] The BnS complement of a BnSS (η,π) is symbolized by $(\eta,\pi)^c$ and is defined $(\eta,\pi)^c = (\eta^c,\pi)$, where $\eta^c : \pi \longrightarrow \beta(\coprod_1)\times\beta(\coprod_2)$, is a mapping given by $\eta^c(\varrho) = (\coprod_1 - \mathcal{X}, \coprod_2 - \mathcal{Y})$ such that $\eta(\varrho) = (\mathcal{X},\mathcal{Y})$. Clearly, $((\eta,\pi)^c)^c = (\eta,\pi)$.

Definition 2.27. [3] A BnSS (η,π) over $\coprod_1,\coprod_2$ is known as a null binary soft set symbolized by $\widetilde{\varnothing}$ if $\eta(\varrho) = (\varnothing,\varnothing)$ for each $\varrho \in \pi$.

Definition 2.28. [3] A BnSS (η,π) over $\coprod_1,\coprod_2$ is known as an absolute binary soft set symbolized by $\widetilde{\coprod}$ if $\eta(\varrho) = (\coprod_1,\coprod_2)$ for each $\varrho \in \pi$.

Definition 2.29. [3] The extended union of two BnSSs (η,π) and (μ,ϑ) over $\coprod_1,\coprod_2$ is the BnSS $(\mho,\mu)$, where $Ḑ = \pi \cup \vartheta$, and for each $\varrho \in Ḑ$,

$$\mho(\varrho) = \begin{cases} (\mathcal{X}_1,\mathcal{Y}_1) & if \quad \varrho \in \pi - \vartheta \\ (\mathcal{X}_2,\mathcal{Y}_2) & if \quad \varrho \in \vartheta - \pi \\ (\mathcal{X}_1 \cup \mathcal{X}_2, \mathcal{Y}_1 \cup \mathcal{Y}_2) & if \quad \varrho \in \pi \cap \vartheta \neq \varnothing \end{cases}$$

such that $\eta(\varrho) = (\mathcal{X}_1,\mathcal{Y}_1)$ for each $\varrho \in \pi$ and $\mu(\varrho) = (\mathcal{X}_2,\mathcal{Y}_2)$ for each $\varrho \in \vartheta$. We indicate it $(\eta,\pi)\, \widetilde{\cup}_{\mathcal{E}}\, (\mu,\vartheta) = (\mho, Ḑ)$.

Definition 2.30. [3] The extended intersection of two BnSSs (η,π) and (μ,ϑ) over $\coprod_1,\coprod_2$ is the BnSS $(\mho,Ḑ)$, where $Ḑ = \pi \cup \vartheta$, and for each $\varrho \in Ḑ$,

$$\mho(\varrho) = \begin{cases} (\mathcal{X}_1,\mathcal{Y}_1) & if \quad \varrho \in \pi - \vartheta \\ (\mathcal{X}_2,\mathcal{Y}_2) & if \quad \varrho \in \vartheta - \pi \\ (\mathcal{X}_1 \cap \mathcal{X}_2, \mathcal{Y}_1 \cap \mathcal{Y}_2) & if \quad \varrho \in \pi \cap \vartheta \neq \varnothing \end{cases}$$

such that $\eta(\varrho) = (\mathcal{X}_1,\mathcal{Y}_1)$ for each $\varrho \in \pi$ and $\mu(\varrho) = (\mathcal{X}_2,\mathcal{Y}_2)$ for each $\varrho \in \vartheta$. We indicate it $(\eta,\pi)\, \widetilde{\cap}_{\mathcal{E}}\, (\mu,\vartheta) = (\mho, Ḑ)$.

Definition 2.31. [3] The BnS union of two BnSSs (η,π) and (μ,ϑ) over $\coprod_1,\coprod_2$ is the BnSS $(\mho,Ḑ) = (\eta,\pi)\, \widetilde{\cup}\, (\mu,\vartheta)$, where $Ḑ = \pi \cap \vartheta \neq \varnothing$, and $\mho(\varrho) = (\mathcal{X}_1 \cup \mathcal{X}_2, \mathcal{Y}_1 \cup \mathcal{Y}_2)$ for each $\varrho \in Ḑ$ such that $\eta(\varrho) = (\mathcal{X}_1,\mathcal{Y}_1)$ for each $\varrho \in \pi$ and $\mu(\varrho) = (\mathcal{X}_2,\mathcal{Y}_2)$ for each $\varrho \in \vartheta$.

Definition 2.32. [3] The BnS intersection of two BnSSs (η,π) and (μ,ϑ) over $\coprod_1,\coprod_2$ is the BnSS $(\mho,Ḑ) = (\eta,\pi)\, \widetilde{\cap}\, (\mu,\vartheta)$, where $Ḑ = \pi \cap \vartheta \neq \varnothing$, and $\mho(\varrho) = (\mathcal{X}_1 \cap \mathcal{X}_2, \mathcal{Y}_1 \cap \mathcal{Y}_2)$ for each $\varrho \in Ḑ$ such that $\eta(\varrho) = (\mathcal{X}_1,\mathcal{Y}_1)$ for each $\varrho \in \pi$ and $\mu(\varrho) = (\mathcal{X}_2,\mathcal{Y}_2)$ for each $\varrho \in \vartheta$.

Definition 2.33. [16] The BnSS $(\eta, ₠)$ is known as a BnS point over $\coprod_1,\coprod_2$, symbolized by ϱ_η, if for the element $\varrho \in ₠$, $\eta(\varrho) \neq (\varnothing,\varnothing)$ and $\eta(\varrho') = (\varnothing,\varnothing)$, for each $\varrho' \in ₠ - \{\varrho\}$.

Definition 2.34. [12] Let $\tau_{\mathcal{B}}$ be the collection of BnSSs over $\coprod_1,\coprod_2$, then $\tau_{\mathcal{B}}$ is claimed to be a binary soft topology (BnST) on $\coprod_1,\coprod_2$ if

1. $\widetilde{\varnothing}, \widetilde{\coprod} \in \tau_{\mathcal{B}}$.
2. The BnS intersection of two BnSSs in $\tau_{\mathcal{B}}$ is belong to $\tau_{\mathcal{B}}$.
3. The BnS union of any number of BnSSs in $\tau_{\mathcal{B}}$ is belong to $\tau_{\mathcal{B}}$.

Then $(\coprod_1,\coprod_2,\tau_{\mathcal{B}}, ₠)$ is known as a binary soft topological space (BnSTS) over $\coprod_1,\coprod_2$. The members of $\tau_{\mathcal{B}}$ are claimed to be BnS open sets in $\coprod_1,\coprod_2$.
A BnSS (η,π) over $\coprod_1,\coprod_2$ is claimed to be a BnS closed set in $\coprod_1,\coprod_2$, if its BnS complement $(\eta,\pi)^c$ belongs to $\tau_{\mathcal{B}}$.

Definition 2.35. [16] Let $(\coprod_1, \coprod_2, \tau_{\mathcal{B}}, ₤)$ be a BnSTS over the common universe sets $\coprod_1, \coprod_2$ and $\mathcal{V}_1, \mathcal{V}_2$ be non empty subsets of $\coprod_1, \coprod_2$. Then $\tau_{(\mathcal{V}_1,\mathcal{V}_2)} = \{((\mathcal{V}_1, \mathcal{V}_2)_\eta, ₤) \mid (\eta, \pi) \in \tau_{\mathcal{B}}\}$ is claimed to be the BnS relative topology over $\mathcal{V}_1, \mathcal{V}_2$ and $(\mathcal{V}_1, \mathcal{V}_2, \tau_{(\mathcal{V}_1,\mathcal{V}_2)}, ₤)$ is known as a BnS subspace of $(\coprod_1, \coprod_2, \tau_{\mathcal{B}}, ₤)$. We can readily confirm that $\tau_{(\mathcal{V}_1,\mathcal{V}_2)}$ is in fact a BnST over $\mathcal{V}_1, \mathcal{V}_2$.

Definition 2.36. [16] A binary soft set (η, π) in a BnSTS $(\coprod_1, \coprod_2, \tau_{\mathcal{B}}, ₤)$ is known as a BnS neighborhood of the BnS point ϱ_η over $\coprod_1, \coprod_2$, if there exists a BnS open set (μ, π) such that $\varrho_\eta \in (\mu, \pi) \subseteq (\eta, \pi)$.
The BnS neighborhood system of BnS point ϱ_η, symbolized by $\mathcal{N}_\tau(\varrho_\eta)$, is the family of all its BnS neighborhoods.

Definition 2.37. [16] A BnSS (η, π) in a BnSTS $(\coprod_1, \coprod_2, \tau_{\mathcal{B}}, ₤)$ is known as a BnS neighborhood of the BnS set $(\mho, \pi)$, if there exists a BnS open set (μ, π) such that $(\mu, \pi) \subseteq (\mho, \pi) \subseteq (\eta, \pi)$.

Definition 2.38. [12] Let $(\coprod_1, \coprod_2, \tau_{\mathcal{B}}, ₤)$ be a BnSTS over $\coprod_1, \coprod_2$ and (η, π) be the BnSS over $\coprod_1, \coprod_2$. Then the BnS closure of (η, π) symbolized by $Cl(\eta, \pi)$ is the BnS intersection of all BnS closed sets containing (η, π).
In other words, $Cl(\eta, \pi) = \widetilde{\cap}\{(\mu, \pi) \mid (\mu, \pi)^c \in \tau_{\mathcal{B}}, (\eta, \pi) \widetilde{\subseteq} (\mu, \pi)\}$.
Thus, $Cl(\eta, \pi)$ is the smallest BnS closed sets over $\coprod_1, \coprod_2$ which contains (η, π).

Definition 2.39. [12] Let $(\coprod_1, \coprod_2, \tau_{\mathcal{B}}, ₤)$ be a BnSTS over $\coprod_1, \coprod_2$ and (η, π) be the BnSS over $\coprod_1, \coprod_2$. Then the BnS interior of BnSS (η, π) over $\coprod_1, \coprod_2$ is symbolized by $Int(\eta, \pi)$ and is described as the BnS union of all BnS open sets contained in (η, π).
In other words, $Int(\eta, \pi) = \widetilde{\cup}\{(\mu, \pi) \mid (\mu, \pi) \in \tau_{\mathcal{B}}, (\mu, \pi) \widetilde{\subseteq} (\eta, \pi)\}$.
Thus, $Int(\eta, \pi)$ is the largest BnS open set contained in (η, π).

3. Binary hypersoft sets

In this section, we define a novel extension of hypersoft sets called the binary hypersoft sets (BnHySSs) which is an advanced generalization of binary soft sets over two universal sets and a parameter set. Later, we present some operations on binary hypersoft sets such as subset, superset, equality, complement, null, absolute, extended union, extended intersection, union, intersection, difference, AND and OR. Moreover, we investigate some of their basic properties.

Let $\coprod_1, \coprod_2$ be two non empty initial universe sets and the non empty set ₤ be an entire set of parameters. Let $\beta(\coprod_1), \beta(\coprod_2)$ indicate the power set of $\coprod_1, \coprod_2$, respectively. Also, let $\varnothing \neq \pi_i, \vartheta_i \subseteq ₤$ with $i = 1, 2, ..., n$. To make things simpler, we write the symbol π for $\pi_1 \times \pi_2 \times \cdots \times \pi_n$ and ϱ for an element of the set π. We also suppose that none of the set π_i is empty for each i.

Definition 3.1. A pair (η, π) is claimed to be a BnHySS over $\coprod_1, \coprod_2$, where η is described as below:

$$\eta : \pi_1 \times \pi_2 \times ... \times \pi_n \longrightarrow \beta(\textstyle\coprod_1) \times \beta(\textstyle\coprod_2),$$

where $\eta(\varrho) = (\mathcal{X}, \mathcal{Y})$ for each $\varrho \in \pi_1 \times \pi_2 \times ... \times \pi_n$ such that $\mathcal{X} \subseteq \coprod_1$ and $\mathcal{Y} \subseteq \coprod_2$.

Example 3.2. Let's say *Mr. X* wishes to purchase a tablet and a phone from a mobile market. The collection of discourse consists of six different kinds of mobiles (options) $\coprod_1=\{m_1,m_2,m_3,m_4,m_5,m_6\}$, and the four types of tablets that make up the discourse set $\coprod_2 = \{\kappa_1, \kappa_2, \kappa_3, \kappa_4\}$. By looking at the qualities, one may choose which choice is better. i.e. ϱ_1 =Company, ϱ_2 =Camera Resolution, ϱ_3 =Size, ϱ_4 =Ram and ϱ_5 =Battery Power. The attribute-valued sets corresponding to these attributes are: ₤$= \{\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5\}$.
Let $\pi_1 = \{\varrho_1, \varrho_2\}$, $\pi_2 = \{\varrho_3, \varrho_4\}$ and $\pi_3 = \{\varrho_5\}$, where $\pi = \pi_1 \times \pi_2 \times \pi_3$.
Then BnHySS (η, π) can be written as follow:
$(\eta, \pi) = \{((\varrho_1, \varrho_3, \varrho_5), (\{m_1, m_2\}, \{\kappa_2\})), ((\varrho_1, \varrho_4, \varrho_5), (\{m_4, m_5, m_6\}, \{\kappa_1, \kappa_3\})), ((\varrho_2, \varrho_3, \varrho_5), (\{m_2, m_4, m_6\}, \{\kappa_2, \kappa_4\})), ((\varrho_2, \varrho_4, \varrho_5), (\{m_1, m_5\}, \{\kappa_3\}))\}$.

Proposition 3.3. *Let (η,π) be a BnHySS over $\coprod_1, \coprod_2$. If we write it into two parts as $\eta_1(\alpha) = \mathcal{X}$ and $\eta_2(\alpha) = \mathcal{Y}$ for each $\alpha \in \pi_1 \times \pi_2 \times ... \times \pi_n$ such that $\mathcal{X} \subseteq \coprod_1$ and $\mathcal{Y} \subseteq \coprod_2$. Then (η_1,π) and (η_2,π) become HySSs over $\coprod_1$ and $\coprod_2$ respectively.*

Example 3.4. Let us consider the BnHySS (η,π) over $\coprod_1, \coprod_2$ in Example 3.2.
We can write the BnHySS (η,π) as follow:
$(\eta_1,\pi) = \{((\varrho_1,\varrho_3,\varrho_5),\{m_1,m_2\}),((\varrho_1,\varrho_4,\varrho_5),\{m_4,m_5,m_6\}),((\varrho_2,\varrho_3,\varrho_5),\{m_2,m_4,m_6\}),((\varrho_2,\varrho_4,\varrho_5),\{m_1,m_5\})\}$ and
$(\eta_2,\pi) = \{((\varrho_1,\varrho_3,\varrho_5),\{\kappa_2\}),((\varrho_1,\varrho_4,\varrho_5),\{\kappa_1,\kappa_3\}),((\varrho_2,\varrho_3,\varrho_5),\{\kappa_2,\kappa_4\}),((\varrho_2,\varrho_4,\varrho_5),\{\kappa_3\})\}$.
Then (η_1,π) and (η_2,π) are HySSs over $\coprod_1$ and $\coprod_2$ respectively.

Remark 3.5. In Definition 3.1, if $n = 1$, then (η,π) becomes BnSS over $\coprod_1, \coprod_2$.

Definition 3.6. (**Subset**) Let $(\eta,\pi),(\mu,\vartheta)$ be two BnHySSs over $\coprod_1, \coprod_2$. (η,π) is known as a binary hypersoft (BnHyS) subset of (μ,ϑ) if

1. $\pi \subseteq \vartheta$, where $\vartheta = \vartheta_1 \times \vartheta_2 \times ... \times \vartheta_n$, that is $\pi_i \subseteq \vartheta_i$ for each $i = 1,2,...,n$.
2. $\mathcal{X}_1 \subseteq \mathcal{X}_2 \subseteq \coprod_1$ and $\mathcal{Y}_1 \subseteq \mathcal{Y}_2 \subseteq \coprod_2$ such that $\eta(\varrho) = (\mathcal{X}_1,\mathcal{Y}_1)$, $\mu(\varrho) = (\mathcal{X}_2,\mathcal{Y}_2)$ for each $\varrho \in \pi$.

We indicate it $(\eta,\pi) \widetilde{\subseteq} (\mu,\vartheta)$, briefly.
(η,π) is known as a BnHyS super set of (μ,ϑ) if (μ,ϑ) is a BnHyS subset of (η,π). We write $(\eta,\pi) \widetilde{\supseteq} (\mu,\vartheta)$.

Definition 3.7. (**Equality**) Let $(\eta,\pi),(\mu,\vartheta)$ be two BnHySSs over $\coprod_1, \coprod_2$. (η,π) is known as a BnHyS equal of (μ,ϑ) if (η,π) is a BnHyS subset of (μ,ϑ) and (μ,ϑ) is a BnHyS subset of (η,π). We indicate it $(\eta,\pi) = (\mu,\vartheta)$.

Definition 3.8. (**Complement**) The BnHyS complement of BnHySS (η,π) is symbolized by $(\eta,\pi)^c$ and is defined $(\eta,\pi)^c = (\eta^c,\pi)$, where $\eta : \pi_1 \times \pi_2 \times ... \times \pi_n \longrightarrow \beta(\coprod_1) \times \beta(\coprod_2)$ is a mapping given by $\eta^c(\varrho) = (\coprod_1 - \mathcal{X}, \coprod_2 - \mathcal{Y})$ such that $\eta(\varrho) = (\mathcal{X},\mathcal{Y})$ for each $\varrho \in (\pi_1 \times \pi_2 \times ... \times \pi_n)$, $\mathcal{X} \subseteq \coprod_1$ and $\mathcal{Y} \subseteq \coprod_2$. Clearly, $((\eta,\pi)^c)^c = (\eta,\pi)$.

Definition 3.9. (**Null**) A BnHySS (η,π) over $\coprod_1, \coprod_2$ is known as a null BnHySS symbolized by $\widetilde{\varnothing}$ if $\eta(\varrho) = (\varnothing,\varnothing)$ for each $\varrho \in \pi$.

Definition 3.10. (**Absolute**) A BnHySS (η,π) over $\coprod_1, \coprod_2$ is known as an absolute BnHySS symbolized by $\widetilde{\coprod}$ if $\eta(\varrho) = (\coprod_1, \coprod_2)$ for each $\varrho \in \pi$.

Example 3.11. Let $\coprod_1 = \{\kappa_1,\kappa_2,\kappa_3,\kappa_4,\kappa_5\}$, $\coprod_2 = \{\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5\}$ and ₤ $= \{\varrho_1,\varrho_2,\varrho_3,\varrho_4,\varrho_5,\varrho_6\}$.
Let $\pi_1 = \{\varrho_1,\varrho_2\}$, $\pi_2 = \{\varrho_4,\varrho_5\}$ and $\pi_3 = \{\varrho_6\}$, where $\pi = \pi_1 \times \pi_2 \times \pi_3$.
Let $\vartheta_1 = \{\varrho_1,\varrho_2,\varrho_3\}$, $\vartheta_2 = \{\varrho_4,\varrho_5\}$ and $\vartheta_3 = \{\varrho_6\}$, where $\vartheta = \vartheta_1 \times \vartheta_2 \times \vartheta_3$.
Then, $(\eta,\pi),(\mu,\vartheta)$ are two BnHySSs over $\coprod_1, \coprod_2$ defined as follow:
$(\eta,\pi) = \{((\varrho_1,\varrho_4,\varrho_6),(\{\kappa_1,\kappa_2\},\{\sigma_1\})),((\varrho_1,\varrho_5,\varrho_6),(\{\kappa_3\},\{\sigma_3,\sigma_4\})),((\varrho_2,\varrho_4,\varrho_6),(\{\kappa_1,\kappa_4\},\{\sigma_1,\sigma_2\})),((\varrho_2,\varrho_5,\varrho_6),(\{\kappa_5\},\{\sigma_4\}))\}$.
$(\mu,\vartheta) = \{(((\varrho_1,\varrho_4,\varrho_6),(\{\kappa_1,\kappa_2,\kappa_3\},\{\sigma_1\})),((\varrho_1,\varrho_5,\varrho_6),(\{\kappa_1,\kappa_3\},\{\sigma_3,\sigma_4,\sigma_5\})),((\varrho_2,\varrho_4,\varrho_6),(\{\kappa_1,\kappa_3,\kappa_4\},\coprod_2)),((\varrho_2,\varrho_5,\varrho_6),(\coprod_1,\coprod_2)),((\varrho_3,\varrho_4,\varrho_6),(\{\kappa_1,\kappa_3,\kappa_4\},\{\sigma_3,\sigma_4\})),((\varrho_3,\varrho_5,\varrho_6),(\{\kappa_1,\kappa_4\},\{\sigma_2,\sigma_3\}))\}$.
Therefore, $(\eta,\pi) \subseteq (\mu,\vartheta)$.

The BnHyS complement of BnHySS (η,π) is:

Table 1: BnHyS complement of BnHySS (η,π)

$\varrho \in \pi$	(η,π)	$(\eta,\pi)^c$
$(\varrho_1,\varrho_4,\varrho_6)$	$(\{\kappa_1,\kappa_2\},\{\sigma_1\})$	$(\{\kappa_3,\kappa_4,\kappa_5\},\{\sigma_2,\sigma_3,\sigma_4,\sigma_5\})$
$(\varrho_1,\varrho_5,\varrho_6)$	$(\{\kappa_3\},\{\sigma_3,\sigma_4\})$	$(\{\kappa_1,\kappa_2,\kappa_4,\kappa_5\},\{\sigma_1,\sigma_2,\sigma_5\})$
$(\varrho_2,\varrho_4,\varrho_6)$	$(\{\kappa_1,\kappa_4\},\{\sigma_1,\sigma_2\})$	$(\{\kappa_2,\kappa_3,\kappa_5\},\{\sigma_3,\sigma_4,\sigma_5\})$
$(\varrho_2,\varrho_5,\varrho_6)$	$(\{\kappa_5\},\{\sigma_4\})$	$(\{\kappa_1,\kappa_2,\kappa_3,\kappa_4\},\{\sigma_1,\sigma_2,\sigma_3,\sigma_5\})$

Let $(\mho, \pi)$ and $(\c{K}, \pi)$ be two BnHySSs as follow:
$(\mho, \pi) = \{((\varrho_1, \varrho_4, \varrho_6), (\emptyset, \emptyset)), ((\varrho_1, \varrho_5, \varrho_6), (\emptyset, \emptyset)), ((\varrho_2, \varrho_4, \varrho_6), (\emptyset, \emptyset)), ((\varrho_2, \varrho_5, \varrho_6), (\emptyset, \emptyset))\}$.
Then, $(\mho, \pi)$ is a null BnHySS.
$(\c{K}, \pi) = \{((\varrho_1, \varrho_4, \varrho_6), (\coprod_1, \coprod_2)), ((\varrho_1, \varrho_5, \varrho_6), (\coprod_1, \coprod_2)), ((\varrho_2, \varrho_4, \varrho_6), (\coprod_1, \coprod_2)), ((\varrho_2, \varrho_5, \varrho_6), (\coprod_1, \coprod_2))\}$.
Then, $(\c{K}, \pi)$ is an absolute BnHySS.

Definition 3.12. (**Extended Union**) Let (η, π) and (μ, π) be two BnHySSs over $\coprod_1, \coprod_2$. Then the extended union of (η, π) and (μ, π) is symbolized by $(\mho, \c{D}) = (\eta, \pi) \mathbin{\widetilde{\cup}_{\mathcal{E}}} (\mu, \vartheta)$, with $\c{D} = \c{D}_1 \times \c{D}_2 \times ... \times \c{D}_n$, where $\c{D}_i = \pi_i \cup \vartheta_i$ with $i = 1, 2, ..., n$, and $\mho$ can be characterized by,

$$\mho(\varrho) = \begin{cases} (\mathcal{X}_1, \mathcal{Y}_1) & if \quad \varrho \in \pi - \vartheta \\ (\mathcal{X}_2, \mathcal{Y}_2) & if \quad \varrho \in \vartheta - \pi \\ (\mathcal{X}_1 \cup \mathcal{X}_2, \mathcal{Y}_1 \cup \mathcal{Y}_2) & if \quad \varrho \in \pi \cap \vartheta \neq \emptyset \end{cases}$$

for each $\varrho \in \c{D}$ such that $\eta(\varrho) = (\mathcal{X}_1, \mathcal{Y}_1)$ for each $\varrho \in \pi$ and $\mu(\varrho) = (\mathcal{X}_2, \mathcal{Y}_2)$ for each $\varrho \in \vartheta$.

Definition 3.13. (**Extended Intersection**) Let (η, π) and (μ, π) be two BnHySSs over $\coprod_1, \coprod_2$. Then the extended intersection of (η, π) and (μ, π) is symbolized by $(\mho, \c{D}) = (\eta, \pi) \mathbin{\widetilde{\cap}_{\mathcal{E}}} (\mu, \vartheta)$, with $\c{D} = \c{D}_1 \times \c{D}_2 \times ... \times \c{D}_n$, where $\c{D}_i = \pi_i \cup \vartheta_i$ with $i = 1, 2, ..., n$, and $\mho$ can be characterized by,

$$\mho(\varrho) = \begin{cases} (\mathcal{X}_1, \mathcal{Y}_1) & if \quad \varrho \in \pi - \vartheta \\ (\mathcal{X}_2, \mathcal{Y}_2) & if \quad \varrho \in \vartheta - \pi \\ (\mathcal{X}_1 \cap \mathcal{X}_2, \mathcal{Y}_1 \cap \mathcal{Y}_2) & if \quad \varrho \in \pi \cap \vartheta \neq \emptyset \end{cases}$$

for each $\varrho \in \c{D}$ such that $\eta(\varrho) = (\mathcal{X}_1, \mathcal{Y}_1)$ for each $\varrho \in \pi$ and $\mu(\varrho) = (\mathcal{X}_2, \mathcal{Y}_2)$ for each $\varrho \in \vartheta$.

Definition 3.14. (**Union**) Let (η, π) and (μ, ϑ) be two BnHySSs over $\coprod_1, \coprod_2$. Then the BnHyS union of (η, π) and (μ, ϑ) is symbolized by $(\mho, \c{D}) = (\eta, \pi) \mathbin{\widetilde{\cup}} (\mu, \vartheta)$, with $\c{D} = \c{D}_1 \times \c{D}_2 \times ... \times \c{D}_n$, where $\c{D}_i = \pi_i \cap \vartheta_i \neq \emptyset$ with $i = 1, 2, ..., n$, and $\mho$ can be characterized by,

$$\mho(\varrho) = \eta(\varrho) \mathbin{\widetilde{\cup}} \mu(\varrho) = (\mathcal{X}_1 \cup \mathcal{X}_2, \mathcal{Y}_1 \cup \mathcal{Y}_2)$$

for each $\varrho \in \c{D}$ such that $\eta(\varrho) = (\mathcal{X}_1, \mathcal{Y}_1)$ for each $\varrho \in \pi$ and $\mu(\varrho) = (\mathcal{X}_2, \mathcal{Y}_2)$ for each $\varrho \in \vartheta$.

Definition 3.15. Let $\{(\eta_j, \pi_j) : j \in J\}$ be an infinite family of BnHySSs over $\coprod_1, \coprod_2$. The BnHyS union of this family is defined as the BnHySS (η, π) (That is $(\eta, \pi) = \widetilde{\bigcup}_{j \in J}(\eta_j, \pi_j)$), where the parameter set is $\pi = \bigcap_{j \in J} \pi_j \neq \emptyset$ where $\pi_j = \pi_{j1} \times \pi_{j2} \times ... \times \pi_{jn}$ for each $j \in J$, and the mapping

$$\eta : \pi_1 \times \pi_2 \times ... \times \pi_n \longrightarrow \beta(\textstyle\coprod_1) \times \beta(\coprod_2)$$

is defined for each parameter $\varrho \in \pi_1 \times \pi_2 \times ... \times \pi_n$ by $\eta(\varrho) = \widetilde{\bigcup}_{\substack{j \in J \\ \varrho \in \pi_j}} \eta_j(\varrho)$.

Definition 3.16. (**Intersection**) Let (η, π) and (μ, ϑ) be two BnHySSs over $\coprod_1, \coprod_2$. Then the BnHyS intersection of (η, π) and (μ, ϑ) is symbolized by $(\mho, \c{D}) = (\eta, \pi) \mathbin{\widetilde{\cap}} (\mu, \vartheta)$, with $\c{D} = \c{D}_1 \times \c{D}_2 \times ... \times \c{D}_n$, where $\c{D}_i = \pi_i \cap \vartheta_i \neq \emptyset$ with $i = 1, 2, ..., n$, and $\mho$ can be characterized by,

$$\mho(\varrho) = \eta(\varrho) \mathbin{\widetilde{\cap}} \mu(\varrho) = (\mathcal{X}_1 \cap \mathcal{X}_2, \mathcal{Y}_1 \cap \mathcal{Y}_2)$$

for each $\varrho \in \c{D}$ such that $\eta(\varrho) = (\mathcal{X}_1, \mathcal{Y}_1)$ for each $\varrho \in \pi$ and $\mu(\varrho) = (\mathcal{X}_2, \mathcal{Y}_2)$ for each $\varrho \in \vartheta$.

Definition 3.17. Let $\{(\eta_j, \pi_j) : j \in J\}$ be an infinite family of BnHySSs over $\coprod_1, \coprod_2$. The BnHyS intersection of this family is defined as the BnHySS (η, π) (That is $(\eta, \pi) = \widetilde{\bigcap}_{j \in J}(\eta_j, \pi_j)$), where the parameter set is $\pi = \bigcap_{j \in J} \pi_j \neq \emptyset$ where $\pi_j = \pi_{j1} \times \pi_{j2} \times ... \times \pi_{jn}$ for each $j \in J$, and the mapping

$$\eta : \pi_1 \times \pi_2 \times ... \times \pi_n \longrightarrow \beta(\coprod_1) \times \beta(\coprod_2)$$

is defined for each parameter $\varrho \in \pi_1 \times \pi_2 \times ... \times \pi_n$ by $\eta(\varrho) = \widetilde{\bigcap}_{\substack{j\in J \\ \varrho\in\pi_j}} \eta_j(\varrho)$.

Definition 3.18. (**Difference**) Let (η, π) and (μ, ϑ) be two BnHySSs over $\coprod_1, \coprod_2$. Then BnHyS difference of (η, π) and (μ, ϑ), symbolized by $(\mho, Ḑ) = (\eta, \pi) \widetilde{\setminus} (\mu, \vartheta)$, with $Ḑ = Ḑ_1 \times Ḑ_2 \times ... \times Ḑ_n$, where $Ḑ_i = \pi_i \cap \vartheta_i \neq \emptyset$ with $i = 1, 2, ..., n$, and $\mho$ can be characterized by

$$\mho(\varrho) = \eta(\varrho) \setminus \mu(\varrho) = (\mathcal{X}_1 - \mathcal{X}_2, \mathcal{Y}_1 - \mathcal{Y}_2)$$

for each $\varrho \in \pi \cap \vartheta$ such that $\eta(\varrho) = (\mathcal{X}_1, \mathcal{Y}_1)$ and $\mu(\varrho) = (\mathcal{X}_2, \mathcal{Y}_2)$.

Example 3.19. Let $\coprod_1 = \{\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5\}$, $\coprod_2 = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5\}$ and $₠ = \{\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6\}$.
Suppose that $\pi_1 = \{\rho_1, \rho_2\}$, $\pi_2 = \{\rho_3, \rho_4\}$, $\pi_3 = \{\rho_5\}$, $\vartheta_1 = \{\rho_1\}$, $\vartheta_2 = \{\rho_3, \rho_4\}$, and $\vartheta_3 = \{\rho_5, \rho_6\}$. That is $\pi_i, \vartheta_i \subseteq ₠$ for each $i = 1, 2, 3$.
Let the BnHySSs (η, π) and (μ, ϑ) be defined by
$(\eta, \pi) =\{((\rho_1, \rho_3, \rho_5), (\{\kappa_1, \kappa_3\}, \{\sigma_1, \sigma_2\})), ((\rho_1, \rho_4, \rho_5), (\{\kappa_2, \kappa_3, \kappa_4\}, \{\sigma_2\})), ((\rho_2, \rho_3, \rho_5), (\{\kappa_4, \kappa_5\}, \{\sigma_3, \sigma_4\})), ((\rho_2, \rho_4, \rho_5), (\{\kappa_1, \kappa_2, \kappa_3\}, \{\sigma_1, \sigma_4\}))\}$,
and
$(\mu, \vartheta) =\{((\rho_1, \rho_3, \rho_5), (\{\kappa_1\}, \{\sigma_2, \sigma_3\})), ((\rho_1, \rho_4, \rho_5), (\{\kappa_2, \kappa_4, \kappa_5\}, \{\sigma_2\}), ((\rho_1, \rho_3, \rho_6), (\{\kappa_4\}, \{\sigma_1, \sigma_4\})), ((\rho_1, \rho_4, \rho_6), (\{\kappa_3, \kappa_4\}, \{\sigma_3, \sigma_4\}))\}$.

Then the extended union and extended intersections of both (η, π) and (μ, ϑ) are given by:

Table 2: Extended union and extended intersections of both (η, π) and (μ, ϑ)

$\rho \in \pi \cup \vartheta$	$(\eta, \pi) \widetilde{\cup}_{\mathcal{E}} (\mu, \vartheta)$	$(\eta, \pi) \widetilde{\cap}_{\mathcal{E}} (\mu, \vartheta)$
(ρ_1, ρ_3, ρ_5)	$(\{\kappa_1, \kappa_3\}, \{\sigma_1, \sigma_2, \sigma_3\})$	$(\{\kappa_1\}, \{\sigma_2\})$
(ρ_1, ρ_4, ρ_5)	$(\{\kappa_2, \kappa_3, \kappa_4, \kappa_5\}, \{\sigma_2\})$	$(\{\kappa_2, \kappa_4\}, \{\sigma_2\})$
(ρ_2, ρ_3, ρ_5)	$(\{\kappa_4, \kappa_5\}, \{\sigma_3, \sigma_4\})$	$(\{\kappa_4, \kappa_5\}, \{\sigma_3, \sigma_4\})$
(ρ_2, ρ_4, ρ_5)	$(\{\kappa_1, \kappa_2, \kappa_3\}, \{\sigma_1, \sigma_4\})$	$(\{\kappa_1, \kappa_2, \kappa_3\}, \{\sigma_1, \sigma_4\})$
(ρ_1, ρ_3, ρ_6)	$(\{\kappa_4\}, \{\sigma_1, \sigma_4\}))$	$(\{\kappa_4\}, \{\sigma_1, \sigma_4\})$
(ρ_1, ρ_4, ρ_6)	$(\{\kappa_3, \kappa_4\}, \{\sigma_3, \sigma_4\})$	$(\{\kappa_3, \kappa_4\}, \{\sigma_3, \sigma_4\})$

Then the BnHyS union and BnHyS intersections of both (η, π) and (μ, ϑ) are given by:

Table 3: BnHyS union and BnHyS intersections of both (η, π) and (μ, ϑ)

$\rho \in \pi \cap \vartheta$	$(\eta, \pi) \widetilde{\cup} (\mu, \vartheta)$	$(\eta, \pi) \widetilde{\cap} (\mu, \vartheta)$
(ρ_1, ρ_3, ρ_5)	$(\{\kappa_1, \kappa_3\}, \{\sigma_1, \sigma_2, \sigma_3\})$	$(\{\kappa_1\}, \{\sigma_2\})$
(ρ_1, ρ_4, ρ_5)	$(\{\kappa_2, \kappa_3, \kappa_4, \kappa_5\}, \{\sigma_2\})$	$(\{\kappa_2, \kappa_4\}, \{\sigma_2\})$

The BnHyS differences $(\eta, \pi) \widetilde{\setminus} (\mu, \vartheta)$ and $(\mu, \vartheta) \widetilde{\setminus} (\eta, \pi)$ are the following:
$(\eta, \pi) \widetilde{\setminus} (\mu, \vartheta) = \{((\rho_1, \rho_3, \rho_5), (\{\kappa_3\}, \{\sigma_1\})), ((\rho_1, \rho_4, \rho_5), (\{\kappa_3\}, \emptyset))\}$.
$(\mu, \vartheta) \widetilde{\setminus} (\eta, \pi) = \{((\rho_1, \rho_3, \rho_5), (\emptyset, \{\sigma_3\})), ((\rho_1, \rho_4, \rho_5), (\{\kappa_5\}, \emptyset))\}$.

Table 4: BnHyS difference of both (η, π) and (μ, ϑ)

$\rho \in \pi \cap \vartheta$	$(\eta, \pi) \widetilde{\setminus} (\mu, \vartheta)$	$(\mu, \vartheta) \widetilde{\setminus} (\eta, \pi)$
(ρ_1, ρ_3, ρ_5)	$(\{\kappa_3\}, \{\sigma_1\})$	$(\emptyset, \{\sigma_3\})$
(ρ_1, ρ_4, ρ_5)	$(\{\kappa_3\}, \emptyset)$	$(\{\kappa_5\}, \emptyset)$

Definition 3.20. (**AND**) If (η,π) and (μ,ϑ) are two BnHySSs, then (η,π)AND(μ,ϑ), symbolized by $(\eta,\pi)\,\widetilde{\wedge}\,(\mu,\vartheta)$, can be characterized by $(\mho,Ḑ) = (\eta,\pi)\,\widetilde{\wedge}\,(\mu,\vartheta)$, with $Ḑ = Ḑ_1\times Ḑ_2\times\ldots\times Ḑ_n$, where $Ḑ_i = \pi_i\times\vartheta_i$ with $i = 1,2,\ldots,n$, and $\mho$ can be characterized by, $\mho(\varrho) = \eta(\varrho)\cap\mu(\varrho) = (\mathcal{X}_1\cap\mathcal{X}_2,\mathcal{Y}_1\cap\mathcal{Y}_2)$ for each $\varrho\in Ḑ$ such that $\eta(\varrho) = (\mathcal{X}_1,\mathcal{Y}_1)$ for each $\varrho\in\pi$ and $\mu(\varrho) = (\mathcal{X}_2,\mathcal{Y}_2)$ for each $\varrho\in\vartheta$.

Definition 3.21. (**OR**) If (η,π) and (μ,ϑ) are two BnHySSs, then (η,π)OR(μ,ϑ), symbolized by $(\eta,\pi)\,\widetilde{\vee}\,(\mu,\vartheta)$, can be characterized by $(\mho,Ḑ) = (\eta,\pi)\,\widetilde{\vee}\,(\mu,\vartheta)$, with $Ḑ = Ḑ_1\times Ḑ_2\times\ldots\times Ḑ_n$, where $Ḑ_i = \pi_i\times\vartheta_i$ with $i = 1,2,\ldots,n$, and $\mho$ can be characterized by, $\mho(\varrho) = \eta(\varrho)\cup\mu(\varrho) = (\mathcal{X}_1\cup\mathcal{X}_2,\mathcal{Y}_1\cup\mathcal{Y}_2)$ for each $\varrho\in Ḑ$ such that $\eta(\varrho) = (\mathcal{X}_1,\mathcal{Y}_1)$ for each $\varrho\in\pi$ and $\mu(\varrho) = (\mathcal{X}_2,\mathcal{Y}_2)$ for each $\varrho\in\vartheta$.

Example 3.22. Let $\coprod_1 = \{\kappa_1,\kappa_2,\kappa_3,\kappa_4,\kappa_5\}$, $\coprod_2 = \{\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5\}$ and $₠ = \{\rho_1,\rho_2,\rho_3,\rho_4,\rho_5,\rho_6\}$. Suppose that
$\pi_1 = \{\rho_1,\rho_2\}$, $\pi_2 = \{\rho_3,\rho_4\}$, $\pi_3 = \{\rho_5\}$, $\vartheta_1 = \{\rho_1\}$, $\vartheta_2 = \{\rho_3,\rho_4\}$, and $\vartheta_3 = \{\rho_5,\rho_6\}$.
That is $\pi_i,\vartheta_i\subseteq ₠$ for each $i = 1,2,3$. Let the BnHySSs (η,π) and (μ,ϑ) be defined by
$(\eta,\pi) = \{((\rho_1,\rho_3,\rho_5),(\{\kappa_1,\kappa_3\},\{\sigma_1,\sigma_2\})),((\rho_1,\rho_4,\rho_5),(\{\kappa_2,\kappa_3,\kappa_4\},\{\sigma_2\})),((\rho_2,\rho_3,\rho_5),(\{\kappa_4,\kappa_5\},\{\sigma_3,\sigma_4\})),((\rho_2,\rho_4,\rho_5),(\{\kappa_1,\kappa_2,\kappa_3\},\{\sigma_1,\sigma_4\}))\}$,
and
$(\mu,\vartheta) = \{((\rho_1,\rho_3,\rho_5),(\{\kappa_1\},\{\sigma_2,\sigma_3\})),((\rho_1,\rho_4,\rho_5),(\{\kappa_2,\kappa_4,\kappa_5\},\{\sigma_2\}),((\rho_1,\rho_3,\rho_6),(\{\kappa_4\},\{\sigma_1,\sigma_4,\})),((\rho_1,\rho_4,\rho_6),(\{\kappa_3,\kappa_4\},\{\sigma_3,\sigma_4\}))\}$.

Then $(\mho,\pi\times\vartheta) = (\eta,\pi)\,\widetilde{\wedge}\,(\mu,\vartheta)$ and $(Ọ,\pi\times\vartheta) = (\eta,\pi)\widetilde{\vee}(\mu,\vartheta)$ are the BnHySSs as shown below:

Table 5: BnHyS AND and BnHyS OR of both (η,π) and (μ,ϑ)

$\rho\in\pi\times\vartheta$	$(\eta,\pi)\,\widetilde{\wedge}\,(\mu,\vartheta)$	$(\eta,\pi)\,\widetilde{\vee}\,(\mu,\vartheta)$
$((\rho_1,\rho_3,\rho_5),(\rho_1,\rho_3,\rho_5))$	$(\{\kappa_1\},\{\sigma_2\})$	$(\{\kappa_1,\kappa_3\},\{\sigma_1,\sigma_2,\sigma_3\})$
$((\rho_1,\rho_3,\rho_5),(\rho_1,\rho_4,\rho_5))$	$(\emptyset,\{\sigma_2\})$	$(\coprod_1,\{\sigma_1,\sigma_2\})$
$((\rho_1,\rho_3,\rho_5),(\rho_1,\rho_3,\rho_6))$	$(\emptyset,\{\sigma_1\})$	$(\{\kappa_1,\kappa_3,\kappa_4\},\{\sigma_1,\sigma_2,\sigma_4\})$
$((\rho_1,\rho_3,\rho_5),(\rho_1,\rho_4,\rho_6))$	$(\{\kappa_3\},\emptyset)$	$(\{\kappa_1,\kappa_3,\kappa_4\},\{\sigma_1,\sigma_2,\sigma_3,\sigma_4\})$
$((\rho_1,\rho_4,\rho_5),(\rho_1,\rho_3,\rho_5))$	$(\emptyset,\{\sigma_2\})$	$(\{\kappa_1,\kappa_2,\kappa_3,\kappa_4\},\{\sigma_2,\sigma_3\})$
$((\rho_1,\rho_4,\rho_5),(\rho_1,\rho_4,\rho_5))$	$(\{\kappa_2,\kappa_4\},\{\sigma_2\})$	$(\{\kappa_2,\kappa_3,\kappa_4,\kappa_5\},\{\sigma_2\})$
$((\rho_1,\rho_4,\rho_5),(\rho_1,\rho_3,\rho_6))$	$(\{\kappa_4\},\emptyset)$	$(\{\kappa_2,\kappa_3,\kappa_4\},\{\sigma_1,\sigma_2,\sigma_4\})$
$((\rho_1,\rho_4,\rho_5),(\rho_1,\rho_4,\rho_6))$	$(\{\kappa_3,\kappa_4\},\emptyset)$	$(\{\kappa_2,\kappa_3,\kappa_4\},\{\sigma_2,\sigma_3,\sigma_4\})$
$((\rho_2,\rho_3,\rho_5),(\rho_1,\rho_3,\rho_5))$	$(\emptyset,\{\sigma_3\})$	$(\{\kappa_1,\kappa_4,\kappa_5\},\{\sigma_2,\sigma_3,\sigma_4\})$
$((\rho_2,\rho_3,\rho_5),(\rho_1,\rho_4,\rho_5))$	$(\{\kappa_4,\kappa_5\},\emptyset)$	$(\{\kappa_2,\kappa_4,\kappa_5\},\{\sigma_2,\sigma_3,\sigma_4\})$
$((\rho_2,\rho_3,\rho_5),(\rho_1,\rho_3,\rho_6))$	$(\{\kappa_4\},\{\sigma_4\})$	$(\{\kappa_4,\kappa_5\},\{\sigma_1,\sigma_3,\sigma_4\})$
$((\rho_2,\rho_3,\rho_5),(\rho_1,\rho_4,\rho_6))$	$(\{\kappa_4\},\{\sigma_3,\sigma_4\})$	$(\{\kappa_3,\kappa_4,\kappa_5\},\{\sigma_3,\sigma_4\})$
$((\rho_2,\rho_4,\rho_5),(\rho_1,\rho_3,\rho_5))$	$(\{\kappa_1\},\emptyset)$	$(\{\kappa_1,\kappa_2,\kappa_3\},\{\sigma_1,\sigma_2,\sigma_3,\sigma_4\})$
$((\rho_2,\rho_4,\rho_5),(\rho_1,\rho_4,\rho_5))$	$(\{\kappa_2\},\emptyset)$	$(\coprod_1,\{\sigma_1,\sigma_2,\sigma_4\})$
$((\rho_2,\rho_4,\rho_5),(\rho_1,\rho_3,\rho_6))$	$(\emptyset,\{\sigma_1,\sigma_4\})$	$(\{\kappa_1,\kappa_2,\kappa_3,\kappa_4\},\{\sigma_1,\sigma_4\})$
$((\rho_2,\rho_4,\rho_5),(\rho_1,\rho_4,\rho_6))$	$(\{\kappa_3\},\{\sigma_4\})$	$(\{\kappa_1,\kappa_2,\kappa_3,\kappa_4\},\{\sigma_1,\sigma_3,\sigma_4\})$

Definition 3.23. The BnHySS (η,π) is known as a BnHyS point over $\coprod_1,\coprod_2$, symbolized by ϱ_η, if for the element $\varrho\in\pi_1\times\pi_2\times\ldots\times\pi_n$, $\eta(\varrho)\neq(\emptyset,\emptyset)$ and $\eta(\varrho') = (\emptyset,\emptyset)$, for each $\varrho'\in(\pi_1\times\pi_2\times\ldots\times\pi_n)-\{\varrho\}$.

Definition 3.24. The BnHyS point ϱ_η is claimed to be in the BnHySS (μ,ϑ), symbolized by $\varrho_\eta\in(\mu,\vartheta)$ if for the element $\varrho\in\pi_1\times\pi_2\times\ldots\times\pi_n$, $\eta(\varrho)\subseteq\mu(\varrho)$.

Example 3.25. Let $\coprod_1 = \{x_1,x_2\}$, $\coprod_2 = \{y_1,y_2\}$, $₠ = \{\varrho_1,\varrho_2,\varrho_3\}$, $\pi_1 = \{\varrho_1\}$, $\pi_2 = \{\varrho_2,\varrho_3\}$.

The BnHyS points over $\coprod_1,\coprod_2$ are defined as follow:
$(\eta_1,\pi) = \{((\varrho_1,\varrho_2),(\emptyset,\{y_1\})),((\varrho_1,\varrho_3),(\emptyset,\emptyset))\} = (\varrho_1,\varrho_2)_{\eta_1}$

$(\eta_2, \pi) = \{((\varrho_1, \varrho_2), (\varnothing, \{y_2\})), ((\varrho_1, \varrho_3), (\varnothing, \varnothing))\} = (\varrho_1, \varrho_2)_{\eta_2}$
$(\eta_3, \pi) = \{((\varrho_1, \varrho_2), (\varnothing, \coprod_2)), ((\varrho_1, \varrho_3), (\varnothing, \varnothing))\} = (\varrho_1, \varrho_2)_{\eta_3}$
$(\eta_4, \pi) = \{((\varrho_1, \varrho_2), (\{x_1\}, \varnothing)), ((\varrho_1, \varrho_3), (\varnothing, \varnothing))\} = (\varrho_1, \varrho_2)_{\eta_4}$
$(\eta_5, \pi) = \{((\varrho_1, \varrho_2), (\{x_1\}, \{y_1\})), ((\varrho_1, \varrho_3), (\varnothing, \varnothing))\} = (\varrho_1, \varrho_2)_{\eta_5}$
$(\eta_6, \pi) = \{((\varrho_1, \varrho_2), (\{x_1\}, \{y_2\})), ((\varrho_1, \varrho_3), (\varnothing, \varnothing))\} = (\varrho_1, \varrho_2)_{\eta_6}$
$(\eta_7, \pi) = \{((\varrho_1, \varrho_2), (\{x_1\}, \coprod_2)), ((\varrho_1, \varrho_3), (\varnothing, \varnothing))\} = (\varrho_1, \varrho_2)_{\eta_7}$
$(\eta_8, \pi) = \{((\varrho_1, \varrho_2), (\{x_2\}, \varnothing)), ((\varrho_1, \varrho_3), (\varnothing, \varnothing))\} = (\varrho_1, \varrho_2)_{\eta_8}$
$(\eta_9, \pi) = \{((\varrho_1, \varrho_2), (\{x_2\}, \{y_1\})), ((\varrho_1, \varrho_3), (\varnothing, \varnothing))\} = (\varrho_1, \varrho_2)_{\eta_9}$
$(\eta_{10}, \pi) = \{((\varrho_1, \varrho_2), (\{x_2\}, \{y_2\})), ((\varrho_1, \varrho_3), (\varnothing, \varnothing))\} = (\varrho_1, \varrho_2)_{\eta_{10}}$
$(\eta_{11}, \pi) = \{((\varrho_1, \varrho_2), (\{x_2\}, \coprod_2)), ((\varrho_1, \varrho_3), (\varnothing, \varnothing))\} = (\varrho_1, \varrho_2)_{\eta_{11}}$
$(\eta_{12}, \pi) = \{((\varrho_1, \varrho_2), (\coprod_1, \varnothing)), ((\varrho_1, \varrho_3), (\varnothing, \varnothing))\} = (\varrho_1, \varrho_2)_{\eta_{12}}$
$(\eta_{13}, \pi) = \{((\varrho_1, \varrho_2), (\coprod_1, \{y_1\})), ((\varrho_1, \varrho_3), (\varnothing, \varnothing))\} = (\varrho_1, \varrho_2)_{\eta_{13}}$
$(\eta_{14}, \pi) = \{((\varrho_1, \varrho_2), (\coprod_1, \{y_2\})), ((\varrho_1, \varrho_3), (\varnothing, \varnothing))\} = (\varrho_1, \varrho_2)_{\eta_{14}}$
$(\eta_{15}, \pi) = \{((\varrho_1, \varrho_2), (\coprod_1, \coprod_2)), ((\varrho_1, \varrho_3), (\varnothing, \varnothing))\} = (\varrho_1, \varrho_2)_{\eta_{15}}$
$(\eta_{16}, \pi) = \{((\varrho_1, \varrho_2), (\varnothing, \varnothing)), ((\varrho_1, \varrho_3), (\varnothing, \{y_1\}))\} = (\varrho_1, \varrho_3)_{\eta_{16}}$
$(\eta_{17}, \pi) = \{((\varrho_1, \varrho_2), (\varnothing, \varnothing)), ((\varrho_1, \varrho_3), (\varnothing, \{y_2\}))\} = (\varrho_1, \varrho_3)_{\eta_{17}}$
$(\eta_{18}, \pi) = \{((\varrho_1, \varrho_2), (\varnothing, \varnothing)), ((\varrho_1, \varrho_3), (\varnothing, \coprod_2))\} = (\varrho_1, \varrho_3)_{\eta_{18}}$
$(\eta_{19}, \pi) = \{((\varrho_1, \varrho_2), (\varnothing, \varnothing)), ((\varrho_1, \varrho_3), (\{x_1\}, \varnothing))\} = (\varrho_1, \varrho_3)_{\eta_{19}}$
$(\eta_{20}, \pi) = \{((\varrho_1, \varrho_2), (\varnothing, \varnothing)), ((\varrho_1, \varrho_3), (\{x_1\}, \{y_1\}))\} = (\varrho_1, \varrho_3)_{\eta_{20}}$
$(\eta_{21}, \pi) = \{((\varrho_1, \varrho_2), (\varnothing, \varnothing)), ((\varrho_1, \varrho_3), (\{x_1\}, \{y_2\}))\} = (\varrho_1, \varrho_3)_{\eta_{21}}$
$(\eta_{22}, \pi) = \{((\varrho_1, \varrho_2), (\varnothing, \varnothing)), ((\varrho_1, \varrho_3), (\{x_1\}, \coprod_2))\} = (\varrho_1, \varrho_3)_{\eta_{22}}$
$(\eta_{23}, \pi) = \{((\varrho_1, \varrho_2), (\varnothing, \varnothing)), ((\varrho_1, \varrho_3), (\{x_2\}, \varnothing))\} = (\varrho_1, \varrho_3)_{\eta_{23}}$
$(\eta_{24}, \pi) = \{((\varrho_1, \varrho_2), (\varnothing, \varnothing)), ((\varrho_1, \varrho_3), (\{x_2\}, \{y_1\}))\} = (\varrho_1, \varrho_3)_{\eta_{24}}$
$(\eta_{25}, \pi) = \{((\varrho_1, \varrho_2), (\varnothing, \varnothing)), ((\varrho_1, \varrho_3), (\{x_2\}, \{y_2\}))\} = (\varrho_1, \varrho_3)_{\eta_{25}}$
$(\eta_{26}, \pi) = \{((\varrho_1, \varrho_2), (\varnothing, \varnothing)), ((\varrho_1, \varrho_3), (\{x_2\}, \coprod_2))\} = (\varrho_1, \varrho_3)_{\eta_{26}}$
$(\eta_{27}, \pi) = \{((\varrho_1, \varrho_2), (\varnothing, \varnothing)), ((\varrho_1, \varrho_3), (\coprod_1, \varnothing))\} = (\varrho_1, \varrho_3)_{\eta_{27}}$
$(\eta_{28}, \pi) = \{((\varrho_1, \varrho_2), (\varnothing, \varnothing)), ((\varrho_1, \varrho_3), (\coprod_1, \{y_1\}))\} = (\varrho_1, \varrho_3)_{\eta_{28}}$
$(\eta_{29}, \pi) = \{((\varrho_1, \varrho_2), (\varnothing, \varnothing)), ((\varrho_1, \varrho_3), (\coprod_1, \{y_2\}))\} = (\varrho_1, \varrho_3)_{\eta_{29}}$
$(\eta_{30}, \pi) = \{((\varrho_1, \varrho_2), (\varnothing, \varnothing)), ((\varrho_1, \varrho_3), (\coprod_1, \coprod_2))\} = (\varrho_1, \varrho_3)_{\eta_{30}}$.

Some main properties of BnHySSs are given below:

Proposition 3.26. *Let $(\eta, \pi), (\mu, \vartheta)$ and $(\mho, Ḑ)$ be three BnHySSs. Then we have the following results:*

1. $(\eta, \pi) \widetilde{\cup} \widetilde{\varnothing} = (\eta, \pi)$.
2. $(\eta, \pi) \widetilde{\cup} \widetilde{\coprod} = \widetilde{\coprod}$.
3. $(\eta, \pi) \widetilde{\cup} (\eta, \pi) = (\eta, \pi)$.
4. $(\eta, \pi) \widetilde{\cup} (\mu, \vartheta) = (\mu, \vartheta) \widetilde{\cup} (\eta, \pi)$.
5. $(\eta, \pi) \widetilde{\cup} ((\mu, \vartheta) \widetilde{\cup} (\mho, Ḑ)) = ((\eta, \pi) \widetilde{\cup} (\mu, \vartheta)) \widetilde{\cup} (\mho, Ḑ)$.
6. $(\eta, \pi) \widetilde{\subseteq} (\eta, \pi) \widetilde{\cup} (\mu, \vartheta)$ *and* $(\mu, \vartheta) \widetilde{\subseteq} (\eta, \pi) \widetilde{\cup} (\mu, \vartheta)$.
7. $(\eta, \pi) \widetilde{\cup} (\mu, \vartheta) = \widetilde{\varnothing}$ *if and only if* $(\eta, \pi) = \widetilde{\varnothing}$ *and* $(\mu, \vartheta) = \widetilde{\varnothing}$.
8. $(\eta, \pi) \widetilde{\subseteq} (\mu, \vartheta)$ *if and only if* $(\eta, \pi) \widetilde{\cup} (\mu, \vartheta) = (\mu, \vartheta)$.

Proof. 5. Suppose that $(\mu, \vartheta) \widetilde{\cup} (\mho, Ḑ) = (\mathcal{K}_1, \vartheta \cap Ḑ)$. Then for all $\alpha \in \vartheta \cap Ḑ$, $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3 \subseteq \coprod_1$ and $\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3 \subseteq \coprod_2$ such that $\eta(\alpha) = (\mathcal{X}_1, \mathcal{Y}_1)$, $\mu(\alpha) = (\mathcal{X}_2, \mathcal{Y}_2)$, $\mho(\alpha) = (\mathcal{X}_3, \mathcal{Y}_3)$ we have the following
$\mathcal{K}_1(\alpha) = \mu(\alpha) \cup \mho(\alpha) = (\mathcal{X}_2 \cup \mathcal{X}_3, \mathcal{Y}_2 \cup \mathcal{Y}_3)$.
Assume $(\eta, \pi) \cup (\mathcal{K}_1, \vartheta \cap Ḑ) = (\mathcal{K}_2, \pi \cap (\vartheta \cap Ḑ))$ then for all $\alpha \in \pi \cap (\vartheta \cap Ḑ)$ we have the following
$\mathcal{K}_2(\alpha) = \eta(\alpha) \cup \mathcal{K}_1(\alpha) = (\mathcal{X}_1 \cup (\mathcal{X}_2 \cup \mathcal{X}_3), \mathcal{Y}_1 \cup (\mathcal{Y}_2 \cup \mathcal{Y}_3)) = ((\mathcal{X}_1 \cup \mathcal{X}_2) \cup \mathcal{X}_3, (\mathcal{Y}_1 \cup \mathcal{Y}_2) \cup \mathcal{Y}_3)$ by associative property. Hence, $\mathcal{K}_2(\alpha) = (\eta(\alpha) \cup \mu(\alpha)) \cup \mho(\alpha)$. Then, we get for all $\alpha \in \pi \cap (\vartheta \cap Ḑ) = (\pi \cap \vartheta) \cap Ḑ$
$(\eta, \pi) \widetilde{\cup} ((\mu, \vartheta) \widetilde{\cup} (\mho, Ḑ)) = ((\eta, \pi) \widetilde{\cup} (\mu, \vartheta)) \widetilde{\cup} (\mho, Ḑ)$.

8. Suppose that $(\eta,\pi)\ \widetilde{\subseteq}\ (\mu,\vartheta)$, then $\pi \subseteq \vartheta$ for all $\alpha \in \pi$, $\mathcal{X}_1 \subseteq \mathcal{X}_2 \subseteq \coprod_1$ and $\mathcal{Y}_1 \subseteq \mathcal{Y}_2 \subseteq \coprod_2$ such that $\eta(\alpha) = (\mathcal{X}_1, \mathcal{Y}_1)$, $\mu(\alpha) = (\mathcal{X}_2, \mathcal{Y}_2)$. Now, $\eta(\alpha) \cup \mu(\alpha) = (\mathcal{X}_1 \cup \mathcal{X}_2, \mathcal{Y}_1 \cup \mathcal{Y}_2) = (\mathcal{X}_2, \mathcal{Y}_2) = \mu(\alpha)$. Therefore, $(\eta,\pi)\ \widetilde{\cup}\ (\mu,\vartheta) = (\mu,\vartheta)$.

Conversely, suppose $(\eta,\pi)\ \widetilde{\cup}\ (\mu,\vartheta) = (\mu,\vartheta)$, then we have the following $\eta(\alpha) \cup \mu(\alpha) = \mu(\alpha)$ for all $\alpha \in \pi$ and $\eta(\alpha) = (\mathcal{X}_1, \mathcal{Y}_1)$, $\mu(\alpha) = (\mathcal{X}_2, \mathcal{Y}_2)$. Let $h \in \mathcal{X}_1$ and $k \in \mathcal{Y}_1$, then $h \in \mathcal{X}_1 \cup \mathcal{X}_2$ and $k \in \mathcal{Y}_1 \cup \mathcal{Y}_2$, we have $\mathcal{X}_1 \cup \mathcal{X}_2 = \mathcal{X}_2$ and $\mathcal{Y}_1 \cup \mathcal{Y}_2 = \mathcal{Y}_2$, therefore $h \in \mathcal{X}_2$ and $k \in \mathcal{Y}_2$ and $\mathcal{X}_1 \subseteq \mathcal{X}_2$ and $\mathcal{Y}_1 \subseteq \mathcal{Y}_2$. Hence, $\eta(\alpha) \subseteq \mu(\alpha)$. Thus, $(\eta,\pi)\ \widetilde{\subseteq}\ (\mu,\vartheta)$.

The remaining parts can be proved with the same method. □

Proposition 3.27. *Let $(\eta,\pi), (\mu,\vartheta)$ and $(\mho,Ḑ)$ be three BnHySSs. Then we have the following results:*

1. $(\eta,\pi)\ \widetilde{\cap}\ (\eta,\pi) = (\eta,\pi)$.
2. $(\eta,\pi)\ \widetilde{\cap}\ (\mu,\vartheta) = (\mu,\vartheta)\ \widetilde{\cap}\ (\eta,\pi)$.
3. $(\eta,\pi)\widetilde{\cap}\ ((\mu,\vartheta)\ \widetilde{\cap}\ (\mho,Ḑ)) = ((\eta,\pi)\ \widetilde{\cap}\ (\mu,\vartheta))\ \widetilde{\cap}\ (\mho,Ḑ)$.
4. $(\eta,\pi)\ \widetilde{\cap}\ \widetilde{\varnothing} = \widetilde{\varnothing}$.
5. $(\eta,\pi)\ \widetilde{\cap}\ \widetilde{\coprod} = (\eta,\pi)$.
6. $(\eta,\pi)\ \widetilde{\cap}\ (\mu,\vartheta)\ \widetilde{\subseteq}\ (\eta,\pi)$ *and* $(\eta,\pi)\ \widetilde{\cap}\ (\mu,\vartheta)\ \widetilde{\subseteq}\ (\mu,\vartheta)$.
7. $(\eta,\pi)\ \widetilde{\subseteq}\ (\mu,\vartheta)$ *if and only if* $(\eta,\pi)\ \widetilde{\cap}\ (\mu,\vartheta) = (\eta,\pi)$.

Proof. We can prove all parts by the same way in the above proposition. □

Proposition 3.28. *Let (η,π) and (μ,ϑ) be two BnHySSs. Then we have the following results:*

1. $(\eta,\pi)\ \widetilde{\cup}\ (\eta,\pi)^c = \widetilde{\coprod}$.
2. $(\eta,\pi)\ \widetilde{\cap}\ (\eta,\pi)^c = \widetilde{\varnothing}$.
3. $(\eta,\pi)\ \widetilde{\subseteq}\ (\mu,\vartheta)$ *if and only if* $(\mu,\vartheta)^c\ \widetilde{\subseteq}\ (\eta,\pi)^c$.
4. $((\eta,\pi)\ \widetilde{\cup}\ (\mu,\vartheta))^c = (\eta,\pi)^c\ \widetilde{\cap}\ (\mu,\vartheta)^c$.
5. $((\eta,\pi)\ \widetilde{\cap}\ (\mu,\vartheta))^c = (\eta,\pi)^c\ \widetilde{\cup}\ (\mu,\vartheta)^c$.

Proof. 4. Let $(\eta,\pi)\ \widetilde{\cup}\ (\mu,\vartheta) = (\mho,Ḑ)$ where $\mho(\alpha) = \eta(\alpha) \cup \mu(\alpha)$ for all $\alpha \in Ḑ = \pi \cap \vartheta$. Since $((\eta,\pi)\ \widetilde{\cup}\ (\mu,\vartheta))^c = (\mho,Ḑ)^c$ we have $\mho^c(\alpha) = (\eta(\alpha) \cup \mu(\alpha))^c = \widetilde{\coprod} - (\eta(\alpha) \cup \mu(\alpha)) = \widetilde{\coprod} - \eta(\alpha) \cap \widetilde{\coprod} - \mu(\alpha) = \eta^c(\alpha) \cap \mu^c(\alpha)$. Now, $(\mho,Ḑ)^c = (\eta,\pi)^c\ \widetilde{\cap}\ (\mu,\vartheta)^c$.

Hence, $((\eta,\pi)\ \widetilde{\cup}\ (\mu,\vartheta))^c = (\eta,\pi)^c\ \widetilde{\cap}\ (\mu,\vartheta)^c$.

The remaining parts can be proved with the same method. □

Proposition 3.29. *Let $(\eta,\pi), (\mu,\vartheta)$ and $(\mho,Ḑ)$ be three BnHySSs. Then we have the following results:*

1. $(\eta,\pi)\ \widetilde{\cap}\ ((\mu,\vartheta)\ \widetilde{\cup}\ (\mho,Ḑ)) = ((\eta,\pi)\ \widetilde{\cap}\ (\mu,\vartheta))\ \widetilde{\cup}\ ((\eta,\pi)\widetilde{\cap}\ (\mho,Ḑ))$.
2. $(\eta,\pi)\ \widetilde{\cup}\ ((\mu,\vartheta)\ \widetilde{\cap}\ (\mho,Ḑ)) = ((\eta,\pi)\ \widetilde{\cup}\ (\mu,\vartheta))\ \widetilde{\cap}\ ((\eta,\pi)\ \widetilde{\cup}\ (\mho,Ḑ))$.

Proof. 1. Suppose that $(\mu,\vartheta)\ \widetilde{\cup}\ (\mho,Ḑ) = (\mathcal{K}_1, \vartheta \cap Ḑ)$ for all $\alpha \in \vartheta \cap Ḑ$, $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3 \subseteq \coprod_1$ and $\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3 \subseteq \coprod_2$ such that $\eta(\alpha) = (\mathcal{X}_1, \mathcal{Y}_1)$, $\mu(\alpha) = (\mathcal{X}_2, \mathcal{Y}_2)$, $\mho(\alpha) = (\mathcal{X}_3, \mathcal{Y}_3)$ we have the following
$\mathcal{K}_1(\alpha) = \mu(\alpha) \cup \mho(\alpha) = (\mathcal{X}_2 \cup \mathcal{X}_3, \mathcal{Y}_2 \cup \mathcal{Y}_3)$.
Assume $(\eta,\pi)\ \widetilde{\cap}\ (\mathcal{K}_1, \vartheta \cap Ḑ) = (\mathcal{K}_2, \pi \cap (\vartheta \cap Ḑ))$ for all $\alpha \in \pi \cap (\vartheta \cap Ḑ)$
$\mathcal{K}_2(\alpha) = \eta(\alpha) \cap \mathcal{K}_1(\alpha) = (\mathcal{X}_1 \cap (\mathcal{X}_2 \cup \mathcal{X}_3), \mathcal{Y}_1 \cap (\mathcal{Y}_2 \cup \mathcal{Y}_3)) = ((\mathcal{X}_1 \cap \mathcal{X}_2) \cup (\mathcal{X}_1 \cap \mathcal{X}_3), (\mathcal{Y}_1 \cap \mathcal{Y}_2) \cup (\mathcal{Y}_1 \cap \mathcal{Y}_3))$ by distribution property $\mathcal{K}_2(\alpha) = ((\eta(\alpha) \cap \mu(\alpha) \cup (\eta(\alpha) \cap \mho(\alpha))$ for all $\alpha \in \pi \cap (\vartheta \cap Ḑ) = (\pi \cap \vartheta) \cap (\pi \cap Ḑ)$.
Hence, $(\eta,\pi)\ \widetilde{\cap}\ ((\mu,\vartheta)\ \widetilde{\cup}\ (\mho,Ḑ)) = ((\eta,\pi)\ \widetilde{\cap}\ (\mu,\vartheta))\ \widetilde{\cup}\ ((\eta,\pi)\widetilde{\cap}\ (\mho,Ḑ))$. □

Proposition 3.30. *Let $(\eta,\pi), (\mu,\vartheta)$ and $(\mho,Ḑ)$ be three BnHySSs. Then we have the following results:*

1. $((\eta,\pi)\ \widetilde{\vee}\ (\mu,\vartheta))^c = (\eta,\pi)^c\ \widetilde{\wedge}\ (\mu,\vartheta)^c$.

2. $((\eta,\pi)\,\widetilde{\wedge}\,(\mu,\vartheta))^c = (\eta,\pi)^c\,\widetilde{\vee}\,(\mu,\vartheta)^c$.
3. $((\eta,\pi)\,\widetilde{\vee}\,(\mu,\vartheta))\,\widetilde{\vee}\,(\mho,\c{D}) = (\eta,\pi)\,\widetilde{\vee}\,((\mu,\vartheta)\,\widetilde{\vee}\,(\mho,\c{D}))$.
4. $((\eta,\pi)\,\widetilde{\wedge}\,(\mu,\vartheta))\,\widetilde{\wedge}\,(\mho,\c{D}) = (\eta,\pi)\,\widetilde{\wedge}\,((\mu,\vartheta)\,\widetilde{\wedge}\,(\mho,\c{D}))$.
5. $(\eta,\pi)\,\widetilde{\wedge}\,((\mu,\vartheta)\,\widetilde{\vee}\,(\mho,\c{D})) = ((\eta,\pi)\,\widetilde{\wedge}\,(\mu,\vartheta))\,\widetilde{\vee}\,((\eta,\pi)\,\widetilde{\wedge}\,(\mho,\c{D}))$.
6. $(\eta,\pi)\,\widetilde{\vee}\,((\mu,\vartheta)\,\widetilde{\wedge}\,(\mho,\c{D})) = ((\eta,\pi)\,\widetilde{\vee}\,(\mu,\vartheta))\,\widetilde{\wedge}\,((\eta,\pi)\,\widetilde{\vee}\,(\mho,\c{D}))$.

Proof. It is obvious. □

4. Binary hypersoft topology

In this section, we introduce the concept of binary hypersoft topology and we present the concept of binary hypersoft subspace.

Let $\coprod_1, \coprod_2$ be two initial universe sets and the non empty set $\c{€}$ be an entire set of parameters. Let $\beta(\coprod_1), \beta(\coprod_2)$ indicate the power set of $\coprod_1, \coprod_2$, respectively. Also, let $\varnothing \neq \pi_i \subseteq \c{€}$ with $i = 1, 2, ..., n$. To make things simpler, we write the symbol π for $\pi_1 \times \pi_2 \times \cdots \times \pi_n$ and ϱ for an element of the set π. We also suppose that none of the set π_i is empty for each i.

Definition 4.1. Let $\tau_{\mathcal{B}n\mathcal{H}y}$ be the collection of BnHySSs over $\coprod_1, \coprod_2$, then $\tau_{\mathcal{B}n\mathcal{H}y}$ is claimed to be a binary hypersoft topology on $\coprod_1, \coprod_2$ if the following three conditions hold:

1. $\widetilde{\varnothing}, \widetilde{\coprod} \,\widetilde{\in}\, \tau_{\mathcal{B}n\mathcal{H}y}$.
2. The BnHyS intersection of two BnHySSs in $\tau_{\mathcal{B}n\mathcal{H}y}$ belongs to $\tau_{\mathcal{B}n\mathcal{H}y}$.
3. The BnHyS union of any number of BnHySSs in $\tau_{\mathcal{B}n\mathcal{H}y}$ belongs to $\tau_{\mathcal{B}n\mathcal{H}y}$.

Then $(\coprod_1, \coprod_2, \tau_{\mathcal{B}n\mathcal{H}y}, \pi)$ is known as a binary hypersoft topological space (BnHySTS) over $\coprod_1, \coprod_2$.

Definition 4.2. Let $(\coprod_1, \coprod_2, \tau_{\mathcal{B}n\mathcal{H}y}, \pi)$ be a BnHySTS over $\coprod_1, \coprod_2$, then the members of $\tau_{\mathcal{B}n\mathcal{H}y}$ are claimed to be a BnHyS open sets in $\coprod_1, \coprod_2$.

Definition 4.3. Let $(\coprod_1, \coprod_2, \tau_{\mathcal{B}n\mathcal{H}y}, \pi)$ be a BnHySTS over $\coprod_1, \coprod_2$, then the BnHySS (η, π) over $\coprod_1, \coprod_2$ is claimed to be a BnHyS closed set in $\coprod_1, \coprod_2$, if its BnHyS complement is belong to $\tau_{\mathcal{B}n\mathcal{H}y}$.

Definition 4.4. Let $\tau_{\mathcal{B}n\mathcal{H}y} = \{\widetilde{\varnothing}, \widetilde{\coprod}\}$. Then $\tau_{\mathcal{B}n\mathcal{H}y}$ is known as the BnHyS indiscrete topology on $\coprod_1, \coprod_2$ and $(\coprod_1, \coprod_2, \tau^{ind}_{\mathcal{B}n\mathcal{H}y}, \pi)$ is known as a BnHyS indiscrete space over $\coprod_1, \coprod_2$.

Definition 4.5. Assume that the collection of all BnHySSs that can be specified over $\coprod_1, \coprod_2$ is $\tau^{dis}_{\mathcal{B}n\mathcal{H}y}$. Then $\tau^{dis}_{\mathcal{B}n\mathcal{H}y}$ is known as the BnHyS discrete topology on $\coprod_1, \coprod_2$ and $(\coprod_1, \coprod_2, \tau^{dis}_{\mathcal{B}n\mathcal{H}y}, \pi)$ is known as a BnHyS discrete space over $\coprod_1, \coprod_2$.

Example 4.6. Consider the following sets: $\coprod_1 = \{\rho_1, \rho_2, \rho_3, \rho_4, \rho_5\}$, $\coprod_2 = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$,
$\c{€} = \{\varrho_1, \varrho_2, \varrho_3, \varrho_4\}$, $\pi_1 = \{\varrho_1\}$, $\pi_2 = \{\varrho_2\}$, $\pi_3 = \{\varrho_3, \varrho_4\}$ and
$\tau_{\mathcal{B}n\mathcal{H}y} = \{\widetilde{\varnothing}, \widetilde{\coprod}, (\eta_1, \pi), (\eta_2, \pi), (\eta_3, \pi), (\eta_4, \pi)\}$
where $(\eta_1, \pi), (\eta_2, \pi), (\eta_3, \pi), (\eta_4, \pi)$ are BnHySSs defined as follow:
$(\eta_1, \pi) = \{((\varrho_1, \varrho_2, \varrho_3), (\{\rho_2\}, \{\sigma_2\})), ((\varrho_1, \varrho_2, \varrho_4), (\{\rho_3\}, \{\sigma_3\}))\}$.
$(\eta_2, \pi) = \{((\varrho_1, \varrho_2, \varrho_3), (\{\rho_3\}, \{\sigma_1\})), ((\varrho_1, \varrho_2, \varrho_4), (\{\rho_3, \rho_5\}, \{\sigma_1, \sigma_2\}))\}$.
$(\eta_3, \pi) = \{((\varrho_1, \varrho_2, \varrho_3), (\{\rho_2, \rho_3\}, \{\sigma_1, \sigma_2\})), ((\varrho_1, \varrho_2, \varrho_4), (\{\rho_3, \rho_5\}, \{\sigma_1, \sigma_2, \sigma_3\}))\}$.
$(\eta_4, \pi) = \{((\varrho_1, \varrho_2, \varrho_3), (\varnothing, \varnothing)), ((\varrho_1, \varrho_2, \varrho_4), (\{\rho_3\}, \varnothing))\}$.
Clearly $\tau_{\mathcal{B}n\mathcal{H}y}$ is a BnHyST over $\coprod_1, \coprod_2$.

Theorem 4.7. *Let $(\coprod_1, \coprod_2, \tau_{\mathcal{B}n\mathcal{H}y}, \pi)$ and $(\coprod_1, \coprod_2, \tau'_{\mathcal{B}n\mathcal{H}y}, \pi)$ be two BnHySTSs over the same universal sets $\coprod_1, \coprod_2$, then $(\coprod_1, \coprod_2, \tau_{\mathcal{B}n\mathcal{H}y} \cap \tau'_{\mathcal{B}n\mathcal{H}y}, \pi)$ is a BnHySTS over $\coprod_1, \coprod_2$.*

Proof. (1) $\widetilde{\emptyset}$, $\widetilde{\coprod}$ are belongs to $\tau_{\mathcal{B}n\mathcal{H}y} \cap \tau'_{\mathcal{B}n\mathcal{H}y}$.
(2) Let the two BnHySSs $(\eta,\pi),(\mu,\pi) \in \tau_{\mathcal{B}n\mathcal{H}y} \cap \tau'_{\mathcal{B}n\mathcal{H}y}$. Then $(\eta,\pi),(\mu,\pi) \in \tau_{\mathcal{B}n\mathcal{H}y}$ and $(\eta,\pi),(\mu,\pi) \in \tau'_{\mathcal{B}n\mathcal{H}y}$. Since $(\eta,\pi)\,\widetilde{\cap}\,(\mu,\pi) \in \tau_{\mathcal{B}n\mathcal{H}y}$ and $(\eta,\pi)\,\widetilde{\cap}\,(\mu,\pi) \in \tau'_{\mathcal{B}n\mathcal{H}y}$, so $(\eta,\pi)\,\widetilde{\cap}\,(\mu,\pi) \in \tau_{\mathcal{B}n\mathcal{H}y} \cap \tau'_{\mathcal{B}n\mathcal{H}y}$.
(3) Let $\{(\eta_i,\pi) \mid i \in I\}$ be a family of BnHySSs in $\tau_{\mathcal{B}n\mathcal{H}y} \cap \tau'_{\mathcal{B}n\mathcal{H}y}$. Then $(\eta_i,\pi) \in \tau_{\mathcal{B}n\mathcal{H}y}$ and $(\eta_i,\pi) \in \tau'_{\mathcal{B}n\mathcal{H}y}$, for each $i \in I$, so $\widetilde{\cup}_{(i\in I)}(\eta_i,\pi) \in \tau_{\mathcal{B}n\mathcal{H}y}$ and $\widetilde{\cup}_{(i\in I)}(\eta_i,\pi) \in \tau'_{\mathcal{B}n\mathcal{H}y}$.
Thus, $\widetilde{\cup}_{(i\in I)}(\eta_i,\pi) \in \tau_{\mathcal{B}n\mathcal{H}y} \cap \tau'_{\mathcal{B}n\mathcal{H}y}$.
Hence, $\tau_{\mathcal{B}n\mathcal{H}y} \cap \tau'_{\mathcal{B}n\mathcal{H}y}$ defines the BnHyST on $\coprod_1,\coprod_2$ and $(\coprod_1,\coprod_2,\tau_{\mathcal{B}n\mathcal{H}y}\,\widetilde{\cap}\,\tau'_{\mathcal{B}n\mathcal{H}y},\pi)$ is a BnHySTS over $\coprod_1,\coprod_2$. □

Remark 4.8. Let $(\coprod_1,\coprod_2,\tau_{\mathcal{B}n\mathcal{H}y},\pi)$ and $(\coprod_1,\coprod_2,\tau'_{\mathcal{B}n\mathcal{H}y},\pi)$ be two BnHySTS over the same universal sets $\coprod_1,\coprod_2$, then $(\coprod_1,\coprod_2,\tau_{\mathcal{B}n\mathcal{H}y} \cup \tau'_{\mathcal{B}n\mathcal{H}y},\pi)$ may not be BnHySTS over $\coprod_1,\coprod_2$.

Example 4.9. Let $\coprod_1 = \{\rho_1,\rho_2,\rho_3,\rho_4,\rho_5\}$, $\coprod_2 = \{\sigma_1,\sigma_2,\sigma_3,\sigma_4\}$, $₤= \{\varrho_1,\varrho_2,\varrho_3,\varrho_4\}$,
$\pi_1 = \{\varrho_1,\varrho_2\}$, $\pi_2 = \{\varrho_3,\varrho_4\}$ and $\tau_{\mathcal{B}n\mathcal{H}y} = \{\widetilde{\emptyset}\ ,\ \widetilde{\coprod},(\eta_1,\pi),(\eta_2,\pi),(\eta_3,\pi),(\eta_4,\pi)\}$,
where $(\eta_1,\pi),(\eta_2,\pi),(\eta_3,\pi),(\eta_4,\pi)$ are BnHySSs defined as follow:
$(\eta_1,\pi)=\{((\varrho_1,\varrho_3),(\{\rho_1\},\{\sigma_1\})),((\varrho_1,\varrho_4),(\{\rho_2\},\{\sigma_2\})),((\varrho_2,\varrho_3),(\emptyset,\emptyset)),((\varrho_2,\varrho_4),(\{\rho_3\},\{\sigma_3\}))\}$.
$(\eta_2,\pi) = \{((\varrho_1,\varrho_3),(\{\rho_4\},\{\sigma_4\})),((\varrho_1,\varrho_4),(\{\rho_3\},\{\sigma_1\})),((\varrho_2,\varrho_3),(\{\rho_1,\rho_2\},\{\sigma_3\})),((\varrho_2,\varrho_4),(\{\rho_3,\rho_5\},\{\sigma_1,\sigma_2\}))\}$.
$(\eta_3,\pi)=\{((\varrho_1,\varrho_3),(\{\rho_1,\rho_4\},\{\sigma_1,\sigma_4\})),((\varrho_1,\varrho_4),(\{\rho_2,\rho_3\},\{\sigma_1,\sigma_2\})),((\varrho_2,\varrho_3),(\{\rho_1,\rho_2\},\{\sigma_3\})),((\varrho_2,\varrho_4),(\{\rho_3,\rho_5\},\{\sigma_1,\sigma_2,\sigma_3\}))\}$.
$(\eta_4,\pi) = \{((\varrho_1,\varrho_3),(\emptyset,\emptyset)),((\varrho_1,\varrho_4),(\emptyset,\emptyset)),((\varrho_2,\varrho_3),(\emptyset,\emptyset)),((\varrho_2,\varrho_4),(\{\rho_3\},\emptyset))\}$.
and
$\tau'_{\mathcal{B}n\mathcal{H}y} = \{\widetilde{\emptyset},\widetilde{\coprod},(\mu_1,\pi),(\mu_2,\pi),(\mu_3,\pi),(\mu_4,\pi)\}$,
where $(\mu_1,\pi),(\mu_2,\pi),(\mu_3,\pi),(\mu_4,\pi)$ are BnHySSs defined as follow:
$(\mu_1,\pi)=\{((\varrho_1,\varrho_3),(\{\rho_2\},\{\sigma_2\})),((\varrho_1,\varrho_4),(\{\rho_3,\rho_4\},\{\sigma_1,\sigma_3\})),((\varrho_2,\varrho_3),(\{\rho_1,\rho_3\},\{\sigma_2\})),((\varrho_2,\varrho_4),(\emptyset,\emptyset))\}$.
$(\mu_2,\pi)=\{((\varrho_1,\varrho_3),(\{\rho_3\},\{\sigma_1\})),((\varrho_1,\varrho_4),(\{\rho_5\},\{\sigma_3\})),((\varrho_2,\varrho_3),(\{\rho_1,\rho_2\},\{\sigma_3\})),((\varrho_2,\varrho_4),(\{\rho_3,\rho_5\},\{\sigma_1,\sigma_2\}))\}$.
$(\mu_3,\pi)=\{((\varrho_1,\varrho_3),(\{\rho_2,\rho_3\},\{\sigma_1,\sigma_2\})),((\varrho_1,\varrho_4),(\{\rho_3,\rho_4,\rho_5\},\{\sigma_1,\sigma_3\})),((\varrho_2,\varrho_3),(\{\rho_1,\rho_2,\rho_3\},\{\sigma_2,\sigma_3\})),((\varrho_2,\varrho_4),(\{\rho_3,\rho_5\},\{\sigma_1,\sigma_2\}))\}$.
$(\mu_4,\pi)=\{((\varrho_1,\varrho_3),(\emptyset,\emptyset)),((\varrho_1,\varrho_4),(\emptyset,\{\sigma_3\})),((\varrho_2,\varrho_3),(\{\rho_1\},\emptyset)),((\varrho_2,\varrho_4),(\emptyset,\emptyset))\}$.
Clearly, $\tau_{\mathcal{B}n\mathcal{H}y}$ and $\tau'_{\mathcal{B}n\mathcal{H}y}$ are BnHySTS.
Then
$\tau_{\mathcal{B}n\mathcal{H}y} \cup \tau'_{\mathcal{B}n\mathcal{H}y} = \{\widetilde{\emptyset}\ ,\ \widetilde{\coprod},(\eta_1,\pi),(\eta_2,\pi),(\eta_3,\pi),(\eta_4,\pi),(\mu_1,\pi),(\mu_2,\pi),(\mu_3,\pi),(\mu_4,\pi)\}$.
Clearly,
$(\eta_1,\pi)\widetilde{\cup}(\mu_1,\pi) = \{((\varrho_1,\varrho_3),(\{\rho_1,\rho_2\},\{\sigma_1,\sigma_2\})),((\varrho_1,\varrho_4),(\{\rho_2,\rho_3,\rho_4\},\{\sigma_1,\sigma_2,\sigma_3\})),((\varrho_2,\varrho_3),(\{\rho_1,\rho_3\},\{\sigma_2\})),((\varrho_2,\varrho_4),(\{\rho_3\},\{\sigma_3\}))\}$.
Since $(\eta_1,\pi)\widetilde{\cup}(\mu_1,\pi) \notin \tau_{\mathcal{B}n\mathcal{H}y} \cup \tau'_{\mathcal{B}n\mathcal{H}y}$.
Thus, $\tau_{\mathcal{B}n\mathcal{H}y} \cup \tau'_{\mathcal{B}n\mathcal{H}y}$ is not BnHyST.

Proposition 4.10. *Let $(\coprod_1,\coprod_2,\tau_{\mathcal{B}n\mathcal{H}y},\pi)$ be a BnHySTS over $\coprod_1,\coprod_2$ and $(\eta,\pi) \in \tau_{\mathcal{B}n\mathcal{H}y}$, where $\eta(\alpha) = (\mathcal{X},\mathcal{Y})$ for each $\alpha \in \pi_1\times\pi_2\times\ldots\times\pi_n$, $\mathcal{X} \subseteq \coprod_1$ and $\mathcal{Y} \subseteq \coprod_2$. Suppose that the collections $\tau_{\mathcal{H}} = \{(\lambda,\pi) \mid \lambda(\alpha) = \mathcal{X}\}$ and $\tau^*_{\mathcal{H}} = \{(\delta,\pi) \mid \delta(\alpha) = \mathcal{Y}\}$. Then $\tau_{\mathcal{H}}$ and $\tau^*_{\mathcal{H}}$ are defines HySTs on $\coprod_1$ and $\coprod_2$ respectively.*

Example 4.11. Consider the following sets: $\coprod_1 = \{\rho_1,\rho_2,\rho_3,\rho_4,\rho_5\}$, $\coprod_2 = \{\sigma_1,\sigma_2,\sigma_3,\sigma_4\}$,
$₤ = \{\varrho_1,\varrho_2,\varrho_3,\varrho_4\}$, $\pi_1 = \{\varrho_1\}$, $\pi_2 = \{\varrho_2\}$, $\pi_3 = \{\varrho_3,\varrho_4\}$ and $\tau_{\mathcal{B}n\mathcal{H}y} = \{\widetilde{\emptyset},\widetilde{\coprod},(\eta_1,\pi),(\eta_2,\pi),(\eta_3,\pi),(\eta_4,\pi)\}$
where $(\eta_1,\pi),(\eta_2,\pi),(\eta_3,\pi),(\eta_4,\pi)$ are BnHySSs defined as follow:
$(\eta_1,\pi) = \{((\varrho_1,\varrho_2,\varrho_3),(\{\rho_2\},\{\sigma_2\})),((\varrho_1,\varrho_2,\varrho_4),(\{\rho_3\},\{\sigma_3\}))\}$,
$(\eta_2,\pi) = \{((\varrho_1,\varrho_2,\varrho_3),(\{\rho_3\},\{\sigma_1\})),((\varrho_1,\varrho_2,\varrho_4),(\{\rho_3,\rho_5\},\{\sigma_1,\sigma_2\}))\}$,
$(\eta_3,\pi) = \{((\varrho_1,\varrho_2,\varrho_3),(\{\rho_2,\rho_3\},\{\sigma_1,\sigma_2\})),((\varrho_1,\varrho_2,\varrho_4),(\{\rho_3,\rho_5\},\{\sigma_1,\sigma_2,\sigma_3\}))\}$ and
$(\eta_4,\pi) = \{((\varrho_1,\varrho_2,\varrho_3),(\emptyset,\emptyset)),((\varrho_1,\varrho_2,\varrho_4),(\{\rho_3\},\emptyset))\}$.
Clearly $\tau_{\mathcal{B}n\mathcal{H}y}$ is a BnHyST over $\coprod_1,\coprod_2$. It can be easily seen that

$\tau_{\mathcal{H}} = \{\widetilde{\varnothing}, \widetilde{\coprod}, (\lambda_1, \pi), (\lambda_2, \pi), (\lambda_3, \pi), (\lambda_4, \pi)\}$
where $(\lambda_1, \pi), (\lambda_2, \pi), (\lambda_3, \pi), (\lambda_4, \pi)$ are HySSs defined as follow:
$(\lambda_1, \pi) = \{((\varrho_1, \varrho_2, \varrho_3), \{\rho_2\}), ((\varrho_1, \varrho_2, \varrho_4), \{\rho_3\})\}$
$(\lambda_2, \pi) = \{((\varrho_1, \varrho_2, \varrho_3), \{\rho_3\}), ((\varrho_1, \varrho_2, \varrho_4), \{\rho_3, \rho_5\})\}$
$(\lambda_3, \pi) = \{((\varrho_1, \varrho_2, \varrho_3), \{\rho_2, \rho_3\}), ((\varrho_1, \varrho_2, \varrho_4), \{\rho_3, \rho_5\})\}$
$(\lambda_4, \pi) = \{((\varrho_1, \varrho_2, \varrho_3), \varnothing), ((\varrho_1, \varrho_2, \varrho_4), \{\rho_3\})\}$
and
$\tau^*_{\mathcal{H}} = \{\widetilde{\varnothing}, \widetilde{\coprod}, (\delta_1, \pi), (\delta_2, \pi), (\delta_3, \pi), (\delta_4, \pi)\}$
where $(\delta_1, \pi), (\delta_2, \pi), (\delta_3, \pi), (\delta_4, \pi)$ are HySSs defined as follow:
$(\delta_1, \pi) = \{((\varrho_1, \varrho_2, \varrho_3), \{\sigma_2\}), ((\varrho_1, \varrho_2, \varrho_4), \{\sigma_3\})\}$
$(\delta_2, \pi) = \{((\varrho_1, \varrho_2, \varrho_3), \{\sigma_1\}), ((\varrho_1, \varrho_2, \varrho_4), \{\sigma_1, \sigma_2\})\}$
$(\delta_3, \pi) = \{((\varrho_1, \varrho_2, \varrho_3), \{\sigma_1, \sigma_2\}), ((\varrho_1, \varrho_2, \varrho_4), \{\sigma_1, \sigma_2, \sigma_3\})\}$
$(\delta_4, \pi) = \{((\varrho_1, \varrho_2, \varrho_3), \varnothing), ((\varrho_1, \varrho_2, \varrho_4), \varnothing)\}$
Then, $\tau_{\mathcal{H}}$ is an HyST on $\coprod_1$ and $\tau^*_{\mathcal{H}}$ is an HyST on $\coprod_2$.

Now we give an example to show that the converse of Proposition 4.10 does not hold.

Example 4.12. Consider the following sets: $\coprod_1 = \{\rho_1, \rho_2, \rho_3, \rho_4, \rho_5\}$, $\coprod_2 = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$, $\pounds_{\mathfrak{z}} = \{\varrho_1, \varrho_2, \varrho_3, \varrho_4\}$, $\pi_1 = \{\varrho_1\}$, $\pi_2 = \{\varrho_2\}$ and $\pi_3 = \{\varrho_3, \varrho_4\}$.
Suppose that $\tau_{\mathcal{H}} = \{\widetilde{\varnothing}, \widetilde{\coprod}, (\lambda_1, \pi), (\lambda_2, \pi), (\lambda_3, \pi), (\lambda_4, \pi)\}$
where $(\lambda_1, \pi), (\lambda_2, \pi), (\lambda_3, \pi), (\lambda_4, \pi)$ are HySSs defined as follow:
$(\lambda_1, \pi) = \{((\varrho_1, \varrho_2, \varrho_3), \{\rho_2\}), ((\varrho_1, \varrho_2, \varrho_4), \{\rho_3\})\}$,
$(\lambda_2, \pi) = \{((\varrho_1, \varrho_2, \varrho_3), \{\rho_3\}), ((\varrho_1, \varrho_2, \varrho_4), \{\rho_3, \rho_5\})\}$,
$(\lambda_3, \pi) = \{((\varrho_1, \varrho_2, \varrho_3), \{\rho_2, \rho_3\}), ((\varrho_1, \varrho_2, \varrho_4), \{\rho_3, \rho_5\})\}$ and
$(\lambda_4, \pi) = \{((\varrho_1, \varrho_2, \varrho_3), \varnothing), ((\varrho_1, \varrho_2, \varrho_4), \{\rho_3\})\}$.
Also,
$\tau^*_{\mathcal{H}} = \{\widetilde{\varnothing}, \widetilde{\coprod}, (\delta_1, \pi), (\delta_2, \pi), (\delta_3, \pi), (\delta_4, \pi)\}$
where $(\delta_1, \pi), (\delta_2, \pi), (\delta_3, \pi), (\delta_4, \pi)$ are HySSs defined as follow:
$(\delta_1, \pi) = \{((\varrho_1, \varrho_2, \varrho_3), \varnothing), ((\varrho_1, \varrho_2, \varrho_4), \varnothing)\}$,
$(\delta_2, \pi) = \{((\varrho_1, \varrho_2, \varrho_3), \{\sigma_1, \sigma_2\}), ((\varrho_1, \varrho_2, \varrho_4), \{\sigma_1, \sigma_2, \sigma_3\})\}$,
$(\delta_3, \pi) = \{((\varrho_1, \varrho_2, \varrho_3), \{\sigma_1\}), ((\varrho_1, \varrho_2, \varrho_4), \{\sigma_1, \sigma_2\})\}$ and
$(\delta_4, \pi) = \{((\varrho_1, \varrho_2, \varrho_3), \{\sigma_2\}), ((\varrho_1, \varrho_2, \varrho_4), \{\sigma_3\})\}$.
Clearly, $\tau_{\mathcal{H}}$ is an HyST on $\coprod_1$ and $\tau^*_{\mathcal{H}}$ is an HyST on $\coprod_2$.
Now, $\tau_{\mathcal{B}n\mathcal{H}y} = \{\widetilde{\varnothing}, \widetilde{\coprod}, (\eta_1, \pi), (\eta_2, \pi), (\eta_3, \pi), (\eta_4, \pi)\}$ where
$(\eta_1, \pi) = \{((\varrho_1, \varrho_2, \varrho_3), (\{\rho_2\}, \varnothing)), ((\varrho_1, \varrho_2, \varrho_4), (\{\rho_3\}, \varnothing))\}$.
$(\eta_2, \pi) = \{((\varrho_1, \varrho_2, \varrho_3), (\{\rho_3\}, \{\sigma_1, \sigma_2\})), ((\varrho_1, \varrho_2, \varrho_4), (\{\rho_3, \rho_5\}, \{\sigma_1, \sigma_2, \sigma_3\}))\}$.
$(\eta_3, \pi) = \{((\varrho_1, \varrho_2, \varrho_3), (\{\rho_2, \rho_3\}, \{\sigma_1\})), ((\varrho_1, \varrho_2, \varrho_4), (\{\rho_3, \rho_5\}, \{\sigma_1, \sigma_2\}))\}$.
$(\eta_4, \pi) = \{((\varrho_1, \varrho_2, \varrho_3), (\varnothing, \{\sigma_2\})), ((\varrho_1, \varrho_2, \varrho_4), (\{\rho_3\}, \{\sigma_3\}))\}$.
Since $(\eta_1, \pi) \widetilde{\cup} (\eta_2, \pi) \notin \tau_{\mathcal{B}n\mathcal{H}y}$.
Then, $\tau_{\mathcal{B}n\mathcal{H}y}$ is not a BnHyST over $\coprod_1, \coprod_2$.

The converse of Proposition 4.10 holds under certain conditions, as stated in the following proposition.

Proposition 4.13. *Suppose that $\tau_{\mathcal{B}n\mathcal{H}y} = \{(\eta, \pi) \mid \eta(\alpha) = (\mathcal{X}, \mathcal{Y})$ for each $\alpha \in \pi_1 \times \pi_2 \times ... \times \pi_n$ such that $\mathcal{X} \subseteq \coprod_1$ and $\mathcal{Y} \subseteq \coprod_2\}$. Suppose that $\tau_{\mathcal{H}} = \{(\lambda, \pi) \mid \lambda(\alpha) = \mathcal{X}\}$ is an HyST on $\coprod_1$ and $\tau^*_{\mathcal{H}} = \{(\delta, \pi) \mid \delta(\alpha) = \mathcal{Y}\}$ is an HyST on $\coprod_2$. If $(\lambda_p(\alpha), \delta_p(\alpha)) = \eta_0(\alpha) \in \tau_{\mathcal{B}n\mathcal{H}y}$ where $\lambda_p(\alpha) = \bigcap_{i=1,...,n} \lambda_i(\alpha) \in \tau_{\mathcal{H}}$ and $\delta_p(\alpha) = \bigcap_{i=1,...,n} \delta_i(\alpha) \in \tau^*_{\mathcal{H}}$ and also if $(\lambda_q(\alpha), \delta_q(\alpha)) = \eta_c(\alpha) \in \tau_{\mathcal{B}n\mathcal{H}y}$ where $\lambda_q(\alpha) = \bigcup_{j\in J} \lambda_j(\alpha) \in \tau_{\mathcal{H}}$ and $\delta_q(\alpha) = \bigcup_{j\in J} \delta_j(\alpha) \in \tau^*_{\mathcal{H}}$ for each $\alpha \in \pi$, then $\tau_{\mathcal{B}n\mathcal{H}y}$ is define a BnHyST over $\coprod_1, \coprod_2$.*

Proof. Obvious. □

Definition 4.14. Let $(\coprod_1, \coprod_2, \tau_{\mathcal{B}n\mathcal{H}y}, \pi)$ be a BnHySTS over the common universe sets $\coprod_1, \coprod_2$ and $\mathcal{V}_1, \mathcal{V}_2$ be non empty subsets of $\coprod_1, \coprod_2$, respectively. Then $\tau_{(\mathcal{V}_1, \mathcal{V}_2)} = \{(((\mathcal{V}_1, \mathcal{V}_2)_\eta, \pi) \mid (\eta, \pi) \in \tau_{\mathcal{B}n\mathcal{H}y}\}$ is claimed to be the BnHyS relative topology over $\mathcal{V}_1, \mathcal{V}_2$ and $(\mathcal{V}_1, \mathcal{V}_2, \tau_{(\mathcal{V}_1,\mathcal{V}_2)}, \pi)$ is known as a BnHyS subspace of $(\coprod_1, \coprod_2, \tau_{\mathcal{B}n\mathcal{H}y}, \pi)$. We can easily verify that $\tau_{(\mathcal{V}_1,\mathcal{V}_2)}$ is infact a BnHyST over $\mathcal{V}_1, \mathcal{V}_2$.

Remark 4.15. Any BnHyS subspace of a BnHyS discrete topological space is a BnHyS discrete topological space.

Remark 4.16. Any BnHyS subspace of a BnHyS indiscrete topological space is a BnHyS indiscrete topological space.

Example 4.17. Let $\coprod_1 = \{\rho_1, \rho_2, \rho_3, \rho_4, \rho_5\}$, $\coprod_2 = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$, $\mathcal{V}_1 = \{\rho_1, \rho_3, \rho_4\}$, $\mathcal{V}_2 = \{\sigma_2, \sigma_4\}$, $\mathcal{E} = \{\varrho_1, \varrho_2, \varrho_3, \varrho_4\}$, $\pi_1 = \{\varrho_1, \varrho_2\}, \pi_2 = \{\varrho_3, \varrho_4\}$ and
$\tau_{\mathcal{B}n\mathcal{H}y} = \{\widetilde{\varnothing}, \widetilde{\coprod}, (\eta_1, \pi), (\eta_2, \pi), (\eta_3, \pi), (\eta_4, \pi)\}$,
where $(\eta_1, \pi), (\eta_2, \pi), (\eta_3, \pi), (\eta_4, \pi)$ are BnHySSs defined as follow:
$(\eta_1, \pi) = \{((\varrho_1, \varrho_3), (\{\rho_1\}, \{\sigma_1\})), ((\varrho_1, \varrho_4), (\{\rho_2\}, \{\sigma_2\})), ((\varrho_2, \varrho_3), (\varnothing, \varnothing)), ((\varrho_2, \varrho_4), (\{\rho_3\}, \{\sigma_3\}))\}$.
$(\eta_2, \pi) = \{((\varrho_1, \varrho_3), (\{\rho_4\}, \{\sigma_4\})), ((\varrho_1, \varrho_4), (\{\rho_3\}, \{\sigma_1\})), ((\varrho_2, \varrho_3), (\{\rho_1, \rho_2\}, \{\sigma_3\})), ((\varrho_2, \varrho_4), (\{\rho_3, \rho_5\}, \{\sigma_1, \sigma_2\}))\}$.
$(\eta_3, \pi) = \{((\varrho_1, \varrho_3), (\{\rho_1, \rho_4\}, \{\sigma_1, \sigma_4\})), ((\varrho_1, \varrho_4), (\{\rho_2, \rho_3\}, \{\sigma_1, \sigma_2\})), ((\varrho_2, \varrho_3), (\{\rho_1, \rho_2\}, \{\sigma_3\})), ((\varrho_2, \varrho_4), (\{\rho_3, \rho_5\}, \{\sigma_1, \sigma_2, \sigma_3\}))\}$.
$(\eta_4, \pi) = \{((\varrho_1, \varrho_3), (\varnothing, \varnothing)), ((\varrho_1, \varrho_4), (\varnothing, \varnothing)), ((\varrho_2, \varrho_3), (\varnothing, \varnothing)), ((\varrho_2, \varrho_4), (\{\rho_3\}, \varnothing))\}$.
Then the BnHyS subspace $(\mathcal{V}_1, \mathcal{V}_2, \tau_{(\mathcal{V}_1,\mathcal{V}_2)}, \pi)$ is described as follow:
$\tau_{(\mathcal{V}_1,\mathcal{V}_2)} = \{\widetilde{\varnothing}, \widetilde{\mathcal{V}}, ((\mathcal{V}_1, \mathcal{V}_2)_{\eta_1}, \pi), ((\mathcal{V}_1, \mathcal{V}_2)_{\eta_2}, \pi), ((\mathcal{V}_1, \mathcal{V}_2)_{\eta_3}, \pi), ((\mathcal{V}_1, \mathcal{V}_2)_{\eta_4}, \pi)\}$,
where $((\mathcal{V}_1, \mathcal{V}_2)_{\eta_1}, \pi), ((\mathcal{V}_1, \mathcal{V}_2)_{\eta_2}, \pi), ((\mathcal{V}_1, \mathcal{V}_2)_{\eta_3}, \pi), ((\mathcal{V}_1, \mathcal{V}_2)_{\eta_4}, \pi)$ are BnHySSs defined as follow:
$((\mathcal{V}_1, \mathcal{V}_2)_{\eta_1}, \pi) = \{((\varrho_1, \varrho_3), (\{\rho_1\}, \varnothing)), ((\varrho_1, \varrho_4), (\varnothing, \{\sigma_2\})), ((\varrho_2, \varrho_3), (\varnothing, \varnothing)), ((\varrho_2, \varrho_4), (\{\rho_3\}, \varnothing))\}$
$((\mathcal{V}_1, \mathcal{V}_2)_{\eta_2}, \pi) = \{((\varrho_1, \varrho_3), (\{\rho_4\}, \{\sigma_4\})), ((\varrho_1, \varrho_4), (\{\rho_3\}, \varnothing)), ((\varrho_2, \varrho_3), (\{\rho_1\}, \varnothing)), ((\varrho_2, \varrho_4), (\{\rho_3\}, \{\sigma_2\}))\}$.
$((\mathcal{V}_1, \mathcal{V}_2)_{\eta_3}, \pi) = \{((\varrho_1, \varrho_3), (\{\rho_1, \rho_4\}, \{\sigma_4\})), ((\varrho_1, \varrho_4), (\{\rho_3\}, \{\sigma_2\})), ((\varrho_2, \varrho_3), (\{\rho_1\}, \varnothing)), ((\varrho_2, \varrho_4), (\{\rho_3\}, \{\sigma_2\}))\}$.
$((\mathcal{V}_1, \mathcal{V}_2)_{\eta_4}, \pi) = \{((\varrho_1, \varrho_3), (\varnothing, \varnothing)), ((\varrho_1, \varrho_4), (\varnothing, \varnothing)), ((\varrho_2, \varrho_3), (\varnothing, \varnothing)), ((\varrho_2, \varrho_4), (\{\rho_3\}, \varnothing))\}$.

5. Binary hypersoft operators

In this section, we investigate binary hypersoft limit points, binary hypersoft neighborhood, binary hypersoft closure, binary hypersoft interior and binary hypersoft boundary.

Definition 5.1. Let $(\coprod_1, \coprod_2, \tau_{\mathcal{B}n\mathcal{H}y}, \pi)$ be a BnHySTS over $\coprod_1, \coprod_2$ and let (η, π) be a BnHySS over $\coprod_1, \coprod_2$. A BnHyS point $\varrho_\eta \widetilde{\in} \widetilde{\coprod}$ is called a binary hypersoft limit point of (η, π) if $(\eta, \pi) \widetilde{\cap} ((\mu, \vartheta) \setminus \varrho_\eta) \neq \widetilde{\varnothing}$ for every BnHyS open set (μ, ϑ) containing BnHyS point ϱ_η.
The set of all binary hypersoft limit points of (η, π) is called the binary hypersoft derived set of (η, π) and is denoted by $(\eta, \pi)^d$.

Proposition 5.2. *Let $(\coprod_1, \coprod_2, \tau_{\mathcal{B}n\mathcal{H}y}, \pi)$ be a BnHySTS over $\coprod_1, \coprod_2$ and let (η_1, π) and (η_2, π) be two BnHySSs over $\coprod_1, \coprod_2$. Then*

1. $(\eta_1, \pi) \widetilde{\subseteq} (\eta_2, \pi)$ *implies* $(\eta_1, \pi)^d \widetilde{\subseteq} (\eta_2, \pi)^d$.
2. $((\eta_1, \pi) \widetilde{\cap} (\eta_2, \pi))^d \widetilde{\subseteq} (\eta_1, \pi)^d \widetilde{\cap} (\eta_2, \pi)^d$.
3. $((\eta_1, \pi) \widetilde{\cup} (\eta_2, \pi))^d = (\eta_1, \pi)^d \widetilde{\cup} (\eta_2, \pi)^d$.

Proof. (1.) Let $\varrho_\eta \in (\eta_1, \pi)^d$, so that ϱ_η is a binary hypersoft limit point of (η_1, π). Then, $(\eta_1, \pi) \widetilde{\cap} ((\mu, \vartheta) \setminus \varrho_\eta \neq \widetilde{\varnothing}$ for every BnHyS open set (μ, ϑ) containing ϱ_η. But $(\eta_1, \pi) \widetilde{\subseteq} (\eta_2, \pi)$, it follows that $(\eta_2, \pi) \widetilde{\cap} ((\mu, \vartheta) \setminus \varrho_\eta \neq \widetilde{\varnothing}$. Thus, $\varrho_\eta \in (\eta_2, \pi)^d$. Therefore, $(\eta_1, \pi)^d \widetilde{\subseteq} (\eta_2, \pi)^d$.

(2.) Since $(\eta_1, \pi) \widetilde{\cap} (\eta_2, \pi) \widetilde{\subseteq} (\eta_1, \pi)$ and $(\eta_1, \pi) \widetilde{\cap} (\eta_2, \pi) \widetilde{\subseteq} (\eta_2, \pi)$. It follows from (1.) that $((\eta_1, \pi) \widetilde{\cap} (\eta_2, \pi))^d \widetilde{\subseteq} (\eta_1, \pi)^d$ and $((\eta_1, \pi) \widetilde{\cap} (\eta_2, \pi))^d \widetilde{\subseteq} (\eta_2, \pi)^d$. Hence $((\eta_1, \pi) \widetilde{\cap} (\eta_2, \pi))^d \widetilde{\subseteq} (\eta_1, \pi)^d \widetilde{\cap} (\eta_2, \pi)^d$.

(3.) Since $(\eta_1,\pi) \widetilde{\subseteq} (\eta_1,\pi) \widetilde{\cup} (\eta_2,\pi)$ and $(\eta_2,\pi) \widetilde{\subseteq} (\eta_1,\pi) \widetilde{\cup} (\eta_2,\pi)$. By (1.) we have $(\eta_1,\pi)^d \widetilde{\subseteq} ((\eta_1,\pi) \widetilde{\cup} (\eta_2,\pi))^d$ and $(\eta_2,\pi)^d \widetilde{\subseteq} ((\eta_1,\pi) \widetilde{\cup} (\eta_2,\pi))^d$. So, $(\eta_1,\pi)^d \widetilde{\cup} (\eta_2,\pi)^d \widetilde{\subseteq} ((\eta_1,\pi) \widetilde{\cup} (\eta_2,\pi))^d$.
Now, let $\varrho_\eta \in ((\eta_1,\pi) \widetilde{\cup} (\eta_2,\pi))^d$. Then $((\eta_1,\pi) \widetilde{\cup} (\eta_2,\pi)) \widetilde{\cap} ((\mu,\vartheta) \setminus \varrho_\eta) \neq \widetilde{\varnothing}$ for every BnHyS open set (μ,ϑ) containing ϱ_η. Therefore, either $(\eta_1,\pi) \widetilde{\cap} ((\mu,\vartheta) \setminus \varrho_\eta) \neq \widetilde{\varnothing}$ or $(\eta_2,\pi) \widetilde{\cap} ((\mu,\vartheta) \setminus \varrho_\eta) \neq \widetilde{\varnothing}$. Thus, either $\varrho_\eta \in (\eta_1,\pi)^d$ or $\varrho_\eta \in (\eta_2,\pi)^d$ and hence $\varrho_\eta \in ((\eta_1,\pi)^d \widetilde{\cup} (\eta_2,\pi)^d)$. Therefore, $((\eta_1,\pi) \widetilde{\cup} (\eta_2,\pi))^d \widetilde{\subseteq} (\eta_1,\pi)^d \widetilde{\cup} (\eta_2,\pi)^d$. So, $((\eta_1,\pi) \widetilde{\cup} (\eta_2,\pi))^d = (\eta_1,\pi)^d \widetilde{\cup} (\eta_2,\pi)^d$. □

Remark 5.3. The following example shows that the equality in Proposition 5.2 (2.) does not hold in general.

Example 5.4. Let $\coprod_1 = \{a_1, a_2\}$, $\coprod_2 = \{b_1, b_2\}$, $₤ = \{\varrho_1, \varrho_2, \varrho_3\}$, $\pi_1 = \{\varrho_1\}$, $\pi_2 = \{\varrho_2, \varrho_3\}$.
Suppose that $\tau_{\mathcal{B}n\mathcal{H}y} = \{\widetilde{\varnothing}, \widetilde{\coprod}, (\eta_1,\pi), (\eta_2,\pi)\}$, where BnHySSs $(\eta_1,\pi), (\eta_2,\pi)$ are defined as below
$(\eta_1,\pi) = \{((\varrho_1,\varrho_2),(\{a_1\},\{b_2\})), ((\varrho_1,\varrho_3),(\varnothing,\{b_1\}))\}$ and $(\eta_2,\pi) = \{((\varrho_1,\varrho_2),(\{a_1\},\{b_2\})), ((\varrho_1,\varrho_3),(\{a_2\},\{b_1\}))\}$.
If we take two BnHySSs $(\mho,\pi)$ and (μ,π) defined as follow
$(\mho,\pi) = \{((\varrho_1,\varrho_2),(\{a_1\},\varnothing)), ((\varrho_1,\varrho_3),(\{a_2\},\{b_1\}))\}$ and $(\mu,\pi) = \{((\varrho_1,\varrho_2),(\{a_2\},\{b_2\})), ((\varrho_1,\varrho_3),(\varnothing,\{b_2\}))\}$.
Then

$$(\mho,\pi)^d = \begin{cases} (\varrho_1,\varrho_2)_{k_i} & \text{if } \ i = 1,2,\ldots,15 \\ (\varrho_1,\varrho_3)_{k_j} & \text{if } \ j = 1,2,\ldots,15, \end{cases}$$

and

$$(\mu,\pi)^d = \begin{cases} (\varrho_1,\varrho_2)_{k_i} & \text{if } \ i = 1,3,4,5,7,8,\ldots,15 \\ (\varrho_1,\varrho_3)_{k_j} & \text{if } \ j = 1,2,\ldots,15. \end{cases}$$

Hence

$$(\mho,\pi)^d \widetilde{\cap} (\mu,\pi)^d = \begin{cases} (\varrho_1,\varrho_2)_{k_i} & \text{if } \ i = 1,3,4,5,7,8,\ldots,15 \\ (\varrho_1,\varrho_3)_{k_j} & \text{if } \ j = 1,2,\ldots,15. \end{cases}$$

Now, since $(\mho,\pi) \widetilde{\cap} (\mu,\pi) = \{((\varrho_1,\varrho_2),(\varnothing,\varnothing)), ((\varrho_1,\varrho_3),(\varnothing,\varnothing))\} = \widetilde{\varnothing}$; so, $((\mho,\pi) \widetilde{\cap} (\mu,\pi))^d = (\varnothing,\varnothing)$. Therefore, $((\mho,\pi) \widetilde{\cap} (\mu,\pi))^d \neq (\mho,\pi)^d \widetilde{\cap} (\mu,\pi)^d$.

The BnHyS points $(\varrho_1,\varrho_2)_{k_i}$ and $(\varrho_1,\varrho_3)_{k_j}$, where $i = 1,2,\ldots,15$ and $j = 1,2,\ldots,15$, over $\coprod_1, \coprod_2$ can be written as follow
$(\varrho_1,\varrho_2)_{k_1} = \{((\varrho_1,\varrho_2),(\varnothing,\{b_1\})), ((\varrho_1,\varrho_3),(\varnothing,\varnothing))\}$
$(\varrho_1,\varrho_2)_{k_2} = \{((\varrho_1,\varrho_2),(\varnothing,\{b_2\})), ((\varrho_1,\varrho_3),(\varnothing,\varnothing))\}$
$(\varrho_1,\varrho_2)_{k_3} = \{((\varrho_1,\varrho_2),(\varnothing,\coprod_2)), ((\varrho_1,\varrho_3),(\varnothing,\varnothing))\}$
$(\varrho_1,\varrho_2)_{k_4} = \{((\varrho_1,\varrho_2),(\{a_1\},\varnothing)), ((\varrho_1,\varrho_3),(\varnothing,\varnothing))\}$
$(\varrho_1,\varrho_2)_{k_5} = \{((\varrho_1,\varrho_2),(\{a_1\},\{b_1\})), ((\varrho_1,\varrho_3),(\varnothing,\varnothing))\}$
$(\varrho_1,\varrho_2)_{k_6} = \{((\varrho_1,\varrho_2),(\{a_1\},\{b_2\})), ((\varrho_1,\varrho_3),(\varnothing,\varnothing))\}$
$(\varrho_1,\varrho_2)_{k_7} = \{((\varrho_1,\varrho_2),(\{a_1\},\coprod_2)), ((\varrho_1,\varrho_3),(\varnothing,\varnothing))\}$
$(\varrho_1,\varrho_2)_{k_8} = \{((\varrho_1,\varrho_2),(\{a_2\},\varnothing)), ((\varrho_2,\varrho_3),(\varnothing,\varnothing))\}$
$(\varrho_1,\varrho_2)_{k_9} = \{((\varrho_1,\varrho_2),(\{a_2\},\{b_1\})), ((\varrho_2,\varrho_3),(\varnothing,\varnothing))\}$
$(\varrho_1,\varrho_2)_{k_{10}} = \{((\varrho_1,\varrho_2),(\{a_2\},\{b_2\})), ((\varrho_2,\varrho_3),(\varnothing,\varnothing))\}$
$(\varrho_1,\varrho_2)_{k_{11}} = \{((\varrho_1,\varrho_2),(\{a_2\},\coprod_2)), ((\varrho_2,\varrho_3),(\varnothing,\varnothing))\}$
$(\varrho_1,\varrho_2)_{k_{12}} = \{((\varrho_1,\varrho_2),(\coprod_1,\varnothing)), ((\varrho_1,\varrho_3),(\varnothing,\varnothing))\}$
$(\varrho_1,\varrho_2)_{k_{13}} = \{((\varrho_1,\varrho_2),(\coprod_1,\{b_1\})), ((\varrho_1,\varrho_3),(\varnothing,\varnothing))\}$
$(\varrho_1,\varrho_2)_{k_{14}} = \{((\varrho_1,\varrho_2),(\coprod_1,\{b_2\})), ((\varrho_1,\varrho_3),(\varnothing,\varnothing))\}$
$(\varrho_1,\varrho_2)_{k_{15}} = \{((\varrho_1,\varrho_2),(\coprod_1,\coprod_2)), ((\varrho_1,\varrho_3),(\varnothing,\varnothing))\}$
$(\varrho_1,\varrho_3)_{k_1} = \{((\varrho_1,\varrho_2),(\varnothing,\varnothing)), ((\varrho_1,\varrho_3),(\varnothing,\{b_1\}))\}$
$(\varrho_1,\varrho_3)_{k_2} = \{((\varrho_1,\varrho_2),(\varnothing,\varnothing)), ((\varrho_1,\varrho_3),(\varnothing,\{b_2\}))\}$
$(\varrho_1,\varrho_3)_{k_3} = \{((\varrho_1,\varrho_2),(\varnothing,\varnothing)), ((\varrho_1,\varrho_3),(\varnothing,\coprod_2))\}$

$(\varrho_1, \varrho_3)_{k_4} = \{((\varrho_1, \varrho_2), (\emptyset, \emptyset)), ((\varrho_1, \varrho_3), (\{a_1\}, \emptyset))\}$
$(\varrho_1, \varrho_3)_{k_5} = \{((\varrho_1, \varrho_2), (\emptyset, \emptyset)), ((\varrho_1, \varrho_3), (\{a_1\}, \{b_1\}))\}$
$(\varrho_1, \varrho_3)_{k_6} = \{((\varrho_1, \varrho_2), (\emptyset, \emptyset)), ((\varrho_1, \varrho_3), (\{a_1\}, \{b_2\}))\}$
$(\varrho_1, \varrho_3)_{k_7} = \{((\varrho_1, \varrho_2), (\emptyset, \emptyset)), ((\varrho_1, \varrho_3), (\{a_1\}, \coprod_2))\}$
$(\varrho_1, \varrho_3)_{k_8} = \{((\varrho_1, \varrho_2), (\emptyset, \emptyset)), ((\varrho_1, \varrho_3), (\{a_2\}, \emptyset))\}$
$(\varrho_1, \varrho_3)_{k_9} = \{((\varrho_1, \varrho_2), (\emptyset, \emptyset)), ((\varrho_1, \varrho_3), (\{a_2\}, \{b_1\}))\}$
$(\varrho_1, \varrho_3)_{k_{10}} = \{((\varrho_1, \varrho_2), (\emptyset, \emptyset)), ((\varrho_1, \varrho_3), (\{a_2\}, \{b_2\}))\}$
$(\varrho_1, \varrho_3)_{k_{11}} = \{((\varrho_1, \varrho_2), (\emptyset, \emptyset)), ((\varrho_1, \varrho_3), (\{a_2\}, \coprod_2))\}$
$(\varrho_1, \varrho_3)_{k_{12}} = \{((\varrho_1, \varrho_2), (\emptyset, \emptyset)), ((\varrho_1, \varrho_3), (\coprod_1, \emptyset))\}$
$(\varrho_1, \varrho_3)_{k_{13}} = \{((\varrho_1, \varrho_2), (\emptyset, \emptyset)), ((\varrho_1, \varrho_3), (\coprod_1, \{b_1\}))\}$
$(\varrho_1, \varrho_3)_{k_{14}} = \{((\varrho_1, \varrho_2), (\emptyset, \emptyset)), ((\varrho_1, \varrho_3), (\coprod_1, \{b_2\}))\}$
$(\varrho_1, \varrho_3)_{k_{15}} = \{((\varrho_1, \varrho_2), (\emptyset, \emptyset)), ((\varrho_1, \varrho_3), (\coprod_1, \coprod_2))\}$.

Definition 5.5. A BnHySS (η, π) in a BnHySTS $(\coprod_1, \coprod_2, \tau_{\mathcal{B}n\mathcal{H}y}, \pi)$ is known as a BnHyS neighborhood of the BnHyS point ϱ_η over $\coprod_1, \coprod_2$, if there exists a BnHyS open set (μ, π) such that $\varrho_\eta \in (\mu, \pi) \mathrel{\widetilde{\subseteq}} (\eta, \pi)$.
The BnHyS neighborhood system of BnHyS point ϱ_η, symbolized by $\mathcal{N}_\tau(\varrho_\eta)$, is the family of all its BnHyS neighborhoods.

Definition 5.6. A BnHySS (η, π) in a BnHySTS $(\coprod_1, \coprod_2, \tau_{\mathcal{B}n\mathcal{H}y}, \pi)$ is known as a BnHyS neighborhood of the BnHySS $(\mho, \pi)$, if there exists a BnHyS open set (μ, π) such that $(\mho, \pi) \mathrel{\widetilde{\subseteq}} (\mu, \pi) \mathrel{\widetilde{\subseteq}} (\eta, \pi)$.

Theorem 5.7. *Let $(\coprod_1, \coprod_2, \tau_{\mathcal{B}n\mathcal{H}y}, \pi)$ be a BnHySTS over $\coprod_1, \coprod_2$, then*

1. *If (η, π) is a BnHyS neighborhood of ϱ_η over $\coprod_1, \coprod_2$, then $\varrho_\eta \in (\eta, \pi)$.*
2. *Each $\varrho_\eta \mathrel{\widetilde{\subseteq}} \widetilde{\coprod}$ has a BnHyS neighborhood.*
3. *If (η, π) and (μ, π) are BnHyS neighborhoods of $\varrho_\eta \mathrel{\widetilde{\subseteq}} \widetilde{\coprod}$, then $(\eta, \pi) \mathrel{\widetilde{\cap}} (\mu, \pi)$ is also a BnHyS neighborhood of $\varrho_\eta \mathrel{\widetilde{\subseteq}} \widetilde{\coprod}$.*
4. *If (η, π) is a BnHyS neighborhood of $\varrho_\eta \mathrel{\widetilde{\subseteq}} \widetilde{\coprod}$, and $(\eta, \pi) \mathrel{\widetilde{\subseteq}} (\mu, \pi)$, then (μ, π) is also a BnHyS neighborhood of $\varrho_\eta \mathrel{\widetilde{\subseteq}} \widetilde{\coprod}$.*

Proof. 1. If (η, π) is a BnHyS neighborhood of ϱ_η, then there is a BnHyS open set $(\mu, \pi) \in \tau_{\mathcal{B}n\mathcal{H}y}$ such that $\varrho_\eta \in (\mu, \pi) \mathrel{\widetilde{\subseteq}} (\eta, \pi)$. Therefore, we have $\varrho_\eta \in (\eta, \pi)$.

2. For any $\varrho_\eta \in \widetilde{\coprod}$, so $\varrho_\eta \in \widetilde{\coprod} \mathrel{\widetilde{\subseteq}} \widetilde{\coprod}$. Thus, $\widetilde{\coprod}$ is a BnHyS neighborhood of ϱ_η.

3. Let $\varrho_\eta \in \widetilde{\coprod}$ be any BnHyS point and let (η, π) and (μ, π) be any two BnHyS neighborhoods of ϱ_η. Now to prove $(\eta, \pi) \cap (\mu, \pi)$ is also a BnHyS neighborhood of ϱ_η. Now (η, π) is a BnHyS neighborhood of ϱ_η implies that there exists a BnHyS open set $(\mho, \pi$ such that $\varrho_\eta \in (\mho, \pi) \subseteq (\eta, \pi)$. Also (μ, π) is a BnHyS neighborhood of ϱ_η implies that there exists a BnHyS open set $(Ķ, \pi)$ such that $\varrho_\eta \in (Ķ, \pi) \subseteq (\mu, \pi)$.
Now $(\mho, \pi) \cap (Ķ, \pi)$ is BnHyS open set, also we have

$$\varrho_\eta \in [(\mho, \pi) \cap (Ķ, \pi)] \subseteq [(\eta, \pi) \cap (\mu, \pi)].$$

Thus, there exists a BnHyS open set $[(\mho, \pi) \cap (Ķ, \pi)]$ such that

$$\varrho_\eta \in [(\mho, \pi) \cap (Ķ, \pi)] \subseteq [(\eta, \pi) \cap (\mu, \pi)].$$

From the definition of BnHyS neighborhood, it follows $[(\eta, \pi) \cap (\mu, \pi)]$ is a BnHyS neighborhood of ϱ_η. Thus, the intersection of any two BnHyS neighborhoods is again BnHyS neighborhood.

4. Let $\varrho_\eta \in \widetilde{\coprod}$ be any BnHyS point and let (η, π) be an BnHyS neighborhood of ϱ_η. Let (μ, π) be any BnHyS superset of (η, π). Now, since (μ, π) is also a BnHyS neighborhood of ϱ_η; therefore, there exists a BnHyS open set $(\mho, \pi)$ such that $\varrho_\eta \in (\mho, \pi) \subseteq (\eta, \pi)$. Now, (η, π) is a BnHyS subset of (μ, π) this implies $(\eta, \pi) \subseteq (\mu, \pi)$.

Therefore, we have $\varrho_\eta \in (\mho,\pi) \subseteq (\eta,\pi) \subseteq (\mu,\pi)$, which implies $\varrho_\eta \in (\mho,\pi) \subseteq (\mu,\pi)$. Thus, there exists a BnHyS open set $(\mho,\pi)$ such that $\varrho_\eta \in (\mho,\pi) \subseteq (\mu,\pi)$. Therefore, (μ,π) is a BnHyS neighborhood of ϱ_η. Thus, every BnHyS superset of a BnHyS neighborhood is again a BnHyS neighborhood of that point. □

Definition 5.8. Let $(\coprod_1,\coprod_2,\tau_{\mathcal{B}n\mathcal{H}y},\pi)$ be a BnHySTS and (η,π) be a BnHySS over $\coprod_1,\coprod_2$. The BnHyS intersection of all BnHyS closed super sets of (η,π) is known as the BnHyS closure of (η,π) and is symbolized by $Cl(\eta,\pi)$.
In other words, $Cl(\eta,\pi) = \{\widetilde{\cap}(\mu,\pi) \mid (\mu,\pi)^c \in \tau_{\mathcal{B}n\mathcal{H}y}, (\eta,\pi)\,\widetilde{\subseteq}\,(\mu,\pi)\}$.
Thus, $Cl(\eta,\pi)$ is the smallest BnHyS closed set containing (η,π).

Example 5.9. Let $\coprod_1 = \{\rho_1,\rho_2,\rho_3,\rho_4,\rho_5\}$, $\coprod_2 = \{\sigma_1,\sigma_2,\sigma_3,\sigma_4\}$, $\text{₠} = \{\varrho_1,\varrho_2,\varrho_3,\varrho_4\}$, $\pi_1 = \{\varrho_1,\varrho_2\}$, $\pi_2 = \{\varrho_3,\varrho_4\}$ and $\tau_{\mathcal{B}n\mathcal{H}y} = \{\widetilde{\varnothing},\widetilde{\coprod},(\eta_1,\pi),(\eta_2,\pi),(\eta_3,\pi),(\eta_4,\pi)\}$,
where $(\eta_1,\pi),(\eta_2,\pi),(\eta_3,\pi),(\eta_4,\pi)$ are BnHySSs defined as follow:
$(\eta_1,\pi) =\{((\varrho_1,\varrho_3),(\{\rho_1\},\{\sigma_1\})),((\varrho_1,\varrho_4),(\{\rho_2\},\{\sigma_2\})),((\varrho_2,\varrho_3),(\varnothing,\varnothing)),((\varrho_2,\varrho_4),(\{\rho_3\},\{\sigma_3\}))\}$.
$(\eta_2,\pi) =\{((\varrho_1,\varrho_3),(\{\rho_4\},\{\sigma_4\})),((\varrho_1,\varrho_4),(\{\rho_3\},\{\sigma_1\})),((\varrho_2,\varrho_3),(\{\rho_1,\rho_2\},\{\sigma_3\})),((\varrho_2,\varrho_4),(\{\rho_3,\rho_5\},\{\sigma_1,\sigma_2\}))\}$.
$(\eta_3,\pi) =\{((\varrho_1,\varrho_3),(\{\rho_1,\rho_4\},\{\sigma_1,\sigma_4\})),((\varrho_1,\varrho_4),(\{\rho_2,\rho_3\},\{\sigma_1,\sigma_2\})),((\varrho_2,\varrho_3),(\{\rho_1,\rho_2\},\{\sigma_3\})),((\varrho_2,\varrho_4),(\{\rho_3,\rho_5\},\{\sigma_1,\sigma_2,\sigma_3\}))\}$.
$(\eta_4,\pi) =\{((\varrho_1,\varrho_3),(\varnothing,\varnothing)),((\varrho_1,\varrho_4),(\varnothing,\varnothing)),((\varrho_2,\varrho_3),(\varnothing,\varnothing)),((\varrho_2,\varrho_4),(\{\rho_3\},\varnothing))\}$.

Let $(\mu,\pi) = \{((\varrho_1,\varrho_3),(\{\rho_3,\rho_5\},\{\sigma_2\})),((\varrho_1,\varrho_4),(\{\rho_5\},\{\sigma_1,\sigma_3\})),((\varrho_2,\varrho_3),(\{\rho_4,\rho_5\},\{\sigma_2\})),((\varrho_2,\varrho_4),(\{\rho_2,\rho_4\},\{\sigma_1,\sigma_2\}))\}$.
Then BnHyS closure of (μ,π) is:
$Cl(\mu,\pi) = (\eta_1,\pi)^c = \{((\varrho_1,\varrho_3),(\{\rho_2,\rho_3,\rho_4,\rho_5\},\{\sigma_2,\sigma_3,\sigma_4\})),((\varrho_1,\varrho_4),(\{\rho_1,\rho_3,\rho_4,\rho_5\},\{\sigma_1,\sigma_3,\sigma_4\})),((\varrho_2,\varrho_3),(\coprod_1,\coprod_2)),((\varrho_2,\varrho_4),(\{\rho_1,\rho_2,\rho_4,\rho_5\},\{\sigma_1,\sigma_2,\sigma_4\}))\}$.

Proposition 5.10. *Let $(\coprod_1,\coprod_2,\tau_{\mathcal{B}n\mathcal{H}y},\pi)$ be a BnHySTS and (η,π),(μ,π) be two BnHySSs over $\coprod_1,\coprod_2$. Then*

1. $Cl(\widetilde{\varnothing},\pi) = (\widetilde{\varnothing},\pi)$ *and* $Cl(\widetilde{\coprod},\pi) = (\widetilde{\coprod},\pi)$.
2. $(\eta,\pi)\,\widetilde{\subseteq}\,Cl(\eta,\pi)$.
3. (η,π) *is BnHyS closed if and only if* $Cl(\eta,\pi) = (\eta,\pi)$.
4. $Cl(Cl(\eta,\pi)) = Cl(\eta,\pi)$.
5. *If* $(\eta,\pi)\,\widetilde{\subseteq}\,(\mu,\pi)$, *then* $Cl(\eta,\pi)\,\widetilde{\subseteq}\,Cl(\mu,\pi)$.
6. $Cl((\eta,\pi)\,\widetilde{\cup}\,(\mu,\pi)) = Cl(\eta,\pi)\,\widetilde{\cup}\,Cl(\mu,\pi)$.
7. $Cl((\eta,\pi)\,\widetilde{\cap}\,(\mu,\pi))\,\widetilde{\subseteq}\,Cl(\eta,\pi)\,\widetilde{\cap}\,Cl(\mu,\pi)$.

Proof. (1.) and (2.) are obvious.

3. If (η,π) is a BnHyS closed set over $\coprod_1,\coprod_2$, then (η,π) is itself a BnHyS closed set over $\coprod_1,\coprod_2$ which contains (η,π). So (η,π) is the smallest BnHyS closed set containing (η,π) and $(\eta,\pi) = Cl(\eta,\pi)$.
Conversely, suppose that $(\eta,\pi) = Cl(\eta,\pi)$. Since $Cl(\eta,\pi)$ is BnHyS closed, so (η,π) is a BnHyS closed set over $\coprod_1,\coprod_2$.

4. Since $Cl(\eta,\pi)$ is a BnHyS closed set, therefore by part (3.), we have $Cl(Cl(\eta,\pi)) = Cl(\eta,\pi)$.

5. Since $(\eta,\pi)\,\widetilde{\subseteq}\,(\mu,\pi)$ and $(\mu,\pi)\,\widetilde{\subseteq}\,Cl(\mu,\pi)$ then $(\eta,\pi)\,\widetilde{\subseteq}\,(\mu,\pi)\,\widetilde{\subseteq}\,Cl(\mu,\pi)$ and hence $(\eta,\pi)\,\widetilde{\subseteq}\,Cl(\mu,\pi)$, so $Cl(\eta,\pi)\,\widetilde{\subseteq}\,Cl(Cl(\mu,\pi))$ which implies that $Cl(\eta,\pi)\,\widetilde{\subseteq}\,Cl(\mu,\pi)$.

6. Since $(\eta,\pi)\,\widetilde{\subseteq}\,(\eta,\pi)\,\widetilde{\cup}\,(\mu,\pi)$ and $(\mu,\pi)\,\widetilde{\subseteq}\,(\eta,\pi)\,\widetilde{\cup}\,(\mu,\pi)$. So, by part (5.)
$Cl(\eta,\pi)\,\widetilde{\subseteq}\,Cl((\eta,\pi)\,\widetilde{\cup}\,(\mu,\pi))$ and $Cl(\mu,\pi)\,\widetilde{\subseteq}\,Cl((\eta,\pi)\,\widetilde{\cup}\,(\mu,\pi))$.
Thus, $Cl(\eta,\pi)\,\widetilde{\cup}\,Cl(\mu,\pi)\,\widetilde{\subseteq}\,Cl((\eta,\pi)\,\widetilde{\cup}\,(\mu,\pi))$.
Conversely, suppose that $(\eta,\pi)\,\widetilde{\subseteq}\,Cl(\eta,\pi)$ and $(\mu,\pi)\,\widetilde{\subseteq}\,Cl(\mu,\pi)$.
So, $(\eta,\pi)\,\widetilde{\cup}\,(\mu,\pi)\,\widetilde{\subseteq}\,Cl(\eta,\pi)\,\widetilde{\cup}\,Cl(\mu,\pi)$. Since $Cl(\eta,\pi)$ and $Cl(\mu,\pi)$ are BnHyS closed sets, then $Cl(\eta,\pi)\,\widetilde{\cup}\,Cl(\mu,\pi)$

is a BnHyS closed set. By part (4.) $Cl(Cl(\eta,\pi)\,\widetilde{\cup}\,Cl(\mu,\pi)) = Cl(\eta,\pi)\,\widetilde{\cup}\,Cl(\mu,\pi)$ implies that $Cl((\eta,\pi)\,\widetilde{\cup}\,(\mu,\pi)) \,\widetilde{\subseteq}\, Cl(\eta,\pi)\,\widetilde{\cup}\,Cl(\mu,\pi)$. Thus, $Cl((\eta,\pi)\,\widetilde{\cup}\,(\mu,\pi)) = Cl(\eta,\pi)\,\widetilde{\cup}\,Cl(\mu,\pi)$.

7. Since $(\eta,\pi)\,\widetilde{\cap}\,(\mu,\pi) \,\widetilde{\subseteq}\, (\eta,\pi)$ and $(\eta,\pi)\,\widetilde{\cap}\,(\mu,\pi) \,\widetilde{\subseteq}\, (\mu,\pi)$. So, by part (5.) $Cl((\eta,\pi)\,\widetilde{\cap}\,(\mu,\pi)) \,\widetilde{\subseteq}\, Cl(\eta,\pi)$ and $Cl((\eta,\pi)\,\widetilde{\cap}\,(\mu,\pi)) \,\widetilde{\subseteq}\, Cl(\mu,\pi)$. Hence, $Cl((\eta,\pi)\,\widetilde{\cap}\,(\mu,\pi)) \,\widetilde{\subseteq}\, Cl(\eta,\pi)\,\widetilde{\cap}\,Cl(\mu,\pi)$. □

Remark 5.11. This example demonstrates that the equivalence in Proposition 5.10 (7.) is not often true.

Example 5.12. Let $\coprod_1 = \{\rho_1,\rho_2,\rho_3\}$, $\coprod_2 = \{\sigma_1,\sigma_2,\sigma_3\}$, $\notin = \{\varrho_1,\varrho_2,\varrho_3\}$, $\pi_1 = \{\varrho_1\}$, $\pi_2 = \{\varrho_2,\varrho_3\}$ and $\tau_{\mathcal{B}n\mathcal{H}y} = \{\widetilde{\emptyset}, \widetilde{\coprod}, (\eta_1,\pi), (\eta_2,\pi), (\eta_3,\pi), (\eta_4,\pi), (\eta_5,\pi)\}$ be a BnHyST defined over $\coprod_1, \coprod_2$,
where $(\eta_1,\pi), (\eta_2,\pi), (\eta_3,\pi), (\eta_4,\pi), (\eta_5,\pi)$ are defined as follow:
$(\eta_1,\pi) = \{((\varrho_1,\varrho_2),(\{\rho_2\},\{\sigma_2\})),((\varrho_1,\varrho_3),(\{\rho_1\},\{\sigma_1\}))\}$.
$(\eta_2,\pi) = \{((\varrho_1,\varrho_2),(\{\rho_2,\rho_3\},\{\sigma_2,\sigma_3\})),((\varrho_1,\varrho_3),(\{\rho_1,\rho_2\},\{\sigma_1,\sigma_2\}))\}$.
$(\eta_3,\pi) = \{((\varrho_1,\varrho_2),(\{\rho_1,\rho_2\},\{\sigma_1,\sigma_2\})),((\varrho_1,\varrho_3),(\coprod_1,\coprod_2))\}$.
$(\eta_4,\pi) = \{((\varrho_1,\varrho_2),(\{\rho_1,\rho_2\},\{\sigma_1,\sigma_2\})),((\varrho_1,\varrho_3),(\{\rho_1,\rho_3\},\{\sigma_1,\sigma_3\}))\}$.
$(\eta_5,\pi) = \{((\varrho_1,\varrho_2),(\{\rho_2\},\{\sigma_2\})),((\varrho_1,\varrho_3),(\{\rho_1,\rho_2\},\{\sigma_1,\sigma_2\}))\}$.
Clearly, we consider the BnHyS closed sets are:
$(\eta_1,\pi)^c = \{((\varrho_1,\varrho_2),(\{\rho_1,\rho_3\},\{\sigma_1,\sigma_3\})),((\varrho_1,\varrho_3),(\{\rho_2,\rho_3\},\{\sigma_2,\sigma_3\}))\}$.
$(\eta_2,\pi)^c = \{((\varrho_1,\varrho_2),(\{\rho_1\},\{\sigma_1\})),((\varrho_1,\varrho_3),(\{\rho_3\},\{\sigma_3\}))\}$.
$(\eta_3,\pi)^c = \{((\varrho_1,\varrho_2),(\{\rho_3\},\{\sigma_3\})),((\varrho_1,\varrho_3),(\emptyset,\emptyset))\}$.
$(\eta_4,\pi)^c = \{((\varrho_1,\varrho_2),(\{\rho_3\},\{\sigma_3\})),((\varrho_1,\varrho_3),(\{\rho_2\},\{\sigma_2\}))\}$.
$(\eta_5,\pi)^c = \{((\varrho_1,\varrho_2),(\{\rho_1,\rho_3\},\{\sigma_1,\sigma_3\})),((\varrho_1,\varrho_3),(\{\rho_3\},\{\sigma_3\}))\}$.
Now we consider the BnHySSs (μ,π) and $(\mho,\pi)$,
$(\mu,\pi) = \{((\varrho_1,\varrho_2),(\{\rho_1\},\{\sigma_3\})),((\varrho_1,\varrho_3),(\{\rho_2\},\{\sigma_2\}))\}$.
$(\mho,\pi) = \{((\varrho_1,\varrho_2),(\{\rho_2\},\{\sigma_3\})),((\varrho_1,\varrho_3),(\{\rho_1,\rho_2\},\{\sigma_3\}))\}$.
$(\mu,\pi)\,\widetilde{\cap}\,(\mho,\pi) = \{((\varrho_1,\varrho_2),(\emptyset,\{\sigma_3\})),((\varrho_1,\varrho_3),(\{\rho_2\},\emptyset))\}$.
Then,
$Cl(\mu,\pi) = \{((\varrho_1,\varrho_2),(\{\rho_1,\rho_3\},\{\sigma_1,\sigma_3\})),((\varrho_1,\varrho_3),(\{\rho_2,\rho_3\},\{\sigma_2,\sigma_3\}))\}$ and
$Cl(\mho,\pi) = \{((\varrho_1,\varrho_2),(\coprod_1,\coprod_2)),((\varrho_1,\varrho_3),(\coprod_1,\coprod_2))\}$.
Hence, $Cl(\mu,\pi)\,\widetilde{\cap}\,Cl(\mho,\pi) = \{((\varrho_1,\varrho_2),(\{\rho_1,\rho_3\},\{\sigma_1,\sigma_3\})),((\varrho_1,\varrho_3),(\{\rho_2,\rho_3\},\{\sigma_2,\sigma_3\}))\}$ and
$Cl((\mu,\pi)\,\widetilde{\cap}\,(\mho,\pi)) = \{((\varrho_1,\varrho_2),(\{\rho_3\},\{\sigma_3\})),((\varrho_1,\varrho_3),(\{\rho_2\},\{\sigma_2\}))\}$.
Thus, $Cl((\eta,\pi)\,\widetilde{\cap}\,(\mu,\pi)) \,\widetilde{\subseteq}\, Cl(\eta,\pi)\,\widetilde{\cap}\,Cl(\mu,\pi)$.
But $Cl((\eta,\pi)\,\widetilde{\cap}\,(\mu,\pi)) \,\widetilde{\not\supseteq}\, Cl(\eta,\pi)\,\widetilde{\cap}\,Cl(\mu,\pi)$.

Definition 5.13. Let $(\coprod_1, \coprod_2, \tau_{\mathcal{B}n\mathcal{H}y}, \pi)$ be a BnHySTS and (η,π) be a BnHySS over $\coprod_1, \coprod_2$. Then BnHyS interior of BnHySS (η,π) over $\coprod_1, \coprod_2$ is symbolized by $Int(\eta,\pi)$ and is described as the BnHyS union of all BnHyS open sets contained in (η,π).
In other words, $Int(\eta,\pi) = \widetilde{\cup}\{(\mu,\pi) \mid (\mu,\pi) \in \tau_{\mathcal{B}n\mathcal{H}y}\ (\mu,\pi) \,\widetilde{\subseteq}(\eta,\pi)\}$.
Thus, $Int(\eta,\pi)$ is the largest BnHyS open set contained in (η,π).

Proposition 5.14. *Let $(\coprod_1, \coprod_2, \tau_{\mathcal{B}n\mathcal{H}y}, \pi)$ be a BnHySTS and (η,π), (μ,π) be two BnHySSs over $\coprod_1, \coprod_2$. Then*

1. $Int(\widetilde{\emptyset},\pi) = (\widetilde{\emptyset},\pi)$ *and* $Int(\widetilde{\coprod},\pi) = (\widetilde{\coprod},\pi)$.
2. $Int(\eta,\pi) \,\widetilde{\subseteq}\, (\eta,\pi)$.
3. $Int(Int(\eta,\pi)) = Int(\eta,\pi)$.
4. (η,π) *is BnHyS open if and only if* $Int(\eta,\pi) = (\eta,\pi)$.
5. *If* $(\eta,\pi) \,\widetilde{\subseteq}\, (\mu,\pi)$, *then* $Int(\eta,\pi) \,\widetilde{\subseteq}\, Int(\mu,\pi)$.
6. $Int((\eta,\pi)\,\widetilde{\cap}\,(\mu,\pi)) = Int(\eta,\pi)\,\widetilde{\cap}\,Int(\mu,\pi)$.
7. $Int(\eta,\pi)\,\widetilde{\cup}\,Int(\mu,\pi) \,\widetilde{\subseteq}\, Int((\eta,\pi)\,\widetilde{\cup}\,(\mu,\pi))$.

Proof. (1.) and (2.) are obvious.

3. Since $Int(\eta,\pi)$ is BnHyS open and $Int(Int(\eta,\pi))$ is the BnHyS union of all BnHyS open subsets over $\coprod_1,\coprod_2$ contained in $Int(\eta,\pi)$, then $Int(\eta,\pi)\,\widetilde{\subseteq}\,Int(Int(\eta,\pi))$. But in general $Int(Int(\eta,\pi))\,\widetilde{\subseteq}\,Int(\eta,\pi)$. Hence $Int(Int(\eta,\pi)) = Int(\eta,\pi)$.

4. If (η,π) is a BnHyS open set, then (η,π) is itself a BnHyS open set contains (η,π). So $Int(\eta,\pi)$ is the largest BnHyS open set contained in (η,π) and $(\eta,\pi) = Int(\eta,\pi)$.
Conversely, suppose that $(\eta,\pi) = Int(\eta,\pi)$. Since $Int(\eta,\pi)$ is a BnHyS open set, so (η,π) is BnHyS open set.

5. Suppose that $(\eta,\pi)\,\widetilde{\subseteq}\,(\mu,\pi)$. Since $Int(\eta,\pi)\,\widetilde{\subseteq}\,(\eta,\pi)\,\widetilde{\subseteq}\,(\mu,\pi)$. $Int(\eta,\pi)$ is a BnHyS open subset of (μ,π), so by definition of $Int(\mu,\pi)$, $Int(\eta,\pi)\,\widetilde{\subseteq}\,Int(\mu,\pi)$.

6. Since $(\eta,\pi)\,\widetilde{\cap}\,(\mu,\pi)\,\widetilde{\subseteq}\,(\eta,\pi)$ and $(\eta,\pi)\,\widetilde{\cap}\,(\mu,\pi)\,\widetilde{\subseteq}\,(\mu,\pi)$, then we have by (5.), $Int((\eta,\pi)\,\widetilde{\cap}\,(\mu,\pi))\,\widetilde{\subseteq}\,Int(\eta,\pi)$ and $Int((\eta,\pi)\,\widetilde{\cap}\,(\mu,\pi))\,\widetilde{\subseteq}\,Int(\mu,\pi)$. This implies that $Int((\eta,\pi)\,\widetilde{\cap}\,(\mu,\pi))\,\widetilde{\subseteq}\,Int(\eta,\pi)\,\widetilde{\cap}\,Int(\mu,\pi)$.

Conversely, $Int(\eta,\pi)\,\widetilde{\subseteq}\,(\eta,\pi)$ and $Int(\mu,\pi)\,\widetilde{\subseteq}\,(\mu,\pi)$ implies that
$Int(\eta,\pi)\,\widetilde{\cap}\,Int(\mu,\pi)\,\widetilde{\subseteq}\,(\eta,\pi)\,\widetilde{\cap}\,(\mu,\pi)$. Therefore $Int(\eta,\pi)\,\widetilde{\cap}\,Int(\mu,\pi)$ is a BnHyS open subset of $(\eta,\pi)\,\widetilde{\cap}\,(\mu,\pi)$.
Hence $Int(\eta,\pi)\,\widetilde{\cap}\,Int(\mu,\pi)\,\widetilde{\subseteq}\,Int((\eta,\pi)\,\widetilde{\cap}\,(\mu,\pi))$.
Thus, $Int((\eta,\pi)\,\widetilde{\cap}\,(\mu,\pi)) = Int(\eta,\pi)\,\widetilde{\cap}\,Int(\mu,\pi)$.

7. Since $(\eta,\pi)\,\widetilde{\subseteq}\,(\eta,\pi)\,\widetilde{\cup}\,(\mu,\pi)$ and $(\mu,\pi)\,\widetilde{\subseteq}\,(\eta,\pi)\,\widetilde{\cup}\,(\mu,\pi)$. So by part (5),
$Int(\eta,\pi)\,\widetilde{\subseteq}\,Int((\eta,\pi)\,\widetilde{\cup}\,(\mu,\pi))$ and $Int(\mu,\pi)\,\widetilde{\subseteq}\,Int((\eta,\pi)\,\widetilde{\cup}\,(\mu,\pi))$.
Thus $Int(\eta,\pi)\,\widetilde{\cup}\,Int(\mu,\pi)\,\widetilde{\subseteq}\,Int((\eta,\pi)\,\widetilde{\cup}\,(\mu,\pi))$. □

Remark 5.15. This example demonstrates that the equivalence in Proposition 5.14 (7.) is not often true.

Example 5.16. Let's think about the BnHySTS $(\coprod_1,\coprod_2,\tau_{\mathcal{B}n\mathcal{H}y},\pi)$ over $\coprod_1,\coprod_2$ in Example 5.12 and the BnHySSs $(\mho,\pi)$ and (μ,π) defined as follow:
$(\mho,\pi) = \{((\varrho_1,\varrho_2),(\{\rho_2\},\{\sigma_2\})),((\varrho_1,\varrho_3),(\{\rho_1,\rho_3\},\{\sigma_1,\sigma_3\}))\}$ and
$(\mu,\pi) = \{((\varrho_1,\varrho_2),(\{\rho_1,\rho_3\},\{\sigma_1,\sigma_3\})),((\varrho_1,\varrho_3),(\coprod_1,\coprod_2))\}$.
Then $(\mho,\pi)\,\widetilde{\cup}\,(\mu,\pi) = \{((\varrho_1,\varrho_2),(\coprod_1,\coprod_2)),((\varrho_1,\varrho_3),(\coprod_1,\coprod_2))\}$,
$Int(\mho,\pi) = \{((\varrho_1,\varrho_2),(\{\rho_2\},\{\sigma_2\})),((\varrho_1,\varrho_3),(\{\rho_1\},\{\sigma_1\}))\}$ and
$Int(\mu,\pi) = \{((\varrho_1,\varrho_2),(\varnothing,\varnothing)),((\varrho_1,\varrho_3),(\varnothing,\varnothing))\}$.
Hence, $Int(\mho,\pi)\,\widetilde{\cup}\,Int(\mu,\pi) = \{((\varrho_1,\varrho_2),(\{\rho_2\},\{\sigma_2\})),((\varrho_1,\varrho_3),(\{\rho_1\},\{\sigma_1\}))\}$ and
$Int((\mho,\pi)\,\widetilde{\cup}\,(\mu,\pi)) = \{((\varrho_1,\varrho_2),(\coprod_1,\coprod_2)),((\varrho_1,\varrho_3),(\coprod_1,\coprod_2))\}$.
So that $Int(\mho,\pi)\,\widetilde{\cup}\,Int(\mu,\pi)\,\widetilde{\subseteq}\,Int((\mho,\pi)\,\widetilde{\cup}\,(\mu,\pi))$.
But $Int(\mho,\pi)\,\widetilde{\cup}\,Int(\mu,\pi)\,\widetilde{\nsupseteq}\,Int((\eta,\pi)\,\widetilde{\cup}\,(\mu,\pi))$.

Proposition 5.17. *Let $(\coprod_1,\coprod_2,\tau_{\mathcal{B}n\mathcal{H}y},\pi)$ be a BnHySTS and (η,π) be a BnHySS over $\coprod_1,\coprod_2$. Then*

1. $(Cl(\eta,\pi))^c = Int((\eta,\pi)^c)$.
2. $Cl((\eta,\pi)^c) = (Int(\eta,\pi))^c$.
3. $Cl(\eta,\pi) = (Int((\eta,\pi)^c))^c$.
4. $Int(\eta,\pi) = (Cl((\eta,\pi)^c))^c$.

Proof. From the definitions of BnHyS closure and BnHyS interior, we have
1. $Cl(\eta,\pi) = \widetilde{\cap}\,\{(\mu,\pi) \mid (\mu,\pi)^c \in \tau_{\mathcal{B}n\mathcal{H}y},(\eta,\pi)\,\widetilde{\subseteq}\,(\mu,\pi)\}$. Then
$(Cl(\eta,\pi))^c = [\widetilde{\cap}\,\{(\mu,\pi) \mid (\mu,\pi)^c \in \tau_{\mathcal{B}n\mathcal{H}y},(\eta,\pi)\,\widetilde{\subseteq}\,(\mu,\pi)\}]^c$ and hence
$(Cl(\eta,\pi))^c = \widetilde{\cup}\,\{(\mu,\pi)^c \mid (\mu,\pi)^c \in \tau_{\mathcal{B}n\mathcal{H}y},(\mu,\pi)^c\,\widetilde{\subseteq}\,(\eta,\pi)^c\} = Int((\eta,\pi)^c)$. □

We can prove (2.), (3.) and (4.) by the same way.

Definition 5.18. Let $(\coprod_1, \coprod_2, \tau_{\mathcal{B}n\mathcal{H}y}, \pi)$ be a BnHySTS and (η, π) be a BnHySS over $\coprod_1, \coprod_2$, then BnHyS boundary of BnHySS (η, π) over $\coprod_1, \coprod_2$ is symbolized by $b(\eta, \pi)$ and is described as $b(\eta, \pi) = Cl(\eta, \pi) \,\widetilde{\cap}\, Cl((\eta, \pi)^c)$.

Proposition 5.19. *Let $(\coprod_1, \coprod_2, \tau_{\mathcal{B}n\mathcal{H}y}, \pi)$ be a BnHySTS and (η, π) be a BnHySS over $\coprod_1, \coprod_2$. Then*

1. $b(\eta, \pi) \,\widetilde{\subseteq}\, Cl(\eta, \pi)$.
2. $b(\eta, \pi) = Cl(\eta, \pi) \,\widetilde{\cap}\, (Int(\eta, \pi))^c = Cl(\eta, \pi) \,\widetilde{\setminus}\, Int(\eta, \pi)$.
3. $Int(\eta, \pi) = (\eta, \pi) \,\widetilde{\setminus}\, b(\eta, \pi)$.
4. $b(Int(\eta, \pi)) \,\widetilde{\subseteq}\, b(\eta, \pi)$.
5. $b(Cl(\eta, \pi)) \,\widetilde{\subseteq}\, b(\eta, \pi)$.

Proof. 1. $b(\eta, \pi) = Cl(\eta, \pi) \,\widetilde{\cap}\, Cl((\eta, \pi)^c) \,\widetilde{\subseteq}\, Cl(\eta, \pi)$.

2. $b(\eta, \pi) = Cl(\eta, \pi) \,\widetilde{\cap}\, Cl((\eta, \pi)^c) = Cl(\eta, \pi) \,\widetilde{\cap}\, (Int(\eta, \pi))^c = Cl(\eta, \pi) \,\widetilde{\setminus}\, Int(\eta, \pi)$.

3. $(\eta, \pi) \,\widetilde{\setminus}\, b(\eta, \pi) = (\eta, \pi) \,\widetilde{\cap}\, (b(\eta, \pi))^c = (\eta, \pi) \,\widetilde{\cap}\, (Int((\eta, \pi)^c) \,\widetilde{\cup}\, Int(\eta, \pi))$
$= ((\eta, \pi) \,\widetilde{\cap}\, Int((\eta, \pi)^c)) \,\widetilde{\cup}\, ((\eta, \pi) \,\widetilde{\cap}\, Int(\eta, \pi)) = \widetilde{\varnothing} \,\widetilde{\cup}\, Int(\eta, \pi) = Int(\eta, \pi)$.

4. $b(Int(\eta, \pi)) = Cl(Int(\eta, \pi)) \,\widetilde{\cap}\, Cl((Int(\eta, \pi))^c) \,\widetilde{\subseteq}\, Cl(\eta, \pi) \,\widetilde{\cap}\, Cl((\eta, \pi)^c) = b(\eta, \pi)$.

5. $b(Cl(\eta, \pi)) = Cl(Cl(\eta, \pi)) \,\widetilde{\cap}\, Cl((Cl(\eta, \pi))^c) \,\widetilde{\subseteq}\, Cl(\eta, \pi) \,\widetilde{\cap}\, Cl((\eta, \pi)^c) = b(\eta, \pi)$. □

Proposition 5.20. *Let $(\coprod_1, \coprod_2, \tau_{\mathcal{B}n\mathcal{H}y}, \pi)$ be a BnHySTS and $(\eta, \pi), (\mu, \pi)$ be two BnHySSs over $\coprod_1, \coprod_2$. Then*

1. $b((\eta, \pi) \,\widetilde{\cup}\, (\mu, \pi)) \,\widetilde{\subseteq}\, b(\eta, \pi) \,\widetilde{\cup}\, b(\mu, \pi)$.
2. $b((\eta, \pi) \,\widetilde{\cap}\, (\mu, \pi)) \,\widetilde{\subseteq}\, b(\eta, \pi) \,\widetilde{\cup}\, b(\mu, \pi)$.

Proposition 5.21. *Let $(\coprod_1, \coprod_2, \tau_{\mathcal{B}n\mathcal{H}y}, \pi)$ be a BnHySTS and (η, π) be a BnHySSs over $\coprod_1, \coprod_2$. Then*

$$Int(\eta, \pi) \,\widetilde{\cup}\, b(\eta, \pi) = Cl(\eta, \pi).$$

6. Conclusion

In this paper, we proposed a novel extension of hypersoft sets, referred to as binary hypersoft sets, which provide a more comprehensive generalization of binary soft sets by operating over two universal sets and a parameter set. We defined and explored several fundamental operations on BnHySSs, including subset, superset, equality, complement, null and absolute sets, as well as extended and standard versions of union, intersection, difference, AND, and OR. In addition to establishing basic properties of BnHySSs, we conducted a comparative analysis with existing frameworks such as HySSs and BnSSs. Furthermore, we introduced the concept of binary hypersoft topology and the related notion of binary hypersoft subspace. Finally, we examined key topological constructs—limit points, neighborhoods, closures, interiors and boundaries—within the context of BnHySSs, thereby demonstrating the potential of this new structure for further theoretical exploration and practical applications.

The introduction of BnHySSs opens up several promising directions for future research. One potential area is the development of decision-making models and algorithms based on BnHySSs, particularly in environments involving multiple universal sets and complex parameter dependencies. Further investigation into the algebraic and categorical properties of BnHySSs could deepen the theoretical foundations and reveal new connections with other soft set extensions. Additionally, expanding the framework to incorporate fuzzy, intuitionistic, or rough elements may enhance its applicability to real-world problems characterized by uncertainty and vagueness. From a topological perspective, exploring continuity, compactness, connectedness, and other advanced topological properties within the BnHySS context could yield richer mathematical structures. Finally, practical applications in areas such as data science, artificial intelligence, medical diagnosis, and engineering decision systems remain a compelling direction for future exploration.

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