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Periodic evolution problems driven by Leray-Lions type operators with fully nonlinear boundary conditions

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Abstract. The purpose of this work is to investigate some classes of periodic evolution problems involving Leray-Lions type operators with nonlinear boundary conditions. Two key results are presented concerning weak solutions. For cases where the source term does not depend on the solution, a general abstract method is utilized, leveraging the time-periodic condition to provide both existence and uniqueness. For scenarios involving a nonlinear source term strongly linked to the solution, the existence and uniqueness of weak solutions are achieved without requiring any sign constraints on the nonlinearity. The methodology heavily relies on Leray-Schauder topological degree, supported by innovative technical estimates.

1. Introduction

The study of partial differential equations, like periodic evolutionary equations, is motivated by describing several relevant real phenomena. For instance, such periodic equations can be used to model periodic behaviors of biological, economic, ecological phenomena and physical processes, like for example the distribution biological model of two interacting species [26, 27], see also lagoon ecological interactions model [2, 3]. Furthermore, scientific evolution has given birth to different types of PDEs, we talk here about those involving nonlinear terms. These type of boundary value problems has the ability to predict certain complex phenomena, like for example the model of nonlinear heat transfer after the enthalpy and Kirchhoff transformations [33, 35, 36]. Several works have been dedicated to investigating the properties of solutions to nonlinear problems (existence, uniqueness, bifurcation, asymptotic behavior, etc). Topological methods are one of the most powerful and elegant approaches for studying nonlinear partial differential equations. It provides new tools commensurate with different application fields. For more details about these methods, we refer interested readers to see the works [11, 16–19, 27, 34].

In this work, we are concerned with a class of periodic PDEs having nonlinear boundary conditions

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modeled by the following problem

$$\begin{cases} \partial_t u - \operatorname{div}(A(t,x,\nabla u)) = f(t,x,u) & \text{in } Q_T \\ u(0,\cdot) = u(T,\cdot) & \text{in } \Omega \\ -A(t,x,\nabla u).\vec{n} = g(t,x,u) & \text{on } \Sigma_T, \end{cases} \tag{1}$$

where Ω is an open regular bounded subset of $\mathbb{R}^N$, with smooth boundary $\partial\Omega$, $\vec{n}$ is the outer unit normal vector on $\partial\Omega$, $T > 0$ is the period, $Q_T = (0,T) \times \Omega$, $\Sigma_T = (0,T) \times \partial\Omega$, $A : Q_T \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfies the standard conditions of Leray-Lions operator with growth $p \geq 2$. $f : Q_T \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, periodic in time with period T and enjoying some growth assumptions. The function $g : \Sigma_T \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, periodic in time with period T and satisfies some growth conditions to be specified later.

Related to the existing literature, the periodic PDEs have gained more attention from several workers [5, 7, 8, 13, 14, 21, 29]. We start by referring the readers to see the book [29] for a major and comprehensive introduction to periodic parabolic equations with regular data. Amann was also studied periodic parabolic equations with regular data in their work [4]. The author assured the existence of a classical periodic solution to the considered problem via the sub- and super-solution-method. In [31] Lions studied the well-posedness of the weak to a class of periodic parabolic equations involving Leray-Lions type operators. The author used the theory of maximal monotone operators to prove existence, uniqueness and regularity properties for the obtained solutions. Deuel & Hess in [21] interested by a quasilinear periodic equation having critical growth nonlinearity with respect to the gradient. They established the existence and regularity property of a weak periodic solution by employing the technics of sub- and super-solutions. The work [21] was generalized by Alaa et al in their paper [12], the authors examined the existence of a weak periodic solution to a nonlinear parabolic equation with L^1 data. They combined the truncation method with the sub- and super-solution technics to obatin SOLA solution (Solution Obtained as the Limit of Approximation). All the early mentioned works involves Dirichlet boundary conditions. However, the literature of periodic parabolic equations with nonlinear boundary condition are more limited. In [6] Badii's considered the equation (1) in the case $p = 2$ with bounded nonlinearity. Under this special choice, He established the existence and uniquness of weak bounded solution. Alaa et al [24] examined a quasilinear periodic equation involving Laplacian operator with nonlinear boundary conditions. The authors proved the existence of a weak periodic solution when the nonlinearity has critical growth with respect to the gradient and involves sign condition. Their proof was based on the utilization of Schauder's fixed point Theorem combined with the truncation method. Fragnelli [27] studied a system of $(p(x), q(x))$-Laplacian time-periodic equations with nonlocal terms under certain growth conditions in the L^2 framework. Utilizing the Leray-Schauder topological degree theory, Fragnelli demonstrated the existence of a positive *weak* periodic solution, assuming $\inf_{x\in\overline{\Omega}} p(x) > 2$ and $\inf_{x\in\overline{\Omega}} q(x) > 2$. In [9], Charkaoui addressed nonlinear periodic evolution equations featuring variable growth conditions and nonlinear source terms. The author introduced an innovative formulation, converting the periodic problem into an equivalent fixed-point problem within an appropriately chosen Banach space. By employing Leray-Schauder's topological degree theory, the existence and uniqueness of *weak* periodic solutions to the studied problems were rigorously established.

In the present work, we propose to investigate the existence and uniqueness of weak periodic solution to (1) with general assumptions. We will show two interesting existence and uniqueness results of the periodic solutions to (1). In the first one, we will study problem (1) when the source term does not depend on the solution u. For this setting, we shall use the monotone operators theory [31] to establish the existence of a periodic solution in a weak sense. Based on the monotony property of Leray-Lions operators, we prove that the obtained weak periodic solution is unique. Secondly, we study problem (1) by imposing general assumptions on the source term $f(t,x,u)$ involving monotony and growth conditions. We start by reformulating the studied problem into an equivalent fixed-point problem posed on a suitable Banach space. Then, we use Leray Shauder topological degree to demonstrate the existence of a weak periodic solution to (1). After, we combine the monotony property of the considered operator with the monotony of the source term $f(t,x,\cdot)$ to prove the uniqueness result. Furthermore, we shall show interesting compactness results which will be applicable to more general classes of periodic PDEs with nonlinear boundary conditions. Let us stress that our work is original in the sense that it generalizes the existing works in the literature [1, 6, 10, 13, 15, 21, 24, 29]. Moreover, it can be seen that our work affords the readers a new topological methods application for nonlinear periodic PDEs with general boundary conditions. Furthermore, the result of our paper can be applied to the periodic p-Laplacian equations with generalized Robin boundary condition modeled by

$$\begin{cases} \partial_t u - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = f(t,x,u) & \text{in } Q_T \\ u(0,\cdot) = u(T,\cdot) & \text{in } \Omega \\ -|\nabla u|^{p-2}\dfrac{\partial u}{\partial \vec{n}} = \gamma(t,x)u|u|^{p-2} + g(t,x,u) & \text{on } \Sigma_T, \end{cases}$$

with γ is a measurable function satisfying $0 < \gamma_0 < \gamma(t,x) < \gamma_1$ for some constants γ_0, γ_1 and for almost (t,x) in Σ_T. On the other view, we can easy to check that our obtained theoretical results include the case of mean curvature periodic PDEs modeled as follows

$$\begin{cases} \partial_t u - \operatorname{div}\left(\left(1+|\nabla u|^2\right)^{\frac{(p-2)}{2}} \nabla u\right) = f(t,x,u) & \text{in } Q_T \\ u(0,\cdot) = u(T,\cdot) & \text{in } \Omega \\ -\left(1+|\nabla u|^2\right)^{\frac{(p-2)}{2}} \dfrac{\partial u}{\partial \vec{n}} = \gamma(t,x)|u|^{p-2}u + g(t,x,u) & \text{on } \Sigma_T. \end{cases}$$

We have structured the rest of this work as follows. We begin Section 2 by presenting the necessary assumptions to solve problem (1). Thereafter, we introduce the functional framework involving our theoretical results and we recall an interesting surjectivity result from the monotone operators theory. Section 3 gives a plain matter for solving problem (1) with linear source term ($f(t,x,u)$ does not depends on the solution u). We will divide it into two subsections, in the first one, we establish the existence of a weak periodic solution via monotone operator theory. In the second subsection, we will show that the weak periodic solution is unique. Section 4 will be reserved to prove the main result of our paper. We will define the notion of a weak periodic solution to problem (1). We divide again the proof of our main result into two paragraphs. For the first one, we reformulate the periodic problem into a well-posedness fixed point problem in a suitable Banach space. Thereafter, we shall show the existence of a fixed point to the latter problem by using the Leray-Schauder topological degree. In the second paragraph, we will demonstrate that the obtained weak periodic solution to (1) is unique. In Section 5, we present an Appendix in which we prove some interesting compactness results that we will frequently use in the proof of our main results.

2. Assumptions and Mathematical Preliminaries

The aim of this section is to build the notion of a weak periodic solution to (1). First of all, let us introduce the necessary hypotheses to study the problem (1).

2.1. Assumptions

Throughout this paper, we assume that $A : Q_T \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfies the following assumptions:

($\mathbb{H}_1$): There exist a nonnegative function $H \in L^{p'}(Q_T)$ with ($p' = \frac{p}{p-1}$) and a nonnegative constant α_0 such that for all $\xi \in \mathbb{R}^N$, we have

$$|A(t,x,\xi)| \le H(t,x) + \alpha_0|\xi|^{p-1} \quad \text{for a.e. } (t,x) \in Q_T.$$

($\mathbb{H}_2$): There exists $d > 0$ such that for all $\xi \in \mathbb{R}^N$

$$A(t,x,\xi)\xi \ge d\,|\xi|^p \quad \text{for a.e. } (t,x) \in Q_T.$$

($\mathbb{H}_3$): For all ξ, ξ^* in $\mathbb{R}^N$ such that ($\xi \neq \xi^*$), we have

$$\langle A(t,x,\xi) - A(t,x,\xi^*), \xi - \xi^* \rangle > 0 \quad \text{for a.e. } (t,x) \in Q_T.$$

We assume that $g : \Sigma_T \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, periodic in time with period T such that

($\mathbb{H}_4$): There exist a constant $\alpha_1 \geq 0$ and a measurable function $K \in L^{p'}(\Sigma_T)$ such that for all $s \in \mathbb{R}$, we have

$$|g(t,x,s)| \leq K(t,x) + \alpha_1 |s|^{p-1} \quad \text{for a.e. } (t,x) \in \Sigma_T.$$

($\mathbb{H}_5$): There exist a measurable, bounded function β periodic in time, such that

$$g(t,x,s).s \geq \beta(t,x)|s|^p, \quad \text{for all } s \in \mathbb{R}, \text{ a.e. } (t,x) \in \Sigma_T. \tag{2}$$

with

$$0 < \beta_0 \leq \beta(t,x) \leq \beta_1, \quad \text{a.e. } (t,x) \in \Sigma_T.$$

($\mathbb{H}_6$): $s \longmapsto g(t,x,s)$ is nondecreasing for a.e. $(t,x) \in \Sigma_T$.

2.2. Functional framework and definitions

To study the existence result of a weak periodic solution to (1), it is necessary to clarify in which sense we want to solve problems (1). Let $T > 0$ and let $\mathfrak{X}$ be a Banach space. For any $\sigma \in [1, \infty]$, we denote by $L^\sigma(0,T;\mathfrak{X})$ the space of measurable functions $u : [0,T] \to \mathfrak{X}$ such that the following norm is finite:

$$\|u\|_{L^\sigma(0,T;\mathfrak{X})} = \begin{cases} \left(\int_0^T \|u(t)\|_{\mathfrak{X}}^\sigma \, dt\right)^{1/\sigma}, & \text{if } 1 \leq \sigma < \infty, \\ \operatorname{ess\,sup}_{t\in[0,T]} \|u(t)\|_{\mathfrak{X}}, & \text{if } \sigma = \infty. \end{cases}$$

We also introduce the space $C([0,T];\mathfrak{X})$, consisting of all strongly continuous mappings from $[0,T]$ into $\mathfrak{X}$, endowed with the norm:

$$\|u\|_{C([0,T];\mathfrak{X})} = \sup_{t\in[0,T]} \|u(t)\|_{\mathfrak{X}}.$$

Equipped with their respective norms, both $L^\sigma(0,T;\mathfrak{X})$ and $C([0,T];\mathfrak{X})$ form Banach spaces. These functional spaces play a central role in our analysis and provide the appropriate framework for studying time-dependent problems. For further foundational details, the reader is referred to classical sources such as [22, 25].

Let us introduce the functional framework involving our work. For any $p \geq 2$, we set

$$\mathcal{V}_T := L^p(0,T;W^{1,p}(\Omega)).$$

We equip with the following norm

$$\|u\|_{\mathcal{V}_T} := \Big(\iint\limits_{Q_T} |\nabla u|^p \, dxdt + \iint\limits_{\Sigma_T} \beta(t,x)|u|^p \, d\sigma_x dt \Big)^{\frac{1}{p}},$$

which is equivalent to the standard norm of $\mathcal{V}_T$. Here σ_x is the measure on the boundary $\partial\Omega$. Furthermore, we set

$$\mathcal{V}_T^* := L^{p'}(0,T;(W^{1,p}(\Omega))^*),$$

the dual space of $\mathcal{V}_T$. These spaces lead to introduce the following functional space

$$\mathcal{W}_T := \{u \in \mathcal{V}_T, \quad \partial_t u \in \mathcal{V}_T^*\}.$$

We equip with the graph norm

$$\|u\|_{\mathcal{W}_T} := \|u\|_{\mathcal{V}_T} + \|\partial_t u\|_{\mathcal{V}_T^*}.$$

We will denote by $\langle \cdot, \cdot \rangle$ the duality pairing between $(W^{1,p}(\Omega))^*$ and $W^{1,p}(\Omega)$. Before closing this paragraph, we recall an interesting Theorem of [31] which will be used in what follows.

Theorem 2.1 ([31]). *Let $\mathbb{L}$ be a linear closed, densely defined operator from the reflexive space $\mathcal{V}_T$ to $\mathcal{V}_T^*$, $\mathbb{L}$ maximal monotone and let $\mathbb{A}$ be a bounded hemicontinuous monotone mapping from $\mathcal{V}_T$ to $\mathcal{V}_T^*$, then $\mathbb{L} + \mathbb{A}$ is maximal monotone in $\mathcal{V}_T \times \mathcal{V}_T^*$. Moreover, if $\mathbb{A}$ is coercive, then $(\mathbb{L} + \mathbb{A})$ is surjective.*

In the whole of the paper, we will denote by C any generic nonnegative constant where its value may be changed from line to line and sometimes in the same line.

Let r and s be two positive constants such that $\frac{1}{r} + \frac{1}{s} = 1$. Then, for any $\varepsilon > 0$, the following inequality holds:

$$ab \leq \varepsilon a^r + C_\varepsilon b^s, \qquad \text{for all } a, b \geq 0. \tag{3}$$

Here, $C_\varepsilon = \varepsilon^{-\frac{1}{r-1}}$, and this result is widely known as the elementary form of Young's inequality with $\varepsilon > 0$.

3. Existence and uniqueness result with $\mathcal{V}_T^*$ source term

This section tackles the existence and uniqueness of a weak periodic solution to (1) when the source term does not depend on the solution u. Let h be an element of $\mathcal{V}_T^*$, we consider the following periodic boundary value problem

$$\begin{cases} \partial_t u - \operatorname{div}(A(t,x,\nabla u)) = h(t,x) & \text{in } Q_T \\ u(0,\cdot) = u(T,\cdot) & \text{in } \Omega \\ -A(t,x,\nabla u).\vec{n} = g(t,x,u) & \text{on } \Sigma_T. \end{cases} \tag{4}$$

Let us mention that the results of this section (existence and uniqueness) can be viewed as an auxiliary step to establish the well-posedness of the proposed model (equation (1)). In the following definition, we will define the notion of a weak periodic solution used to solve problem (4).

Definition 3.1. *A measurable function $u : Q_T \to \mathbb{R}$ is said to be a weak periodic solution to (4) if it satisfies*

$$u \in \mathcal{W}_T, \quad u(0) = u(T) \text{ in } L^2(\Omega)$$

$$\int_0^T \langle \partial_t u, \varphi \rangle \, dt + \iint_{Q_T} A(t,x,\nabla u)\nabla\varphi \, dxdt + \iint_{\Sigma_T} g(t,x,u)\varphi \, d\sigma_x dt = \int_0^T \langle h, \varphi \rangle \, dt, \tag{5}$$

for every test function $\varphi \in \mathcal{V}_T$.

In this manuscript, it is important to note that all integrals over the boundary $\partial\Omega$ are taken with respect to the $(N-1)$-dimensional Hausdorff measure, denoted by σ_x, which serves as the surface measure.

Remark 3.2. *Let us remark that the periodic condition in Definition 3.1 makes sense. Indeed, from [31] the following continuous embedding holds true*

$$\mathcal{W}_T \hookrightarrow C\left([0,T]; L^2(\Omega)\right), \tag{6}$$

which gives that the mapping $t \mapsto u(t,x)$ is continuous from $[0,T]$ into $L^2(\Omega)$ and therefore we have

$$\int_\Omega u(0,x)\phi(x) \, dx = \int_\Omega u(T,x)\phi(x) \, dx,$$

for all ϕ in $L^2(\Omega)$.

In the following theorem, we enunciate the main result of this section.

Theorem 3.3. *Assume that* $(\mathbb{H}_1)$-$(\mathbb{H}_6)$ *hold. Then, for any h belonging to* $\mathcal{V}_T^*$*, problem* (4) *has a unique weak periodic solution u in the sens of Definition 3.1.*

We divide the proof of Theorem 3.3 into two steps. In the first step, we establish the existence of a weak periodic solution to (4) via the result of Theorem 2.1. While the second step will be reserved to establishing the uniqueness of the obtained solution.

3.1. Existence result

To show the existence of a weak periodic solution to 4, we shall use the result of Theorem 2.1. To do this, we must introduce two mappings $\mathbb{L}$ and $\mathbb{A}$. Let us define

$$\mathbb{D}(\mathbb{L}) := \Big\{ u \in \mathcal{W}_T \text{ such that } u(0) = u(T) \Big\}.$$

By employing the density of $C_c^\infty(Q_T)$ in $\mathcal{V}_T$ and using the fact that $C_c^\infty(Q_T) \subset \mathbb{D}(\mathbb{L})$, one may conclude that $\mathbb{D}(\mathbb{L})$ is dense in $\mathcal{V}_T$. Let us define the mapping $\mathbb{L} : \mathbb{D}(\mathbb{L}) \to \mathcal{V}_T^*$ such as

$$\langle \mathbb{L}u, \varphi \rangle := \int_0^T \langle \partial_t u, \varphi \rangle \, dt, \quad \forall \varphi \in \mathcal{V}_T.$$

In view to the result of [[31], Lemma 1.1, p. 313], we can deduce that $\mathbb{L}$ is closed, skew-adjoint and maximal monotone operator. Thereafter, we will define the operator $\mathbb{A} : \mathcal{V}_T \longrightarrow \mathcal{V}_T^*$ by the following form

$$\langle \mathbb{A}u, \varphi \rangle := \iint_{Q_T} A(t, x, \nabla u) \nabla \varphi \, dxdt + \iint_{\Sigma_T} g(t, x, u) \varphi \, d\sigma_x dt,$$

for all $\varphi \in \mathcal{V}_T$. Hence, it is easily verified that the problem (4) is equivalent to the research of a solution to following abstract equation

$$\mathbb{L}u + \mathbb{A}u = \mathbb{F}, \tag{7}$$

where $\mathbb{F}$ is an element of $\mathcal{V}_T^*$ defined as

$$\langle \mathbb{F}, \varphi \rangle := \int_0^T \langle h, \varphi \rangle \, dt, \quad \forall \varphi \in \mathcal{V}_T.$$

To apply the result of Theorem 2.1, we must verify that $\mathbb{A}$ is hemicontinuous, monotone and coercive operator.

- **Hemicontinuity of the operator $\mathbb{A}$:** For any $u, \varphi \in \mathcal{V}_T$, we employ the growth assumptions $(\mathbb{H}_1)$ and $(\mathbb{H}_4)$ to obtain

$$|\langle \mathbb{A}u, \varphi \rangle| \leq \iint_{Q_T} \Big(H(t, x) + \alpha_0 |\nabla u|^{p-1} \Big) |\nabla \varphi| \, dxdt + \iint_{\Sigma_T} \Big(K(t, x) + \alpha_1 |u|^{p-1} \Big) |\varphi| \, d\sigma_x dt.$$

  By applying Hölder's inequality, one gets

$$|\langle \mathbb{A}u, \varphi \rangle| \leq \Big(\|H\|_{L^{p'}(Q_T)} + \alpha_0 \|\nabla u\|_{L^p(Q_T)}^{p-1} \Big) \|\nabla \varphi\|_{L^p(Q_T)} + \Big(\|K\|_{L^{p'}(\Sigma_T)} + \alpha_1 \|u\|_{L^p(\Sigma_T)}^{p-1} \Big) \|\varphi\|_{L^p(\Sigma_T)}.$$

Which permit us to deduce

$$|\langle \mathbb{A}u, \varphi\rangle| \leq \left(\|H\|_{L^{p'}(Q_T)} + \|K\|_{L^{p'}(\Sigma_T)} + C\|u\|^{p-1}_{\mathcal{V}_T}\right)\|\varphi\|_{\mathcal{V}_T}, \tag{8}$$

where C is a constant dependent on $T, \Omega, \alpha_0, \alpha_1$ and β_1. Hence, from (8) it follows that

$$\|\mathbb{A}u\|_{\mathcal{V}_T^*} \leq \left(\|H\|_{L^{p'}(Q_T)} + \|K\|_{L^{p'}(\Sigma_T)} + C\|u\|^{p-1}_{\mathcal{V}_T}\right).$$

Consequently, by using the result of [[30], Theorems 2.1. and 2.3], it follows that $\mathbb{A}$ is a hemicontinuous bounded operator.

- **Monotonicity of the operator $\mathbb{A}$:** Let $u, v \in \mathcal{V}_T$, we have

$$\begin{aligned}\langle \mathbb{A}(u) - \mathbb{A}(v), u - v\rangle = &\iint\limits_{Q_T} (A(t,x,\nabla u) - A(t,x,\nabla v))(\nabla u - \nabla v)\, dxdt \\ &+ \iint\limits_{\Sigma_T} (g(t,x,u) - g(t,x,v))(u - v)\, d\sigma_x dt.\end{aligned}$$

Due to the assumptions **(**$\mathbb{H}_3$**)** and **(**$\mathbb{H}_6$**)**, one may deduce that

$$\langle \mathbb{A}(u) - \mathbb{A}(v), u - v\rangle \geq 0.$$

Which gives us the monotony property of the operator $\mathbb{A}$.

- **Coercivity of the operator $\mathbb{A}$:** Let us employ the assumptions **(**$\mathbb{H}_2$**)** and **(**$\mathbb{H}_5$**)**, one has

$$\begin{aligned}\langle \mathbb{A}u, u\rangle &= \iint\limits_{Q_T} A(t,x,\nabla u)\nabla u\, dxdt + \iint\limits_{\Sigma_T} g(t,x,u)u\, d\sigma_x dt \\ &\geq d \iint\limits_{Q_T} |\nabla u|^p\, dxdt + \iint\limits_{\Sigma_T} \beta(t,x)|u|^p\, d\sigma_x dt \\ &\geq \min(1,d)\|u\|^p_{\mathcal{V}_T}.\end{aligned}$$

Then we get

$$\lim_{\|u\|_{\mathcal{V}_T}\to\infty} \frac{\langle \mathbb{A}u, u\rangle}{\|u\|_{\mathcal{V}_T}} \geq \min(1,d) \lim_{\|u\|_{\mathcal{V}_T}\to\infty} \|u\|^{p-1}_{\mathcal{V}_T} = \infty.$$

Hence, the operator $\mathbb{A}$ is coercive.

Consequently, one may use the result of Theorem 2.1 to deduce the existence of $u \in \mathbb{D}(\mathbb{L})$ a solution to the abstract problem (7), which permit us to conclude the existence of a weak periodic solution to (4) in the sense of Definition 3.1.

3.2. Uniquness result

To establish the uniqueness of the weak periodic solution (4), we take u_1 and u_2 two weak periodic solutions to (4). Let us choose $\varphi = u_1 - u_2$ in the weak formulation (5) of u_1 and also for the weak formulation of u_2, one obtains

$$\int_0^T \langle \partial_t u_1, u_1 - u_2\rangle\, dt + \iint\limits_{Q_T} A(t,x,\nabla u_1)\nabla(u_1 - u_2) + \iint\limits_{\Sigma_T} g(t,x,u_1)(u_1 - u_2)\, d\sigma_x dt = \int_0^T \langle h, (u_1 - u_2)\rangle\, dt, \tag{9}$$

and

$$\int_0^T \langle \partial_t u_2, u_1 - u_2\rangle \, dt + \iint_{Q_T} A(t,x,\nabla u_2)\nabla(u_1 - u_2) + \iint_{\Sigma_T} g(t,x,u_2)(u_1 - u_2)\, d\sigma_x dt = \int_0^T \langle h, (u_1 - u_2)\rangle \, dt. \quad (10)$$

After, we subtract the two relations (9) and (10), one gets

$$\begin{aligned} &\int_0^T \langle \partial_t(u_1 - u_2), u_1 - u_2\rangle \, dt + \iint_{Q_T} (A(t,x,\nabla u_1) - A(t,x,\nabla u_2))\nabla(u_1 - u_2)\, dxdt \\ &\qquad + \iint_{\Sigma_T} (g(t,x,u_1) - g(t,x,u_2))(u_1 - u_2)\, d\sigma_x dt = 0. \end{aligned} \quad (11)$$

Let us remark that $(u_1 - u_2)(0,\cdot) = (u_1 - u_2)(T,\cdot)$ in Ω, we then have

$$\int_0^T \langle \partial_t(u_1 - u_2), u_1 - u_2\rangle \, dt = 0.$$

Therefore, the relation (11) becomes

$$\iint_{Q_T} (A(t,x,\nabla u_1) - A(t,x,\nabla u_2))\nabla(u_1 - u_2)\, dxdt + \iint_{\Sigma_T} (g(t,x,u_1) - g(t,x,u_2))(u_1 - u_2)\, d\sigma_x dt = 0. \quad (12)$$

From (12), it is not difficult to see that

$$\iint_{Q_T} (A(t,x,\nabla u_1) - A(t,x,\nabla u_2))\nabla(u_1 - u_2)\, dxdt = 0 \quad (13)$$

$$\iint_{\Sigma_T} (g(t,x,u_1) - g(t,x,u_2))(u_1 - u_2)\, d\sigma_x dt = 0. \quad (14)$$

Therefore, using assumptions **(**$\mathbb{H}_3$**)** and **(**$\mathbb{H}_6$**)**, we can deduce from the relations (13) and (14) the existence of a constant c such that

$$u_1 - u_2 = c \text{ a.e. in } Q_T \text{ and } u_1 - u_2 = 0 \text{ a.e. in } \Sigma_T. \quad (15)$$

From (15), it comes that

$$u_1 = u_2 \text{ a.e. in } Q_T.$$

Which proves the uniqueness of the weak periodic solution to (4).

4. Main results

In this section, we take interest in the existence and uniqueness of the weak periodic solution to (1). We will start by specifying some hypotheses on the nonlinear source term $f(t,x,u)$. Under the assumption that $f : Q_T \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, periodic in time with period T, we assume that

($\mathbb{H}_7$**):** f satisfies the following growth condition

$$\left| f(t,x,s) \right| \leq \mathfrak{a}(t,x)|s|^{(\delta-1)} + \mathfrak{b}(t,x), \quad (16)$$

for a.e $(t,x) \in Q_T$ and for all $s \in \mathbb{R}$, where $\delta \in [1,p]$ and $(\mathfrak{a},\mathfrak{b})$ are nonnegative functions belonging to $L^\infty(Q_T) \times L^{p'}(Q_T)$.

($\mathbb{H}_8$): $s \longmapsto f(t,x,s)$ is nondecreasing for a.e. (t,x) in Q_T.

Without loss of clarity, we propose to enunciate in the following definition the means of a weak periodic solution to (1).

Definition 4.1. *We call weak periodic solution to* (1)*, all measurable function* $u : Q_T \to \mathbb{R}$ *that satisfies*

$$u \in \mathcal{W}_T, \quad u(0) = u(T) \text{ in } L^2(\Omega)$$

$$\int_0^T \langle \partial_t u, \varphi \rangle \, dt + \iint_{Q_T} A(t,x,\nabla u)\nabla\varphi \, dxdt + \iint_{\Sigma_T} g(t,x,u)\varphi \, d\sigma_x dt = \iint_{Q_T} f(t,x,u)\varphi \, dxdt, \tag{17}$$

for every test function $\varphi \in \mathcal{V}_T$.

Remark 4.2. *It is important to highlight that assumption* ($\mathbb{H}_7$) *guarantees that, for every* $u \in L^p(Q_T)$*, the nonlinear function* $f(\cdot,\cdot,u)$ *belongs to* $L^{p'}(Q_T)$*. This result stems directly from* ($\mathbb{H}_7$)*, particularly when combined with the classical Hölder's inequality. Specifically, we derive the following estimate*

$$\iint_{Q_T} |f(t,x,u)|^{p'} dx\,dt \leq C\left(\|\mathfrak{a}\|^{p'}_{L^\infty(Q_T)} \iint_{Q_T} |u|^{\frac{p(\delta-1)}{p-1}} \, dx\,dt + \iint_{Q_T} |\mathfrak{b}(t,x)|^{p'} \, dx\,dt \right).$$

Applying Hölder's inequality once more with carefully chosen exponents, we further obtain

$$\iint_{Q_T} |f(t,x,u)|^{p'} \, dx\,dt \leq C\left((T \operatorname{meas}(\Omega))^{1-\theta} \, \|\mathfrak{a}\|^{p'}_{L^\infty(Q_T)} \|u\|^{\theta}_{L^p(Q_T)} + \|\mathfrak{b}\|^{p'}_{L^{p'}(Q_T)} \right). \tag{18}$$

where $\theta = \frac{\delta-1}{p-1} \in [0,1]$*, and C is a nonnegative constant depending on p. Moreover, taking into account the embedding* $\mathcal{V}_T \hookrightarrow L^p(Q_T)$ *and utilizing the result from* (18)*, we conclude that the right-hand side of* (17) *is well-defined.*

In the following theorem, we state our main result of this section.

Theorem 4.3. *Assume that* ($\mathbb{H}_1$)-($\mathbb{H}_8$) *hold. Additionally, if either of the following conditions is fulfilled:*

(*i*) $1 \leq \delta < p$*, or*

(*ii*) $\delta = p$ *and* $\|\mathfrak{a}\|_{L^\infty(Q_T)}$ *is sufficiently small,*

then problem (1) *admits a unique weak periodic solution u satisfying the conditions of Definition 4.1.*

Similarly to the above theorem, we will divide the proof of Theorem 4.3 into two main parts. In the first one, we will establish the existence of a weak periodic solution to the problem (1). For the second part, we shall show the uniqueness of the weak periodic solution to (1).

4.1. Existence result

The aim of this paragraph is to show the existence of a weak periodic solution to (1). To do this, we propose to use Leray Schauder topological degree. We start by introducing the following mapping

$$\begin{aligned} \mathcal{G} : [0,1] \times L^p(Q_T) &\longrightarrow L^p(Q_T) \\ (\lambda, v) &\longmapsto u, \end{aligned}$$

where u is a weak periodic solution to the following problem

$$\begin{cases} \partial_t u - \operatorname{div}(A(t,x,\nabla u)) = \lambda f(t,x,v) & \text{in } Q_T \\ u(0,\cdot) = u(T,\cdot) & \text{in } \Omega \\ -A(t,x,\nabla u).\vec{n} = g(t,x,u) & \text{on } \Sigma_T. \end{cases} \tag{19}$$

From Remark 4.2, we argue that for any fixed (λ, v) in $[0,1] \times L^p(Q_T)$ the source term $\lambda f(t,x,v)$ belongs to $L^{p'}(Q_T)$. Moreover, we know that $L^{p'}(Q_T) \hookrightarrow \mathcal{V}_T^*$. Then, according to the existence and uniqueness results of Theorem 2.1, we derive that for any fixed $(\lambda, v) \in [0,1] \times L^p(Q_T)$ problem (19) has a unique weak periodic solution u which satisfies the following conditions

$$\begin{gathered} u \in \mathcal{W}_T, \quad u(0,x) = u(T,x) \text{ in } L^2(\Omega) \\ \int_0^T \langle \partial_t u, \varphi \rangle \, dt + \iint_{Q_T} A(t,x,\nabla u)\nabla\varphi \, dxdt + \iint_{\Sigma_T} g(t,x,u)\varphi \, d\sigma_x dt = \lambda \iint_{Q_T} f(t,x,v)\varphi \, dxdt, \end{gathered} \tag{20}$$

for every test function $\varphi \in \mathcal{V}_T$. Thus, the mapping $\mathcal{G}$ is well defined for all (λ, v) belonging to $[0,1] \times L^p(Q_T)$. As well known, to apply Leray Schauder's topological degree, we need to proclaim that $\mathcal{G}$ satisfies some properties. This will be done in the succeeding subsections.

4.1.1. The mapping $\mathcal{G}$ is continuous

Our aim is to show that the mapping $\mathcal{G}$ is continuous. To do this, let us consider a sequence (λ_n, v_n) in $[0,1] \times L^p(Q_T)$ such that

$$(\lambda_n, v_n) \to (\lambda, v) \text{ strongly in } [0,1] \times L^p(Q_T). \tag{21}$$

By setting

$$u_n = \mathcal{G}(\lambda_n, v_n), \quad u = \mathcal{G}(\lambda, v), \tag{22}$$

the continuity of $\mathcal{G}$ requires only that (u_n) converges strongly to u in $L^p(Q_T)$, where u is the weak periodic solution of (19) and (u_n) is a sequence of weak periodic solution to the following problem

$$\begin{cases} \partial_t u_n - \operatorname{div}(A(t,x,\nabla u_n)) = \lambda_n f(t,x,v_n) & \text{in } Q_T \\ u_n(0,\cdot) = u_n(T,\cdot) & \text{in } \Omega \\ -A(t,x,\nabla u_n).\vec{n} = g(t,x,u_n) & \text{on } \Sigma_T. \end{cases} \tag{23}$$

It is clear that (u_n) satisfies the following weak formulation

$$\begin{gathered} u_n \in \mathcal{W}_T, \quad u_n(0) = u_n(T) \text{ in } L^2(\Omega) \\ \int_0^T \langle \partial_t u_n, \varphi \rangle \, dt + \iint_{Q_T} A(t,x,\nabla u_n)\nabla\varphi \, dxdt + \iint_{\Sigma_T} g(t,x,u_n)\varphi \, d\sigma_x dt = \lambda_n \iint_{Q_T} f(t,x,v_n)\varphi \, dxdt, \end{gathered} \tag{24}$$

for every test function $\varphi \in \mathcal{V}_T$. Particularly for $\varphi = u_n$ in (24), one gets

$$\int_0^T \langle \partial_t u_n, u_n \rangle \, dt + \iint_{Q_T} A(t,x,\nabla u_n)\nabla u_n \, dxdt + \iint_{\Sigma_T} g(t,x,u_n)u_n \, d\sigma_x dt = \lambda_n \iint_{Q_T} f(t,x,v_n)u_n \, dxdt. \tag{25}$$

By using the periodicity properties of (u_n), we obtain

$$\int_0^T \langle \partial_t u_n, u_n \rangle \, dt = \frac{1}{2}\int_\Omega (u_n)^2(T)\, dx - \frac{1}{2}\int_\Omega (u_n)^2(0)\, dx = 0. \tag{26}$$

Furthermore, by using hypotheses $(\mathbb{H}_2)$ and $(\mathbb{H}_5)$, we derive from (25) that

$$d \iint_{Q_T} |\nabla u_n|^p \, dxdt + \iint_{\Sigma_T} \beta(t,x)|u_n|^p \, d\sigma_x dt \le \iint_{Q_T} \left| f(t,x,v_n)u_n \right| \, dxdt \tag{27}$$

To deal with the right-hand side of (27), we employ Hölder's inequality, one gets

$$\min(1,d)\|u_n\|^p_{\mathcal{V}_T} \le \left(\iint_{Q_T} \left|f(t,x,v_n)\right|^{p'} dx\,dt\right)^{\frac{1}{p'}} \times \left(\iint_{Q_T} |u_n|^p\,dx\,dt\right)^{\frac{1}{p}}. \tag{28}$$

From (21), we deduce that the sequence (v_n) is bounded in $L^p(Q_T)$. Moreover, using the result from Remark 4.2, we can infer the existence of a nonnegative constant C, independent of n, such that

$$\left(\iint_{Q_T} \left|f(t,x,v_n)\right|^{p'} dx\,dt\right)^{\frac{1}{p'}} \le C. \tag{29}$$

Therefore inequality (28) becomes

$$\min(1,d)\|u_n\|^p_{\mathcal{V}_T} \le C\|u_n\|_{L^p(Q_T)}. \tag{30}$$

By combining the continuous embedding $\mathcal{V}_T \hookrightarrow L^p(Q_T)$ with Young's inequality, we obtain

$$\|u_n\|^p_{\mathcal{V}_T} \le C, \tag{31}$$

where C is a nonnegative constant independent of n. Which proves that (u_n) is bounded in $\mathcal{V}_T$. In addition, by taking into account the growth condition $(\mathbb{H}_4)$, we derive that $(g(t,x,u_n))$ is bounded in $L^{p'}(\Sigma_T)$. Thus, by following a standard argument, we can show that

$$\|\partial_t u_n\|_{\mathcal{V}^*_T} \le C. \tag{32}$$

Thanks to the compactness results (i)-(iii) of Lemma 5.2, we deduce the existence of a measurable function $u^\star : Q_T \to \mathbb{R}$ and a subsequence of (u_n) still denoted by (u_n) for simplicity such that

$$(u_n) \to u^\star \text{ strongly in } L^p(Q_T) \text{ and a.e. in } Q_T, \tag{33}$$

$$(u_n) \to u^\star \text{ strongly in } L^p(\Sigma_T) \text{ and a.e. in } \Sigma_T, \tag{34}$$

$$(\nabla u_n) \to \nabla u^\star \text{ a.e. in } Q_T. \tag{35}$$

Furthermore, it results that

$$\lambda_n f(t,x,v_n) \rightharpoonup \lambda f(t,x,v) \text{ weakly in } L^{p'}(Q_T) \text{ and a.e. in } Q_T, \tag{36}$$

$$g(t,x,u_n) \rightharpoonup g(t,x,u^\star) \text{ weakly in } L^{p'}(\Sigma_T) \text{ and a.e. in } \Sigma_T. \tag{37}$$

On the other hand, we combine assumption $(\mathbb{H}_1)$ with (31), we conclude that $(A(t,x,\nabla u_n))$ is bounded in $L^{p'}(Q_T)^N$. In addition, the almost everywhere convergence (35) and (32) gives us

$$(A(t,x,\nabla u_n)) \rightharpoonup A(t,x,\nabla u^\star) \text{ weakly in } L^{p'}(Q_T)^N, \tag{38}$$

$$\partial_t u_n \rightharpoonup \partial_t u^\star \text{ weakly in } \mathcal{V}^*_T. \tag{39}$$

With the help of the convergences (36)-(39), we pass to the limit in (24) as n goes to $+\infty$, one obtains

$$u^\star \in \mathcal{W}_T, \quad u^\star(0) = u^\star(T) \text{ in } L^2(\Omega)$$

$$\int_0^T \langle \partial_t u^\star, \varphi\rangle\,dt + \iint_{Q_T} A(t,x,\nabla u^\star)\nabla\varphi\,dxdt + \iint_{\Sigma_T} g(t,x,u^\star)\varphi\,d\sigma_x dt = \lambda \iint_{Q_T} f(t,x,v)\varphi\,dxdt,$$

for all test function $\varphi \in \mathcal{V}_T$. Which proves that $u^\star$ solves (19) and since problem (19) has a unique weak periodic solution, we deduce that $u^\star = u$. Hence, the mapping $\mathcal{G}$ is continuous.

4.1.2. The mapping $\mathcal{G}$ is compact

In this paragraph, we will show that the mapping $\mathcal{G}$ is compact, which means that the image of each bounded set of $[0,1]\times L^p(Q_T)$ is a relatively compact set in $L^p(Q_T)$. To do this, let us take (λ_n, v_n) a bounded sequence in $[0,1]\times L^p(Q_T)$, we shall show that the sequence $(\mathcal{G}(\lambda_n, v_n))$ is relatively compact in $L^p(Q_T)$. By setting $u_n = \mathcal{G}(\lambda_n, v_n)$, it is clear that u_n solves problem (23). With the help of the boundness of (λ_n, v_n) in $[0,1]\times L^p(Q_T)$, we follow the same reasoning as of the continuity proof, we can establish directly the following estimates

$$\|u_n\|_{\mathcal{W}_T} \le C, \quad \|f(t,x,v_n)\|_{L^{p'}(Q_T)} \le C, \quad \|g(t,x,u_n)\|_{L^{p'}(\Sigma_T)} \le C. \tag{40}$$

Consequently, using again the compactness result (i) of Lemma 5.2, we deduce that (u_n) is relatively compact in $L^p(Q_T)$, which is quivalent to say that the mapping $\mathcal{G}$ is compact.

4.1.3. Existence of a positive radius $\mathcal{R}$ such that $\boldsymbol{deg}(u-\mathcal{G}(\lambda,u),\mathcal{B}_{\mathcal{R}},0)=1$

To continue, we need to establish the existence of a radius $\mathcal{R}>0$ such that the homotopy $\mathcal{G}$ is admissible within the ball

$$\mathcal{B}_{\mathcal{R}} := \left\{ v \in L^p(Q_T), \quad \|v\|_{L^p(Q_T)} < \mathcal{R} \right\}.$$

This is equivalent to proving that there exists a radius $\mathcal{R}>0$, independent of λ, such that $u \neq \mathcal{G}(\lambda,u)$ for any $u \in \partial\mathcal{B}_{\mathcal{R}}$ and $\lambda \in [0,1]$. Assuming the contrary, let us suppose there exists a sequence $(u_n) \in L^p(Q_T)$ such that $\|u_n\|_{L^p(Q_T)} \to +\infty$ and $u_n = \mathcal{G}(\lambda_n, u_n)$ for some $\lambda_n \in [0,1]$. This implies that (u_n) solves the following weak formulation

$$u_n \in \mathcal{W}_T, \quad u_n(0) = u_n(T) \text{ in } L^2(\Omega)$$
$$\int_0^T \langle \partial_t u_n, \varphi \rangle \, dt + \iint_{Q_T} A(t,x,\nabla u_n)\nabla\varphi \, dxdt + \iint_{\Sigma_T} g(t,x,u_n)\varphi \, d\sigma_x dt = \lambda_n \iint_{Q_T} f(t,x,u_n)\varphi \, dxdt, \tag{41}$$

for every test function $\varphi \in \mathcal{V}_T$. By choosing $\varphi = u_n$ as a test function in (41) and employing $(\mathbb{H}_2)$ and $(\mathbb{H}_5)$, we obtain

$$\int_0^T \langle \partial_t u_n, u_n \rangle \, dt + d \iint_{Q_T} |\nabla u_n|^p \, dxdt + \iint_{\Sigma_T} \beta(t,x)|u_n|^p \, d\sigma_x dt \le \iint_{Q_T} \left| f(t,x,u_n)u_n \right| dxdt. \tag{42}$$

Since u_n is periodic in time, one gets

$$\int_0^T \langle \partial_t u_n, u_n \rangle \, dt = \frac{1}{2}\int_\Omega (u_n)^2(T)\, dx - \frac{1}{2}\int_\Omega (u_n)^2(0)\, dx = 0. \tag{43}$$

Taking advantage of the growth assumption (16), inequation (42) becomes

$$\min(1,d)\,\|u_n\|^p_{\mathcal{V}_T} \le \iint_{Q_T} \mathfrak{a}(t,x)|u_n|^\delta \, dxdt + \iint_{Q_T} \mathfrak{b}(t,x)|u_n|\, dxdt. \tag{44}$$

By employing Hölder's inequality and invoking the continuous embedding $\mathcal{V}_T \hookrightarrow L^p(Q_T)$, one achieves

$$\mathcal{S}^p \min(1,d)\,\|u_n\|^p_{L^p(Q_T)} \le (T\,\mathrm{meas}\,(\Omega))^{\frac{p-\delta}{p}} \|\mathfrak{a}\|_{L^\infty(Q_T)}\|u_n\|^\delta_{L^p(Q_T)} + \|\mathfrak{b}\|_{L^{p'}(Q_T)}\|u_n\|_{L^p(Q_T)}, \tag{45}$$

where $\mathcal{S}$ is the embedding constant which depending on Ω, T and N. We have two cases to discuss.
Case 1: If $1 \leq \delta < p$. A direct application of Young's inequality (3) yields, for some $\varepsilon, \varepsilon' > 0$, the following result holds

$$\mathcal{S}^p \min(1, d) \|u_n\|^p_{L^p(Q_T)} \leq (\varepsilon + \varepsilon') \|u_n\|^p_{L^p(Q_T)} + C_\varepsilon (T \operatorname{meas}(\Omega)) \|\mathfrak{a}\|^{\frac{p}{p-\delta}}_{L^\infty(Q_T)} + C_{\varepsilon'} \|\mathfrak{b}\|^{p'}_{L^{p'}(Q_T)}. \tag{46}$$

By choosing $0 < (\varepsilon + \varepsilon') < \mathcal{S}^p \min(1, d)$ in (46), we derive that u_n is bounded $L^p(Q_T)$.
Case 2: If $\delta = p$. Inequality (45) becomes

$$\mathcal{S}^p \min(1, d) \|u_n\|^p_{L^p(Q_T)} \leq \|\mathfrak{a}\|_{L^\infty(Q_T)} \|u_n\|^p_{L^p(Q_T)} + \|\mathfrak{b}\|_{L^{p'}(Q_T)} \|u_n\|_{L^p(Q_T)}. \tag{47}$$

Using again Young's inequality (3) with the choice $\varepsilon = \frac{\mathcal{S}^p \min(1,d)}{2}$, it implies from (47) that

$$\frac{\mathcal{S}^p \min(1, d)}{2} \|u_n\|^p_{L^p(Q_T)} \leq \|\mathfrak{a}\|_{L^\infty(Q_T)} \|u_n\|^p_{L^p(Q_T)} + C_\varepsilon \|\mathfrak{b}\|^{p'}_{L^{p'}(Q_T)}. \tag{48}$$

Then, when $\|\mathfrak{a}\|_{L^\infty(Q_T)}$ is small enough such that $\|\mathfrak{a}\|_{L^\infty(Q_T)} < \frac{\mathcal{S}^p \min(1,d)}{2}$, we deduce from (48) that (u_n) is bounded $L^p(Q_T)$. Thus taking advantage to the analysis of two above cases, we derive a contradiction with the fact that $\|u_n\|_{L^p(Q_T)} \to +\infty$. We establish that there exists a nonnegative radius $\mathcal{R}$ for which the Leray-Schauder topological degree is admissible within the ball $\mathcal{B}_\mathcal{R}$ in the space $L^p(Q_T)$. Consequently, we conclude that the topological degree remains invariant under homotopy, which yields the relation

$$\mathbf{deg}(\mathcal{I}_d(\cdot) - \mathcal{G}(1, \cdot), \mathcal{B}_\mathcal{R}, 0) = \mathbf{deg}(\mathcal{I}_d(\cdot) - \mathcal{G}(0, \cdot), \mathcal{B}_\mathcal{R}, 0), \tag{49}$$

where $\mathcal{I}_d$ is the identity mapping defined from $L^p(Q_T)$ into $L^p(Q_T)$. Our strategy now is to show that

$$\mathbf{deg}(\mathcal{I}_d(\cdot) - \mathcal{G}(0, \cdot), \mathcal{B}_\mathcal{R}, 0) \neq 0. \tag{50}$$

Thus, to check property (50), we set $\theta = \mathcal{G}(0, \theta)$ which means that θ satisfies the following conditions

$$\begin{gathered} \theta \in \mathcal{W}_T, \quad \theta(0) = \theta(T) \text{ in } L^2(\Omega) \\ \int_0^T \langle \partial_t \theta, \varphi \rangle \, dt + \iint_{Q_T} A(t, x, \nabla\theta) \nabla\varphi \, dxdt + \iint_{\Sigma_T} g(t, x, \theta)\varphi \, d\sigma_x dt = 0, \end{gathered} \tag{51}$$

for all $\varphi \in \mathcal{V}_T$. We choose $\varphi = \theta$ as a test function in (51), one gets

$$\int_0^T \langle \partial_t \theta, \theta \rangle \, dt + \iint_{Q_T} A(t, x, \nabla\theta) \nabla\theta \, dxdt + \iint_{\Sigma_T} g(t, x, \theta)\theta \, d\sigma_x dt = 0. \tag{52}$$

To deal with the first integral, we use the periodicity properties of θ, we have

$$\int_0^T \langle \partial_t \theta, \theta \rangle \, dt = \frac{1}{2} \int_\Omega (\theta)^2(T) \, dx - \frac{1}{2} \int_\Omega (\theta)^2(0) \, dx = 0.$$

Furthermore, by using the hypotheses $(\mathbb{H}_2)$ and $(\mathbb{H}_5)$, the equality (52) becomes

$$d \iint_{Q_T} |\nabla\theta|^p \, dxdt + \iint_{\Sigma_T} \beta(t, x) |\theta|^p \, d\sigma_x dt \leq 0. \tag{53}$$

Which implies that

$$\|\theta\|^p_{\mathcal{V}_T} = 0,$$

and consequently $\theta = 0$. By taking into account the homotopy invariance property (49), we arrive at

$$\mathbf{deg}\,(\mathcal{I}_d(\cdot) - \mathcal{G}(1,\cdot), \mathcal{B}_{\mathcal{R}}, 0) = \mathbf{deg}\,(\mathcal{I}_d(\cdot), \mathcal{B}_{\mathcal{R}}, 0) = 1.$$

Hence, by employing the Leray–Schauder topological degree (see e.g [23])), we deduce that problem (1) has a weak periodic solution u which satisfies the conditions of Definition 4.1. Which ends the existence proof.

4.2. Uniquness result

Let us consider u_1 and u_2 two weak periodic solutions to (1) which satisfy the weak formulation (20). By taking $\varphi = u_1 - u_2$ as a test function in the difference between the two weak formulations of u_1 and u_2, we get

$$\begin{aligned} &\int_0^T \langle \partial_t(u_1 - u_2), u_1 - u_2\rangle \, dt + \iint_{Q_T} (A(t,x,\nabla u_1) - A(t,x,\nabla u_2))\nabla(u_1 - u_2)\, dxdt \\ &+ \iint_{\Sigma_T} (g(t,x,u_1) - g(t,x,u_2))(u_1 - u_2)\, d\sigma_x dt = \iint_{Q_T} (f(t,x,u_1) - f(t,x,u_2))(u_1 - u_2)\, dxdt. \end{aligned} \tag{54}$$

From the monotonicity assumption $(\mathbb{H}_8)$, the relation (54) becomes

$$\begin{aligned} \int_0^T \langle \partial_t(u_1 - u_2), u_1 - u_2\rangle \, dt &+ \iint_{Q_T} (A(t,x,\nabla u_1) - A(t,x,\nabla u_2))\nabla(u_1 - u_2)\, dxdt \\ &+ \iint_{\Sigma_T} (g(t,x,u_1) - g(t,x,u_2))(u_1 - u_2)\, d\sigma_x dt \le 0. \end{aligned}$$

Therefore, the rest of the uniqueness proof immediately follows the same reasoning as that of Theorem 3.3, which finishes Theorem 4.3 proof.

5. Appendix

This section focuses on proving a compactness result relevant to periodic parabolic problems with general nonlinear boundary conditions. To begin, we recall a fundamental lemma commonly known as Lebesgue's generalized convergence theorem.

Lemma 5.1. *[20] Let (F_n) be a sequence of measurable functions and F a measurable function such that $F_n \to F$ a.e. in Q_T. Let $(G_n) \subset L^1(Q_T)$ such that for all $n \in \mathbb{N}$, we have $|F_n| \le G_n$ a.e. in Q_T and $G_n \to G$ in $L^1(Q_T)$. Then*

$$\iint_{Q_T} F_n \, dxdt \to \iint_{Q_T} F\, dxdt.$$

Lemma 5.2. *Assume that $(\mathbb{H}_1)$-$(\mathbb{H}_6)$ hold and let (u_n) be the weak periodic solution of the following problem*

$$\begin{cases} \partial_t u_n - \operatorname{div}(A(t,x,\nabla u_n)) = f_n(t,x,u_n) & in \;\; Q_T \\ u_n(0,\cdot) \;=\; u_n(T,\cdot) & in \;\; \Omega \\ -A(t,x,\nabla u_n).\vec{n} = g_n(t,x,u_n) & on \;\; \Sigma_T, \end{cases} \tag{55}$$

in the sense that

$$u_n \in \mathcal{W}_T, \quad u_n(0) = u_n(T) \text{ in } L^2(\Omega)$$
$$\int_0^T \langle \partial_t u_n, \varphi \rangle \, dt + \iint_{Q_T} A(t,x,\nabla u_n)\nabla\varphi \, dxdt + \iint_{\Sigma_T} g_n(t,x,u_n)\varphi \, d\sigma_x dt = \iint_{Q_T} f_n(t,x,u_n)\varphi \, dxdt, \tag{56}$$

for every test function $\varphi \in \mathcal{V}_T$. If we assume that (u_n) is bounded in $\mathcal{V}_T$, $(f_n(t,x,u_n))$ is bounded in $L^{p'}(Q_T)$ and $(g_n(t,x,u_n))$ is bounded in $L^{p'}(\Sigma_T)$. Then, we have (up to a subsequence)

(i) *$(u_n) \to u$ strongly in $L^p(Q_T)$ and a.e. in Q_T,*
(ii) *$(u_n) \to u$ strongly in $L^p(\Sigma_T)$ and a.e. in Σ_T,*
(iii) *$(\nabla u_n) \to \nabla u$ a.e. in Q_T.*
(iv) *Moreover, if we have*

$$\lim_{n\to\infty} \iint_{Q_T} (A(t,x,\nabla u_n) - A(t,x,\nabla u))(\nabla u_n - \nabla u) \, dxdt \leq 0. \tag{57}$$

Then

$$u_n \to u \text{ strongly in } \mathcal{V}_T.$$

Proof.

(i) Since (u_n) is bounded in $\mathcal{V}_T$, $(f_n(t,x,u_n))$ is bounded in $L^{p'}(Q_T)$ and $(g_n(t,x,u_n))$ is bounded in $L^{p'}(\Sigma_T)$, one may deduce from the equation (55) that $(\partial_t u_n)$ is bounded in $\mathcal{V}_T^*$. By using Aubin's compactness lemma (see [31])), we derive that

$$\mathcal{W}_T \overset{\text{compact}}{\hookrightarrow} L^p(Q_T).$$

Therefore, we deduce that (up to a subsequence), one has

$$u_n \to u \text{ strongly in } L^p(Q_T) \text{ and a.e. in } Q_T.$$

(ii) With the help of the above boundness results, one may use Theorem 3.4.1 from [32], we get

$$u_n \to u \text{ strongly in } L^p(\Sigma_T) \text{ and a.e. in } \Sigma_T.$$

(iii) Our strategy is to prove that (∇u_n) is a Cauchy sequence in measure which permits to deduce that (∇u_n) converges to ∇u almost everywhere in Q_T (up to a subsequence). To do this, we are aiming to show that

$$\forall \delta > 0, \forall \varepsilon > 0, \exists n_0 \text{ such that } \forall n,\ m \geq n_0, \text{meas}\left\{(t,x), |(\nabla u_n - \nabla u_m)(t,x)| \geq \delta\right\} \leq \varepsilon. \tag{58}$$

Let $\delta > 0$ and $\varepsilon > 0$, we remark that for $k > 0$ and $\eta > 0$ we have the following inequality

$$\text{meas}\left\{(t,x), |(\nabla u_n - \nabla u_m)(t,x)| \geq \delta\right\} \leq \text{meas}(\omega_1) + \text{meas}(\omega_2) + \text{meas}(\omega_3) + \text{meas}(\omega_4),$$

where,

$$\omega_1 = \{(t,x), |\nabla u_n| \geq k\}, \ \omega_2 = \{(t,x), |\nabla u_m| \geq k\}, \ \omega_3 = \{(t,x), |u_n - u_m| \geq \eta\},$$
$$\omega_4 = \{(t,x), |(\nabla u_n - \nabla u_m)| \geq \delta, |\nabla u_n| \leq k, |\nabla u_m| \leq k, |u_n - u_m| \leq \eta.\}$$

Let us start by the first set ω_1, we have

$$k \operatorname{meas}(\omega_1) \leq \iint_{\omega_1} |\nabla u_n| \, dxdt \leq \iint_{Q_T} |\nabla u_n| \, dxdt.$$

Sine (u_n) is bounded in $\mathcal{V}_T$ and by using the continuous embedding $\mathcal{V}_T \hookrightarrow L^1\left(0, T; W^{1,1}(\Omega)\right)$, we deduce that

$$\operatorname{meas}(\omega_1) \leq \frac{1}{k} \|\nabla u_n\|_{L^1(Q_T)} \leq \frac{C}{k} \|u_n\|_{\mathcal{V}_T} \leq \frac{C}{k}.$$

By the same manner, we obtain that

$$\operatorname{meas}(\omega_2) \leq \frac{C}{k}.$$

Consequently, one may choose k large enough such that

$$\operatorname{meas}(\omega_1) + \operatorname{meas}(\omega_2) \leq 2\varepsilon. \tag{59}$$

Now, let us fix k large enough such that (59) holds true and passing to estimate meas(ω_3). We remark that

$$\eta \operatorname{meas}(\omega_3) \leq \iint_{\omega_3} |(u_n - u_m)| \, dxdt \leq \iint_{Q_T} |(u_n - u_m)| \, dxdt.$$

By employing Hölder's inequality, one obtains

$$\operatorname{meas}(\omega_3) \leq \frac{C}{\eta} \|u_n - u_m\|_{L^p(Q_T)}.$$

Using the strong convergence of (u_n) to u in $L^p(Q_T)$, we deduce that for a given η, there exists n_0 such that for n, $m \geq n_0$ we have

$$\operatorname{meas}(\omega_3) \leq \varepsilon.$$

To finalize the proof, we need to estimate meas (ω_4) and to choose η. To this end, we must use the equation (55) satisfied by u_n. Due to the monotonicity property $(\mathbb{H}_3)$, one has $(A(t,x,\xi_1) - A(t,x,\xi_2))(\xi_1 - \xi_2) > 0$ for $\xi_1 \neq \xi_2$. Moreover, let us remark that the set

$$\mathcal{K} := \left\{(\xi_1, \xi_2) \in \mathbb{R}^N \times \mathbb{R}^N \text{ such that } |\xi_1| \leq k, \, |\xi_2| \leq k \text{ and } |\xi_1 - \xi_2| \geq \delta\right\}$$

is compact and the mapping $\xi \longmapsto A(t,x,\xi)$ is continuous for almost (t,x) in Q_T. This fact permits to conclude that $(A(t,x,\xi_1) - A(t,x,\xi_2))(\xi_1 - \xi_2)$ reaches its minimum on $\mathcal{K}$ which will be denoted by $\gamma(t,x)$. Therefore, it is clearly that $\gamma(t,x) > 0$ a.e in Q_T. Employing this result, we deduce the existence of ε' such that, for all measurable set $\omega \subset Q_T$,

$$\iint_{\omega} \gamma(t,x) \, dxdt \leq \varepsilon' \quad \Rightarrow \quad \operatorname{meas}(\omega) \leq \varepsilon. \tag{60}$$

Thus, to obtain meas$(\omega_4) \leq \varepsilon$, it suffices to prove that $\iint_{\omega_4} \gamma(t,x) \, dxdt \leq \varepsilon'$. In accordance with the definitions of γ and ω_4, we have

$$\iint_{\omega_4} \gamma(t,x) \, dxdt \leq \iint_{\omega_4} (A(t,x,\nabla u_n) - A(t,x,\nabla u_m))(\nabla u_n - \nabla u_m) \mathbf{1}_{\{|u_n - u_m| \leq \eta\}} \, dxdt.$$

In addition, let us remark that the integrated term is nonnegative and

$$\nabla T_\eta (u_n - u_m) = (\nabla u_n - \nabla u_m) \mathbf{1}_{\{|u_n - u_m| \le \eta\}},$$

where $T_\eta(s) = \min(\eta, \max(s, -\eta))$. As a consequence, we obtain

$$\iint\limits_{\omega_4} \gamma(t,x)\, dxdt \le \iint\limits_{Q_T} (A(t,x,\nabla u_n) - A(t,x,\nabla u_m)) \nabla T_\eta (u_n - u_m)\, dxdt. \tag{61}$$

Subtracting the equation (56) for different indexes n and m, one gets

$$\begin{aligned}\int_0^T \langle \partial_t(u_n - u_m), \varphi\rangle\, dt + \iint\limits_{Q_T} (A(t,x,\nabla u_n) - A(t,x,\nabla u_m))\nabla\varphi\, dxdt + \iint\limits_{\Sigma_T} (g_n(t,x,u_n) - g_m(t,x,u_m))\varphi\, d\sigma_x dt \\ = \iint\limits_{Q_T} (f_n(t,x,u_n) - f_m(t,x,u_m))\varphi\, dxdt.\end{aligned} \tag{62}$$

By taking $\varphi = T_\eta (u_n - u_m) \in \mathcal{V}_T$ in the equation (62), one has

$$\begin{aligned}\int_0^T \left\langle \partial_t (u_n - u_m), T_\eta (u_n - u_m)\right\rangle dt + \iint\limits_{Q_T} (A(t,x,\nabla u_n) - A(t,x,\nabla u_m)) \nabla T_\eta (u_n - u_m)\, dxdt \\ + \iint\limits_{\Sigma_T} (g_n(t,x,u_n) - g_m(t,x,u_m)) T_\eta (u_n - u_m)\, d\sigma_x dt \\ = \iint\limits_{Q_T} (f_n(t,x,u_n) - f_m(t,x,u_m)) T_\eta (u_n - u_m)\, dxdt.\end{aligned} \tag{63}$$

Using the fact that $(u_n - u_m)(0) = (u_n - u_m)(T)$, it follows that

$$\int_0^T \left\langle \partial_t (u_n - u_m), T_\eta (u_n - u_m)\right\rangle dt = \int_\Omega S_\eta(u_n - u_m)(T)\, dx - \int_\Omega S_\eta(u_n - u_m)(0)\, dx = 0,$$

where $S_\eta(r) = \int_0^r T_\eta(s)ds$. Hence, the equation (63) becomes

$$\begin{aligned}&\iint\limits_{Q_T} (A(t,x,\nabla u_n) - A(t,x,\nabla u_m)) \nabla T_\eta (u_n - u_m)\, dxdt \\ &+ \iint\limits_{\Sigma_T} (g_n(t,x,u_n) - g_m(t,x,u_m)) T_\eta (u_n - u_m)\, d\sigma_x dt \\ &= \iint\limits_{Q_T} (f_n(t,x,u_n) - f_m(t,x,u_m)) T_\eta (u_n - u_m)\, dxdt.\end{aligned} \tag{64}$$

Using the fact that $|T_\eta(s)| \leq \eta$, one gets

$$\begin{aligned}&\iint\limits_{Q_T} \left(A\left(t,x,\nabla u_n\right) - A\left(t,x,\nabla u_m\right)\right) \nabla T_\eta \left(u_n - u_m\right)\, dxdt \\ &\qquad \leq 2\eta \left(\iint\limits_{\Sigma_T} \left|g_n(t,x,u_n)\right|\, d\sigma_x dt + \iint\limits_{Q_T} \left|f_n(t,x,u_n)\right|\, dxdt \right).\end{aligned} \tag{65}$$

On the other hand, since $(f_n(t,x,u_n))$ is bounded in $L^{p'}(Q_T)$ and $(g_n(t,x,u_n))$ is bounded in $L^{p'}(\Sigma_T)$, we deduce from (65) that

$$\iint\limits_{Q_T} \left(A\left(t,x,\nabla u_n\right) - A\left(t,x,\nabla u_m\right)\right) \nabla T_\eta \left(u_n - u_m\right)\, dxdt \leq 2\eta C. \tag{66}$$

Consequently, by employing (66) we can estimate (61) as

$$\iint\limits_{\omega_4} \gamma(t,x)\, dxdt \leq 2\eta C. \tag{67}$$

Therefore, by choosing $\eta \leq \dfrac{\varepsilon'}{2C}$, one obtains $\displaystyle\iint\limits_{\omega_4} \gamma(t,x)\, dxdt \leq \varepsilon'$ and due to the result of (60), we conclude that meas $(\omega_4) \leq \varepsilon$.
Finally, we have η is fixed and from the upper bound of meas (ω_3) we obtain the existence of n_0 such that for all n, $m \geq n_0$, we have

$$\text{meas}\left\{(t,x), \left|\left(\nabla u_n - \nabla u_m\right)(t,x)\right| \geq \delta\right\} \leq 4\varepsilon.$$

This proves that (∇u_n) converges to ∇u a.e in Q_T (up to a subsequence).

(*iv*) Let us split the succeeding integral as follows

$$\iint\limits_{Q_T} (A(t,x,\nabla u_n) - A(t,x,\nabla u))(\nabla u_n - \nabla u)\, dxdt = I_{1,n} - I_{2,n} - I_{3,n}, \tag{68}$$

where,

$$I_{1,n} = \iint\limits_{Q_T} A(t,x,\nabla u_n)\nabla u_n\, dxdt, \quad I_{2,n} = \iint\limits_{Q_T} A(t,x,\nabla u_n)\nabla u\, dxdt,$$
$$I_{3,n} = \iint\limits_{Q_T} A(t,x,\nabla u)(\nabla u_n - \nabla u)\, dxdt.$$

From the result of (*iii*) and the weak convergence of (u_n) in $\mathcal{V}_T$, it is clear that

$$\lim_{n\to\infty} I_{2,n} = \iint\limits_{Q_T} A(t,x,\nabla u)\nabla u\, dxdt, \quad \lim_{n\to\infty} I_{3,n} = 0.$$

Hence, employing the assumption (57) and passing to the limit in (68), one obtains

$$\lim_{n\to\infty} I_{1,n} \leq \iint\limits_{Q_T} A(t,x,\nabla u)\nabla u\, dxdt. \tag{69}$$

On the other hand, it comes from the result of *iii*) that

$$A(t,x,\nabla u_n)\nabla u_n \to A(t,x,\nabla u)\nabla u \text{ a.e in } Q_T$$

Then, after a direct application of Fatou's Lemma one deduce that

$$\iint_{Q_T} A(t,x,\nabla u)\nabla u\, dxdt \leq \lim_{n\to\infty} I_{1,n}. \tag{70}$$

Combining (69), (70) and ($\mathbb{H}_2$), we arrive at

$$A(t,x,\nabla u_n)\nabla u_n \to A(t,x,\nabla u)\nabla u \text{ strongly in } L^1(Q_T).$$

Now, with the help of the assumption ($\mathbb{H}_2$) one may use the result of Lemma 5.1 with the following choice

$$F_n = d\,|\nabla u_n|^p, \quad F = d\,|\nabla u|^p, \quad G_n = A(t,x,\nabla u_n)\nabla u_n, \quad G = A(t,x,\nabla u)\nabla u.$$

We deduce that,

$$\iint_{Q_T} |\nabla u_n|^p\, dxdt \to \iint_{Q_T} |\nabla u|^p\, dxdt.$$

On the other hand, by using the strong convergence result of (*i*), we arrive at

$$u_n \to u \text{ strongly in } \mathcal{V}_T.$$

□

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