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Generalised solution of fractional diffusion-wave equation

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Abstract. This article is devoted to establishing the existence and uniqueness of solutions to the fractional problem of diffusion waves in the following Colombeau algebra:

$$\begin{cases} D_t^{\alpha} u(x,t) + \Delta_x u(x,t) &= f(t, u(t,x)); \qquad (x,t) \in \Omega \times [0,T] \\ u(0,x) &= \psi_0(x) = \delta(x); \\ \partial_t u(0,x) &= \psi_1(x). \end{cases}$$

Where D_t^{α} is the fractionnal derivative with $1 < \alpha < 2$, Δ is the Laplace operator, ψ_0, ψ_1 are generalized functions, δ is distributions and $\Omega \subset \mathbb{R}^n$. This study is based on the integral solution of this problem using the Gronwall's lemma. Finally we study the association concept with the classical solution.

1. Introduction

A fractional diffusion-wave equation is an integro-linear partial differential formula, extracted from the classical diffusion or wave equation by substituting the first- or second-order time derivative with a fractional α derivative, $1 < \alpha < 2$. These equations get up in continuous-time random walks [5], modeling of anomalous diffusive and sub-diffusive systems [3], unification of diffusion and wave propagation phenomenon [2]. For trendy scattering $\alpha = 1$, while for abnormal sub-scattering $\alpha < 1$, and for abnormal super-scattering $\alpha > 1$. Fractional diffusion equations are of great importance because of their link with several fields such as physics, chemistry, engineering, finance and other sciences that have been developed in the last decade. In 1999, Oldham, Spanier, took into consideration a fractional diffusion equation that carries a first-order derivative in space and a half-order derivative in time[4].In 1986, Nigmatullin talked about that a few of the regular electromagnetic, and mechanical responses may be modeled appropriately the usage of the fractional diffusion-wave equation [6]. In 1990, Fujita, supplied the life and uniqueness of the answer of the Cauchy trouble of the fallowing type

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \frac{\partial^{\beta} u(x,t)}{\partial x^{\beta}}, 1 \leq \alpha, \beta \leq 2$$

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The effects supplied provide an interpretation to phenomena among the warmth equation ($\alpha = 1, \beta = 2$) and the wave equation ($\alpha = \beta = 2$). In 1990, Fujita taken into consideration integro-differential equations which show off warmth diffusion and wave propagation properties [10, 11]. In 1996, Mainardi offered analytical research of the time-fractional diffusion wave equations. Using Laplace remodel method, he acquired the essential answers of the simple Cauchy and signalling issues and expressed them in phrases of an auxiliary characteristic $M(z; y)$, wherein $z = \frac{|x|}{y}$ is the similarity variable. He similarly confirmed that this kind of characteristic is an whole characteristic of Wright type [12, 13]. In 1997, Mainardi furnished a complete evaluate of studies at the utility of calculus in continuum and statistical mechanics which include studies on fractional diffusion-wave solutions [2]. In 2001, Agarwal used the identical method to gain a general answer for a fourth-order fractional diffusion-wave equation [15]. In 2002, Agarwal offered a fashionable answer for a time-fractional diffusion-wave equation described in a bounded area area. His answer relies upon upon the usage of the finite sine remodel method to transform fractional diffusion-wave equation from a area area to a wave quantity area, then the Laplace remodel is used to lessen the ensuing equation to an normal algebraic equation, finally, the inverse Laplace and inverse sine transforms are used to gain the preferred solutions [16]. In 2005, Al-Khaled and Momani used the decomposition technique to acquire an approximate answer for the generalized time-fractional diffusion-wave equation. Their consequences showed the transition from a natural diffusion process ($\alpha = 1$) to a natural wave process ($\alpha = 2$) [17]. This short assessment of fractional diffusion-wave equations and their programs is with the aid of using no way complete. References to different papers similar to fractional diffusion-wave equaitons can be discovered in $[2, 4, 6, 7, 14, 20 - -22]$.

In this paper, we consider the following time-fractional diffusion-wave equation of order $1 < \alpha < 2$:

$$\begin{cases} \mathrm{D}_t^\alpha u(x,t) + \Delta_x u(x,t) = f(t, u(t,x)); & (x,t) \in \Omega \times [0,T] \\ u(0,x) = \psi_0(x) = \delta(x); \\ \partial_t u(0,x) = \psi_1(x). \end{cases} \tag{1}$$

The motivation for considering the fractional derivative of delta and other distributions was its connection to equations with delay and memory and their applications. The article is organized as follows. In Section 2, we recall some notions of Colombeau's algebra and present the definition of the fractional derivative in Colombeau algebra. In Section 3, we will demonstrate the existence and uniqueness of the solution in Colombeau algebra. In section 4, we study the association.

2. Preliminaries

In this section, we will give notations and definition from Colombeau theory, definitions and results from the fractional calculus used in this paper that we have used while evalu-ating the main results. For more details, see [1, 5, 18, 19]. Let $\mathcal{D}(\mathbb{R}^n)$ be the space of all smooth functions $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ with compact support. For $q \in \mathbb{N}$ we denote

$$\mathcal{A}_q(\mathbb{R}^n) = \left\{ \varphi \in \mathcal{D}(\mathbb{R}^n) \,/ \int_{\mathbb{R}^n} \varphi(x)dx = 1 \text{ and } \int_{\mathbb{R}^n} x^\alpha \varphi(x)dx = 0 \text{ for } 1 \le \alpha \le q \right\}.$$

The elements of the set $\mathcal{A}_q$ are called test functions. It is obvious that $\mathcal{A}_1 \supset \mathcal{A}_2 \supset \mathcal{A}_3 \dots$ Also, $A_i \neq \emptyset$ for $i \in \mathbb{N}$, For $\varphi \in \mathcal{A}_q(\mathbb{R}^n)$ and $\epsilon > 0$ it is denoted as

$$\varphi_\epsilon(x) = \frac{1}{\varepsilon^n}\varphi\left(\frac{x}{\varepsilon}\right) \text{ for } \varphi \in \mathcal{D}(\mathbb{R}^n).$$

We denote by

$$\mathcal{E}(\mathbb{R}^n) = \{u : \mathcal{A}_1 \times \mathbb{R}^n \to \mathbb{C} / \quad \text{with} \quad u(\varphi, x) \quad \text{is} \quad C^\infty \quad \text{to the second variable} \quad x\}.$$

With $\forall\varphi \in \mathcal{A}_1$

$$u(\varphi_\varepsilon, x) = u_\varepsilon(x), \quad \forall x \in \mathbb{R}^n.$$

In this work we construct an extended Colombeau algebra of generalized functions in L^2 norm containing fractional derivatives following the approach given in [7, 8] for the purposes of the problem under consideration in this document.

Let Ω subset of $\mathbb{R}^n$, $I = (0,1)$

$$\mathcal{E}_2(\Omega) = \left\{u = (u_\varepsilon)_\varepsilon \in (C^\infty(\Omega))' \quad \text{such that} \quad u_\varepsilon \in W^{2,\infty}(\Omega), \forall \varepsilon \in I\right\};$$

$$\mathcal{E}_{M,2}(\Omega) = \left\{u \in \mathcal{E}_2(\Omega) \quad \text{such that} \quad \forall\gamma \in \mathbb{N}^n \; \exists N \in \mathbb{N} \quad \text{such that } \|\partial^\gamma u_\varepsilon(x)\|_{L^2} = O(\varepsilon^{-N}), \varepsilon \to 0\right\};$$

$$\mathcal{N}_2(\Omega) = \{u \in \mathcal{E}_2(\Omega) \quad \text{such that} \quad \forall\gamma \in \mathbb{N}^n \; \forall q \in \mathbb{N} \quad \text{such that } \|\partial^\gamma u_\varepsilon(x)\|_{L^2} = O(\varepsilon^{q}), \varepsilon \to 0\};$$

$$\mathcal{G}_2(\Omega) = \mathcal{E}_{M,2}(\Omega)/\, \mathcal{N}_2(\Omega).$$

Let $i_\psi : W^{2,\infty}(\mathbb{R}^n) \longrightarrow \mathcal{E}_2(\mathbb{R}^n)$, $w \mapsto (w * \phi_\varepsilon)_{\varepsilon\in I}$. With, $W^{2,\infty}(\mathbb{R}^n)$ is a sub-algebra of $\mathcal{G}_2(\mathbb{R}^n)$. For compact imbedding $i : \mathcal{D}' \longrightarrow \mathcal{G}(R^n)$, $\;i(u) = [u * (\mu\psi_\varepsilon)_{\varepsilon\in I}]$ in which μ is a few compactly supported smooth features identically identical to at least one in a community of zero, [8, 9]. Define the Colombeau space $\mathcal{G}_{W^{2,2}}([0,T] \times \mathbb{R}^n)$, $T > 0$, within the algebra $\mathcal{G}_2([0,T] \times \mathbb{R}^n)$. Let

$$\mathcal{E}_{M,W^{2,2}}([0,T] \times \mathbb{R}^n) = \{\, (u_\varepsilon)_\varepsilon \in \mathcal{E}_2([0,T] \times \mathbb{R}^n) \quad \text{such that} \quad \forall x \in \Omega \; \forall T > 0 \; \exists N \exists C > 0 \quad \text{and } \varepsilon_0 \in I,$$
$$\|\partial^\gamma u_\varepsilon(t,x)\|_{L^2} \leq C\varepsilon^{-N}), \gamma \in \{0,1,2\}, \varepsilon < \varepsilon_0\}.$$

$$\mathcal{N}_{W^{2,2}}([0,T] \times \mathbb{R}^n) = \{(u_\varepsilon)_\varepsilon \in \mathcal{E}_2([0,T] \times \mathbb{R}^n) \quad \text{such that} \quad \forall x \in \Omega \; \forall T > 0 \; \forall p \in \mathbb{N} \exists C > 0 \quad \text{and } \varepsilon_0 \in I,$$
$$\|\partial^\gamma u_\varepsilon(t,x)\|_{L^2} \leq C\varepsilon^{p}), \gamma \in \{0,1,2\}, \varepsilon < \varepsilon_0\}.$$

This norm is negligible for any $\gamma \in \{0,1,2\}$. Colombeau algebra generalized functions defined by

$$\mathcal{G}_{W^{2,2}}([0,T] \times \mathbb{R}^n) = \mathcal{E}_{M,W^{2,2}}([0,T] \times \mathbb{R}^n)/\, \mathcal{N}_{W^{2,2}}([0,T] \times \mathbb{R}^n).$$

Define the Colombeau space $\mathcal{G}^e_{W^{2,2}}([0,T] \times \mathbb{R}^n)$, $\quad T > 0$, [7] within the algebra $\mathcal{G}_{W^{2,2}}([0,T] \times \mathbb{R}^n)$, with $\mathrm{D}^\gamma u_\varepsilon$ is a fractional derivative, where u_ε is a representative of u.

$$\mathcal{E}^e_{M,W^{2,2}}([0,T] \times \mathbb{R}^n) = \{(u_\varepsilon)_\varepsilon \in \mathcal{E}_{M,W^{2,2}}([0,T] \times \mathbb{R}^n) \quad \text{such that} \quad \forall x \in \Omega \; \forall T > 0 \; \exists N \exists C > 0 \quad \text{and } \varepsilon_0 \in I,$$
$$\|\mathrm{D}^\gamma u_\varepsilon(t,x)\|_{L^2} \leq C\varepsilon^{-N}), \gamma \in [1,2], \varepsilon < \varepsilon_0\}.$$

$$\mathcal{N}^e_{W^{2,2}}([0,T] \times \mathbb{R}^n) = \}(u_\varepsilon)_\varepsilon \in \mathcal{E}_{M,W^{2,2}}([0,T] \times \mathbb{R}^n) \quad \text{such that} \quad \forall x \in \Omega \; \forall T > 0 \; \forall p \in \mathbb{N} \exists C > 0 \quad \text{and } \varepsilon_0 \in I,$$
$$\|\mathrm{D}^\gamma u_\varepsilon(t,x)\|_{L^2} \leq C\varepsilon^{p}), \gamma \in [1,2], \varepsilon < \varepsilon_0\}.$$

This norm is negligible for any $\gamma \in \{0,1,2\}$. Colombeau algebra generalized functions defined by

$$\mathcal{G}^e_{W^{2,2}}([0,T] \times \mathbb{R}^n) = \mathcal{E}^e_{M,W^{2,2}}([0,T] \times \mathbb{R}^n)/\, \mathcal{N}^e_{W^{2,2}}([0,T] \times \mathbb{R}^n).$$

The meaning of the term association in Colombeau's algebra is given as follows:
Let $u, v \in \mathcal{G}_{L^\infty}(\mathbb{R}^n)$, We say that u and v are associated and we note $u \approx v$, if

$$lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} (u_\varepsilon - v_\varepsilon)(x)\varphi(x)dx = 0.$$

For all $\varphi \in \mathcal{D}(\mathbb{R}^n)$.

3. Existence and uniqueness of the generalized solution in $\mathcal{G}^e_{W^{2,2}}$

In this section, we need the following definition and proposition in what follows:

Definition 3.1. *[1] We say that f generalized function is of logarithmic type, if there exists a representative (f_ε) of f such that*

$$\sup_{x\in K}|f_\varepsilon(x)| = O\left(\ln\left(\varepsilon^{-N}\right)\right) \quad when \quad \varepsilon \longrightarrow 0.$$

Proposition 3.2. *[1] If f is of logarithmic type, then for any representative of (f_ε)*

$$\sup_{x\in K}|f_\varepsilon(x)| = O\left(\ln\left(\varepsilon^{-N}\right)\right) \quad when \quad \varepsilon \longrightarrow 0.$$

Theorem 3.3. *Suppose that $1 < \alpha < 2$, if $f \in L^\infty_{Loc}([0,T]\times\mathbb{R}^n)$ is of logarithmic type. Then, there exist a unique solution to the problem (1) in the extended Colombeau space $\mathcal{G}^e_{W^{2,2}}([0,T]\times\mathbb{R}^n)$, if the initial data $\psi_{0,\epsilon} \in \mathcal{G}^e_{W^{2,2}}([0,T]\times\mathbb{R}^n)$.*

Proof. **Existence:**
The integral solution of the equation (1) is

$$\begin{aligned} u(t,x) &= E_\alpha(t^\alpha\Delta)\phi_0(x) + JE_\alpha(t^\alpha\Delta)\phi_1(x) \\ &\quad + \int_0^1 JE_\alpha((t-\tau)^\alpha\Delta)D_t^{2-\alpha}f(\tau,u(\tau,x))d(\tau) \ \text{ with } 1<\alpha<2 \end{aligned}$$

Fourier transform of the integral representation is

$$\begin{aligned} \hat{u}(t,\xi) &= \hat{E}_\alpha(t^\alpha A(\xi))\phi_0(\xi) + J\hat{E}_\alpha(t^\alpha A(\xi))\phi_1(\xi) \\ &\quad + \int_0^1 J\hat{E}_\alpha((t-\tau)^\alpha A(\xi))D_t^{2-\alpha}\hat{f}(\tau,u(\tau,\xi))d(\tau) \text{ with } 1<\alpha<2. \end{aligned}$$

Using logarithmic boundedness of f and connection of fractional differentiation and integration, we obtain by Hölder in-equality, for $0<b<1$

$$\begin{aligned} \|\hat{u}(t,\xi)\|^2_{L^2} &\le \left\|\hat{E}_\alpha(t^\alpha A(\xi))\phi_0(\xi)\right\|^2_{L^2} + \left\|J\hat{E}_\alpha(t^\alpha A(\xi))\phi_1(\xi)\right\|^2_{L^2} \\ &\quad + \left\|\int_0^1 J\hat{E}_\alpha((t-\tau)^\alpha A(\xi))D_t^{2-\alpha}\hat{u}(\tau,u(\tau,\xi))d(\tau)\right\|^2_{L^2} \end{aligned}$$

assuming that

$$\|I_1\|^2_{L^2} = \left\|\hat{E}_\alpha(t^\alpha A(\xi))\phi_0(\xi)\right\|^2_{L^2} + \left\|J\hat{E}_\alpha(t^\alpha A(\xi))\phi_1(\xi)\right\|^2_{L^2}$$

and

$$\|I_2\|^2_{L^2} = \left\|\int_0^1 J\hat{E}_\alpha((t-\tau)^\alpha A(\xi))D_t^{2-\alpha}\hat{u}(\tau,u(\tau,\xi))d(\tau)\right\|^2_{L^2}.$$

For $\|I_1\|^2_{L^2}$ the estimate is give in details in [28, 29], so we have a moderateness of I_1 due to the moderateness of the initial data and $L2$-boundedness of the Mittag-Leffler function on compact set.
For

$$\left\|\hat{I}_2\right\|^2_{L^2} \le C|\ln\epsilon|^b \left\|\int_0^t J^{2-\alpha}\hat{E}_\alpha((t-\tau)^\alpha A(\xi))D_t^{2-\alpha}\hat{u}(\tau,u(\tau,\xi))d(\tau)\right\|^2_{L^2}.$$

By Buniakowsky equality

$$\left\|\hat{I}_2\right\|^2_{L^2} \le C|\ln\epsilon|^b \int_{\mathbb{R}^n}\left\{\left(\int_0^t \left(J\hat{E}_\alpha((t-\tau)^\alpha A(\xi)\right)^2 d\tau\right)\left(\int_0^t \left(J^{\alpha-2}\hat{u}(\tau,\xi)\right)^2 d(\tau)\right)\right\} d\xi.$$

We denote these two multipliers by $\int_{\mathbb{R}^n} ABd\xi$, we calculate the parts A, B separately, we have

$$|A| \le \int_0^t \left(J\hat{E}_\alpha((t-\tau)^\alpha A(\xi)\right)^2 d\tau \le \int_0^t (\tau-s)^2 \hat{E}_\alpha\left((\tau-s)^\alpha A(\xi)ds\right)^2 d\tau.$$

By Buniakowsky inequality

$$|A| \le \int_0^t \left(\int_0^\tau (\tau-s)^2 ds\right) \int_0^\tau \left(|\hat{E}_\alpha\left(\tau-s\right)^\alpha A(\xi))\, ds|^2\right) d\tau \le C_N \frac{t^2}{2}.$$

On the other hand

$$|B| \le \int_0^t \left(J^{\alpha-2}\hat{u}(\tau,\xi)\right)^2 d(\tau) \le \int_0^t \left(\int_0^\tau (\tau-s)^{\alpha-3}\hat{u}(\tau,\xi)ds\right)^2 d(\tau).$$

By Buniakowsky inequality

$$\begin{aligned} |B| &\le \int_0^t \left(\int_0^\tau (\tau-s)^{2\alpha-6} ds\right)\left(\int_0^\tau |\hat{u}(s,\xi)^2| ds\right) d\tau \\ &\le \int_0^\tau \frac{\tau^{2\alpha-4}}{2\alpha-5} \sup_\tau |\hat{u}(s,\xi)|^2 d\tau. \end{aligned}$$

So, we obtain using Fubini theorem

$$\begin{aligned} \left\|\hat{I}_2\right\|_{L^2}^2 &\le C|\ln\epsilon|^b \int_{\mathbb{R}^n} \left[\frac{t^2}{2}\left(\int_0^\tau \frac{\tau^{2\alpha-4}}{2\alpha-5} \sup_\tau |\hat{u}(s,\xi)|^2 d\tau\right)\right] d\xi \\ &\le C_N^2 C|\ln\epsilon|^b \frac{T^2}{2} \int_0^\tau \frac{\tau^{2\alpha-4}}{2\alpha-5} \sup_\tau \|\hat{u}(s,\xi)\|_{L^2}^2 d\tau. \end{aligned}$$

So

$$\sup_t \|\hat{u}(t,\xi)\|_{L^2}^2 \le C|\ln\epsilon|^{a_k n} + C_N^2 C|\ln\epsilon|^b \frac{T^2}{2} \int_0^\tau \frac{\tau^{2\alpha-4}}{2\alpha-5} \sup_\tau \|\hat{u}(s,\xi)\|_{L^2}^2 d\tau.$$

By the Gronwall's inequality

$$\sup \|\hat{u}(t,\xi)\|_{L^2}^2 \le C|\ln\epsilon|^{a_k n} \exp\left(C_N^2 C|\ln\epsilon|^b \frac{T^{2\alpha-1}}{2(2\alpha-1)(2\alpha-5)}\right).$$

And moderateness of the L_2^2-norm follows with a great speed, by Plancharel equality we obtain

$$\sup_t \|u_\varepsilon(t,x)\|_{L^2}^2 \le C_N^2 C\varepsilon^{-N}, \quad \exists N>0, x \in \mathbb{R}^n.$$

Uniqueness:
Let's say there are two solutions $u_{1,\varepsilon}(t,.)$, $u_{2,\varepsilon}(t,.)$ to the problem (1), consequently

$$\begin{cases} D_t^\alpha u_{1,\varepsilon}(x,t) + \Delta_x u_{1,\varepsilon}(x,t) &= f_\varepsilon(t, u_{1,\varepsilon}(t,x)); \qquad (x,t) \in \Omega \times [0,T] \\ u_{1,\varepsilon}(0,x) &= \psi_{0,\varepsilon}(x) = \delta_\varepsilon(x); \\ \partial_t u_{1,\varepsilon}(0,x) &= \psi_{1,\varepsilon}(x). \end{cases}$$

And

$$\begin{cases} D_t^\alpha u_{2,\varepsilon}(x,t) + \Delta_x u_{2,\varepsilon}(x,t) &= f_\varepsilon(t, u_{2,\varepsilon}(t,x)); \qquad (x,t) \in \Omega \times [0,T] \\ u_{2,\varepsilon}(0,x) &= \psi_{0,\varepsilon}(x) = \delta_\varepsilon(x); \\ \partial_t u_{2,\varepsilon}(0,x) &= \psi_{1,\varepsilon}(x). \end{cases}$$

Then

$$\begin{cases} \mathrm{D}_t^\alpha(u_{1,\varepsilon}-u_{2,\varepsilon})(x,t)+\Delta_x(u_{1,\varepsilon}-u_{2,\varepsilon})(x,t)=f_\varepsilon(t,u_{1,\varepsilon}(t,x))-f_\varepsilon(t,u_{2,\varepsilon}(t,x)); \\ (u_{1,\varepsilon}-u_{2,\varepsilon})(0,x)=n_{0,\varepsilon}; \\ \partial_t(u_{1,\varepsilon}-u_{2,\varepsilon})(0,x)=n_{1,\varepsilon}; \end{cases} \tag{2}$$

with $n_{0,1}, n_{0,2} \in \mathcal{N}_{W^{2,2}}([0,T]\times\mathbb{R}^n)$. We consider the integral solution of problem (2)

$$u_{1,\varepsilon}(t,x)-u_{2,\varepsilon}(t,x)=n_{0,\varepsilon}+n_{1,\varepsilon}\int_0^1 JE_\alpha((t-\tau)^\alpha\Delta)D_t^{2-\alpha}\left(f_\varepsilon(\tau,u_{1,\varepsilon}(\tau,x))-f_\varepsilon(\tau,u_{2,\varepsilon}(\tau,x))\right)d(\tau)$$

with $1<\alpha<2$, we pose that $n_\varepsilon = n_{0,\varepsilon}+n_{1,\varepsilon}$. Then

$$u_{1,\varepsilon}(t,x)-u_{2,\varepsilon}(t,x)=n_\varepsilon+\int_0^1 JE_\alpha((t-\tau)^\alpha\Delta)D_t^{2-\alpha}(f_\varepsilon(\tau,u_{1,\varepsilon}(\tau,x))-f_\varepsilon(\tau,u_{2,\varepsilon}(\tau,x)))d(\tau)$$

$$\begin{aligned}\|u_{1,\varepsilon}(t,x)-u_{2,\varepsilon}(t,x)\| &\le \|n_\varepsilon\| \\ &\quad +\left\|\int_0^1 JE_\alpha((t-\tau)^\alpha\Delta)D_t^{2-\alpha}(f_\varepsilon(\tau,u_{1,\varepsilon}(\tau,x))-f_\varepsilon(\tau,u_{2,\varepsilon}(\tau,x)))d(\tau)\right\|\end{aligned}$$

$$\begin{aligned}\|u_{1,\varepsilon}(t,x)-u_{2,\varepsilon}(t,x)\| &\le \|n_\varepsilon\| \\ &\quad +\int_0^1 \|JE_\alpha((t-\tau)^\alpha\Delta)\|a\|f_\varepsilon(\tau,u_{1,\varepsilon}(\tau,x))-f_\varepsilon(\tau,u_{2,\varepsilon}(\tau,x)))\|d(\tau)\end{aligned}$$

$$\begin{aligned}\|u_{1,\varepsilon}(t,x)-u_{2,\varepsilon}(t,x)\| &\le \|n_\varepsilon\| \\ &\quad +a\|\nabla_x f_\varepsilon\|\int_0^1 \|JE_\alpha((t-\tau)^\alpha\Delta)\|\|u_{1,\varepsilon}(\tau,x)-u_{2,\varepsilon}(\tau,x)\|d(\tau).\end{aligned}$$

According to the Gronwall's inequality, we have

$$\|u_{1,\varepsilon}(t,x)-u_{2,\varepsilon}(t,x)\| \le |n_\varepsilon|\exp\left(a\|\nabla_x f_\varepsilon\|\int_0^1 \|JE_\alpha((t-\tau)^\alpha\Delta)\|d(\tau)\right)$$

$$\|u_{1,\varepsilon}(t,x)-u_{2,\varepsilon}(t,x)\| \le |n_\varepsilon|\exp\left(\frac{a\|\nabla_x f_\varepsilon\|C_N t^2}{2}\right).$$

As f is logarithmic type, $n_\varepsilon \in \mathcal{N}^e_{W^{2,2}}([0,T]\times\mathbb{R}^n)$, it follows that

$$\|u_{1,\varepsilon}(t,x)-u_{2,\varepsilon}(t,x)\| = \mathcal{O}(\varepsilon^q), \forall q\in\mathbb{N}, \text{ when } \varepsilon\to 0.$$

□

4. Association with classical solution

Let v the solution to

$$\begin{cases} (\mathrm{D}_t^\alpha+\Delta_x)v(x,t)=0; \qquad (x,t)\in\Omega\times[0,T] \\ v(x,0)=\psi_0(x); \quad \partial_t v(x,0)=\psi_1(x) \quad x\in\mathbb{R}; \end{cases} \tag{3}$$

and w the solution to

$$\begin{cases} (\mathrm{D}_t^\alpha+\Delta_x)w(x,t)=f(t,w(t,x)); \qquad (x,t)\in\Omega\times[0,T] \\ w(x,0)=0; \quad \partial_t w(x,0)=0 \quad x\in\mathbb{R}. \end{cases} \tag{4}$$

Proposition 4.1. *The generalized solution u_ε of (1) in $\mathcal{G}_{W^{2,2}}([0,T]\times\mathbb{R}^n)$ is associated with $v+w$.*

Proof. Let v_ε by the classical solution to

$$\begin{cases} (\mathrm{D}_t^\alpha + \Delta_x)v_\varepsilon(x,t) = 0; & (x,t) \in \Omega \times [0,T] \\ v_\varepsilon(x,0) = \psi_{0,\varepsilon}(x); \quad \partial_t v_\varepsilon(x,0) = \psi_{1,\varepsilon}(x) & x \in \mathbb{R}. \end{cases} \tag{5}$$

Then

$$\begin{cases} (\mathrm{D}_t^\alpha + \Delta_x)(v_\varepsilon + w)(x,t) = f(t, w(t,x)); & (x,t) \in \Omega \times [0,T] \\ (v_\varepsilon + w)(x,0) = \psi_{0,\varepsilon}(x); \quad \partial_t(v_\varepsilon + w)(x,0) = \psi_{1,\varepsilon}(x) & x \in \mathbb{R}. \end{cases} \tag{6}$$

We have

$$\begin{cases} (\mathrm{D}_t^\alpha + \Delta_x)(u_\varepsilon - v_\varepsilon - w)(x,t) = f_\varepsilon(t, u_\varepsilon(t,x)) - f(t, w(t,x)); & (x,t) \in \Omega \times [0,T] \\ (u_\varepsilon - v_\varepsilon - w)(x,0) = 0; \quad \partial_t(u_\varepsilon - v_\varepsilon - w)(x,0) = 0 & x \in \mathbb{R}. \end{cases} \tag{7}$$

Let u_ε the integral solution generalized the problem (1)

$$\begin{aligned} u_\varepsilon(t,x) &= E_\alpha(t^\alpha \Delta)\psi_{0,\varepsilon}(x) + JE_\alpha(t^\alpha \Delta)\psi_{1,\varepsilon}(x) \\ &\quad + \int_0^1 JE_\alpha((t-\tau)^\alpha \Delta) D_t^{2-\alpha} f(\tau, u_\varepsilon(\tau,x)) d(\tau) \ \text{ with } 1 < \alpha < 2. \end{aligned}$$

And $v_\varepsilon + w$ the integral solution classic of the equation

$$\begin{aligned} (v_\varepsilon + w)(t,x) &= E_\alpha(t^\alpha \Delta)\psi_{0,\varepsilon}(x) + JE_\alpha(t^\alpha \Delta)\psi_{1,\varepsilon}(x) \\ &\quad + \int_0^1 JE_\alpha((t-\tau)^\alpha \Delta) D_t^{2-\alpha} f(\tau, w(\tau,x)) d(\tau) \ \text{ with } 1 < \alpha < 2. \end{aligned}$$

The integral solution the problem (7) with $1 < \alpha < 2$ is

$$(u_\varepsilon - v_\varepsilon - w)(t,x) = \int_0^1 JE_\alpha((t-\tau)^\alpha \Delta) D_t^{2-\alpha}(f_\varepsilon(\tau, u_\varepsilon(\tau,x)) - f(\tau, w(\tau,x))) d(\tau)$$

$$\|(u_\varepsilon - v_\varepsilon - w)(t,x)\|_{L^2} = \|\int_0^1 JE_\alpha((t-\tau)^\alpha \Delta) D_t^{2-\alpha}(f_\varepsilon(\tau, u_\varepsilon(\tau,x)) - f(\tau, w(\tau,x))) d(\tau)\|_{L^2}$$

$$\begin{aligned} \|(u_\varepsilon - v_\varepsilon - w)(t,x)\|_{L^2} \leq \int_0^1 &\|JE_\alpha((t-\tau)^\alpha \Delta)\|_{L^2} D_t^{2-\alpha} \|(f_\varepsilon(\tau, u_\varepsilon(\tau,x)) - f_\varepsilon(\tau, (v_\varepsilon + w)(\tau,x)) \\ &+ f_\varepsilon(\tau, (v_\varepsilon + w)(\tau,x)) - f_\varepsilon(\tau, w(\tau,x)))\| d(\tau) \end{aligned}$$

$$\begin{aligned} \|(u_\varepsilon - v_\varepsilon - w)(t,x)\|_{L^2} &\leq \int_0^1 \|JE_\alpha((t-\tau)^\alpha \Delta)\| D_t^{2-\alpha} \|(f_\varepsilon(\tau, u_\varepsilon(\tau,x)) - f_\varepsilon(\tau, (v_\varepsilon + w)(\tau,x))\|_{L^2} d(\tau) \\ &\quad + \int_0^1 \|JE_\alpha((t-\tau)^\alpha \Delta)\|_{L^2} D_t^{2-\alpha} \|f_\varepsilon(\tau, (v_\varepsilon + w)(\tau,x)) - f_\varepsilon(\tau, w(\tau,x)))\|_{L^2} d(\tau). \end{aligned}$$

Since

$$\|f_\varepsilon(\tau, (v_\varepsilon + w)(\tau,x)) - f_\varepsilon(\tau, w(\tau,x)))\|_{L^2} \to 0, \quad \forall (\tau,x) \in [0,T] \times \mathbb{R}.$$

Then

$$\|(u_\varepsilon - v_\varepsilon - w)(t,x)\|_{L^2} \leq \frac{C_N T^2 \|f_\varepsilon\|_{L^2}}{2} \int_0^1 \|u_\varepsilon(\tau,x) - (v_\varepsilon + w)(\tau,x)\|_{L^2} d\tau.$$

By the Granwall's inequality

$$\|(u_\varepsilon - v_\varepsilon - w)(t,x)\|_{L^2} \leq \exp\left(\frac{C_N T^2 \|f_\varepsilon\|_{L^2}}{2}\right).$$

Such as f_ε is of logarithmic type

$$\|(u_\varepsilon - v_\varepsilon - w)(t,x)\|_{L^2} = O(\varepsilon^q), \quad \text{as} \quad \varepsilon \to 0 \quad \forall q \in \mathbb{N}.$$

Then,

$$u_\varepsilon \approx v + w.$$

□

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