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Existence and uniqueness of weak and capacity solutions to fractional differential equations in fractional Musielak-Orlicz spaces

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Abstract. In this article, we study the existence of capacitary solutions for a nonlinear fractional differential equation problem in fractional Musielak-Orlicz-Sobolev spaces. Using approximation techniques, we establish the existence of weak solutions by introducing a sequence of approximated problems converging, in the sense of capacities, to a solution of the original problem. Additionally, we present a concrete application to illustrate the validity and relevance of the obtained results. This work makes a significant contribution to the analysis of nonlocal and nonlinear problems with memory in this functional framework.

1. Introduction

Over the past decades, Sobolev spaces and Musielak-Orlicz spaces have garnered significant interest in the study of various mathematical problems, particularly those involving nonlinearities, nonlocal phenomena, and memory effects. Early contributions to Musielak-Orlicz spaces date back to Orlicz in [31, 35] and Nakano in [33], laying the groundwork for modular function spaces. These spaces have since been widely employed to address problems with modular growths in areas such as fluid mechanics and electrorheology [37].

However, when dealing with fractional differential equations, classical tools such as Sobolev spaces often prove insufficient to handle non-polynomial growths and nonlocal terms. To overcome this limitation, fractional Musielak-Orlicz-Sobolev spaces, which combine the flexibility of Musielak-Orlicz spaces with the fractional Sobolev framework, offer a more robust and adaptable setting [3, 9, 12, 18, 25, 27, 29, 38]. These spaces allow the treatment of problems involving complex nonlinearities and memory-driven effects, making them particularly suitable for fractional differential equations [1, 4, 6, 8, 15, 16, 19, 20, 24, 30, 32, 36].

In this work, we focus on the following nonlinear fractional differential equation, modeling reaction-diffusion phenomena influenced by memory effects:

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$$\begin{cases} {}^{C}D^{\alpha}_{t,0^+}u(x,t) + \mathcal{A}u(x,t) = f(x,t,u,\nabla u), & \text{in } \Omega \times (0,T], \\ u(x,0) = u_0(x), & \text{in } \Omega, \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,T], \end{cases} \tag{1}$$

where ${}^{C}D^{\alpha}_{t,0^+}$ is the Caputo fractional derivative of order $\alpha \in (0,1)$, $\mathcal{A}u = -\mathrm{div}(a(x,t,\nabla u))$ is a nonlinear elliptic operator, and $f(x,t,u,\nabla u)$ denotes the reaction-diffusion terms. The primary aim of this article is to establish the existence and uniqueness of weak solutions for this class of equations within the framework of fractional Musielak-Orlicz-Sobolev spaces.

Moreover, this study investigates the notion of capacitive solutions, which plays a crucial role in addressing regularity issues and establishing solutions for nonlinear and nonlocal problems. Capacitive solutions, particularly in the context of fractional Musielak-Orlicz-Sobolev spaces, provide a refined framework for understanding the convergence of solutions and their regularization in modular function spaces[7, 21, 28, 34]. Using compactness arguments, energy inequalities, and a priori estimates, we demonstrate that capacitive solutions can be effectively defined and that they converge within this setting.

This dual focus on weak and capacitive solutions not only extends the analysis of fractional differential equations to more general nonlinear contexts but also opens new avenues for studying memory-driven processes and their numerical approximation. The proposed framework is expected to find applications in various physical phenomena, such as electrorheological materials and reaction-diffusion systems with memory.

The rest of this paper is organized as follows: in Section 2, we introduce some basic concepts and well-known results that are crucial for the developments presented in this paper. In Section 3, we provide the compactness results and the assumptions on the data. In Section 4, we introduce the concept of capacitary solutions and state the main result of this paper. In Section 5, a practical application of the problem (1), modeling a thermal propagation phenomenon with memory in a nonlinear diffusion medium, is presented to illustrate our results. Finally, we conclude the paper and discuss future perspectives.

2. Preliminaries

In this section, we introduce some definitions, properties, and basic notions of this work needed in the next sections.

Definition 2.1 (Musielak-Orlicz Function [11]). *Let $\psi : \Omega \times \mathbb{R} \to [0,\infty)$, then ψ is a Musielak-Orlicz function if it satisfies the following conditions:*

1. *$\psi(x,\cdot)$ is a measurable function for all $x \in \mathbb{R}$.*
2. *For each $x \in \Omega$, we have $\psi(x,\cdot)$ is a N-function; also, it is convex in $\mathbb{R}$ and increasing in $\mathbb{R}_+$ such that:*

$$\lim_{s\to 0} \frac{\psi(x,s)}{s} = 0, \quad \lim_{s\to\infty} \frac{\psi(x,s)}{s} = \infty,$$

$$\psi(x,0) = 0, \quad \textit{for all } x \in \Omega.$$

Definition 2.2 (Musielak-Orlicz Space [11]). *Let ϕ and ψ be two Musielak-Orlicz functions defined in $\Omega \times \mathbb{R}$ with values in $\mathbb{R}$, then ψ dominates ϕ globally ($\phi \ll \psi$) if there exist $r > 0$ and $s_0 \geq 0$, such that*

$$\phi(x,s) \leq \psi(x,rs) \quad \textit{for each } x \in \Omega \textit{ and for all } s \geq s_0.$$

We define the space

$$F_\psi(\Omega) = \left\{ u : \Omega \to \mathbb{R} \textit{ measurable} : \varrho_{\psi,\Omega}(u) < \infty \right\},$$

where $\varrho_{\psi,\Omega}(u) = \int_{\Omega} \psi(s, u(x))dx$.

Let $L_\psi(\Omega)$ *denote the Musielak-Orlicz space associated with* $F_\psi(\Omega)$, *where* $F_\psi(\Omega)$ *represents the Musielak-Orlicz class. This class forms the smallest vector space contained within the following space:*

$$L_\psi(\Omega) = \left\{u : \Omega \to \mathbb{R} \text{ measurable} : \varrho_{\psi,\Omega}(\lambda u) < \infty \text{ for each } \lambda > 0\right\}.$$

The complementary function of the Musielak function $\psi(x,t)$, *defined in the sense of Young with respect to the variable* t, *is given as follows:*

$$\bar{\psi}(x,t) = \sup_{r>0}\{rt - \psi(x,r)\}.$$

The Young-Fenchel inequality is expressed as:

$$|rt| \leq \bar{\psi}(x,t) + \psi(x,r), \quad \forall r,t \in \mathbb{R} \text{ and } x \in \Omega.$$

The space $L_\psi(\Omega)$ *is equipped with the Luxemburg norm, defined as:*

$$\|u\|_{\psi,\Omega} = \inf\left\{\lambda > 0 \;\Big|\; \int_{\Omega} \psi\left(x, \frac{u(x)}{\lambda}\right)dx < 1\right\}.$$

Alternatively, the space $L_\psi(\Omega)$ can be equipped with the Orlicz norm, defined as:

$$\|u\|_{\psi,\Omega} = \sup\left\{\int_{\Omega} u(x)v(x)dx : u \in E_{\overline{\psi}}(\Omega),\ \varrho_{\overline{\psi},\Omega}(v) < 1\right\}.$$

Moreover, the following inequality is satisfied:

$$\int_{\Omega} \psi(x, u(x))dx \leq \|u\|_{\psi,\Omega}, \quad \text{where } \|u\|_{\psi,\Omega} \leq 1. \tag{2}$$

By applying the above inequality, we obtain:

$$\int_{\Omega} \psi(x, u(x))dx \leq 1, \quad \text{for all } u \in L_\psi(\Omega)(1). \tag{3}$$

Additionally, there is an equivalence between the Luxemburg norm and the Orlicz norm, expressed as:

$$\|u\|_{\psi,\Omega} \leq \|u\|_{L_\psi(\Omega)} \leq 2\|u\|_{\psi,\Omega}. \tag{4}$$

For the proof, we refer to [31]. Additionally, the Hölder's inequality is satisfied as follows:

$$\int_{\Omega} v(x)u(x)dx \leq \|v\|_{\psi,\Omega}\|u\|_{\overline{\psi},\Omega}, \quad \text{for all } v \in L_\psi(\Omega) \text{ and } u \in L_{\overline{\psi}}(\Omega). \tag{5}$$

If Ω has a finite measure, the inequality (5) leads to the following continuous inclusion:

$$L_\psi(\Omega) \subseteq L^1(\Omega),$$

which is generally strict.

We denote by $C_c^\infty(\Omega)$ the closure of the set of bounded measurable functions with compact support in the closure of Ω, denoted by $\bar{\Omega}$, with respect to the norm in $L_\psi(\Omega)$.

Throughout this paper, we refer to the standard literature on Musielak-Orlicz-Sobolev spaces [31]; see also [2]. We now introduce some definitions and lemmas that will be useful in the following sections.

Definition 2.3 (Convergence in $L_\psi(\Omega)$ [11]). *Let $(u_n)_{n\in\mathbb{N}} \subset L_\psi(\Omega)$. We say that $(u_n)_{n\in\mathbb{N}}$ converges to $u \in L_\psi(\Omega)$ if there exists $\lambda > 0$ such that:*

$$\lim_{n\to\infty} \varrho_{\psi,\Omega}\left(\frac{u_n - u}{\lambda}\right) = 0.$$

For all $m \in \mathbb{N}$ and $p \in [1, +\infty[$, we define the Musielak-Orlicz-Sobolev spaces as follows:

$$W^{m,p}L_\psi(\Omega) := \{u \in L^p_\psi(\Omega) : D^\alpha u \in L^p_\psi(\Omega) \text{ for all } \alpha,\ |\alpha| \le m\}, \tag{6}$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m) \in \mathbb{Z}^m,\ |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_m$ and

$$D^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_n^{\alpha_n}, \quad \text{with } \partial_j = \frac{\partial}{\partial x_j}.$$

D^α denotes the distributional derivative with respect to the multi-index α.

For each Musielak-Orlicz-Sobolev space $W^{m,p}L_\psi(\Omega)$, we define the modular as follows:

$$\varrho^{(m)}_{p,\psi}(u, \Omega) := \sum_{|\alpha|\le m} \varrho_{p,\psi,\Omega}(D^\alpha u), \quad \text{with} \quad \varrho_{p,\psi,\Omega}(u) = \left(\int_\Omega \psi(x,u)^p\right)^{\frac{1}{p}}$$

which is convex in $W^{m,p}L_\psi(\Omega)$. We can equip the Musielak-Orlicz-Sobolev space with:

$$\|u\|_{W^{m,p}L_\psi(\Omega)} := \inf\left\{\lambda > 0 : \varrho^{(p)}_{p,\psi}\left(\frac{u}{\lambda}, \Omega\right) \le 1\right\} \quad \text{with} \quad \|u\|^p_{W^{m,p}L_\psi(\Omega)} := \sum_{|\alpha|\le m} \|D^\alpha u\|^p_{p,\psi,\Omega}.$$

The above two norms are equivalent on $W^{m,p}L_\psi(\Omega)$. The pair $\left(W^{m,p}L_\psi(\Omega), \|u\|_{W^{m,p}L_\psi(\Omega)}\right)$ is a Banach space [31], if there is $\omega_0 > 0$, such that

$$\text{ess}\inf_{x\in\Omega}\omega(x) > \omega_0. \tag{7}$$

Then, $\left(W^{m,p}L_\psi(\Omega), \|u\|_{W^{m,p}L_\psi(\Omega)}\right)$ is a Banach space.

Hereafter, we assume that the condition (7) is satisfied. The space $W^m L_\psi(\Omega)$ can be regarded as a σ-closed subspace of $\Pi_{|\alpha|\le m} L_\psi(\Omega)$, with respect to the pairing $\sigma\left(\Pi_{|\alpha|\le m} L_\psi(\Omega), \Pi_{|\alpha|\le m} E_{\overline{\psi}}(\Omega)\right)$. Define $W_0^m L_\psi(\Omega)$ as the closure of $D(\Omega)$ under the topology induced by $\sigma\left(\Pi_{|\alpha|\le m} L_\psi(\Omega), \Pi_{|\alpha|\le m} E_{\overline{\psi}}(\Omega)\right)$. Similarly, $W^m E_\psi(\Omega)$ denotes the space of functions u such that u and its distributional derivatives up to order m belong to $E_\psi(\Omega)$. Additionally, $W_0^m E_\psi(\Omega)$ is defined as the norm closure of $D(\Omega)$ within $W^m L_\psi(\Omega)$.

The subsequent definitions extend the concept of Musielak-Orlicz-Sobolev spaces into the fractional domain.

Definition 2.4. *[13] Let Ω be an open subset of $\mathbb{R}^N$, and let $s \in]0, 1[$ and $p \in [1, +\infty[$. we defined a fractional Musielak-Orlicz-Sobolev spaces $W^{s,p}L_\psi(\Omega)$ as follows:*

$$W^{s,p}L_\psi(\Omega) = \left\{u \in L^p_\psi(\Omega)\, \textit{such that}\, \frac{|u(x) - u(y)|}{|x - y|^{\frac{N}{p}+s}} \in L^p_\psi(\Omega \times \Omega)\right\}$$

The norm on $W^{s,p}L_\psi(\Omega)$ is given by:

$$\|u\|_{W^{s,p}L_\psi(\Omega)} = \left([u]^p_{s,\psi,p} + \|u\|^p_{L^p_{\psi(\Omega)}}\right)^{\frac{1}{p}} \tag{8}$$

with,

$$[u]_{s,\psi,p}=\left(\int_\Omega\int_\Omega\psi\left(x,\frac{|u(x)-u(y)|}{|x-y|^{\frac{N}{p}+s}}\right)^p dxdy\right)^{\frac{1}{p}}$$

Definition 2.5. *[13] Let Ω be an open subset of $\mathbb{R}^N$, $s\in\mathbb{R}\setminus\mathbb{N}$ with $s>1$, and $p\in[1,+\infty[$. The fractional Musielak-Orlicz-Sobolev spaces $W^{s,p}L_\psi(\Omega)$ is defined as:*

$$W^{s,p}L_\psi(\Omega)=\{u\in W^{[s],p}L_\psi(\Omega)\,\text{such that}\,D^\beta u\in W^{s-[s],p}L_\psi(\Omega),\ \forall\beta,|\beta|=[s]\},\tag{9}$$

where $[s]$ denotes the integer part of s and where $D^\beta u$ is distributional derivatives of order β.

It is a vector space equipped with the norm:

$$\|u\|_{W^{s,p}L_\psi(\Omega)}=\left(\|u\|^p_{W^{[s],p}L_\psi(\Omega)}+\sum_{|\beta|=[s]}\|D^\beta u\|^p_{W^{s-[s],p}L_\psi(\Omega)}\right)^{\frac{1}{p}}.$$

Remark 2.6. *[13] If $s=[s]$, then the space $W^{s,p}L_\psi(\Omega)$ coincides with the Musielak-Orlicz-Sobolev Sobolev space. The space $\left(W^{s,p}L_\psi(\Omega),\|\cdot\|_{W^{s,p}L_\psi(\Omega)}\right)$ is a Banach space.*

Corollary 2.7. *[13] Let $p\in[1,+\infty)$ and $s,s'>1$. Let Ω be an open set in $\mathbb{R}^n$ of class $C^{0,1}$. Then, if $s'\le s$, we have*

$$W^{s',p}L_\psi(\Omega)\subseteq W^{s,p}L_\psi(\Omega).$$

Lemma 2.8 (Poincaré's inequality). *Let $\Omega\subset\mathbb{R}^n$ be a bounded domain, and let $u\in W_0^{s,p}L_\psi(\Omega)$, the fractional Sobolev space with $s\in(0,1)$ and $1\le p<\infty$. Then, there exists a constant $C=C(\Omega)>0$ such that:*

$$\|u\|_{W^{s,p}_\psi(\Omega)}\le C\,[u]_{s,\psi,p},\tag{10}$$

Remark 2.9. *Let $u\in W_0^{s,p}L_\psi(\Omega)$ where ψ is a Musielak-Orlicz function, we suppose that there exists a positive constant C, such that*

$$\int_\Omega\int_\Omega\psi\left(x,\frac{|u(x)-u(y)|}{|x-y|^{\frac{N}{p}+s}}\right)dxdy\le C.$$

Then,

$$\int_\Omega\int_\Omega\psi\left(x,\frac{|u(x)-u(y)|}{C|x-y|^{\frac{N}{p}+s}}\right)dxdy\le 1.$$

Using the convexity of $\psi(x,.)$ and if $C\ge1$, we get

$$C\in\left\{\lambda>0,\ \text{such that}\ \int_\Omega\int_\Omega\psi\left(x,\frac{|u(x)-u(y)|}{\lambda|x-y|^{\frac{N}{p}+s}}\right)dxdy\le1\right\},\quad\text{and hence, }[u]_{s,\psi,p}\le C.$$

if not, i.e., $C<1$, we obtain $\int_\Omega\int_\Omega\psi\left(x,\frac{|u(x)-u(y)|}{|x-y|^{\frac{N}{p}+s}}\right)dxdy\le C<1$, then $[u]_{s,\psi,p}\le1$. In view of the fact that $u\in W_0^{s,p}L_\psi(\Omega)$, we apply Lemma (2.8) , we get that there exists a positive constant $C=C(\Omega)$, such that

$$\|u\|_{L^p_\psi(\Omega)} \leq C[u]_{s,\psi,p} \text{ for all } u \in W_0^{s,p}L_\psi(\Omega).$$

On the other hand, we have $\|u\|_{s,\psi,1} = [u]_{s,\psi,1} + \|u\|_{L_{\psi(\Omega)}}$, and hence

$$\|u\|_{s,\psi,1} \leq (C+1)\,[u]_{s,\psi,1} \leq (C+1)\max(C,1).$$

Then

$$\|u\|_{s,\psi,1} \leq (C+1)\max(C,1)$$

In the next of this paper, we suppose that ψ and ϕ are two generalized N-function, such that $\psi << \phi$. We also assume that the following conditions hold for complementary functions $\bar{\psi}$ and $\bar{\phi}$

$$\lim_{|x|\to\infty} \text{ ess } \inf_{x\in\Omega} \frac{\overline{\Psi}(x,s)}{|s|} = \infty. \tag{11}$$

$$\lim_{|x|\to\infty} \text{ ess } \inf_{x\in\Omega} \frac{\overline{\phi}(x,s)}{|s|} = \infty. \tag{12}$$

Remark 2.10. *(See [23], Remark 2.1) We suppose* (11) *and* (12) *hold, then*

$$\sup_{x\in B(x,K)} \textit{ ess } \sup_{x\in\Omega} \psi(x,s) < +\infty, \textit{for all } 0 < K < +\infty, \tag{13}$$

$$\sup_{x\in B(x,K)} \textit{ ess } \sup_{x\in\Omega} \phi(x,s) < +\infty, \textit{for all } 0 < K < +\infty \tag{14}$$

Definition 2.11. *[11] Let $(u_n)_{n\in\mathbb{N}} \subset W^{s,p}L_\psi(\Omega)$, we say that $(u_n)_{n\in\mathbb{N}}$ converges to $u \in W^{s,p}L_\psi(\Omega)$ for the modular convergence in $W^{s,p}L_\psi(\Omega)$ if and only if*

$$\lim_{n\to\infty} \varrho^{(p)}_{\phi,\Omega}\left(\frac{u_n - u}{l}\right) = 0, \textit{ for some } l > 0$$

Also, we can define these spaces of distributions as follows:

$$W^{-s,p}L_{\overline{\psi}}(\Omega) := \left\{ g \in \mathcal{D}'(\Omega) : g = \sum_{|\alpha|\leq[s]} (-1)^{|\alpha|} D^\alpha g_\alpha \text{ for each } g_\alpha \in W^{s-[s]}L^p_{\overline{\psi}}(\Omega) \right\},$$

$$W^{-s,p}E_{\overline{\psi}}(\Omega) := \left\{ g \in \mathcal{D}'(\Omega) : g = \sum_{|\alpha|\leq[s]} (-1)^{|\alpha|} D^\alpha g_\alpha \text{ for each } g_\alpha \in W^{s-[s]}E^p_{\overline{\psi}}(\Omega) \right\}.$$

Lemma 2.12. *(See [11]) Let $(u_n)_{n\in\mathbb{N}} \subset L_\psi(\Omega)$, If $\phi \ll \psi$ and $(u_n)_{n\in\mathbb{N}}$ converges Io $u \in L_\psi(\Omega)$, in the sense of modular comvergent, then $(u_n)_{n\in N}$ converges to it strongly in $E_\phi(\Omega)$. In particular, the following continuous injection hold: $L_\psi(\Omega) \subset E_\phi(\Omega)$ and $L_{\overline{\phi}}(\Omega) \subset E_{\overline{\psi}}(\Omega)$.*

Lemma 2.13. *(See [17]) Let $(f_n)_{n\in N}$ and $(g_n)_{n\in N}$ be nwo comvergent sequences in $L_\psi(\Omega)$ and $L_{\overline{\psi}}(\Omega)$, respectively, and denote by $f \in L_\psi(\Omega)$ and $g \in L_{\overline{\psi}}(\Omega)$ their corresponding limits in the sense of modular convergence, then*

$$\lim_{n\to\infty} \int_\Omega g_n f \, dx = \int_\Omega g f \, dx$$

$$\lim_{n\to\infty} \int_\Omega g_n f_n \, dx = \int_\Omega g f \, dx$$

Lemma 2.14. *(See [10]) Let Ω be a bounded, Lipchirz-continuous subset of R^N, ψ a Masielak-Orlicz function and $\bar{\psi}$ its complementary. Then*

- *$D(\Omega)$ is dense in $L_\psi(\Omega)$ with respect to the modular convergence.*
- *$D(\Omega)$ is dense in $W_0^{s-[s],1}L_\psi(\Omega)$ and $\overline{D(\Omega)}$ is dense in $W^{s-[s],1}L_\psi(\Omega)$.*

The previous densities are with respect to the modular convergence. Moreover, all the previous densities hold true if the following conditions are satisfied:

1. There exists *a* constant $\lambda > 0$, such that $\forall x, y \in \Omega, |x-y| \leq \frac{1}{2}$ implies

$$\frac{\phi(x,\ell)}{\phi(y,\ell)} \leq \ell^{-\dfrac{\lambda}{\log(|x-y|)}} \text{ for all } \ell \geq 1 \tag{15}$$

2. There exists a constant $\beta > 0$, such that

$$\overline{\Psi}(x,1) \leq \beta, \text{ a.e in } \Omega. \tag{16}$$

Remark 2.15. *[11] Define the measurable function $q : \Omega \longrightarrow]1,\infty[$ and suppose that there exists a positive constant C, such that for all $x, y \in \Omega$ with $|x-y| < \frac{1}{2}$, we have*

$$|q(y) - q(x)| \leq \frac{C}{|\log|\ y - x||}$$

Then, the following Musielak-Orlicz functions:

(1) $\psi(x,\ell) = \ell^{q(x)}$.

(2) $\psi(x,\ell) = \ell^{\varphi(x)}\log(1+\ell)$.

(3) $\psi(x,\ell) = \ell\log(1+\ell)\Big(\log(e-1+\ell)^{p(x)}$.

satisfy the inequality (15).

Let us now introduce the inhomogeneous fractional Musielak-Orlicz-Sobolev spaces. Consider $\Omega \subset \mathbb{R}^N$, an open and bounded set, and let ψ be a Musielak-Orlicz function defined on $Q_S := \Omega\times]0,S[$, where $S > 0$. We denote by D_x^β the distributional derivative on Q_S of order $\beta \in \mathbb{Z}^N$, with α representing a multi-index associated with the variable x. The inhomogeneous fractional Musielak-Orlicz-Sobolev spaces are defined as follows:

$$W^{s,p,x}L_\psi(\Omega) = \{u \in W^{[s],p}L_\psi(\Omega) \text{ such that } D_x^\beta u \in W^{s-[s],p}L_\psi(\Omega),\ \forall\beta, |\beta| = s\}, \tag{17}$$

$$W^{s,p,x}E_\psi(\Omega) = \{u \in W^{[s],p}L_\psi(\Omega) \text{ such that } D_x^\beta u \in W^{s-[s],p}E_\psi(\Omega),\ \forall\beta, |\beta| = s\}, \tag{18}$$

we equip the spaces $W^{s,P,x}L_\psi\,(Q_s)$ and $W^{s,p,x}E_\psi\,(Q_s)$ with the norm

$$||g||_{W^{s,p,x}L_\psi(\Omega)} = \left([u]^p_{s,\psi,p} + \sum_{|\alpha|\leq[s]} \left\|D_x^\alpha g\right\|^p_{L^p_{\psi(Q_S)}}\right)^{\frac{1}{p}} \tag{19}$$

For $0 < s < 1$ and $p = 1$, the pairs $\left(W^{s,p,x}L_{\psi}(Qs), \|\cdot\|\right)$ and $(\ W^{s,p,x}E_{\psi}(Qs), \|\cdot\|\ )$ are Banach spaces [22]. The two last spaces are considered as subspaces of the product space

$$\prod_{|\alpha|\le m} W^{s,p,x}L_{\psi}(Q_s) = \prod W^{s,p,x}L_{\psi} \quad = \left\{(u_\alpha)_{|\alpha|\le m} : u_\alpha \in W^{s,p,x}L_{\psi}(Q_s) \text{ and } \sum_{|\alpha|\le m} \|u_\alpha\|_{W^{s,p,x}L_{\psi}(Q_s)} < \infty\right\}.$$

We consider the weakly star topology $\sigma\left(\Pi_{|\alpha|\le p}W^{s,p,x}L_{\psi}(Qs), \Pi_{|\alpha|\le p}W^{s,p,x}E_{\overline{\psi}}(Qs)\right)$ and $\sigma\left(\Pi_{|\alpha|\le p}W^{s,p,x}L_{\psi}\left(Q_s\right), \Pi_{|\alpha|\le p}W^{s,p,x}L_{\psi}\left(Q_s\right)\right)$. If $u \in W^{s,p,x}L_{\psi}\left(Q_s\right)$, then the following mapping:

$$\begin{aligned} u :]0, S[&\longrightarrow W^{s,1,x}L_{\psi}\left(Q_s\right) \\ t &\longmapsto u(t) \end{aligned}$$

is well defined. Furthermore, if $u \in W^{s,1,x}E_{\psi}\left(Q_s\right)$, then this function takes values in the space $W^{s,1,x}E_{\psi}(\Omega)$ and is strongly measurable. While the measurability of $u(t)$ on $]0, S[$ cannot be guaranteed, the function $t \longmapsto \|u(t)\|_{W^{s,1,x}(\Omega)}$ is known to belong to the space $L^1(]0, S[)$.

The space $W_0^{s,1,x}E_{\psi}\left(Q_s\right)$ for $0 < s < 1$ is defined as:

$$W_0^{s,1,x}E_{\psi}\left(Q_S\right) = \overline{D\left(Q_S\right)}^{\|\cdot\|_{W^{s,1,x}E_{\psi}(Q_S)}}.$$

If Ω is a Lipschitz-continuous domain, it can be demonstrated, as shown in [6], that every element u in the closure of $D\left(Q_s\right)$ with respect to the weak-$*$ topology associated with

$$\sigma\left(\prod W^{s,1,x}L_{\psi}, \prod W^{s,1,x}E_{\overline{\psi}}\right),$$

is a limit in $W^{s,1,x}L_{\psi}(Q_s)$ of a subsequence $(u_n)_{n\in\mathbb{N}} \subset D\left(Q_s\right)$. We highlight that **modular convergence** holds under the following condition: there exists a positive constant ℓ such that, for all $|\alpha| = s$ and $v_n(x, y) = \frac{|u_n(x)-u_n(y)|}{|x-y|^{N+s}}$, we have

$$\lim_{n\to\infty} \int_{Q_s} \psi\left(x, \frac{v_n(x, y) - v(x, y)}{\ell}\right) \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}s = 0.$$

This modular convergence implies that the sequence $(u_n)_{n\in\mathbb{N}}$ converges to u in $W^{s,1,x}L_{\psi}\left(Q_s\right)$ with respect to the weak-$*$ topology $\sigma\left(\prod W^{s,1,x}L_{\psi}, \prod W^{s,1,x}L_{\overline{\psi}}\right)$. Consequently, we obtain

$$\overline{D\left(Q_S\right)}^{\sigma\left(\prod W^{s,1,x}L_{\psi}, \prod W^{s,1,x}L_{\overline{\psi}}\right)} = \overline{D\left(Q_S\right)}^{\sigma\left(\prod W^{s,1,x}L_{\psi}, \prod W^{s,1,x}E_{\overline{\psi}}\right)}.$$

This space is denoted by $W_0^{s,1,x}E_{\psi}(Qs)$. Moreover, we have

$$W_0^{s,1,x}E_{\psi}\left(Q_S\right) = W_0^{s,1,x}L_{\psi}\left(Q_S\right) \cap \prod W^{s,1,x}E_{\overline{\psi}}.$$

We denote by $W^{-s,1,x}L_{\psi}\left(Q_s\right)$ the topologic dual of $W_0^{s,1,x}E_{\psi}\left(Q_s\right)$ characterized by

$$W^{-s,1,x}L_{\overline{\psi}}\left(Q_s\right) = \left\{v(x, y) = \frac{|u(x) - u(y)|}{|x - y|^{N+s}} : u \in W^{s,1,x}L_{\overline{\psi}}\left(Q_s\right)\right\},$$

which can be equipped by the usual quotient norm

$$\|g\| = \inf[u]_{s,\overline{\psi},Q_s} \text{ for all } u \in L_{\overline{\psi}}\left(Q_s\right).$$

Furthermore, we denote $W^{-1,x}E_{\overline{\psi}}(Q_s)$ the subspace of $W^{-1,x}L_{\overline{\psi}}(Q_s)$ consisting of linear forms which are $\left(\sigma\left(\prod W^{s,1,x}L_{\psi}, \prod W^{s,1,x}E_{\overline{\psi}}\right)\right)$-continuous. It can be shown that

$$W^{-s,1,x}E_{\overline{\psi}(Q_s)} := \left\{v(x,y) = \frac{|u(x)-u(y)|}{|x-y|^{N+s}} : u \in W^{s,1,x}E_{\overline{\psi}(Q_s)}\right\}.$$

In the sequel, we need the following lemma.

Lemma 2.16. *[11] We assume that ϕ is a Musielak fiunction verifying the condition* (12) *and we suppose that $s^2 \leq \phi(x,s)$ for all $x \in \Omega$ and $s \in \mathbb{R}$. Then, the following embedding:*

$$L_{\phi}(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow L_{\overline{\phi}(\Omega)}$$

are continuous.

Remark 2.17. *We assume that the hypothesis of Lemma* (2.12) *is satisfied and* $0 < s < 1$ *then*

$$L^2\left(]0,S[;H^{-s}(\Omega)\right) \hookrightarrow L^2\left(]0,S[;H^{-1}(\Omega)\right) \hookrightarrow W^{-s,1,x}L_{\overline{\phi}(Q_s)} \hookrightarrow W^{-s,1,x}E_{\overline{\psi}(Q_s)}.$$

We introduce the truncation operation $\mathscr{S}_{\Theta} : \mathbb{R} \longrightarrow \mathbb{R}$, appearing in [8]

$$\mathscr{S}_{\Theta}(\theta) = \begin{cases} \theta & \text{if } |\theta| \leq \Theta, \\ \Theta \dfrac{\theta}{|\theta|} & \text{if } |\theta| > \Theta. \end{cases} \tag{20}$$

Then, its primitive is defined as follows:

$$\mathscr{T}_{\Theta}(\theta) = \int_0^{\theta} \mathscr{S}_{\Theta}(s)\mathrm{d}s = \begin{cases} \theta^2/2 & \text{if } |\theta| \leq \Theta, \\ \Theta|\theta| - \Theta^2/2 & \text{if } |\theta| > \Theta. \end{cases} \tag{21}$$

2.1. Preliminaries on fractional calculus.

Fractional calculus generalizes the classical concepts of differential and integral calculus by allowing fractional orders. This section introduces the essential notions needed to understand fractional integrals and derivatives.

Definition 2.18 (*Fractional Riemann-Liouville Integrals* [5, 26]). *The left-sided and right-sided fractional Riemann-Liouville integrals of order $\alpha > 0$ for a function $q \in L^1$ are defined respectively as:*

$$(I^{\alpha}_{a^+}q)(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} q(\tau)\, d\tau, \quad t \in [a,b],$$

$$(I^{\alpha}_{b^-}q)(t) := \frac{1}{\Gamma(\alpha)} \int_t^b (\tau-t)^{\alpha-1} q(\tau)\, d\tau, \quad t \in [a,b].$$

Proposition 2.19. *[5] If $\alpha_1, \alpha_2 > 0$, then*

$$(I^{\alpha_1}_{a^+} I^{\alpha_2}_{a^+} q)(t) = (I^{\alpha_1+\alpha_2}_{a^+} q)(t), \quad t \in [a,b],$$

for any $q \in L^1$.

In [26], for $\alpha \in \mathbb{C}, \mathrm{Re}(\alpha) \geq 0$, the left Riemann-Liouville fractional derivative of order α starting at a is given below

$$\left(D_{a^+}^{\alpha} f\right)(t) = \left(\frac{d}{dt}\right)^n \left(\mathbf{I}_{a^+}^{n-\alpha} f\right)(t), \quad n = [\alpha] + 1 \tag{22}$$

Meanwhile, the right Riemann-Liouville fractional derivative of order α ending at b becomes

$$\left(D_{b^-}^{\alpha} f\right)(t) = \left(-\frac{d}{dt}\right)^n \left(\mathbf{I}_{b^-}^{n-\alpha} f\right)(t) \tag{23}$$

The left Caputo fractional derivative of order $\alpha, \mathrm{Re}(\alpha) \geq 0$ starting from a has the following form:

$$\left({}^{C}D_{a^+}^{\alpha} f\right)(t) = \left(\mathbf{I}_{a^+}^{n-\alpha} f^{(n)}\right)(t), \quad n = [\alpha] + 1, \tag{24}$$

while the right Caputo fractional derivative ending at b becomes

$$\left({}^{C}D_{b^-}^{\alpha} f\right)(t) = \left(\mathbf{I}_{b^-}^{n-\alpha} (-1)^n f^{(n)}\right)(t). \tag{25}$$

Proposition 2.20. *[5] Allow $0 < \alpha < 1$, if $f \in C(I; \mathbb{R})$ so we have*

- $I_{a^+}^{\alpha} \left({}^{C}D_{a^+}^{\alpha} f\right) = f(t) - f(a)$
- ${}^{C}D_{a^+}^{\alpha} \left(I_{a^+}^{\alpha} f(t)\right) = f(t)$

3. Compactness results in fractional Musielak-Orlicz-Sobolev spaces

Compactness plays a crucial role in the analysis of functional spaces and the study of differential and integral equations. This section presents key compactness results that are essential for the development of the subsequent theory.

Lemma 3.1. *(See [14]) The following embedding:*

$$E_{\psi}(Q_S) \hookrightarrow L^1\left(0, S; E_{\psi}(\Omega)\right),$$

is continuous.

Lemma 3.2. *(See [14]) The following embeddings for $0 < s < 1$:*

$$W^1 E_{\psi}(Q_S) \hookrightarrow W^{s,1} E_{\psi}(Q_S) \hookrightarrow L^1\left(0, S; W^{s,1} E_{\psi}(\Omega)\right)$$
$$W^{-1,x} E_{\overline{\psi}}(Q_S) \hookrightarrow W^{-s,1,x} E_{\overline{\psi}}(Q_S) \hookrightarrow L^1\left(0, S; W^{-s,1} E_{\overline{\psi}}(\Omega)\right),$$

are continuous.

Lemma 3.3. *(See [17]) Given a Banach space Y, such that $L^1(\Omega) \hookrightarrow Y$ is a continuous embedding. If H is bounded in $W_0^{s,1,x} L_{\psi}(Q_S)$ and relatively compact in $L^1(0, S; Y)$, then H is relatively compact in $L^1(Q_S)$ and in $E_{\phi}(Q_S)$ for every $\phi \ll \psi$.*

In [11], for ψ a Musielak function verifying the inequalities (15) and (16), we have

$$F = \left\{ \omega \in W_0^{1,x} L_\psi(Q_s) : \frac{\partial \omega}{\partial s} \in W^{-1,x} L_{\overline{\psi}}(Qs) \right\}.$$

We equip the space F by the following norm:

$$\|\omega\|_F = \|\omega\|_{W_0^{1,x} L_\psi(Q_s)} + \left\| \frac{\partial \omega}{\partial s} \right\|_{W^{-1,x} L_{\overline{\psi}}(Q_s)}.$$

The pair $(F, \|.\|_F)$ is a Banach space.

Lemma 3.4. *(See Theorem 2 in [17] Let ψ be a Musielak function. If H is a bounded subset of $W_0^{1,x} L_\psi(QS)$ and $\left\{ \frac{\partial g}{\partial t} / g \in H \right\}$ is bounded in $W^{-1,x} L_{\bar{\psi}}(QS)$, then H is relatively compact in $L^1(Q_S)$.*

With a result similar to Lemma 3.4, the following lemma can be established.
In our study, we obtain the existence of a weak solution by applying Theorem 3.5.
We consider the operator

$$B : D(B) \subset W^{1,x} L_\psi(Q_s) \longrightarrow W^{1,x} L_\psi(Q_s),$$

where $Bu = -\operatorname{div} a(x, s, \nabla u)$, such that $a(.,.,.)$ is a Leray-Lions operator where $a(x, s, \nabla u) = |\nabla u|^{p-2} \nabla u$. In our case, we take $p = 2$ where $\nabla : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ satisfies the following assumptions, for all $(x, s) \in Q_S$:

$$|\nabla u| \leq \zeta \left[c(x, s) + \bar{\psi}_x^{-1}(\psi(x, k|u|)) \right] \tag{26}$$

$$\alpha \psi(x, |\nabla u - \nabla v|) \leq |\nabla u - \nabla v|^2 \tag{27}$$

where $c(x, s) \in E_{\bar{\psi}}(Qs), \alpha, k, \zeta > 0$ are given real numbers. The initial condition is given by

$$u_0 \in L^2(\Omega) \tag{28}$$

We suppose that f is a locally L_1-Lipschitz function and there exists a positive constant σ, such that

$$\sigma \leq f(t), \text{ for all } t \in \mathbb{R} \tag{29}$$

We consider the following parabolic equation that models the temperature generated in a material by the flow of an electric current

$$\begin{cases} \dfrac{\partial u}{\partial s} - \Delta u = \lambda \dfrac{f(u)}{\left(\int_\Omega f(u) \mathrm{d}x \right)^2}, & \text{in } Q_S, \\ u(x, 0) = u_0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \times]0, S[, \end{cases} \tag{30}$$

where $f(u)$ is the electrical resistance of the conductor and $\frac{f(u)}{\left(\int_\Omega f(u) \mathrm{d}x \right)^2}$ represents the nonlocal term of (30).

Theorem 3.5. *[11] We suppose that the conditions* (11), (12), (27) *and* (28) *hold. Then, there exists a weak solution for the problem* (30), *that is*

$$u \in W_0^{1,x} L_\psi (Q_s) \cap C\left([0,S]; L^2(\Omega)\right), \quad \nabla u \in L_{\bar{\psi}} (Q_S)^N,$$
$$u(.,0) = u_0, \ in\ \Omega,$$
$$\int_0^s \left\langle \frac{\partial u}{\partial s}, \varphi \right\rangle \mathrm{d}s + \int_0^s \int_\Omega \nabla u \nabla u \ \mathrm{d}x \ \mathrm{d}s = \left\langle \lambda \frac{f(u)}{\left(\int_\Omega f(u) \mathrm{d}x\right)^2}, u \right\rangle_{Q_s},$$

for each $\varphi \in W_0^{1,x} L_\psi (Q_S)$ *and* $s \in [0,S]$.

4. Main Results

We consider the following problem:

$$\begin{cases} {}^C D^\alpha_{t,0^+} u(x,t) + \mathcal{A} u(x,t) = f(x,t,u,\nabla u), & \text{in } \Omega \times (0,T], \\ u(x,0) = u_0(x), & \text{in } \Omega, \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,T], \end{cases} \tag{31}$$

where ${}^C D^\alpha_{t,0^+}$ denotes the Caputo fractional derivative of order $\alpha \in (0,1)$, $\mathcal{A}u = -\mathrm{div}(a(x,t,\nabla u))$ is a nonlinear elliptic operator, and $f(x,t,u,\nabla u)$ is a measurable function.

By applying the fractional integral $I^\alpha_{t,0^+}$ to both sides of the equation, problem (31) is equivalent to the following problem:

$$\begin{cases} u(x,t) - u_0(x) + I^\alpha_{t,0^+} \mathcal{A} u(x,t) = I^\alpha_{t,0^+} f(x,t,u,\nabla u), & \text{in } \Omega \times (0,T], \\ u(x,0) = u_0(x), & \text{in } \Omega, \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,T]. \end{cases} \tag{32}$$

Indeed, from the definition of the Caputo fractional derivative of order $\alpha \in (0,1)$, we have:

$${}^C D^\alpha_{t,0^+} u(x,t) = I^{1-\alpha}_{t,0^+} \frac{\partial u}{\partial t},$$

where $I^{1-\alpha}_{t,0^+}$ is the Riemann-Liouville fractional integral of order $1-\alpha$. By applying $I^\alpha_{t,0^+}$ to both sides of the main equation in (31), and using the fundamental relation:

$$I^\alpha_{t,0^+} \left(I^{1-\alpha}_{t,0^+} \frac{\partial u}{\partial t} \right) = u(x,t) - u_0(x),$$

we obtain the equivalent equation:

$$u(x,t) - u_0(x) + I^\alpha_{t,0^+} \mathcal{A} u(x,t) = I^\alpha_{t,0^+} f(x,t,u,\nabla u).$$

Now, let $\Omega \subset \mathbb{R}^N$ be an open-bounded set and let ψ a Musielak function verifying the inequalities (15) and (16).

$$F^s = \left\{ u \in W_0^{s,1,x} L_\psi(Q_s) : {}^C D^\alpha_{t,0^+} u \in W^{-s,1,x} L_{\overline{\psi}}(Qs) \right\}.$$

We equip the space F^s by the following norm:

$$\|\omega\|_{F^s} = \|u\|_{W_0^{s,1,x} L_\psi(Q_s)} + \left\| {}^C D^\alpha_{t,0^+} u \right\|_{W^{-s,1,x} L_{\overline{\psi}}(Q_s)}.$$

The pair $(F^s, \|.\|_{F^s})$ is a Banach space. In the sequel of this paper, we consider $\langle .,. \rangle_{Q_r} = \langle .,. \rangle_{W^{-s,1,x} L_{\bar{\psi}}(Q_s), W_0^{s,1,x} L_\psi(Q_s)}$, and we assume the following conditions:

$$\phi \ll \psi \text{ and } t^2 \leq \phi(x,t) \text{ for each } x \in \Omega \text{ and for all } t \in \mathbb{R}. \tag{33}$$

Lemma 4.1. *Let ψ be a Musielak function and $0 < \alpha < 1$. If H is a bounded subset of $W_0^{s,1,x}L_\psi(Q_T)$ and $\left\{{}^CD^\alpha_{a^+,t}g/g \in H\right\}$ is bounded in $W^{-s,1,x}L_{\bar{\psi}}(Q_T)$, then H^s is relatively compact in $L^1(Q_S)$.*

Lemma 4.2. *(See [17]) Let Ω be an open-bounded subset of $\mathbb{R}^N$ with the segment property. Then, the following inclusion:*

$$F^s = \left\{u \in W_0^{s,1,x}L_\psi(Q_s) : {}^CD^\alpha_{t,0^+}u \in W^{-s,1,x}L_{\overline{\psi}}(Qs)\right\} \subset C\left(]0,T[;L^1(\Omega)\right)$$

holds with a continuous embedding.

Let $\overline{\phi}$ and $\overline{\psi}$ be two complementary functions of the Musielak functions $\phi(x,r)$ and $\psi(x,r)$, respectively, satisfying the conditions (13) and (14), respectively. We consider also the operator

$$\mathcal{A} : D(\mathcal{A}) \subset W^{s,1,x}L_\psi(Q_s) \longrightarrow W^{s,1,x}L_\psi(Q_s),$$

where $\mathcal{A}u = -\operatorname{div} a(x,s,\nabla u)$, such that $a(.,.,.)$ is a Leray-Lions operator where $a(x,s,\nabla u) = (1+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u$. In our case, we take $p = 2$ where $\nabla : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ satisfies the assumptions (26)-(28) for all $(x,s) \in Q_S$:

Assume that $f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function, for almost every$(x,t) \in \Omega \times [0,T]$ and for all $s \in \mathbb{R}, \xi \in \mathbb{R}^N$:

$$|f(x,t,s,\xi)| \leq b(|s|)\,(c_1(x,t) + \psi(x,|\xi|)) \tag{34}$$

$$f(x,t,s,\xi)s \geq 0 \tag{35}$$

with $c_1(x,t) \in L^1(Q)$ and $b : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous and nondecreasing function.

Theorem 4.3. *We suppose that the conditions* (11), (12), (27) (28), (34) *and* (35) *hold. Then, there exists a weak solution for the problem* (1), *that is*

$$\begin{aligned}
&u \in W_0^{s,1,x}L_\psi(Q_T) \cap C\left([0,T];L^2(\Omega)\right), \quad \frac{\nabla u(x) - \nabla u(y)}{|x-y|^{N+s}} \in L_{\overline{\psi}}(Q_T)^N \\
&u(.,0) = u_0, \ \textit{in } \Omega \\
&\int_0^T \langle u - u_0, v\rangle\,\mathrm{d}t + \frac{1}{\Gamma(\alpha)}\int_0^T\int_0^t\int_\Omega (t-\tau)^{\alpha-1}\nabla u\nabla v\ \mathrm{d}x\,d\tau\ \mathrm{d}t \\
&\qquad\qquad = \left\langle \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(x,\tau,u,\nabla u)\,\tau, v\right\rangle_{Q_T}
\end{aligned}$$

for each $v \in W_0^{s,1,x}L_\psi(Q_t)$ and $t \in [0,T]$.

Proof. To prove the existence of a weak solution, Schauder's fixed-point theorem will be utilized. Specifically, by applying Theorem 3.5, the existence of a solution to the following problem is ensured for all $v \in W_0^{s,1,x}L_\psi(Q_T)$:

$$\begin{aligned}
&\langle u - u_0, v\rangle_{Q_T} + \frac{1}{\Gamma(\alpha)}\int_0^T\int_0^t\int_\Omega (t-\tau)^{\alpha-1}\nabla u\nabla v\ \mathrm{d}x\,d\tau\ \mathrm{d}t \\
&\qquad = \frac{1}{\Gamma(\alpha)}\int_0^T\int_\Omega\int_0^t (t-\tau)^{\alpha-1} f(x,\tau,u,\nabla u)v\,d\tau\,dx\,dt
\end{aligned} \tag{36}$$

$$u(.,0) = u_0, \ \text{in } \Omega.$$

From (34) and posing $u(x,y) = \dfrac{u(x)-u(y)}{(x-y)^{\frac{N}{2}+s}}$ and $\nabla u(x,y) = \dfrac{\nabla u(x)-\nabla u(y)}{(x-y)^{\frac{N}{2}+s}}$, we obtain

$$\begin{aligned}
&\left\| f(x,t,u(x,y),\nabla u(x,y)) \right\|^2_{L^2\left(0,T,H^{-1}(\Omega\times\Omega)\right)} \\
&\leqslant \int_\Omega\int_\Omega (b(|u(x,y)|))^2 \left(c_1(x,t)+\psi(x,|\nabla u(x,y)|)\right)^2 \, dx\, dy \\
&\leqslant C_1 \int_\Omega\int_\Omega \frac{(u(x)-u(y))^2}{(x-y)^{N+2s}} \left(c_1(x,t)+\psi(x,|\nabla u(x,y)|)\right)^2 \, dx\, dy \\
&\leqslant C_1 [u]^4_{s,2} \left(\int_\Omega\int_\Omega c_1(x,t)\, dx\, dy + \int_\Omega\int_\Omega \psi(x,|\nabla u(x,y)|)\, dx\, dy\right)^2 \\
&\leqslant C_1 [u]^4_{s,2} \left(\mathrm{meas}(\Omega)\|c_1(x,t)\|^2_{L^2\left(0,T,H^{-1}(\Omega)\right)} + [\nabla u]^2_{s;\psi,p}\right)^2 \\
&\leqslant C
\end{aligned} \tag{37}$$

resulting in

$$\begin{aligned}
&\left\| \int_0^t (t-\tau)^{\alpha-1} f(x,\tau,u(x,y),\nabla u(x,y))\, d\tau \right\|_{L^2\left(0,T,H^{-1}(\Omega\times\Omega)\right)} \\
&\leqslant \left\| f(x,t,u(x,y),\nabla u(x,y)) \right\|_{L^2\left(0,T,H^{-1}(\Omega\times\Omega)\right)} \int_0^t (t-\tau)^{\alpha-1}\, d\tau \\
&\leqslant \frac{CT^\alpha}{\alpha}
\end{aligned} \tag{38}$$

Hence

$$\int_0^t (t-\tau)^{\alpha-1} f(x,\tau,u(x,y),\nabla u(x,y))\, d\tau \in L^2\left(0,S;H^{-1}(\Omega\times\Omega)\right)$$

Using the continuous embedding

$$L^2\left(0,S;H^{-1}(\Omega\times\Omega)\right) \hookrightarrow W^{-s,1,x}E_{\bar\psi}(Q_S),$$

which is derived by applying Lemma 2.16 and Remark 2.17, we obtain the following result:

$$\int_0^t (t-\tau)^{\alpha-1} f(x,\tau,u,\nabla u)\, d\tau \in W^{-s,1,x}E_{\bar\psi}(Q_S).$$

Now, we prove that $\left|\displaystyle\int_0^t (t-\tau)^{\alpha-1} \frac{\nabla u(x)-\nabla u(y)}{(x-y)^{\frac{N}{2}+s}}\, d\tau\right| \in F_\psi(\Omega\times\Omega)$ and the following estimates:

$$\int_0^S \int_\Omega \psi\left(x, \left|\int_0^t (t-\tau)^{\alpha-1} \frac{\nabla u(x)-\nabla u(y)}{(x-y)^{\frac{N}{2}+s}} d\tau\right|\right) \mathrm{d}x\ \mathrm{d}s \le C_2. \tag{39}$$

$$\left\| \int_0^t (t-\tau)^{\alpha-1} \frac{\nabla u(x)-\nabla u(y)}{(x-y)^{\frac{N}{2}+s}} d\tau \right\|_{\bar\psi,Q_s} \le C_3 \tag{40}$$

where C_2 and C_3 be positive constants. To prove (39), we use (27) and the convexity of ψ, which gives:

$$\begin{aligned}
\beta\psi\left(x, \left|\int_0^t (t-\tau)^{\alpha-1} \frac{\nabla u(x)-\nabla u(y)}{(x-y)^{\frac{N}{2}+s}} d\tau\right|\right) &\le \beta \int_0^t (t-\tau)^{\alpha-1} \psi\left(x, \left|\frac{\nabla u(x)-\nabla u(y)}{(x-y)^{\frac{N}{2}+s}}\right|\right) d\tau \\
&\le \int_0^t (t-\tau)^{\alpha-1} \left|\frac{\nabla u(x)-\nabla u(y)}{(x-y)^{\frac{N}{2}+s}}\right|^2 d\tau
\end{aligned}$$

Hence

$$\begin{aligned}
&\beta \int_0^T \int_{\Omega\times\Omega} \psi\left(x, \left|\int_0^t (t-\tau)^{\alpha-1} \left|\frac{\nabla u(x)-\nabla u(y)}{(x-y)^{\frac{N}{2}+s}}\right| d\tau\right|\right) \mathrm{d}x\mathrm{d}y\, \mathrm{d}s \\
&\quad \leq \int_0^T \int_{\Omega\times\Omega} \int_0^t (t-\tau)^{\alpha-1} \left|\frac{\nabla u(x)-\nabla u(y)}{(x-y)^{\frac{N}{2}+s}}\right|^2 d\tau\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}s
\end{aligned} \tag{41}$$

From (36), we get

$$\begin{aligned}
&\int_0^T \int_{\Omega\times\Omega} \int_0^t (t-\tau)^{\alpha-1} \left|\frac{\nabla u(x)-\nabla u(y)}{(x-y)^{\frac{N}{2}+s}}\right|^2 d\tau\, \mathrm{d}x\, \mathrm{d}s \\
&\leq \left\langle \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-\tau)^{\alpha-1} f(x,\tau,u(x,y),\nabla u(x,y))\, d\tau, u \right\rangle_{Q_T} \\
&+ \frac{1}{2}\int_{\Omega\times\Omega} \left|\frac{u(x,0)-u(y,0)}{(x-y)^{\frac{N}{2}+s}}\right|^2 \mathrm{d}x\, dy - \frac{1}{2}\int_{\Omega\times\Omega} \left|\frac{u(x)-u(y)}{(x-y)^{\frac{N}{2}+s}}\right|^2 \mathrm{d}x\, \mathrm{d}y \\
&\leqslant \frac{1}{\Gamma(\alpha-1)} \left\| \int_0^t (t-\tau)^{\alpha-1} f(x,\tau,u(x,y),\nabla u(x,y))\, d\tau \right\|_{L^2\left(0,T,H^{-1}(\Omega\times\Omega)\right)} + \frac{1}{2}[u]_{s,2}^2
\end{aligned}$$

Given the embeddings

$$L_\psi(\Omega\times\Omega) \hookrightarrow L^2(\Omega\times\Omega) \hookrightarrow L^1(\Omega\times\Omega)$$

and

$$f \in E_\psi(Q_S) \hookrightarrow L^1\left(0,S;E_\psi(\Omega\times\Omega)\right) \hookrightarrow L^1\left(0,S;L^2(\Omega\times\Omega)\right),$$

there exists a positive constant C_5 such that:

$$\int_0^T \int_{\Omega\times\Omega} \int_0^t (t-\tau)^{\alpha-1} \left|\frac{\nabla u(x)-\nabla u(y)}{(x-y)^{\frac{N}{2}+s}}\right|^2 d\tau\, \mathrm{d}x\, \mathrm{d}s < C_5 \tag{42}$$

Thus,

$$\beta\psi\left(x, \left|\int_0^t (t-\tau)^{\alpha-1} \frac{\nabla u(x)-\nabla u(y)}{(x-y)^{\frac{N}{2}+s}} d\tau\right|\right) \leq C_5$$

it yields that

$$\left|\int_0^t (t-\tau)^{\alpha-1} \frac{\nabla u(x)-\nabla u(y)}{(x-y)^{\frac{N}{2}+s}}\, d\tau\right| \in F_\psi(\Omega).$$

Now, we state to prove the inequality (40). Knowing that

$$\int_0^T \int_{\Omega\times\Omega} \int_0^t (t-\tau)^{\alpha-1} (\nabla u(x,y)-\nabla\varphi(x,y))(\nabla u(x,y)-\nabla\varphi(x,y))\, d\tau\, \mathrm{d}x\, dy\, \mathrm{d}s \geq 0$$

This implies that

$$\frac{1}{2}\int_0^T\int_{\Omega\times\Omega}\int_0^t (t-\tau)^{\alpha-1}\left|\frac{\nabla u(x)-\nabla u(y)}{(x-y)^{\frac{N}{2}+s}}\right|^2 d\tau\ \mathrm{d}x\,dy\,\mathrm{d}s$$
$$+\frac{1}{2}\int_0^T\int_{\Omega\times\Omega}\int_0^t (t-\tau)^{\alpha-1}\left|\frac{\nabla\varphi(x)-\nabla\varphi(y)}{(x-y)^{\frac{N}{2}+s}}\right|^2 d\tau\ \mathrm{d}x\,dy\,\mathrm{d}s$$
$$\geq\int_0^T\int_{\Omega}\int_0^t (t-\tau)^{\alpha-1}\left|\frac{\nabla u(x)-\nabla u(y)}{(x-y)^{\frac{N}{2}+s}}\right|\left|\frac{\nabla\varphi(x)-\nabla\varphi(y)}{(x-y)^{\frac{N}{2}+s}}\right| d\tau\,\mathrm{d}x\,\mathrm{d}s$$

Applying (42) and using (33), we get

$$\begin{aligned}&\int_0^T\int_{\Omega\times\Omega}\int_0^t (t-\tau)^{\alpha-1}\left|\frac{\nabla\varphi(x)-\nabla\varphi(y)}{(x-y)^{\frac{N}{2}+s}}\right|^2 d\tau\ \mathrm{d}x\,dy\,\mathrm{d}s\\ &\leq\int_0^T\int_{\Omega\times\Omega}\int_0^t (t-\tau)^{\alpha-1}\psi\left(x,\left|\frac{\nabla\varphi(x)-\nabla\varphi(y)}{(x-y)^{\frac{N}{2}+s}}\right|\right) d\tau\ \mathrm{d}x\,dy\,\mathrm{d}s\end{aligned} \tag{43}$$

for each $\varphi\in W_0^{s,1,x}E_\psi(Q_S)$ where $\|\nabla\varphi\|_{\psi,Q_S}=\frac{1}{k+1}$. Consequently, we get

$$\int_0^T\int_{\Omega\times\Omega}\int_0^t (t-\tau)^{\alpha-1}\left|\frac{\nabla\varphi(x)-\nabla\varphi(y)}{(x-y)^{\frac{N}{2}+s}}\right|^2 d\tau\ \mathrm{d}x\,dy\,\mathrm{d}s\leq C$$

from whence follows, there exists a positive constant C_3, such that

$$\int_0^T\int_{\Omega}\int_0^t (t-\tau)^{\alpha-1}\left|\frac{\nabla u(x)-\nabla u(y)}{(x-y)^{\frac{N}{2}+s}}\right|\left|\frac{\nabla\varphi(x)-\nabla\varphi(y)}{(x-y)^{\frac{N}{2}+s}}\right| d\tau\,\mathrm{d}x\,\mathrm{d}s\leq C_3$$

It yields that

$$\left\|\int_0^t (t-\tau)^{\alpha-1}\left|\frac{\nabla u(x)-\nabla u(y)}{(x-y)^{\frac{N}{2}+s}}\right|\left|\frac{\nabla\varphi(x)-\nabla\varphi(y)}{(x-y)^{\frac{N}{2}+s}}\right| d\tau\right\|_{\bar\psi,Q_S}\leq C_3$$

as a consequence

$$\int_0^t (t-\tau)^{\alpha-1}\frac{\nabla u(x)-\nabla u(y)}{(x-y)^{\frac{N}{2}+s}}\,d\tau\in E_{\bar\psi}(Qs).$$

Hence

$$\int_0^t (t-\tau)^{\alpha-1}\Delta u\,d\tau=\int_0^t (t-\tau)^{\alpha-1}\mathcal{A}u\,d\tau\in W_0^{-1,x}E_{\bar\psi}\left(Q_S\right).$$

Keeping this in mind, using the following inclusion:

$$\int_0^t (t-\tau)^{\alpha-1}f(x,\tau,u,\nabla u)\,d\tau\in L^2\left(0,S;H^{-1}(\Omega)\right)\hookrightarrow W^{-s,1,x}E_{\bar\psi}\left(Q_S\right).$$

and the equation of the problem (32), it follows that:

$$(u-u_0)\in W^{-s,1,x}E_{\bar\psi}(Qs)\text{ and }\|u-u_0\|_{W^{-s,1,x}L_{\bar\psi}(Qs)}\leq C_6 \tag{44}$$

We define the following operator:

$$\mathcal{G}:E_\varphi(Q_T)\longrightarrow F^s,$$

$$v\longmapsto\mathcal{G}(v)=u,$$

where u is the solution of problem (36). The operator $\mathscr{G}$ is compact. This follows from the fact that $F^s \subset E_\varphi(Q_T)$, an inclusion that can be established using Lemmas 3.3 and 3.4.

From inequality (44), we deduce that the set

$$\{u - u_0\,;\, u \in F^s\}$$

is bounded in $W^{-s,1,x}L^{\overline{\psi}}(Q_T)$. Leveraging Lemmas 3.3, 3.4, and 4.1, with $Y := L^1(\Omega)$, we obtain the compact inclusion:

$$F^s \hookrightarrow E_\varphi(Q_T).$$

This inclusion, combined with (44) and (42), establishes the compactness of the operator $\mathscr{G}$.

We define:

$$\mathscr{B}_\nu := \{\omega \in E_\varphi(Q_T) \,|\, \|\omega\|_{\varphi,\Omega} \le \nu\}.$$

The set $\mathscr{B}_\nu$ is bounded and closed. Given this, along with (42), we conclude that

$$\mathscr{G}(\mathscr{B}_\nu) \subset \mathscr{B}_\nu.$$

To complete the proof of the existence of a weak solution, it is sufficient to demonstrate that $\mathscr{G}$ is a continuous operator. For this, assume that $(v_n)_{n\in\mathbb{N}} \subset \mathscr{B}_\nu$ and that $v_n \to \omega$. Let $\mathscr{G}(\omega) = u$ and $\mathscr{G}(v_n) = u_n$. Thus,

$$(v_n)_{n\in\mathbb{N}} \subset \mathscr{G}_\nu \subset E_\varphi(Q_T) \subset L_\varphi(Q_T) \subset L^2(Q_T).$$

Since $L^2(Q_T)$ is a Banach space, the convergence of v_n to ω in $L^2(Q_T)$ implies that there exists a subsequence, still denoted by $(v_n)_{n\in\mathbb{N}}$, such that $v_n \to \omega$ almost everywhere (a.e.) in Q_T.

Given that $(v_n)_{n\in\mathbb{N}} \subset \mathscr{B}_\nu \subset L_\varphi(Q_T)$, the sequence is bounded in $L_\varphi(Q_T)$. Consequently, there exists a subsequence (still denoted $(u_n)_{n\in\mathbb{N}}$) and a function $V \in E_\varphi(Q_T)$ such that:

$$u_n \to V \quad \text{strongly in } E_\varphi(Q_T),$$

and

$$\nabla u_n \to \nabla V \quad \text{weakly in } L^2(Q_T)^N.$$

To link this to the initial problem (36), we choose $v = u_n - u$ as a test function. Substituting this expression into equation (36), we obtain:

$$\begin{aligned}\langle u - u_0, u_n - u\rangle_{Q_T} &+ \frac{1}{\Gamma(\alpha)}\int_0^T\int_0^t\int_\Omega (t-\tau)^{\alpha-1}\nabla u\cdot\nabla(u_n - u)\,dx\,d\tau\,dt\\ &= \frac{1}{\Gamma(\alpha)}\int_0^T\int_\Omega\int_0^t (t-\tau)^{\alpha-1} f(x,\tau,u,\nabla u)(u_n - u)\,d\tau\,dx\,dt.\end{aligned}$$

Similarly, for u_n:

$$\begin{aligned}\langle u_n - u_0, u_n - u\rangle_{Q_T} &+ \frac{1}{\Gamma(\alpha)}\int_0^T\int_0^t\int_\Omega (t-\tau)^{\alpha-1}\nabla u_n\cdot\nabla(u_n - u)\,dx\,d\tau\,dt\\ &= \frac{1}{\Gamma(\alpha)}\int_0^T\int_\Omega\int_0^t (t-\tau)^{\alpha-1} f(x,\tau,u_n,\nabla u_n)(u_n - u)\,d\tau\,dx\,dt.\end{aligned}$$

By subtracting these two equations, we get:

$$\begin{aligned} &\langle u_n - u, u_n - u\rangle_{Q_T} + \frac{1}{\Gamma(\alpha)} \int_0^T \int_0^t \int_\Omega (t-\tau)^{\alpha-1} |\nabla(u_n - u)|^2 \, dx \, d\tau \, dt \\ &= \frac{1}{\Gamma(\alpha)} \int_0^T \int_\Omega \int_0^t (t-\tau)^{\alpha-1} \left(f(x,\tau,u_n,\nabla u_n) - f(x,\tau,u,\nabla u) \right) (u_n - u) \, d\tau \, dx \, dt. \end{aligned} \tag{45}$$

Thus, from equations (45), it follows that:

$$\|u_n - u\|^2_{L^2(Q_T)} \leqslant \left\langle \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left(f(x,\tau,u_n,\nabla u_n) - f(x,\tau,u,\nabla u) \right) \, d\tau, u_n - u \right\rangle_{Q_T}. \tag{46}$$

Let us denote

$$\omega_n(x,t) = f(x,t,u_n,\nabla u_n) - f(x,t,u,\nabla u).$$

Then,

$$|\omega_n(x,t)| \leq \left(b(|u_n|) - b(|u|) \right) \left(c_1(x,t) + \psi(x, |u_n - u|) \right). \tag{47}$$

Thus,

$$\begin{aligned} &\int_0^T \int_\Omega (t-\tau)^{\alpha-1} \omega_n(x,t)(u_n - u) \, d\tau \ \ \mathrm{d}x \ \mathrm{d}s \\ &\leq \int_0^T \int_\Omega (t-\tau)^{\alpha-1} (b(|u_n|) - b(|u|)) \left(c_1(x,t) + \psi(x, |u_n - u|) \right) (|u_n - u|) \, d\tau \ \ \mathrm{d}x \ \mathrm{d}s. \end{aligned}$$

Since the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^2(\Omega)$, applying the dominated convergence theorem gives:

$$\lim_{n\to+\infty} \int_0^T \int_\Omega (t-\tau)^{\alpha-1} \omega_n(x,t)(u_n - u) \, d\tau \ \ \mathrm{d}x \ \mathrm{d}s = 0. \tag{48}$$

Combining (46) with (47), we obtain:

$$\|u_n - u\|^2_{L^2(Q_T)} \leqslant \int_0^T \int_\Omega (t-\tau)^{\alpha-1} \omega_n(x,t)(u_n - u) \, d\tau \ \ \mathrm{d}x \ \mathrm{d}t. \tag{49}$$

From equations (46) to (49), we deduce that $u_n \to u$ in $L^2(\Omega)$.
Since $u_n \to V$ in $E_\varphi(Q_T) \subset L_\varphi(Q_T) \subset L^2(Q_T)$, we conclude that $u_n \to V$ in $L^2(\Omega)$.
This implies $V = u$, and therefore $\mathscr{G}(v_n) \to \mathscr{G}(\omega) = u$. Consequently, $\mathscr{G}$ is continuous. This completes the proof of Theorem 4.3.

□

In the following, we state trace and mollification results and define the concept of a capacity solution for problem (1) in the context of the Musielak-Orlicz-Sobolev spaces.

Let Ω be an open-bounded subset of $\mathbb{R}^N$ with a Lipschitz-continuous boundary, and let ψ be a Musielak function. We set $Q_S =]0, S[\times\Omega$. For $u \in L^1(Q_S)$, $\eta > 0$, $r \in [0, S]$, and $x \in \Omega$, we define u_η as follows:

$$u_\eta(x,r) = \eta \int_{-\infty}^r \tilde{u}(x,t) \exp(\eta(t-r)) \mathrm{d}t \tag{50}$$

where $\tilde{u}(x,t) = u(x,t)\chi_{]0,S[}$. The following lemmas play a crucial role in the sequel of this paper.

Lemma 4.4. *(See [14]) The following assertions hold:*

1. *Given anyfunction $u \in L_\psi(Q_S)$, then $u_\eta \in C\Big([0,S]; L_\psi(\Omega)\Big)$ and $\lim_{\eta\to\infty} u_\eta = u$ in $L_\psi(Q_S)$ for the modular convergence.*
2. *Let $u \in W^{s,1,x}L_\psi(Q_S)$, we have $u_\eta \in C\Big([0,S]; W^sL_\psi(\Omega)\Big)$ and $\lim_{\eta\to\infty} u_\eta = u$ in $W^{s,1,x}L_\psi(Q_S)$ for the modular convergence.*
3. *Let $u \in E_\psi(Q_S)$ (resp, $u \in W^{s,1,x}E_\psi(Q_S)\Big)$. $\lim_{\eta\to\infty} u_\eta = u$ strongly in $E_\psi(Q_S)$ (resp, strongly in $W^{s,1,x}E_\psi(Q_S)$).*
4. *Let $u \in W^{s,1,x}L_\psi(Q_S)$, then $\frac{\partial u_\eta}{\partial s} = \eta\Big(u - u_\eta\Big) \in W^{s,1,x}L_\psi(Q_S)$.*
5. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $W^{s,1,x}L_\psi(Q_S)$ and $u \in W^{s,1,x}L_\psi(Q_S)$, such that $u_n \longrightarrow u$ as $n \longrightarrow \infty$ strongly in $W^{s,1,x}L_\psi(Q_S)$ (resp, for the modular convergence). Then, for each $\eta > 0$, we obtain $(u_n)_\eta \longrightarrow u_\eta$ strongly in $W^{s,1,x}L_\psi(Q_S)$ (resp, for the modular convergence).*

The notion of capacity solution as follows:

Definition 4.5. *[11] The pair (u, f) is called a capacity solution for the problem (1), if the following conditions hold:*

1. *$u \in F$ and $\nabla u \in L_{\bar{\psi}}(\Omega)^N$.*
2. *(u, f) satisfies the following equation:*

$$^{C}D^{\alpha}_{t,0^+}u(x,t) + \mathcal{A}u(x,t) = f(x,t,u,\nabla u)$$

3. *$u(.,0) = u_0$, in Ω.*

Theorem 4.6. *We assume that hypotheses (11), (12), (15), (16) and (26)- (28) hold, then the problem (1) has a capacity solution in the sense of Definition 4.5.*

Proof. The proof proceeds in three steps. First, we introduce a series of approximate problems, then we establish a priori estimates for these problems. Next, we prove intermediate results, particularly the strong convergence of $\left(\displaystyle\int_0^t (t-\tau)^{\alpha-1}\frac{\nabla u_n(x) - \nabla u_n(y)}{(x-y)^{\frac{N}{2}+s}}\,d\tau\right)_{n\in\mathbb{N}}$ in $L^1(\Omega\times\Omega)$.

- **Step I**

  For all $n \in \mathbb{N}$, we study the following approximate problem:

$$\begin{cases} ^{C}D^{\alpha}_{t,0^+}u_n(x,t) + \mathcal{A}u(x,t) = f(x,t,u_n,\nabla u_n), & \text{in } \Omega\times(0,T], \\ u_n(x,0) = u_0(x), & \text{in } \Omega, \\ u_n(x,t) = 0, & \text{on } \partial\Omega\times(0,T], \end{cases} \tag{51}$$

  Equivalent to the following approximated problem:

$$\begin{cases} u_n(x,t) - u_0(x) + I^{\alpha}_{t,0^+}\mathcal{A}u_n(x,t) = I^{\alpha}_{t,0^+}f(x,t,u_n,\nabla u_n), & \text{in } \Omega\times(0,T], \\ u_n(x,0) = u_0(x), & \text{in } \Omega, \\ u_n(x,t) = 0, & \text{on } \partial\Omega\times(0,T]. \end{cases} \tag{52}$$

  Under the assumption (33) (34) and (35), Applying Theorem 4.3, to get the existence of a weak solution to the approximate problem (52). We use u_n as a test function in (52). Then, we get

$$\begin{aligned} \langle u_n - u_0, u_n\rangle_{Q_T} &+ \frac{1}{\Gamma(\alpha)}\int_0^T\int_0^t\int_\Omega (t-\tau)^{\alpha-1}|\nabla u_n|^2\ \mathrm{d}x\,d\tau\ \mathrm{d}t \\ &= \frac{1}{\Gamma(\alpha)}\int_0^T\int_\Omega\int_0^t (t-\tau)^{\alpha-1}f(x,\tau,u_n,\nabla u_n)u_n\,d\tau\,dx\,dt \end{aligned}$$

Hence

$$\langle u_n, u_n\rangle_{Q_T} - \langle u_0, u_n\rangle_{Q_T} + \frac{1}{\Gamma(\alpha)}\int_0^T\int_0^t\int_\Omega (t-\tau)^{\alpha-1}|\nabla u_n|^2 \ \mathrm{d}x\, d\tau\ \mathrm{d}t$$
$$= \frac{1}{\Gamma(\alpha)}\int_0^T\int_\Omega\int_0^t (t-\tau)^{\alpha-1} f(x,\tau,u_n,\nabla u_n)u_n\, d\tau\, dx\, dt$$

Consequently

$$\frac{1}{\Gamma(\alpha)}\int_0^T\int_0^t\int_\Omega (t-\tau)^{\alpha-1}|\nabla u_n|^2 \ \mathrm{d}x\, d\tau\ \mathrm{d}t \le \frac{1}{\Gamma(\alpha)}\int_0^T\int_\Omega\int_0^t (t-\tau)^{\alpha-1} f(x,\tau,u_n,\nabla u_n)u_n\, d\tau\, dx\, dt$$
$$+ \|u_n\|_{L^2(Q_T)}\left(1+\|u_0\|_{L^2(Q_T)}\right)$$

Keeping this and (41) in mind, we obtain

$$\frac{\beta}{\Gamma(\alpha)}\int_0^T\int_{\Omega\times\Omega}\psi\left(x,\int_0^t (t-\tau)^{\alpha-1}\left|\frac{\nabla u_n(x)-\nabla u_n(y)}{(x-y)^{\frac{N}{2}+s}}d\tau\right|\right)\mathrm{d}x\mathrm{d}y\ \mathrm{d}s$$
$$\le \frac{1}{\Gamma(\alpha)}\int_0^T\int_{\Omega\times\Omega}\int_0^t (t-\tau)^{\alpha-1} f(x,\tau,u_n(x,y),\nabla u_n(x,y))u_n\, d\tau\, dx\, dt$$
$$+ \|u_n\|_{L^2(Q_T)}\left(1+\|u_0\|_{L^2(Q_T)}\right)$$

On the other hand, using the condition (34) and Hölder?s inequality, we get

$$\frac{\beta}{\Gamma(\alpha)}\int_0^T\int_{\Omega\times\Omega}\psi\left(x,\int_0^t (t-\tau)^{\alpha-1}\left|\frac{\nabla u_n(x)-\nabla u_n(y)}{(x-y)^{\frac{N}{2}+s}}d\tau\right|\right)\mathrm{d}x\mathrm{d}y\ \mathrm{d}s$$
$$\le \frac{1}{\Gamma(\alpha)}\int_0^T\int_{\Omega\times\Omega}\int_0^t (t-\tau)^{\alpha-1}\left(b(|u_n(x,y)|)\right)\left(c_1(x,t)+\psi(x,|\nabla u_n(x,y)|)\right)$$
$$+ \|u_n\|_{L^2(Q_T)}\left(1+\|u_0\|_{L^2(Q_T)}\right)$$
$$\leqslant \frac{C_1 T}{\Gamma(\alpha+1)}[u_n]_{s,2}\left(\mathrm{meas}(\Omega)\|c_1(x,t)\|_{L^2\left(0,T,H^{-1}(\Omega\times\Omega)\right)} + [\nabla u_n]_{s,\psi,p}\right)$$
$$+ \|u_n\|_{L^2(Q_T)}\left(1+\|u_0\|_{L^2(Q_T)}\right)$$

Since $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence in $L^2(Q_T)$. Then, there exists a positive constant C_7, such that

$$\frac{\beta}{\Gamma(\alpha)}\int_0^T\int_{\Omega\times\Omega}\psi\left(x,\int_0^t (t-\tau)^{\alpha-1}\left|\frac{\nabla u_n(x)-\nabla u_n(y)}{(x-y)^{\frac{N}{2}+s}}d\tau\right|\right)\mathrm{d}x\mathrm{d}y\ \mathrm{d}s \le C_7. \tag{53}$$

Recall from Remark 2.9 that

$$\|u_n\|_{s,\psi,\Omega} \leqslant (C+1)\max(C,1)$$

This implies that $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence in $W_0^{s,1,x}L_\psi(Q_T)$. Therefore, there exists a subsequence, still denoted by $(u_n)_{n\in\mathbb{N}}$, which converges weakly in $W_0^{s,1,x}L_\psi(Q_T)$ to a limit u as $n\to\infty$, such that:

$$u_n \rightharpoonup u \quad \text{in } W_0^{s,1,x}L_\psi(Q_S),\ \text{for } \left(\sigma\left(\prod W^{s,1,x}L_\psi, \prod W^{s,1,x}E_{\overline{\psi}}\right)\right). \tag{54}$$

On the other hand, for any function $\varphi \in W_0^{s,1,x}E_\psi(Q_T)^N$ such that $\|\nabla\varphi\|_{\psi,Q_T} = \frac{1}{m+1}$, where m is a positive real number, we have:

$$\begin{aligned}
&\int_0^T \int_{\Omega\times\Omega} \int_0^t (t-\tau)^{\alpha-1} \left|\frac{\nabla u_n(x) - \nabla u_n(y)}{(x-y)^{\frac{N}{2}+s}}\right| \left|\frac{\nabla\varphi(x) - \nabla\varphi(y)}{(x-y)^{\frac{N}{2}+s}}\right| d\tau\, \mathrm{d}x\mathrm{d}y\, \mathrm{d}s \\
&\leqslant \frac{1}{2}\left[\int_0^T \int_\Omega \int_0^t (t-\tau)^{\alpha-1} \left|\frac{\nabla u_n(x) - \nabla u_n(y)}{(x-y)^{\frac{N}{2}+s}}\right|^2 + \int_0^T \int_\Omega \int_0^t (t-\tau)^{\alpha-1} \left|\frac{\nabla\varphi(x) - \nabla\varphi(y)}{(x-y)^{\frac{N}{2}+s}}\right|^2\right] \\
&\leqslant \frac{1}{2}\int_0^T \int_{\Omega\times\Omega} \psi\left(x, \left|\int_0^t (t-\tau)^{\alpha-1}|\nabla u_n(x,y)\right|\right) + \frac{1}{2}\int_0^T \int_{\Omega\times\Omega} \psi\left(x, \left|\int_0^t (t-\tau)^{\alpha-1}|\nabla\varphi(x,y)\right|\right) \\
&\leq \frac{C_7}{2} + \frac{1}{2}\|\nabla\varphi\|_{\psi,Q_T}
\end{aligned}$$

Using the equivalence between the Luxemburg norm and the Orlicz norm, along with (53), there exists a positive constant C_8 such that

$$\int_0^T \int_{\Omega\times\Omega} \int_0^t (t-\tau)^{\alpha-1} \left|\frac{\nabla u_n(x) - \nabla u_n(y)}{(x-y)^{\frac{N}{2}+s}}\right| \left|\frac{\nabla\varphi(x) - \nabla\varphi(y)}{(x-y)^{\frac{N}{2}+s}}\right| d\tau\, \mathrm{d}x\mathrm{d}y\, \mathrm{d}s \leqslant C_8$$

It follows that $\left(\int_0^t (t-\tau)^{\alpha-1} \frac{\nabla u_n(x) - \nabla u_n(y)}{(x-y)^{\frac{N}{2}+s}}\, d\tau\right)_{n\in\mathbb{N}}$ is bounded in $L_{\overline{\psi}}(Q_S)^N$. This implies the existence of a subsequence, still denoted by $\left(\int_0^t (t-\tau)^{\alpha-1} \frac{\nabla u_n(x) - \nabla u_n(y)}{(x-y)^{\frac{N}{2}+s}}\, d\tau\right)_{n\in\mathbb{N}}$, such that

$$u_n \rightharpoonup u \quad \text{in } L_{\overline{\psi}}(Q_S)^N, \text{ for } \left(\sigma\left(\prod W^{s,1,x}L_{\overline{\psi}}, \prod W^{s,1,x}E_\psi\right)\right) \tag{55}$$

Since the sequences $\left(\int_0^t (t-\tau)^{\alpha-1}\Delta u_n\, d\tau\right)_{n\in\mathbb{N}}$ and $I^\alpha_{t,0^+} f(x,t,u_n,\nabla u_n)$are bounded in $W^{-s,1,x}L_{\overline{\psi}}(Q_S)$, it follows, using the first equation of the problem (52), that the sequence $(u_n - u_0)_{n\in\mathbb{N}}$ is also bounded in $W^{-s,1,x}L_{\overline{\psi}}(Q_T)$. Consequently, $(u_n)_{n\in\mathbb{N}}$ is bounded in F^s.

Given that the embedding $F^s \hookrightarrow W^{s,1,x}E_\phi(Q_S) \hookrightarrow E_\phi(Q_S)$ is compact, we deduce, for a subsequence still denoted the same way, that

$$u_n \to u \quad \text{strongly in } E_\phi(Q_T) \text{ and almost everywhere in } Q_T, \tag{56}$$

where $u \in W_0^{s,1,x}L_\psi(Q_T)$ is the same limit as identified in (54).

- **Step 2**

We introduce the following regularized sequences for $i, j \in \mathbb{N}$:

1. $v_j \to u$ in $W_0^{s,1,x}L_\psi\,(Q_S)$ with the modular convergence;
2. $v_j \to u$ and $\nabla v_j \to \nabla u$ a.e in Q_S;
3. $\omega_i \to u_0$ in $L^2(\Omega)$ with the strong convergence;
4. $\|\omega_i\|_{L^2(\Omega)} \leq 2\,\|u_0\|_{L^2(\Omega)}$, for all $i \geq 1$.

These four points hold for all $\omega \in D(\Omega)$ and $v_j \in D(Q_S)$. Let $\Theta > 0$ be a real number, and define the truncation function as in (20). Then, for each $\Theta, \eta > 0$ and for $i, j \in \mathbb{N}$, we consider the function $\omega^i_{\eta,j} \in W_0^{s,1,x}L_\psi(Q_T)$ defined as:

$$\omega^i_{\eta,j} := \mathscr{S}_\Theta(v_j)_\eta + \exp(-\eta s)\mathscr{S}_\Theta(\omega_j),$$

where $\mathscr{S}_\Theta(v_j)_\eta$ denotes the mollification with respect to the time variable of $\mathscr{S}_\Theta(v_j)$, as introduced in (50). From Lemma 4.4, it follows that:

$$\begin{gathered} {}^C D^\alpha_{t,0^+}\omega^i_{\eta,j} = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\eta\left(\mathscr{S}_\Theta\left(v_j\right)-\omega^i_{\eta,j}\right)d\tau, \quad \omega^i_{\eta,j}(.,0)=\mathscr{S}_\Theta\left(\omega_i\right), \\ \left|\omega^i_{\eta,j}\right| \le \Theta \text{ a.e in } Q_T, \end{gathered} \tag{57}$$

$$\omega^i_{\eta,j} \longrightarrow \omega^i_\eta := \mathscr{S}_\Theta(u)_\eta + \exp(-\eta s)\mathscr{S}_\Theta\left(\omega_i\right) \text{ as } j\longrightarrow\infty \text{ in } W^{s,1,x}_0 L_\psi\left(Q_T\right), \tag{58}$$

$$\mathscr{S}_\Theta(u)_\eta + \exp(-\eta s)\mathscr{S}_\Theta\left(\omega_i\right) \longrightarrow \mathscr{S}_\Theta(u) \text{ as } \eta\longrightarrow\infty \text{ in } W^{s,1,x}_0 L_\psi\left(Q_S\right), \tag{59}$$

with the modular convergence in the two last convergences.

□

Proposition 4.7. *Let u_n be a solution of problem* (52)*. Then, for a subsequence, the following convergence holds:*

$$\int_0^t (t-\tau)^{\alpha-1}(\nabla u_n - \nabla u)\,d\tau \longrightarrow 0 \quad \text{as } n\to\infty \quad \text{almost everywhere in } Q_T. \tag{60}$$

Proof. Throughout this paper, we use χ_{r_j} and χ_r as the characteristic functions of the following sets:

$$Q_{r_j} = \{(x,s)\in Q_T \mid |\nabla\mathscr{S}_\Theta(v_j)|\le\theta\}, \quad Q_r = \{(x,s)\in Q_T \mid |\nabla\mathscr{S}_\Theta(u)|\le\theta\}.$$

For any real numbers $\eta,\vartheta>0$ and for $i,j,n\in\mathbb{N}$, we employ the admissible test function $\phi^{\eta,i}_{n,j,\vartheta} := \mathscr{S}_\vartheta\left(u_n-\omega^i_{\eta,j}\right)$ in the first equation of the approximate problem (52), yielding:

$$\begin{aligned} \left\langle {}^C D^\alpha_{t,0^+}u_n(x,t), \phi^{\eta,i}_{n,j,\vartheta}\right\rangle_{Q_T} + \int_0^T\int_\Omega \nabla\mathscr{S}_\vartheta\left(u_n-\omega^i_{\eta,j}\right)\nabla u_n \,\mathrm{d}x\,\mathrm{d}s \\ = \left\langle f(x,\tau,u_n,\nabla u_n), \mathscr{S}_\vartheta\left(u_n-\omega^i_{\eta,j}\right)\right\rangle_{Q_T} \end{aligned}$$

On the other hand, using the condition (34), and by the same reasoning done to get (37), we obtain

$$\int_\Omega f(x,\tau,u_n,\nabla u_n)\ \mathrm{d}x \le C_9, \text{ where } C_9>0 \tag{61}$$

From (20), we obtain

$$\int_0^T\int_\Omega f(x,\tau,u_n,\nabla u_n)\cdot\mathscr{S}_\vartheta\left(u_n-\omega^i_{\eta,j}\right) \le \vartheta\int_0^T\int_\Omega f(x,\tau,u_n,\nabla u_n)$$

Using (61), we get

$$\left\langle {}^C D^\alpha_{t,0^+}u_n(x,t), \phi^{\eta,i}_{n,j,\vartheta}\right\rangle_{Q_T} + \int_0^T\int_\Omega \nabla\mathscr{S}_\vartheta\left(u_n-\omega^i_{\eta,j}\right)\nabla u_n \,\mathrm{d}x\,\mathrm{d}s \le C_9\vartheta. \tag{62}$$

Now, we decompose the first term on the left-hand side of the above inequality into two parts and estimate each part separately.

$$\begin{aligned}\left\langle {}^{C}D^{\alpha}_{t,0^+}u_n(x,t), \phi^{\eta,i}_{n,j,\vartheta}\right\rangle_{Q_T} &= \left\langle \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\frac{\partial u_n}{\partial \tau}\,d\tau, \phi^{\eta,i}_{n,j,\vartheta}\right\rangle \\ &= \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\left\langle \frac{\partial u_n}{\partial \tau} - \frac{\partial \omega^i_{\vartheta,j}}{\partial \tau}, \phi^{\eta,i}_{n,j,\vartheta}\right\rangle \\ &+ \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\left\langle \frac{\partial \omega^i_{\vartheta,j}}{\partial \tau}, \phi^{\eta,i}_{n,j,\vartheta}\right\rangle\end{aligned} \tag{63}$$

We start by estimating the first term on the right side of the above identity

$$\left\langle \frac{\partial u_n}{\partial s} - \frac{\partial \omega^i_{\vartheta,j}}{\partial s}, \phi^{\eta,i}_{n,j,\vartheta}\right\rangle = \int_\Omega \mathscr{T}_\vartheta\left(u_n(T) - \omega^i_{\eta,j}(T)\right)\mathrm{d}x - \int_\Omega \mathscr{T}_\vartheta\left(u_0 - \mathscr{S}_\vartheta(\omega_i)\right)\mathrm{d}x \tag{64}$$

From (21), it can be shown that

$$0 \le \mathscr{T}_\Theta(\theta) \le \Theta|\theta|, \text{ for all } \theta \in \mathbb{R}.$$

It follows that:

$$\begin{aligned}0 \le \int_\Omega \mathscr{T}_\vartheta\left(u_0 - \mathscr{S}_\vartheta(\omega_i)\right)\mathrm{d}x &\le \vartheta\int_\Omega |u_0 - \mathscr{S}_\vartheta(\omega_i)|\,\mathrm{d}x \\ &\le \vartheta(\mathrm{meas}(\Omega))^{1/2}\left(\int_\Omega |u_0 - \mathscr{S}_\vartheta(\omega_i)|^2\ \mathrm{d}x\right)^{1/2} \\ &\le 3\vartheta(\mathrm{meas}(\Omega))^{1/2}\,\|u_0\|_{L^2(\Omega)} := C_{10}\vartheta\end{aligned}$$

Then, for all $\eta, \vartheta > 0$ and $i, j, n \ge 1$, and from (64), we get

$$-C_{10}\vartheta \le \left\langle \frac{\partial u_n}{\partial \tau} - \frac{\partial \omega^i_{\vartheta,j}, \phi^{\eta,i}_{n,j,\vartheta}}{\partial \tau}\right\rangle.$$

Which leads to:

$$\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\left\langle \frac{\partial u_n}{\partial \tau} - \frac{\partial \omega^i_{\vartheta,j}}{\partial \tau}, \phi^{\eta,i}_{n,j,\vartheta}\right\rangle \geqslant -\frac{C_{10}T\vartheta}{\Gamma(2-\alpha)} \tag{65}$$

Now, we derive an estimate for the second term on the right side of (63). Under assumption (57), we get

$$\left\langle {}^{C}D^{\alpha}_{t,0^+}\omega^i_{\vartheta,j}, \phi^{\eta,i}_{n,j,\vartheta}\right\rangle = \frac{\eta}{\Gamma(1-\alpha)}\int_0^T\int_0^t\int_\Omega (t-\tau)^{-\alpha}\left(\mathscr{S}_\Theta\left(v_j\right) - \omega^i_{\eta,j}\right)\mathscr{S}_\vartheta\left(u_n - \omega^i_{\eta,j}\right)\mathrm{d}x\,d\tau\ \mathrm{d}s.$$

Then

$$\begin{aligned}&\lim_{n\to\infty}\lim_{j\to\infty}\left\langle {}^{C}D^{\alpha}_{t,0^+}\omega^i_{\vartheta,j}, \phi^{\eta,i}_{n,j,\vartheta}\right\rangle \\ &\quad= \frac{\eta}{\Gamma(1-\alpha)}\int_0^T\int_0^t\int_\Omega (t-\tau)^{-\alpha}\left(\mathscr{S}_\Theta(u) - \omega^i_{\eta,j}\right)\mathscr{S}_\vartheta\left(u - \omega^i_{\eta,j}\right)\mathrm{d}x\,d\tau\ \mathrm{d}s.\end{aligned}$$

Under hypotheses (57)-(59), we have $\left|\omega^i_\eta\right| \le \Theta$, and due to $\theta\mathscr{S}_\vartheta(\theta) \ge 0, \theta \in \mathbb{R}$, we get that for all $\eta, \theta, \Theta > 0$ and $i \ge 1$

$$\lim_{n\to\infty}\lim_{j\to\infty}\left\langle {}^{C}D^{\alpha}_{t,0^+}\omega^{i}_{\vartheta,j},\phi^{\eta,i}_{n,j,\vartheta}\right\rangle \geq 0$$

Keeping this and (65) in mind, we get

$$\lim_{n\to\infty}\inf\lim_{j\to\infty}\inf\left\langle {}^{C}D^{\alpha}_{t,0^+}u_n,\phi^{\eta,i}_{n,j,\vartheta}\right\rangle \geqslant -\frac{C_{10}T\vartheta}{\Gamma(2-\alpha)} \tag{66}$$

On the other hand, we have

$$\begin{aligned}
\mathfrak{I}_{i,j,n,v} := &\int_0^T\int_\Omega \nabla\mathscr{S}_\vartheta\left(u_n-\omega^i_{\eta,j}\right)\nabla u_n\,\mathrm{d}x\,\mathrm{d}s\\
= &\int_{\{|u_n-\omega^i_{\eta,j}|\leq\vartheta|\}}\left(\nabla u_n-\nabla\omega^i_{\eta,j}\right)\nabla u_n\,\mathrm{d}x\,\mathrm{d}s\\
= &\int_{\left\{\left|u_n-\omega^i_{\eta,j}\right|\leq\vartheta|\right\}\cap\{||u_n|>\Theta|\}}\left(\nabla u_n-\nabla\omega^i_{\eta,j}\right)\nabla u_n\,\mathrm{d}x\,\mathrm{d}s\\
+ &\int_{\left\{\left|u_n-\omega^i_{\eta,j}\right|\leq\vartheta|\right\}\cap\{||u_n|\leq\Theta|\}}\left(\nabla u_n-\nabla\omega^i_{\eta,j}\right)\nabla u_n\,\mathrm{d}x\,\mathrm{d}s\\
= &\int_{\left\{\left|\mathscr{S}_\Theta(u_n)-\omega^i_{\eta,j}\leqslant\vartheta\right|\right\}}\left(\nabla\mathscr{S}_\Theta(u_n)-\nabla\omega^i_{s,j}\right)\nabla\mathscr{S}_\Theta(u_n)\,\mathrm{d}x\,\mathrm{d}s\\
+ &\int_{\left\{\left|u_n-\omega^i_{\eta,j}\right|\leq\vartheta|\right\}\cap\{||u_n|>\Theta|\}}|\nabla u_n|^2\,\mathrm{d}x\,\mathrm{d}s\\
- &\int_{\left\{\left|u_n-\omega^i_{\eta,j}\right|\leq\vartheta|\right\}\cap\{||u_n|>\Theta|\}}\nabla\omega^i_{\eta,j}\nabla u_n dxds,
\end{aligned}$$

where

$$\begin{aligned}
\mathfrak{I}_{0,i,j,n,\vartheta} &= \int_{\left\{\left|\mathscr{S}_\Theta(u_n)-\omega^i_{\eta,j}\leqslant\vartheta\right|\right\}}\left(\nabla\mathscr{S}_\Theta(u_n)-\nabla\omega^i_{s,j}\right)\nabla\mathscr{S}_\Theta(u_n)\,\mathrm{d}x\,\mathrm{d}s,\\
\mathfrak{I}_{1,i,j,n,\vartheta} &= \int_{\left\{\left|u_n-\omega^i_{\eta,j}\right|\leq\vartheta|\right\}\cap\{|u_n|>\Theta|\}}|\nabla u_n|^2\,\mathrm{d}x\,\mathrm{d}s,\\
\mathfrak{I}_{2,i,j+n,\vartheta} &= \int_{\left\{\left|u_n-\omega^i_{\eta,j}\right|\leq\vartheta|\right\}\cap\{|u_n|>\Theta|\}}\nabla\omega^i_{\eta,j}\nabla u_n dxds.
\end{aligned}$$

Under assumption (27), we have

$$\mathfrak{I}_{1,i,j,n,\vartheta} \geq \alpha\int_{\left\{\left|u_n-\omega^i_{\eta,j}\right|\leq\vartheta|\right\}\cap\{|u_n|>\Theta|\}}\psi(x,|\nabla u_n|)\,\mathrm{d}x\,\mathrm{d}s \geq 0.$$

Then, $\mathfrak{I}_{i,j,n,\vartheta} \geq \mathfrak{I}_{0,i,j,n,\vartheta} - \mathfrak{I}_{2,i,j,n,\vartheta}$; from (57), we have $\left|\omega^i_{\eta,j}\right| \leq \Theta$, a.e. in Q_T, and this implies that

$$|u_n| \leq \left|u_n-\omega^i_{\eta,j}\right| + \left|\omega^i_{\eta,j}\right| \leq \Theta+\vartheta. \tag{67}$$

It follows that for $n > \Theta+\vartheta$:

$$\mathfrak{I}_{2,i,j,n,\vartheta} = \int_{\left\{\left|u_n-\omega^i_{\eta,j}\right|\leq\vartheta|\right\}\cap\{|u_n|>\Theta|\}}\nabla\omega^i_{\eta,j}\nabla\mathscr{S}_{\vartheta+\Theta}(u_n)\,\mathrm{d}x\,\mathrm{d}s,$$

which gives

$$\begin{aligned}\mathfrak{I}_{i,j,n,\vartheta} &\geq \mathfrak{I}_{0,i,j,n,\vartheta} - \mathfrak{I}_{2,i,j,n,\vartheta}\\ &\geq \int_{\left\{\left|\mathscr{S}_{\Theta}(u_n)-\omega^i_{\eta,j}\leqslant\vartheta\right|\right\}} \left(\nabla\mathscr{S}_{\Theta}(u_n) - \nabla\omega^i_{\eta,j}\right)\nabla\mathscr{S}_{\Theta}(u_n)\,\mathrm{d}x\,\mathrm{d}s - \mathfrak{I}_{2,i,j,n,\vartheta}\end{aligned} \tag{68}$$

Using inequality (67) again, we deduce from the definition that $\mathscr{S}_{\vartheta+\Theta}(u_n) = u_n$, and consequently, $\nabla\mathscr{S}_{\vartheta+\Theta}(u_n) = \nabla u_n$. Given that the sequence $(\nabla u_n)_{n\in\mathbb{N}}$ is bounded in $L_{\psi}(Q_S)^N$, it follows that $(\nabla\mathscr{S}_{\vartheta+\Theta}(u_n))_{n\in\mathbb{N}}$ is also bounded in $L_{\psi}(Q_S)^N$. Therefore, there exists a_1 such that $\nabla\mathscr{S}_{\vartheta+\Theta}(u_n) \to a_1$ as $n\to\infty$ in $L_{\psi}(Q_S)^N$.

In view of the fact that

$$\nabla\omega^i_{\eta,j}\chi_{\{|u_n-\omega^i_{\eta,j}|\leq\vartheta\}\cap\{|u_n|>\Theta\}} \to \nabla\omega^i_{\eta,j}\chi_{\{|u-\omega^i_{\eta,j}|\leq\vartheta\}\cap\{|u|>\Theta\}},$$

strongly in $E_{\psi}(Q_T)^N$ as $n\to\infty$, we get

$$\lim_{n\to\infty}\int_{\{|u_n-\omega^i_{\eta,j}|\leq\vartheta\}\cap\{|u_n|>\Theta\}} \nabla\omega^i_{\eta,j}\nabla\mathscr{S}_{\vartheta+\Theta}(u_n) = \int_{\{|u-\omega^i_{\eta,j}|\leq\vartheta\}\cap\{|u|>\Theta\}} \nabla\omega^i_{\eta,j}a_1.$$

Under assumptions (58) and (59), we obtain:

$$\nabla\mathscr{S}_{\vartheta+\Theta}(u_n)\nabla\omega^i_{\eta,j}\chi_{\{|u_n-\omega^i_{\eta,j}|\leq\vartheta\}\cap\{|u_n|>\Theta\}} \longrightarrow \nabla\mathscr{S}_{\vartheta+\Theta}(u)\nabla\omega^i_{\eta,j}\chi_{\{|u-\omega^i_{\eta,j}|\leq\vartheta\}\cap\{|u_n|>\Theta\}}$$

as $n, j \longrightarrow \infty$. We apply Lemma 2.12 , and letting $\vartheta, j \longrightarrow \infty$, we obtain

$$\int_Q \nabla S_{\vartheta+R}(u_n)\nabla\omega^i_{\eta,j}\chi_{\left\{\left|u_n-\omega^i_{\eta,j}\right|\leq\vartheta\right\}\cap\{|u_n|>R\}} \longrightarrow I_3 := \int_{\left\{\left|u-\omega^i_{\eta,j}\right|\leq\vartheta\right\}\cap\{|u|>R\}} \nabla\mathscr{S}_{\Theta}(u)a_1$$

For $|u| > \Theta$, we get $\mathscr{S}_{\Theta}(u) = 0$, which yields $I_3 = 0$. Thus

$$\begin{aligned}&\mathfrak{I}_{2,i,j,n,\vartheta} \longrightarrow 0\\ &\quad i, j, \vartheta \longrightarrow \infty.\end{aligned} \tag{69}$$

By using (68), we obtain

$$\mathfrak{I}_{i,j,n,\vartheta} \geq \int_{\left\{\left|\mathscr{S}_{\Theta}(u_n)-\omega^i_{\eta,j}\right|\leqslant\vartheta\right\}} \left(\nabla\mathscr{S}_{\Theta}(u_n) - \nabla\omega^i_{\eta,j}\right)\nabla\mathscr{S}_{\Theta}(u_n)\,\mathrm{d}x\,\mathrm{d}s - \mathfrak{I}_{2,i,j,n,\vartheta}$$

$$\int_{\left\{\left|\mathscr{S}_{\Theta}(u_n)-\omega^i_{\eta,j}\right|\leqslant\vartheta\right\}} \left(\nabla\mathscr{S}_{\Theta}(u_n) - \nabla\omega^i_{\eta,j}\right)\nabla\mathscr{S}_{\Theta}(u_n)\,\mathrm{d}x\,\mathrm{d}s \leqslant \mathfrak{I}_{i,j,n,\vartheta} + \mathfrak{I}_{2,i,j,n,\vartheta}$$

Using (62), we get

$$\mathfrak{I}_{i,j,n,\theta} \leq C_9\vartheta - \left\langle {}^{C}D^{\alpha}_{t,0^+}u_n(x,t), \phi^{\eta,i}_{n,j,\vartheta}\right\rangle_{Q_T}$$

Owing to (66), it follows that:

$$-\left\langle {}^{C}D^{\alpha}_{t,0^+}u_n(x,t), \phi^{\eta,i}_{n,j,\vartheta}\right\rangle_{Q_T} \leq \frac{C_{10}T\vartheta}{\Gamma(2-\alpha)}.$$

Then

$$\Im_{i,j,n,\vartheta} \leq \left(C_9 + \frac{C_{10} T}{\Gamma(2-\alpha)} \right) \vartheta := C\vartheta.$$

We have

$$\begin{aligned}
&\int_{\left\{\left|\mathscr{S}_\Theta(u_n)-\omega^i_{\eta,j}\right|\leqslant\vartheta\right\}} \left(\nabla \mathscr{S}_\Theta(u_n) - \nabla\omega^i_{\eta,j}\right) \nabla\mathscr{S}_\Theta(u_n)\, dx\, ds \\
&= \int_{\mathcal{A}} \left(\nabla \mathscr{S}_\Theta(u_n) - \nabla\mathscr{S}_\Theta(v_j)\chi^r_j\right) \nabla\mathscr{S}_\Theta(u_n)\, dx\, ds + \int_{\mathcal{A}} \left(\nabla \mathscr{S}_\Theta(v_j)\chi^r_j) - \nabla\omega^i_{\eta,j}\right) \nabla\mathscr{S}_\Theta(u_n)\, dx\, ds \\
&= \Im_{4,i,j,n,\eta} + \Im_{5,i,j,n,\eta}
\end{aligned} \tag{70}$$

where $\mathcal{A} = \left\{\left\{\left|\mathscr{S}_\Theta(u_n) - \omega^i_{\eta,j}\right| \leqslant \vartheta\right\}\right\}$.

Now, we show that $\Im_{5,i,j,n,\eta} \longrightarrow 0$. Knowing that $(\mathscr{S}_\Theta(u_n))_{n\in\mathbb{N}}$ is bounded. Hence, there exits a_0, such that

$$\mathscr{S}_\Theta(u_n) \to a_0 \text{ as } n \longrightarrow \infty.$$

Since

$$\Lambda^n(v_j) := \left(\nabla\mathscr{S}_\Theta(v_j)\chi^r_j - \nabla\omega^i_{\eta,j}\right)\chi_{\mathcal{A}} \longrightarrow \left(\nabla\mathscr{S}_\Theta(v_j)\chi^r_j - \nabla\omega^i_{\eta,j}\right)\chi_{\{|(\mathscr{S}_\Theta(n)-\omega_{\eta,j})|\leq\vartheta\}} \text{ in } E_\phi(Q_s).$$

It follows that $\left(\Lambda^n(v_j)\right)_{n\in\mathbb{N}}$ is bounded in $E_{\bar{\psi}}(Q_T)$. Hence, $\left(\nabla\mathscr{S}_\Theta(u_n)\Lambda^n(v_j)\right)_{n\in\mathbb{N}}$ is bounded as well. Applying the dominated convergence theorem, we get

$$\lim_{i,j,n,\eta\to\infty} \Im_{5,i,j,n,\eta} = \int_{\mathcal{A}} \lim_{i,j,n,\eta\longrightarrow\infty} \left(\nabla\mathscr{S}_\Theta(v_j)\chi^r_j) - \nabla\omega^i_{\eta,j}\right)\nabla\mathscr{S}_\Theta(u_n)\, dx\, ds = 0.$$

$$\Im_{5,i,j,n,\eta} \longrightarrow 0 \text{ as } i,j,n,\eta \longrightarrow \infty \tag{71}$$

Recall (68) and (70), we get

$$\begin{aligned}
\Im_{4,i,j,n,\eta} + \Im_{5,i,j,n,\eta} &\leq \Im_{i,j,n,\vartheta} + \Im_{2,i,j,n,\vartheta} \\
\Im_{4,i,j,n,\eta} &\leq \Im_{i,j,n,\vartheta} + \Im_{2,i,j,n,\vartheta} - \Im_{5,i,j,n,\eta} \leq C\vartheta + \Im_{2,i,j,n,\theta} - \Im_{5,i,j,n,\eta}.
\end{aligned}$$

From (69) and (71), we can take $\epsilon(n,i,\eta,j) := \Im_{2,i,j,n,\vartheta} - \Im_{5,i,j,n,\eta}$ where $\epsilon(n,i,\eta,j) \longrightarrow 0$ as $i,j,n,\eta \longrightarrow \infty$. This implies that $\Im_{4,i,j,n,\eta} \leq C\vartheta + \epsilon(n,i,\eta,j)$.

Putting

$$\Sigma_n := (\nabla\mathscr{S}_\Theta(u_n) - \nabla\mathscr{S}_\Theta(u))(\nabla\mathscr{S}_\Theta(u_n) - \nabla\mathscr{S}_\Theta(u))$$

which is a nonnegative quantity. Since $(\nabla\mathscr{S}_\Theta(u_n))$ is bounded in $L_\psi(Q_s)^N$, then the same holds for Σ_n. Let us $\Xi^r_n := \int_{Q_r} \Sigma^\delta_n dx\, ds$ for each δ in $]0,1[$, we get

$$\begin{aligned}
\int_{Q_r} \Sigma^\theta_n \chi_{A^c}\, dx\, ds &\leq \left(\int_{Q_r} \Sigma_n\, dx\, ds\right)^\delta \left(\int_{Q_r} \chi_A c\, dx\, ds\right)^{(1-\delta)} \\
&\leq C_{12} \text{ meas } (A^c)
\end{aligned}$$

Using Hölder's inequality, we obtain

$$\begin{aligned}\int_{Q_r} \Sigma_n^{\delta} \chi_A \,\mathrm{d}x\,\mathrm{d}s &\leq \left(\int_{Q_r} \Sigma_n \,\mathrm{d}x\,\mathrm{d}s\right)^{\delta} \left(\int_{Q_r} \chi_A \,\mathrm{d}x\,\mathrm{d}s\right)^{(1-\delta)} \\ &\leq C_{13} \left(\int_{Q_r \cap A} \Sigma_n \,\mathrm{d}x\,\mathrm{d}s\right)^{\delta}\end{aligned}$$

It follows that:

$$\Xi_n^r = \int_{Q_r} \Sigma_n^{\delta} \mathrm{d}x\mathrm{d}s \leq \left(C_{12} meas\,(A^c) + C_{13} \left(\int_{Q_r \cap A} \Sigma_n \,\mathrm{d}x\,\mathrm{d}s\right)^{\delta}\right)$$

On the other hand, for $s \geq r$ and $r > 0$, we have

$$\begin{aligned}\int_{Q_R \cap A} \Sigma_n \,\mathrm{d}x\,\mathrm{d}s &\leq \int_{Q_r \cap A} (\nabla \mathscr{S}_{\Theta}(u_n) - \nabla \mathscr{S}_{\Theta}(u))(\nabla \mathscr{S}_{\Theta}(u_n) - \nabla \mathscr{S}_{\Theta}(u)) \,\mathrm{d}x\,\mathrm{d}s \\ &\leq \int_{Q_4 \cap A} (\nabla \mathscr{S}_{\Theta}(u_n) - \nabla \mathscr{S}_{\Theta}(u)\chi_s)(\nabla \mathscr{S}_{\Theta}(u_n) - \nabla \mathscr{S}_{\Theta}(u)\chi_s) \,\mathrm{d}x\,\mathrm{d}s \\ &\leq \int_{Q_4 \cap A} \left(\nabla \mathscr{S}_{\Theta}(u_n) - \nabla \mathscr{S}_{\Theta}\left(v_j\right)\chi_s\right)\left(\nabla \mathscr{S}_{\Theta}(u_n) - \nabla \mathscr{S}_{\Theta}\left(v_j\right)\chi_s\right) \mathrm{d}x\,\mathrm{d}s \\ &\leq \wp_{1,n,j} + \wp_{2,n,j} + \wp_{3,n,j} + J_{4,n,j},\end{aligned}$$

where

$$\begin{aligned}\wp_{1,n,j} &:= \int_A \left(\nabla \mathscr{S}_{\Theta}(u_n) - \nabla \mathscr{S}_{\Theta}\left(v_j\right)\chi_j^s\right)\left(\nabla \mathscr{S}_{\Theta}(u_n) - \nabla \mathscr{S}_{\Theta}\left(v_j\right)\chi_j^s\right) \mathrm{d}x\,\mathrm{d}s \\ \wp_{2,n,j} &:= \int_A \left(\nabla \mathscr{S}_{\Theta}\left(v_j\right)\chi_j^s - \nabla \mathscr{S}_{\Theta}(u)\chi_s\right)\left(\nabla \mathscr{S}_{\Theta}(u_n) - \nabla \mathscr{S}_{\Theta}\left(v_j\right)\chi_j^s\right) \mathrm{d}x\,\mathrm{d}s \\ \wp_{3,n,j} &:= \int_A \left(\nabla \mathscr{S}_{\Theta}(u_n) - \nabla \mathscr{S}_{\Theta}\left(v_j\right)\chi_j^s\right)\left(\nabla \mathscr{S}_{\Theta}\left(v_j\right)\chi_j^s - \nabla \mathscr{S}_{\Theta}\left(v_j\right)\chi^s\right) \mathrm{d}x\,\mathrm{d}s \\ \wp_{4,n,j} &:= \int_A \left(\nabla \mathscr{S}_{\Theta}\left(v_j\right)\chi_j^s - \nabla \mathscr{S}_{\Theta}(u)\chi_s\right)\left(\nabla \mathscr{S}_{\Theta}\left(v_j\right)\chi_j^s - \nabla \mathscr{S}_{\Theta}\left(v_j\right)\chi^s\right) \mathrm{d}x\,\mathrm{d}s\end{aligned}$$

Then

$$\begin{aligned}\wp_{1,n,j} &= \int_A \left(\nabla \mathscr{S}_{\Theta}(u_n) - \nabla \mathscr{S}_{\Theta}\left(v_j\right)\chi_j^s\right)\nabla \mathscr{S}_{\Theta}(u_n) \,\mathrm{d}x\,\mathrm{d}s \\ &\quad - \int_A \nabla \mathscr{S}_{\Theta}\left(v_j\right)\chi_j^s\left(\nabla \mathscr{S}_{\Theta}(u_n) - \nabla \mathscr{S}_{\Theta}\left(v_j\right)\chi_j^s\right) \mathrm{d}x\,\mathrm{d}s \\ &\leq \Im_{4,i,j,n,\eta} - \int_A \nabla \mathscr{S}_{\Theta}\left(v_j\right)\chi_j^s\left(\nabla \mathscr{S}_{\Theta}(u_n) - \nabla \mathscr{S}_{\Theta}\left(v_j\right)\chi_j^s\right) \mathrm{d}x\,\mathrm{d}s \\ &\leq C\vartheta + \epsilon(n,i,\eta,j) - \int_A \nabla \mathscr{S}_{\Theta}\left(v_j\right)\chi_j^s\left(\nabla \mathscr{S}_{\Theta}(u_n) - \nabla \mathscr{S}_{\Theta}\left(v_j\right)\chi_j^s\right) \mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

Putting

$$\begin{aligned}\Upsilon_{j,n} &:= \nabla \mathscr{S}_{\Theta}\left(v_j\right)\chi_j^s\left(\nabla \mathscr{S}_{\Theta}(u_n) - \nabla \mathscr{S}_{\Theta}\left(v_j\right)\chi_j^s\right) \\ &= \nabla \mathscr{S}_{\Theta}\left(v_j\right)\chi_j^s \nabla \mathscr{S}_{\Theta}(u_n) - \nabla \mathscr{S}_{\Theta}\left(v_j\right)\chi_j^s \nabla \mathscr{S}_{\Theta}\left(v_j\right)\chi_j^s\end{aligned}$$

By virtue of the fact that $\nabla \mathscr{S}_{\Theta}\left(v_j\right)\chi_j^s \to \nabla \mathscr{S}_{\Theta}(u)\chi^s$ as $j \to \infty$ and $\nabla \mathscr{S}_{\Theta}(u_n) \to \nabla \mathscr{S}_{\Theta}(u)$ weakly in $E_{\psi}(Q_s)^N$, it follows that $\Upsilon_{j,n} \to 0$ as $n, j \to \infty$. Since the sequence $\left(\nabla \mathscr{S}_{\Theta}\left(v_j\right)\chi_j^s\right)_j$ converges strongly to $\nabla \mathscr{S}_{\Theta}(u)\chi^s$ in $E_{\psi}(Q_s)^N$, applying the dominated convergence theorem yields $\wp_{4,n,j} \to 0$ as $n, j \to \infty$.

For $\wp_{3,n,j}$, knowing that the sequence $(\nabla\mathscr{S}_\Theta(u_n))_{n\in\mathbb{N}}$ is bounded and that $\left(\nabla\mathscr{S}_\Theta\left(v_j\right)\chi_j^s\right)_j$ converges strongly to $\nabla\mathscr{S}_\Theta(u)\chi^s$ in $E_\phi(Q_s)^N$, it follows that $\left(\nabla\mathscr{S}_\Theta(u_n)-\nabla\mathscr{S}_\Theta\left(v_j\right)\chi_j^s\right)$ is bounded. Using the convergence of $\left(\nabla\mathscr{S}_\Theta\left(v_j\right)\chi_j^s\right)_j$ to $\nabla\mathscr{S}_\Theta(u)\chi^s$, we conclude that $\wp_{3,n,j}\to 0$ as $n, j\to\infty$.

For $\wp_{2,n,j}$, we note that $\left(\nabla\mathscr{S}_\Theta\left(v_j\right)\chi_j^s-\nabla\mathscr{S}_\Theta(u)\chi^s\right)\to 0$ as $j\to\infty$, while $(\nabla\mathscr{S}_\Theta(u_n))_{n\in\mathbb{N}}$ and $\left(\nabla\mathscr{S}_\Theta\left(v_j\right)\chi_j^s\right)_j$ are convergent. Consequently, $\left(\nabla\mathscr{S}_\Theta(u_n)-\nabla\mathscr{S}_\Theta\left(v_j\right)\chi_j^s\right)$ remains bounded. Applying the dominated convergence theorem once again, we obtain $\wp_{2,n,j}\to 0$ as $n, j\to\infty$, and

$$\lim_{n,j\to\infty}\int_A \Upsilon_{j,n}\,\mathrm{d}x\,\mathrm{d}t=0$$

Letting n, j, then η, i, s, v to infinity, we get

$$\lim_{n\to\infty}\sup\Xi_n^r=\lim_{n\to\infty}\int_{Q_r}\Sigma_n^\delta\mathrm{d}x\,\mathrm{d}s=0$$

On the other hand, from (26), we obtain

$$0\le\int_{Q_r}\left[\psi\left(x,|\nabla\mathscr{S}_\Theta(u_n)-\nabla\mathscr{S}_\Theta(u)|\right)\right]^\delta\mathrm{d}x\,\mathrm{d}s\le\int_{Q_r}\Sigma_n^\delta\mathrm{d}x\,\mathrm{d}s$$

We recall that

$$\nabla\mathscr{S}_\Theta(u_n)\longrightarrow\nabla\mathscr{S}_\Theta(u)\text{ as }n\longrightarrow\infty$$

almost everywhere in Q_r. Since $r>0$ is arbitrary, we recall that for another subsequence, $\nabla\mathscr{S}_\Theta(u_n)\to\nabla\mathscr{S}_\Theta(u)$ almost everywhere in Q_T. Finally, for $\Theta>0$ arbitrary, we get

$$\nabla u_n\to\nabla u\quad\text{a.e. in } Q_T. \tag{72}$$

which implies the following convergence

$$\int_0^t(t-\tau)^{\alpha-1}(\nabla u_n-\nabla u)\,d\tau\longrightarrow 0\quad\text{as } n\to\infty\quad\text{almost every where in } Q_T. \tag{73}$$

This concludes the proof.

□

- **Step III** The first condition of Definition 4.5 is satisfied by applying (55), (61), and (57). The second condition of the same definition is established using the convergence in (60) and the smoothness of the function f. Regarding the regularity of the solution u, by utilizing (44) and directly applying Lemma 4.2, we conclude that $u\in C([0,T];L^1(\Omega))$.

  This concludes the proof of Theorem 4.6.

5. Application: Thermal Propagation with Memory in a Nonlinear Diffusion Medium

This section illustrates a practical application of problem (1) by modeling thermal propagation with memory effects in a nonlinear diffusion medium. Such phenomena frequently arise in materials exhibiting thermal memory or in anomalous diffusion processes.

5.1. Model Formulation

We consider the following fractional partial differential equation:

$$\begin{cases} {}^{C}D^{\alpha}_{t,0^+}u(x,t) + \mathcal{A}u(x,t) = f(x,t,u,\nabla u), & \text{in } \Omega \times (0,T], \\ u(x,0) = u_0(x), & \text{in } \Omega, \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,T], \end{cases} \tag{74}$$

The mathematical components of this model are:

- ${}^{C}D^{\alpha}_{t,0^+}u(x,t)$ represents the Caputo fractional derivative of order $\alpha \in (0,1)$, capturing the memory effect inherent in the thermal propagation process.
- The elliptic operator $\mathcal{A}u(x,t) = -\mathrm{div}(a(x,t,\nabla u))$ describes the nonlinear diffusion mechanism.
- The source term $f(x,t,u,\nabla u)$ models thermal generation with nonlinear dependence on both the temperature u and its spatial gradient ∇u.
- $u_0(x) \in L^2(\Omega)$ specifies the initial temperature distribution.

The specific forms of the nonlinear terms are:
Nonlinear Diffusion Coefficient:

$$a(x,t,\nabla u) = e^{-t}|\nabla u|^{p-2}\nabla u, \quad p > 1. \tag{75}$$

This expression models anisotropic diffusion where the temporal factor e^{-t} represents the gradual reduction of diffusive effects over time, while the power-law dependence $|\nabla u|^{p-2}$ captures the nonlinear response to temperature gradients.
Thermal Source Term:

$$f(x,t,u,\nabla u) = e^{-|u|}\left(1 + |\nabla u|^{p-1}\right). \tag{76}$$

This source decreases exponentially with temperature amplitude while exhibiting sensitivity to local temperature gradients through the term $|\nabla u|^{p-1}$.
Musielak-Orlicz Functions:

$$\psi(x,|\xi|) = |\xi|^{rq(x)}, \quad \phi(x,|\xi|) = |\xi|^{q(x)}, \quad \text{where } q(x) = 1 + \frac{1}{\log(|1+x|)}, \quad r > 1. \tag{77}$$

These functions govern the growth and regularity properties within the functional framework, with the relationship $\phi \ll \psi$ ensuring proper scaling behavior.

5.2. Verification of the Musielak-Orlicz Conditions

We now establish that our choice of functions satisfies the required structural conditions.
Verification of the domination condition $\phi \ll \psi$:
To establish $\phi \ll \psi$, we must show that $\lim_{t\to\infty} \frac{\phi(x,t)}{\psi(x,t)} = 0$. Computing the ratio:

$$\frac{\phi(x,t)}{\psi(x,t)} = \frac{|t|^{q(x)}}{|t|^{rq(x)}} = |t|^{q(x)(1-r)}. \tag{78}$$

Since $r > 1$, we have $1 - r < 0$, which immediately gives:

$$\lim_{t\to\infty} |t|^{q(x)(1-r)} = 0. \tag{79}$$

Thus, the domination condition $\phi \ll \psi$ is satisfied.

5.3. Verification of Growth Conditions

For the N-function $\phi(x,s) = |s|^{q(x)}$, the complementary function is:

$$\bar{\phi}(x,t) = \frac{|t|^{q'(x)}}{q'(x)}, \tag{80}$$

where $q'(x)$ is the conjugate exponent of $q(x)$, given by:

$$q'(x) = \frac{q(x)}{q(x)-1} = \frac{1 + \frac{1}{\log(|1+x|)}}{\frac{1}{\log(|1+x|)}} = \log(|1+x|) + 1. \tag{81}$$

For $\psi(x,s) = |s|^{rq(x)}$, the complementary function is:

$$\bar{\psi}(x,t) = \frac{|t|^{\frac{rq(x)}{rq(x)-1}}}{\frac{rq(x)}{rq(x)-1}}. \tag{82}$$

Verification of condition (11)**:**
We analyze the asymptotic behavior of $\frac{\bar{\psi}(x,s)}{|s|}$:

$$\frac{\bar{\psi}(x,s)}{|s|} \sim \frac{|s|^{\frac{rq(x)}{rq(x)-1}-1}}{\frac{rq(x)}{rq(x)-1}} = \frac{|s|^{\frac{1}{rq(x)-1}}}{\frac{rq(x)}{rq(x)-1}}. \tag{83}$$

Since $q(x) \geq 1 + \frac{1}{\log(2)} > 1$ and $r > 1$, we have $rq(x) > 1$, ensuring $\frac{1}{rq(x)-1} > 0$. Consequently:

$$\lim_{|s|\to\infty} \frac{\bar{\psi}(x,s)}{|s|} = \infty. \tag{84}$$

This verifies condition (11).
Verification of condition (12)**:**
Similarly, for the ϕ-function:

$$\frac{\bar{\phi}(x,s)}{|s|} = \frac{|s|^{q'(x)-1}}{q'(x)} = \frac{|s|^{\log(|1+x|)}}{\log(|1+x|)+1}. \tag{85}$$

Since $\log(|1+x|) \geq 0$ for $x \in \Omega$, we obtain:

$$\lim_{|s|\to\infty} \frac{\bar{\phi}(x,s)}{|s|} = \infty. \tag{86}$$

This establishes condition (12).

5.4. Verification of Regularity Conditions

Verification of condition (15)**:**
For the log-Hölder continuity, we examine:

$$\frac{\phi(x,\ell)}{\phi(y,\ell)} = \frac{\ell^{q(x)}}{\ell^{q(y)}} = \ell^{q(x)-q(y)}. \tag{87}$$

The key estimate involves bounding $|q(x) - q(y)|$:

$$|q(x) - q(y)| = \left| \frac{1}{\log(|1+x|)} - \frac{1}{\log(|1+y|)} \right| \tag{88}$$

$$= \left| \frac{\log(|1+y|) - \log(|1+x|)}{\log(|1+x|)\log(|1+y|)} \right| \tag{89}$$

$$= \left| \frac{\log\left(\frac{|1+y|}{|1+x|}\right)}{\log(|1+x|)\log(|1+y|)} \right|. \tag{90}$$

Using the mean value theorem and the constraint $|x-y| \leq \frac{1}{2}$, there exists a constant $\lambda > 0$ such that:

$$|q(x) - q(y)| \leq \frac{\lambda}{|\log(|x-y|)|}. \tag{91}$$

Therefore:

$$\ell^{q(x)-q(y)} \leq \ell^{-\frac{\lambda}{\log(|x-y|)}}, \tag{92}$$

which verifies condition (15) for $1 < r < 2$.

Verification of condition (16)**:**

We compute:

$$\bar{\psi}(x,1) = \frac{1^{\frac{rq(x)}{rq(x)-1}}}{\frac{rq(x)}{rq(x)-1}} = \frac{rq(x)-1}{rq(x)} = 1 - \frac{1}{rq(x)}. \tag{93}$$

Since $q(x) \geq 1 + \frac{1}{\log(|\Omega|+1)}$ is bounded below on Ω and $r > 1$:

$$0 < 1 - \frac{1}{rq(x)} < 1. \tag{94}$$

Thus $\bar{\psi}(x,1) \leq \beta$ with $\beta = 1$, confirming condition (16).

5.5. *Verification of Technical Conditions*

Parameter Selection: We choose the following parameters to ensure all technical conditions are satisfied:

$$\alpha = \frac{1}{\max(q(x), 2/r)}, \tag{95}$$

$$k = \frac{1}{\|\nabla u\|_\infty}, \tag{96}$$

$$c(x,s) = \bar{\psi}^{-1}(x,s), \tag{97}$$

$$\zeta > 0 \text{ sufficiently large.} \tag{98}$$

Verification of condition (26)**:**

The gradient bound condition $|\nabla u| \leq \zeta[c(x,s) + \bar{\psi}_x^{-1}(\psi(x,k|u|))]$ is satisfied by construction. With $k = \frac{1}{\|\nabla u\|_\infty}$:

$$\bar{\psi}_x^{-1}(\psi(x,k|u|)) = \bar{\psi}_x^{-1}\left(\frac{|u|^{rq(x)}}{\|\nabla u\|_\infty^{rq(x)}}\right). \tag{99}$$

Choosing ζ sufficiently large ensures condition (26).

Verification of condition (27)**:**

With our parameter choice $\alpha = \frac{1}{\max(q(x),2/r)}$ and $\psi(x, |\nabla u - \nabla v|) = |\nabla u - \nabla v|^{rq(x)}$:

$$\alpha\psi(x, |\nabla u - \nabla v|) = \frac{|\nabla u - \nabla v|^{rq(x)}}{\max(q(x), 2/r)}. \tag{100}$$

For $1 < r < 2$, we distinguish two cases:

- If $q(x) \geq \frac{2}{r}$, then $\max(q(x), 2/r) = q(x)$ and $rq(x) \geq 2$. By Young's inequality:

$$\frac{|\nabla u - \nabla v|^{rq(x)}}{q(x)} \leq |\nabla u - \nabla v|^2. \tag{101}$$

- If $q(x) < \frac{2}{r}$, then $\max(q(x), 2/r) = \frac{2}{r}$ and $rq(x) < 2$. For sufficiently large $|\nabla u - \nabla v|$:

$$\frac{|\nabla u - \nabla v|^{rq(x)}}{2/r} = \frac{r}{2}|\nabla u - \nabla v|^{rq(x)} \leq |\nabla u - \nabla v|^2. \tag{102}$$

This establishes condition (27).

5.6. *Verification of Source Term Conditions*

For the thermal source $f(x, t, u, \nabla u) = e^{-|u|}(1 + |\nabla u|^{p-1})$, we verify the required growth and sign conditions.

Growth Control (Condition (34)):

The source term satisfies controlled growth since:

$$|f(x, t, u, \nabla u)| = e^{-|u|}(1 + |\nabla u|^{p-1}) \tag{103}$$
$$\leq 1 + |\nabla u|^{p-1} \tag{104}$$
$$\leq 1 + |\nabla u|^p, \tag{105}$$

where we used $e^{-|u|} \leq 1$ and the inequality $|\nabla u|^{p-1} \leq 1 + |\nabla u|^p$ for $p > 1$. This verifies the controlled growth condition (34).

Sign Condition (Condition (35)):

To ensure the proper sign behavior, we consider the modified source term:

$$f(x, t, u, \nabla u) = e^{-|u|}\,\mathrm{sign}(u)(1 + |\nabla u|^{p-1}), \tag{106}$$

where $\mathrm{sign}(u) = \frac{u}{|u|}$ for $u \neq 0$ and $\mathrm{sign}(0) = 0$.

With this modification:

$$f(x, t, u, \nabla u) \cdot u = e^{-|u|}|u|(1 + |\nabla u|^{p-1}) \geq 0, \tag{107}$$

satisfying the sign condition (35).

All necessary conditions for applying the existence and uniqueness results are verified with the parameter choices:

- $1 < r < 2$ (ensuring proper power-law behavior),
- $\alpha = \frac{1}{\max(q(x),2/r)}$ (balancing fractional and spatial orders),
- $k = \frac{1}{\|\nabla u\|_\infty}$ (normalizing gradient bounds),
- $\zeta > 0$ sufficiently large (accommodating technical estimates),
- $c(x, s) = \bar{\psi}^{-1}(x, s)$ (inverse relationship for complementary functions).

Consequently, the thermal propagation problem with memory effects in a nonlinear diffusion medium admits a weak solution in the appropriate fractional Musielak-Orlicz space. This solution captures both the memory-dependent temporal evolution through the fractional derivative and the complex spatial behavior through the nonlinear diffusion and source mechanisms.

Conclusion

In this work, we have demonstrated the existence of capacitary solutions for a nonlinear fractional differential equation within the framework of fractional Musielak-Orlicz-Sobolev spaces. By employing approximation methods, we were able to establish the existence of weak solutions through the convergence of a sequence of approximated problems to the original problem in terms of capacities. Moreover, we have illustrated the practical relevance of our results through a concrete application, showcasing how our theoretical findings can be applied to real-world problems involving nonlocal and nonlinear phenomena with memory.

The analysis presented here offers a robust framework for tackling complex nonlinear problems with memory in fractional spaces, expanding the understanding of such equations in a variety of applied fields, including thermal propagation, diffusion processes, and other phenomena governed by nonlocal interactions. The results contribute significantly to the study of nonlocal and nonlinear equations, providing both theoretical insights and practical tools for future research in this area.

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