Filomat 39:28 (2025), 9865–9946
https://doi.org/10.2298/FIL2528865S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On preconditioned AOR method for solving linear systems

Yongzhong Song[a]

[a]*Ministry of Education Key Laboratory of NSLSCS, School of Mathematical Sciences, Nanjing Normal University, Nanjing 210023, People's Republic of China*

Abstract. In this paper, we investigate the preconditioned AOR method for solving linear systems. We study two general preconditioners and propose some lower triangular, upper triangular and combination preconditioners. For A being an L-matrix, a nonsingular M-matrix, an irreducible L-matrix and an irreducible nonsingular M-matrix, four types of comparison theorems are presented, respectively. They contain a general comparison result, a strict comparison result and two Stein-Rosenberg type comparison results. Our theorems include and are better than almost all known corresponding results.

1. Introduction

Consider a system of n equations

$$Ax = b, \tag{1}$$

where $A = (a_{i,j}) \in \mathscr{R}^{n\times n}$, b, $x \in \mathscr{R}^n$ with b known and x unknown. In order to solve the system (1) with iterative methods, the coefficient matrix A is split into

$$A = M - N, \tag{2}$$

where M is nonsingular and $N \neq 0$. Then a linear stationary iterative method for solving (1) can be described as

$$x^{k+1} = Tx^k + M^{-1}b, \quad k = 0, 1, 2, \cdots, \tag{3}$$

where $T = M^{-1}N$ is the iteration matrix.

We decompose A into

$$A = D - L - U,$$

where D is a diagonal matrix, L and U are strictly lower and upper triangular matrices, respectively, as usual.

2020 *Mathematics Subject Classification*. Primary 65F10; Secondary 65F08.
Keywords. Linear system, preconditioner, preconditioned AOR method, comparison, M-matrix, L-matrix.
Received: 22 April 2021; Accepted: 26 July 2025
Communicated by Dragan S. Djordjević
Research supported by the National Natural Science Foundation of China (Grant Nos. 11771213 and 11971242), the Priority Academic Program Development of Jiangsu Higher Education Institutions.
Email address: yzsong@njnu.edu.cn (Yongzhong Song)
ORCID iD: https://orcid.org/0000-0002-8204-7616 (Yongzhong Song)

For $\omega \in \mathscr{R} \setminus \{0\}$ and $\gamma \in \mathscr{R}$, let

$$A = M_{\gamma,\omega} - N_{\gamma,\omega}, \tag{4}$$

where

$$M_{\gamma,\omega} = \frac{1}{\omega}(D - \gamma L),\ N_{\gamma,\omega} = \frac{1}{\omega}\left[(1-\omega)D + (\omega - \gamma)L + \omega U\right].$$

Then the AOR method for solving (1) is defined in [16] by

$$x^{k+1} = \mathscr{L}_{\gamma,\omega}x^k + \omega(D - \gamma L)^{-1}b, \quad k = 0, 1, 2, \dots,$$

where

$$\mathscr{L}_{\gamma,\omega} = (D - \gamma L)^{-1}[(1-\omega)D + (\omega - \gamma)L + \omega U]$$

is the AOR iteration matrix. The splitting (4) is also called the AOR splitting of A.

When (γ, ω) is equal to (ω, ω), (1,1) and (0,1), the AOR method reduces respectively to the SOR method, Gauss-Seidel method and Jacobi method, whose iteration matrices are represented by $\mathscr{L}_\omega$, $\mathscr{L}$ and $\mathscr{J}$.

In [16] it is pointed out that, for $\gamma \neq 0$, the AOR method is an extrapolated SOR (ESOR) method with overrelaxation parameter γ and extrapolation one ω/γ, i.e.,

$$\mathscr{L}_{\gamma,\omega} = \left(1 - \frac{\omega}{\gamma}\right)I + \frac{\omega}{\gamma}\mathscr{L}_\gamma,$$

and, hence, if η is an eigenvalue of $\mathscr{L}_\gamma$ and λ, the corresponding one of $\mathscr{L}_{\gamma,\omega}$, then we have

$$\lambda = 1 - \frac{\omega}{\gamma} + \frac{\omega}{\gamma}\eta. \tag{5}$$

It is well known that, if A is nonsingular, then the iterative method (3) is convergent if and only if the spectral radius $\rho(T)$ of the iteration matrix T is less than 1. In this case, the matrix T is also called convergent. However, if A is singular, then we have $\rho(T) \geq 1$, so that we can only require the semiconvergence of the splitting. When the iterative method (semi)converges, the convergence speed is determined by $\rho(T)$, and the smaller it is, the faster the iterative method converges. Therefore $\rho(T)$ is called convergence factor.

In order to decrease the spectral radius of the iteration matrix, an effective method is to precondition the linear system (1). It is well known that the term preconditioning refers to transforming the system (1) into another system with more favorable properties for iterative methods.

If P is a nonsingular matrix, then the preconditioned linear system

$$PAx = Pb$$

has the same solution as (1). Here P is called the preconditioner.

Generally speaking, preconditioning attempts to improve the spectral properties of the coefficient matrix. A good preconditioner P should meet the following requirements:

- The preconditioned system should have more favorable properties for iterative methods, in particular, the iterative methods can be convergent more faster.
- The preconditioner should be cheap to construct.

To choose a good preconditioner P is an interesting problem, which has been investigated widely. In a large number of papers, in particular for the AOR method, a special preconditioner P is proposed by

$$P = I + Q,\ Q > 0.$$

Then we define a matrix splitting as

$$PA = \hat{M} - \hat{N}.$$

A preconditioned iterative method can be defined by

$$x^{k+1} = \hat{T}x^k + \hat{M}^{-1}b, \quad k = 0, 1, 2, \cdots ,$$

where $\hat{T} = \hat{M}^{-1}\hat{N}$ is the iteration matrix.

When A is an L-matrix, a nonsingular M-matrix, an irreducible L-matrix or an irreducible nonsingular M-matrix, the preconditioned AOR, SOR, Gauss-Seidel and Jacobi methods are constructed, generalized and applied by [1, 2, 7–9, 12, 14, 15, 17–20, 22–24, 26, 29–31, 34–47, 50–56, 58–63, 66–72, 74–76, 79, 82–85, 90–94, 96–105, 107–111, 113, 114].

In this paper, we investigate the preconditioned AOR method for solving linear systems. We study two general preconditioners and propose some lower triangular, upper triangular and combination preconditioners. For A being an L-matrix, a nonsingular M-matrix, an irreducible L-matrix and an irreducible nonsingular M-matrix, four types of comparison theorems are presented, respectively. They contain a general comparison result, a strict comparison result and two Stein-Rosenberg type comparison results. Our theorems include and are better than almost all known corresponding results. Some incorrect known results are pointed out.

This paper is organized as follows. In Section 2 we give some concepts and results, which will applied in the next section. In Section 3, we study two general preconditioners, some lower triangular, upper triangular and combination preconditioners for the preconditioned AOR method. Four types of comparison results are proved. In Section 4, we give some explanations and prospects.

2. Some concepts and lemmas

For convenience we recall and give some concepts and lemmas as follows.

A matrix $B \in \mathscr{R}^{n\times m}$ is called nonnegative, semi-positive, positive if each element of B is nonnegative, nonnegative but at least a positive element, positive, which is denoted by $B \geq 0$, $B > 0$ and $B \gg 0$, respectively. When $B_1 - B_2 \geq (>, \gg)0$, we denote $B_1 \geq (>, \gg)B_2$ or $B_2 \leq (<, \ll)B_1$. Similarly, for $y \in \mathscr{R}^n$, by identifying it with $n \times 1$ matrix, we can also define $y \geq (\leq)0$, $y > (<)0$ and $y \gg (\ll)0$. $B \in \mathscr{R}^{n\times n}$ is called monotone, if B is invertible and $B^{-1} > 0$. $B = (b_{i,j}) \in \mathscr{R}^{n\times n}$ is called a Z-matrix if $b_{i,j} \leq 0$ for $i, j = 1, \cdots , n$, $i \neq j$; an L-matrix if it is a Z-matrix with $a_{i,i} > 0$, $i = 1, \cdots , n$; a nonsingular M-matrix if it is a Z-matrix and is monotone. It is well known that a nonsingular M-matrix is an L-matrix.

A matrix $B \in \mathscr{R}^{n\times n}$ is called reducible if there is a permutation matrix V such that

$$VBV^T = \begin{bmatrix} B_{1,1} & B_{1,2} \\ 0 & B_{2,2} \end{bmatrix},$$

where $B_{1,1} \in \mathscr{R}^{r\times r}$, $B_{2,2} \in \mathscr{R}^{(n-r)\times(n-r)}$ with $1 \leq r \leq n-1$. Otherwise, B is irreducible. The directed graph of a matrix $B = (b_{i,j}) \in \mathscr{R}^{n\times n}$ is denoted by $G(B)$. A path in $G(B)$ which leads from the vertex V_i to the vertex V_j is denoted by $\sigma_{i,j}$, i.e., $\sigma_{i,j} = (j_0, j_1, \cdots , j_{l+1})$ with $i = j_0$, $j = j_{l+1}$, $l \geq 0$ and $b_{j_k j_{k+1}} \neq 0$, $k = 0, \cdots , l$. It is well known that a matrix B is irreducible if and only if $G(B)$ is strongly connected, which means that for any $i, j \in \{1, \cdots , n\}$ there exists a path $\sigma_{i,j} \in G(B)$.

Definition 2.1. *The decomposition (2) is called a splitting of A if M is nonsingular. A splitting is called:*

(i) Regular if $M^{-1} \geq 0$ and $N \geq 0$ (cf. [86, Definition 3.28]);

(ii) Weak regular if $M^{-1} \geq 0$ and $M^{-1}N \geq 0$ (cf. [86, Definition 3.28]);

(iii) Nonnegative if $M^{-1}N \geq 0$ (cf. [80, Definition 1.1]).

Lemma 2.2. *[86, Theorems 2.7 and 2.20]*

(i) Let $B \geq 0$. Then B has a nonnegative eigenvalue equal to $\rho(B)$, and there corresponds an eigenvector $x > 0$.

(ii) Let $B \geq 0$ be irreducible. Then B has a positive eigenvalue equal to $\rho(B)$, and there corresponds an eigenvector $x \gg 0$.

Lemma 2.3. *[4, Theorem 2-1.11] Let $B \geq 0$.*

(i) If $Bx \geq \alpha x$ with $x > 0$, then $\rho(B) \geq \alpha$.

(ii) If $Bx \leq \beta x$ with $x \gg 0$, then $\rho(B) \leq \beta$.

(iii) If B is irreducible and $Bx > \alpha x$ with $x > 0$, then $\rho(B) > \alpha$.

(iv) If B is irreducible and $Bx < \beta x$ with $x > 0$, then $\rho(B) < \beta$ and $x \gg 0$.

Here we have made a minor modification to [4, Theorem 2-1.11]. In fact, for (*iii*), $x \gg 0$ cannot be derived.

Lemma 2.4. *Let $B \geq 0$ and $x > 0$.*

(i) If $Bx \gg \alpha x$ then $\rho(B) > \alpha$.

(ii) If $Bx \ll \beta x$ then $\rho(B) < \beta$ and $x \gg 0$.

Proof. Since $B \geq 0$, then there exists $y > 0$ such that $B^T y = \rho(B)y$. Multiply y^T on the left side of $Bx \gg \alpha x$ or $Bx \ll \beta x$ respectively, we can obtain $\rho(B)y^T x > \alpha y^T x$ or $\rho(B)y^T x < \beta y^T x$, which derives $\rho(B) > \alpha$ or $\rho(B) < \beta$ directly. When $Bx \ll \beta x$, $x \gg 0$ is obvious. □

Lemma 2.5. *[4, Theorem 6-2.7] Let B be an irreducible Z-matrix. Then B is a nonsingular M-matrix if and only if $B^{-1} \gg 0$.*

Lemma 2.6. *[4, Theorem 6-2.3] Let B be a Z-matrix. Then the following statements are equivalent:*

(i) B is a nonsingular M-matrix.

(ii) There is a vector $x \gg 0$ such that $Bx \gg 0$.

(iii) The weak regular splitting of B is convergent.

Lemma 2.7. *[86, Theorem 3.37] Any weak regular splitting of B is convergent if and only if B is nonsingular with $B^{-1} > 0$.*

Lemma 2.8. *[86, Exerxise 3.3-6] Let A be an irreducible L-matrix. Then $\rho(\mathscr{L}) > 0$ and it has associated eigenvector $x \gg 0$.*

Lemma 2.9. *Let A be an irreducible L-matrix, and let $0 \leq \gamma \leq \omega \leq 1$ and $\omega > 0$.*

(i) Then $\rho(\mathscr{L}_{\gamma,\omega}) > 0$ holds.

(ii) Assume that $x > 0$ satisfies $\mathscr{L}_{\gamma,\omega} x = \rho(\mathscr{L}_{\gamma,\omega})x$. Then $x \gg 0$.

(iii) Assume that $y > 0$ satisfies $y^T \mathscr{L}_{\gamma,\omega} = \rho(\mathscr{L}_{\gamma,\omega})y^T$. Then $y^T(D - \gamma L)^{-1} \gg 0$.

Proof. Denote $\rho = \rho(\mathscr{L}_{\gamma,\omega})$.

Since A is an irreducible L-matrix, then $(D - \gamma L)^{-1} \geq 0$ and

$$\mathscr{L}_{\gamma,\omega} = (D - \gamma L)^{-1}[(1 - \omega)D + (\omega - \gamma)L + \omega U] = (1 - \omega)I + \omega(D - \gamma L)^{-1}[(1 - \gamma)L + U] \geq 0,$$

so that there exists $u > 0$ such that $\mathscr{L}_{\gamma,\omega} u = \rho u$. Clearly, $\rho = 0$ if and only if $N_{\gamma,\omega} u = 0$. Because A is irreducible L-matrix, it gets that $N_{\gamma,\omega} \geq U > 0$.

When $\omega < 1$, we have $N_{\gamma,\omega}u \geq (1-\omega)/\omega u > 0$. When $\omega = 1$ and $\gamma < 1$, we have that $N_{\gamma,\omega} = (1-\gamma)L + U$ is irreducible so that $N_{\gamma,\omega}u > 0$. Hence, for these two cases we obtain $\rho > 0$. When $\omega = \gamma = 1$, by Lemma 2.8 we have also $\rho > 0$. We have proved (*i*).

If $\mathscr{L}_{\gamma,\omega}x = \rho x$, then we obtain

$$[(\omega+\rho-1)D - (\omega-\gamma+\gamma\rho)L - \omega U]x = 0. \tag{6}$$

Let $\hat{A} = (\omega+\rho-1)D - (\omega-\gamma+\gamma\rho)L - \omega U$. Since $\omega > 0$ and $\omega-\gamma+\gamma\rho > 0$, then $\hat{A}$ is an irreducible Z-matrix. If x has some zero elements, without loss of generality, then we can assume that

$$x = \begin{pmatrix} 0 \\ \hat{x} \end{pmatrix},\ \hat{x} \gg 0 \in \mathscr{R}^r,\ 1 \leq r \leq n-1.$$

We divide $\hat{A}$ accordingly as

$$\hat{A} = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix},\ A_{2,2} \in \mathscr{R}^{r\times r}.$$

Then $A_{1,2} \leq 0$. From (6) we derive $A_{1,2}\hat{x} = 0$ so that $A_{1,2} = 0$, which implies that $\hat{A}$ is reducible. This is a contradiction. Hence, x has no zero elements, i.e., $x \gg 0$. This has proved (*ii*).

When $y^T\mathscr{L}_{\gamma,\omega} = \rho y^T$, let $z^T = y^T(D-\gamma L)^{-1}$. Then $z > 0$. Further, we have $z^T[(1-\omega)D + (\omega-\gamma)L + \omega U] = \rho z^T(D-\gamma L)$, so that $[(\omega+\rho-1)D - (\omega-\gamma+\gamma\rho)L^T - \omega U^T]z = 0$. Since A^T ia also an irreducible L-matrix, then similar to the proof of (*ii*) we can prove $z \gg 0$. Then (*iii*) is proved. □

Lemma 2.10. *Let $P > 0$ be nonsingular, and let the splitting*

$$PA = \hat{M} - \hat{N} \tag{7}$$

be weak regular. Then the following three statements are equivalent:

(i) $A^{-1} > 0$.

(ii) $(PA)^{-1} > 0$.

(iii) The splitting (7) is convergent.

Proof. By Lemma 2.7, (*ii*) and (*iii*) are equivalent, immediately.

The splitting (7) can be rewritten into $A = P^{-1}\hat{M} - P^{-1}\hat{N}$. Clearly, this splitting is weak regular and $(P^{-1}\hat{M})^{-1}(P^{-1}\hat{N}) = \hat{M}^{-1}\hat{N}$. The equivalence between (*i*) and (*iii*) follows directly by Lemma 2.7 again. □

From this lemma, the following lemma is obvious.

Lemma 2.11. *Let A and PA be Z-matrices, where $P > 0$ is nonsingular. Then A is a nonsingular M-matrix if and only if PA is a nonsingular M-matrix.*

We prove two Stein-Rosenberg type comparison theorems.

Lemma 2.12. *Let the splittings $A = M_1 - N_1 = M_2 - N_2$ be respectively weak regular and nonnegative, and let $x \gg 0$, $y > 0$ satisfy $M_2^{-1}N_2x = \rho(M_2^{-1}N_2)x$, $y^TM_1^{-1}N_1 = \rho(M_1^{-1}N_1)y^T$. Suppose that one of the following two conditions is satisfied:*

(i) $M_1^{-1}(N_2 - N_1)x \gg 0$.

(ii) $y^TM_1^{-1} \gg 0$ *and* $(N_2 - N_1)x > 0$.

Then one of the following mutually exclusive relations holds:

(a) $\rho(M_1^{-1}N_1) < \rho(M_2^{-1}N_2) < 1$.

(b) $\rho(M_1^{-1}N_1) = \rho(M_2^{-1}N_2) = 1$.

(c) $\rho(M_1^{-1}N_1) > \rho(M_2^{-1}N_2) > 1$.

Proof. Since $Ax = (M_2 - N_2)x = [1 - \rho(M_2^{-1}N_2)]M_2x$, then by simple operation we have

$$\begin{aligned} M_1^{-1}N_1x - \rho(M_2^{-1}N_2)x &= (M_1^{-1}N_1 - M_2^{-1}N_2)x = (M_2^{-1}A - M_1^{-1}A)x = M_1^{-1}(M_1 - M_2)M_2^{-1}Ax \\ &= [\rho(M_2^{-1}N_2) - 1]M_1^{-1}(M_2 - M_1)x = [\rho(M_2^{-1}N_2) - 1]M_1^{-1}(N_2 - N_1)x \end{aligned} \tag{8}$$

and, therefore,

$$[\rho(M_1^{-1}N_1) - \rho(M_2^{-1}N_2)]y^Tx = [\rho(M_2^{-1}N_2) - 1]y^TM_1^{-1}(N_2 - N_1)x.$$

When one the conditions (*i*) and (*ii*) is satisfied, it derives that $y^TM_1^{-1}(N_2 - N_1)x > 0$. Since $y^Tx > 0$, then we obtain

$$\rho(M_1^{-1}N_1) - \rho(M_2^{-1}N_2)\begin{cases} < \; 0, & if \quad \rho(M_2^{-1}N_2) < 1, \\ = \; 0, & if \quad \rho(M_2^{-1}N_2) = 1, \\ > \; 0, & if \quad \rho(M_2^{-1}N_2) > 1. \end{cases}$$

The proof is completed. □

By the definition of the AOR method, when $\omega = \gamma = 1$ in (4) we derive the Gauss-Seidel method, whose iteration matrix is denoted by $\mathscr{L}$. Now, let

$$PA = \tilde{D} - \tilde{L} - \tilde{U} = \tilde{M}_{1,1} - \tilde{N}_{1,1},$$

where $\tilde{D}$ is a diagonal matrix, $\tilde{L}$ and $\tilde{U}$ are strictly lower and upper triangular matrices respectively and $\tilde{M}_{1,1} = \tilde{D} - \tilde{L}$, $\tilde{N}_{1,1} = \tilde{U}$. Then the preconditioned Gauss-Seidel iteration matrix can be defined as $\tilde{\mathscr{L}} = \tilde{M}_{1,1}^{-1}\tilde{N}_{1,1} = (\tilde{D} - \tilde{L})^{-1}\tilde{U}$.

Lemma 2.13. *Let A and PA be L-matrices, and let $\mathscr{L}x = \rho(\mathscr{L})x$ with $x \gg 0$. Suppose that the second to nth elements of $\tilde{M}_{1,1}^{-1}(N_{1,1} - \tilde{N}_{1,1})x$ are positive. Then one of the following mutually exclusive relations holds:*

(a) $\rho(\tilde{\mathscr{L}}) < \rho(\mathscr{L}) < 1$.

(b) $\rho(\tilde{\mathscr{L}}) = \rho(\mathscr{L}) = 1$.

(c) $\rho(\tilde{\mathscr{L}}) > \rho(\mathscr{L}) > 1$.

Proof. Consider the splittings

$$PA = \tilde{M}_{1,1} - \tilde{N}_{1,1} = PM_{1,1} - PN_{1,1}.$$

Clearly, they are regular and nonnegative respectively.

Since the first column $\tilde{\mathscr{L}}$ is a zero vector, then it can be decomposed as

$$\tilde{\mathscr{L}} = \begin{pmatrix} 0 & \psi_{1,2} \\ 0 & \Psi_{2,2} \end{pmatrix}, \quad \Psi_{2,2} \geq 0 \in \mathscr{R}^{(n-1)\times(n-1)},$$

so that $\rho(\tilde{\mathscr{L}}) = \rho(\Psi_{2,2})$.

Correspondingly, we decompose x and $\tilde{M}_{1,1}^{-1}(N_{1,1} - \tilde{N}_{1,1})x$ as

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \tilde{M}_{1,1}^{-1}(N_{1,1} - \tilde{N}_{1,1})x = \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix}, \quad x_2, \tilde{x}_2 \in \mathscr{R}^{(n-1)}.$$

Then $x_2 \gg 0$ and $\tilde{x}_2 \gg 0$.

Similar to the proof of Lemma 2.12, by (8) we can obtain that

$$\begin{pmatrix} 0 & \psi_{1,2} \\ 0 & \Psi_{2,2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \rho(\mathscr{L})\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = [\rho(\mathscr{L})-1]\tilde{M}_{1,1}^{-1}(N_{1,1}-\tilde{N}_{1,1})x = [\rho(\mathscr{L})-1]\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix},$$

so that

$$\Psi_{2,2}x_2 - \rho(\mathscr{L})x_2 = [\rho(\mathscr{L})-1]\tilde{x}_2.$$

Since $\Psi_{2,2} \geq 0$, then there exists $z > 0$ such that $z^T\Psi_{2,2} = \rho(\Psi_{2,2})z^T$. Hence, we have

$$[\rho(\Psi_{2,2}) - \rho(\mathscr{L})]z^T x_2 = z^T\Psi_{2,2}x_2 - \rho(\mathscr{L})z^T x_2 = [\rho(\mathscr{L})-1]z^T\tilde{x}_2.$$

Because of $x_2 \gg 0$, $\tilde{x}_2 \gg 0$ and $z > 0$, we drives $z^T x_2 > 0$ and $z^T\tilde{x}_2 > 0$, so that

$$\rho(\Psi_{2,2}) - \rho(\mathscr{L})\begin{cases} < & 0, \quad if \quad \rho(\mathscr{L}) < 1, \\ = & 0, \quad if \quad \rho(\mathscr{L}) = 1, \\ > & 0, \quad if \quad \rho(\mathscr{L}) > 1. \end{cases}$$

The proof is completed. □

Similar to [81, Theorem 3.4], we prove a strictly comparison result.

Lemma 2.14. *Let the both splittings $A_1 = M_1 - N_1$ and $A_2 = M_2 - N_2$ be nonnegative and convergent with $A_2^{-1} \gg 0$ and $A_2^{-1} \geq A_1^{-1}$. Suppose that there exists $x > 0$ such that $M_2^{-1}N_2x = \rho(M_2^{-1}N_2)x$ and $M_2x > M_1x \geq 0$. Then $\rho(M_1^{-1}N_1) < \rho(M_2^{-1}N_2)$.*

Proof. We have that

$$A_2^{-1}M_2x \gg A_2^{-1}M_1x \geq A_1^{-1}M_1x,$$

so that

$$\frac{1}{1-\rho(M_2^{-1}N_2)}x = A_2^{-1}M_2x \gg (I - M_1^{-1}N_1)^{-1}x.$$

Clearly, $(I - M_1^{-1}N_1)^{-1} \geq 0$. It follows by Lemma 2.4 that

$$\frac{1}{1-\rho(M_2^{-1}N_2)} > \rho((I - M_1^{-1}N_1)^{-1}) = \frac{1}{1-\rho(M_1^{-1}N_1)}.$$

The required result can be derived. □

3. Preconditioned AOR method and comparison results

In this section, without loss of generality, suppose that all of the diagonal elements of A are 1. In this case, A is an L-matrix if and only if A is a Z-matrix.

For convenience, if the matrix Q is chosen as Q_ν, then we write $P_\nu = I + Q_\nu$, $Q_\nu = (q_{i,j}^{(\nu)})$ and $A^{(\nu)} = P_\nu A = (a_{i,j}^{(\nu)})$. Let

$$A^{(\nu)} = D_\nu - L_\nu - U_\nu = M_{\gamma,\omega}^{(\nu)} - N_{\gamma,\omega}^{(\nu)} \tag{9}$$

with

$$M_{\gamma,\omega}^{(\nu)} = \frac{1}{\omega}(D_\nu - \gamma L_\nu),\ N_{\gamma,\omega}^{(\nu)} = \frac{1}{\omega}\left[(1-\omega)D_\nu + (\omega-\gamma)L_\nu + \omega U_\nu\right],$$

where $D_\nu = diag(A^{(\nu)})$ is a diagonal matrix, L_ν and U_ν are strictly lower and upper triangular matrices respectively. Then the corresponding preconditioned AOR method for solving (1) can be defined as

$$x^{k+1} = \mathscr{L}_{\gamma,\omega}^{(\nu)}x^k + \omega(D_\nu - \gamma L_\nu)^{-1}P_\nu b, \quad k = 0, 1, 2, \ldots,$$

where

$$\mathscr{L}_{\gamma,\omega}^{(\nu)} = (D_\nu - \gamma L_\nu)^{-1}[(1-\omega)D_\nu + (\omega-\gamma)L_\nu + \omega U_\nu]$$

is the preconditioned AOR iteration matrix.

We will propose four types of comparison theorems. They contain a general comparison result, a strict comparison result and two Stein-Rosenberg type comparison results.

We first give them as follows.

Theorem A (Stein-Rosenberg Type Theorem I)

Let A be an L-matrix. Then one of the following mutually exclusive relations is valid:

(i) $\rho(\mathscr{L}_{\gamma,\omega}^{(\nu)}) \le \rho(\mathscr{L}_{\gamma,\omega}) < 1$.

(ii) $\rho(\mathscr{L}_{\gamma,\omega}^{(\nu)}) = \rho(\mathscr{L}_{\gamma,\omega}) = 1$.

(iii) $\rho(\mathscr{L}_{\gamma,\omega}^{(\nu)}) \ge \rho(\mathscr{L}_{\gamma,\omega}) > 1$.

Theorem B (General Comparison Theorem)

Let A be a nonsingular M-matrix. Then

$$\rho(\mathscr{L}_{\gamma,\omega}^{(\nu)}) \le \rho(\mathscr{L}_{\gamma,\omega}) < 1.$$

Theorem C (Stein-Rosenberg Type Theorem II)

Let A be an irreducible L-matrix. Then one of the following mutually exclusive relations is valid:

(i) $\rho(\mathscr{L}_{\gamma,\omega}^{(\nu)}) < \rho(\mathscr{L}_{\gamma,\omega}) < 1$.

(ii) $\rho(\mathscr{L}_{\gamma,\omega}^{(\nu)}) = \rho(\mathscr{L}_{\gamma,\omega}) = 1$.

(iii) $\rho(\mathscr{L}_{\gamma,\omega}^{(\nu)}) > \rho(\mathscr{L}_{\gamma,\omega}) > 1$.

Theorem D (Strict Comparison Theorem)

Let A be an irreducible nonsingular M-matrix. Then

$$\rho(\mathscr{L}_{\gamma,\omega}^{(\nu)}) < \rho(\mathscr{L}_{\gamma,\omega}) < 1.$$

Lemma 3.1. *Suppose that* $\mathscr{L}_{\gamma,\omega} \ge 0$, $\mathscr{L}_{\gamma,\omega}^{(\nu)} \ge 0$. *Assume that one of Theorems A, B, C and D is valid for* $0 \le \gamma \le \omega \le 1$, $\omega > 0$. *Then it is valid for* $0 < \omega \le 1$ *and* $0 \le \gamma \le 1$.

Proof. Assume that Theorem C is valid for $0 \le \gamma \le \omega \le 1$, $\omega > 0$.

We just need to prove that it is also valid for $0 < \omega < \gamma \le 1$. From (5), it is easy to prove that

$$\rho(\mathscr{L}_{\gamma,\omega}) = 1 - \frac{\omega}{\gamma} + \frac{\omega}{\gamma}\rho(\mathscr{L}_\gamma), \quad \rho(\mathscr{L}_{\gamma,\omega}^{(\nu)}) = 1 - \frac{\omega}{\gamma} + \frac{\omega}{\gamma}\rho(\mathscr{L}_\gamma^{(\nu)}).$$

Clearly,

$$\rho(\mathscr{L}_{\gamma,\omega}) \begin{cases} < 1 \\ = 1 \\ > 1 \end{cases} \iff \rho(\mathscr{L}_\gamma) \begin{cases} < 1 \\ = 1 \\ > 1 \end{cases}$$

and

$$\rho(\mathscr{L}_{\gamma,\omega}^{(\nu)}) \begin{cases} < 1 \\ = 1 \\ > 1 \end{cases} \iff \rho(\mathscr{L}_\gamma^{(\nu)}) \begin{cases} < 1 \\ = 1 \\ > 1. \end{cases}$$

Since Theorem C is valid for $\gamma \le \omega$, then we have that

$$\rho(\mathscr{L}_\gamma^{(\nu)}) \begin{cases} < \rho(\mathscr{L}_\gamma) & \text{if} \quad \rho(\mathscr{L}_\gamma) < 1 \\ = \rho(\mathscr{L}_\gamma) & \text{if} \quad \rho(\mathscr{L}_\gamma) = 1 \\ > \rho(\mathscr{L}_\gamma) & \text{if} \quad \rho(\mathscr{L}_\gamma) > 1, \end{cases}$$

so that

$$\rho(\mathscr{L}_{\gamma,\omega}^{(\nu)}) \begin{cases} < \rho(\mathscr{L}_{\gamma,\omega}) & \text{if} \quad \rho(\mathscr{L}_{\gamma,\omega}) < 1 \\ = \rho(\mathscr{L}_{\gamma,\omega}) & \text{if} \quad \rho(\mathscr{L}_{\gamma,\omega}) = 1 \\ > \rho(\mathscr{L}_{\gamma,\omega}) & \text{if} \quad \rho(\mathscr{L}_{\gamma,\omega}) > 1. \end{cases}$$

When one of Theorems A, B and D is valid for $0 \le \gamma \le \omega \le 1$, $\omega > 0$, the proof is completely same. The proof is completed. □

In the following, if there is no special explanation then we always assume that

$$0 < \omega \le 1, \quad 0 \le \gamma \le 1.$$

We will construct some Q_ν (P_ν) to make the above four theorems hold. For simplicity, when we provide the conditions for the establishment of Theorems A, B, C and D we always assume that A is an L-matrix, a nonsingular M-matrix, an irreducible L-matrix and an irreducible nonsingular M-matrix, respectively. We will not elaborate on this point one by one below.

3.1. *General preconditioners*

In [94] we have proposed some general preconditioners. A class of general constructions of Q is given by

$$Q_1 = (q_{i,j}^{(1)})$$

with

$$q_{i,j}^{(1)} \begin{cases} = 0, & i = j = 1, \cdots, n, \\ \ge 0, & i, j = 1, \cdots, n, \ i \ne j, \end{cases} \quad \text{and} \quad \sum_{\substack{i,j=1 \\ i \ne j}}^{n} q_{i,j}^{(1)} \ne 0.$$

Some comparison theorems have been proved.

By direct operation we have

$$a_{i,j}^{(1)} = a_{i,j} + q_{i,j}^{(1)} + \sum_{\substack{k=1 \\ k \ne i,j}}^{n} q_{i,k}^{(1)} a_{k,j}, \ i, j = 1, \cdots, n, i \ne j, \ \ a_{i,i}^{(1)} = 1 + \sum_{\substack{k=1 \\ k \ne i}}^{n} q_{i,k}^{(1)} a_{k,i}, \ i = 1, \cdots, n. \tag{10}$$

We define several decompositions as

$$Q_1 = Q^{(l)} + Q^{(u)}, \ Q^{(l)} U = E_1 + F_1 + G_1, \ Q^{(u)} L = E_2 + F_2 + G_2,$$

where E_1 and E_2 are diagonal matrices, $Q^{(l)}$, F_1 and F_2 are strictly lower triangular matrices, while $Q^{(u)}$, G_1 and G_2 are strictly upper triangular matrices. Then the three matrices in (9) are given by

$$D_1 = I - E_1 - E_2, \ L_1 = L + F_1 + F_2 + Q^{(l)} L - Q^{(l)}, \ U_1 = U + G_1 + G_2 + Q^{(u)} U - Q^{(u)}.$$

Similar to the proof of [94, Theorem 2.6], we prove a lemma.

Lemma 3.2. *(i)*

$$P_1 N_{\gamma,\omega} - N_{\gamma,\omega}^{(1)} = P_1 M_{\gamma,\omega} - M_{\gamma,\omega}^{(1)} = \frac{1}{\omega}[E_1 + E_2 + \gamma(F_1 + F_2) + (1-\gamma) Q^{(l)} + \omega Q^{(u)} M_{\gamma,\omega}].$$

(ii) Assume that λ is an eigenvalue of $\mathscr{L}_{\gamma,\omega}$ and $x \neq 0$ is its associated eigenvector. Then

$$\mathscr{L}^{(1)}_{\gamma,\omega}x - \lambda x = (\lambda - 1)[M^{(1)}_{\gamma,\omega}]^{-1}[E_1 + E_2 + \gamma(F_1 + F_2) + (1-\gamma)Q^{(l)} + \omega Q^{(u)}M_{\gamma,\omega}]x.$$

Proof. Since

$$P_1A = M^{(1)}_{\gamma,\omega} - N^{(1)}_{\gamma,\omega} = P_1M_{\gamma,\omega} - P_1N_{\gamma,\omega},$$

then by direct operation we have

$$\begin{aligned} P_1N_{\gamma,\omega} - N^{(1)}_{\gamma,\omega} = P_1M_{\gamma,\omega} - M^{(1)}_{\gamma,\omega} &= \frac{1}{\omega}[(I + Q^{(l)} + Q^{(u)})(I - \gamma L) - (D_1 - \gamma L_1)] \\ &= \frac{1}{\omega}[E_1 + E_2 + \gamma(F_1 + F_2) + (1-\gamma)Q^{(l)} + Q^{(u)}(I - \gamma L)], \end{aligned}$$

which shows (i).

Similar to (8), we can get

$$\mathscr{L}^{(1)}_{\gamma,\omega}x - \lambda x = (\lambda - 1)[M^{(1)}_{\gamma,\omega}]^{-1}(P_1M_{\gamma,\omega} - M^{(1)}_{\gamma,\omega})x.$$

By (i), we derive (ii). □

Let

$$\Delta^{(1)}(\gamma) = (E_1 + E_2) + \gamma F_1 + \gamma F_2 + \gamma Q^{(u)}U + (1-\gamma)Q = \Delta_{11} + \gamma\Delta_{12} + \gamma\Delta_{13} + \gamma\Delta_{14} + (1-\gamma)Q,$$

where $\Delta_{11} = E_1 + E_2$, $\Delta_{12} = F_1$, $\Delta_{13} = F_2$, $\Delta_{14} = Q^{(u)}U$. Denote

$$\Delta^{(1)}(\gamma) = (\delta^{(1)}_{i,j}(\gamma)),\ \Delta_k = (\delta^{(1k)}_{i,j}),\ k = 1,2,3,4.$$

By direct operation we can obtain that

$$\begin{aligned}
\delta^{(11)}_{i,j} &= \begin{cases} -\sum\limits_{\substack{k=1\\k\neq i}}^{n} q^{(1)}_{i,k}a_{k,i}, & i = j = 1,\cdots,n, \\ 0, & otherwise, \end{cases} \\
\delta^{(12)}_{i,j} &= \begin{cases} -\sum\limits_{k=1}^{j-1} q^{(1)}_{i,k}a_{k,j}, & i = 3,\cdots,n,\ j = 2,\cdots,i-1, \\ 0, & otherwise, \end{cases} \\
\delta^{(13)}_{i,j} &= \begin{cases} -\sum\limits_{k=i+1}^{n} q^{(1)}_{i,k}a_{k,j}, & i = 2,\cdots,n-1,\ j = 1,\cdots,i-1, \\ 0, & otherwise, \end{cases} \\
\delta^{(14)}_{i,j} &= \begin{cases} -\sum\limits_{k=i+1}^{j-1} q^{(1)}_{i,k}a_{k,j}, & i = 1,\cdots,n-2,\ j = i+2,\cdots,n, \\ 0, & otherwise, \end{cases}
\end{aligned}$$

so that

$$\begin{aligned}
\delta^{(1)}_{i,j}(\gamma) &= \delta^{(11)}_{i,j} + \gamma\delta^{(12)}_{i,j} + \gamma\delta^{(13)}_{i,j} + \gamma\delta^{(14)}_{i,j} + (1-\gamma)q^{(1)}_{i,j} \\
&= \begin{cases} -\sum\limits_{\substack{k=1\\k\neq i}}^{n} q^{(1)}_{i,k}a_{k,i}, & i = j = 1,\cdots,n; \\ (1-\gamma)q^{(1)}_{i,j} - \gamma\sum\limits_{k=i+1}^{j-1} q^{(1)}_{i,k}a_{k,j}, & i = 1,\cdots,n-1,\ j = i+1,\cdots,n; \\ (1-\gamma)q^{(1)}_{i,j} - \gamma\sum\limits_{\substack{1\le k\le j-1\\ i+1\le k\le n}} q^{(1)}_{i,k}a_{k,j}, & i = 2,\cdots,n,\ j = 1,\cdots,i-1 \end{cases}
\end{aligned}$$

and

$$\delta_{i,j}^{(1)}(1)=\begin{cases} -\sum\limits_{\substack{1\le k\le j-1\\ i+1\le k\le n}} q_{i,k}^{(1)}a_{k,j}, & i=1,\cdots,n,\ j=1,\cdots,i,\ (i,j)\neq(n,1);\\ -\sum\limits_{k=i+1}^{j-1} q_{i,k}^{(1)}a_{k,j}, & i=1,\cdots,n-2,\ j=i+2,\cdots,n;\\ 0, & i=1,\cdots,n-1,\ j=i+1;\\ 0, & i=n,\ j=1, \end{cases} \tag{11}$$

where the sum is taken to be zero when the upper limit is less than the lower limit.

Lemma 3.3. *Let A be an L-matrix.*

(i) Let $0\le\gamma<1$.

(i_1) If $q_{i,j}^{(1)}>0$ for some $i\neq j$, then $\delta_{i,j}^{(1)}(\gamma)>0$.

(i_2) There exist $i,j\in\{1,\cdots,n\}$, $i\neq j$, such that $\delta_{i,j}^{(1)}(\gamma)>0$.

(ii) Let $\gamma=1$.

(ii_1) Suppose that there exist $i,j\in\{1,\cdots,n\}$ such that $q_{i,j}^{(1)}a_{j,i}<0$. Then $\delta_{i,i}^{(1)}(1)>0$.

(ii_2) Suppose that there exist $i\in\{1,\cdots,n-1\}$ and $j\in\{1,\cdots,i\}$ such that $q_{i,n}^{(1)}a_{n,j}<0$. Then $\delta_{i,j}^{(1)}(1)>0$.

(ii_3) Suppose that there exist $i\in\{1,\cdots,n-1\}$ and $j\in\{i+1,\cdots,n\}$ such that $q_{i,j}^{(1)}a_{j,1}<0$. Then $\delta_{i,1}^{(1)}(1)>0$.

(ii_4) Suppose that there exist $i\in\{1,\cdots,n\}$ and $j\in\{1,\cdots,n-1\}$ such that $q_{i,j}^{(1)}a_{j,j+1}<0$. Then $\delta_{i,j+1}^{(1)}(1)>0$.

(ii_5) Suppose that there exist $i\in\{2,\cdots,n\}$ and $j\in\{2,\cdots,i\}$ such that $q_{i,1}^{(1)}a_{1,j}<0$. Then $\delta_{i,j}^{(1)}(1)>0$.
In addition, suppose that A is irreducible.

(ii_6) If $q_{i,i+1}^{(1)}>0$ for some $i\in\{1,\cdots,n-1\}$, then there exists $j\in\{1,\cdots,n\}\setminus\{i+1\}$ such that $\delta_{i,j}^{(1)}(1)>0$.

(ii_7) If $q_{n,1}^{(1)}>0$, then there exists $j\in\{2,\cdots,n\}$ such that $\delta_{n,j}^{(1)}(1)>0$.

(ii_8) If $a_{n,1}<0$ and $a_{k,k+1}<0$, $k=1,\cdots,n-1$, then there exist $i,j\in\{1,\cdots,n\}$ such that $\delta_{i,j}^{(1)}(1)>0$.

Proof. Since A is an L-matrix, then $\delta_{i,j}^{(1k)}\ge 0$, $k=1,2,3,4$, and $\delta_{i,j}^{(1)}(\gamma)\ge 0$.

Assume that $\gamma<1$. If $q_{i,j}^{(1)}>0$, $i\neq j$, then $\delta_{i,j}^{(1)}(\gamma)\ge(1-\gamma)q_{i,j}^{(1)}>0$, i.e., ($i_1$) holds.

By the definition of Q_1, there exist $i,j\in\{1,\cdots,n\}$ and $i\neq j$, such that $q_{i,j}^{(1)}>0$, it follows by (i_1) that (i_2) holds.

Now we prove (ii). From

$$\delta_{i,i}^{(1)}(1)=-\sum_{\substack{k=1\\k\neq i}}^{n} q_{i,k}^{(1)}a_{k,i},\ i=1,\cdots,n,$$

(ii_1) is obvious.

By (11), when $i=1,\cdots,n-1$, then we have $\delta_{i,j}^{(1)}(1)\ge -q_{i,n}^{(1)}a_{n,j}$ for $j=1,\cdots,i$, which implies (ii_2), while $\delta_{i,1}^{(1)}(1)\ge -q_{i,j}^{(1)}a_{j,1}$, for $j=i+1,\cdots,n$, which implies (ii_3).

Similarly, when $i=2,\cdots,n$, $j=1,\cdots,i-1$ and $i=1,\cdots,n-2$, $j=i+1,\cdots,n-1$, we get $\delta_{i,j+1}^{(1)}(1)\ge -q_{i,j}^{(1)}a_{j,j+1}$, which implies ($ii_4$).

While, when $i=2,\cdots,n$ and $j=2,\cdots,i$, then $\delta_{i,j}^{(1)}(1)\ge -q_{i,1}^{(1)}a_{1,j}$, which implies ($ii_5$).

Assume that A is irreducible.

By the irreducibility of A, we have $\sum_{j=1,j\neq i+1}^{n} a_{i+1,j} < 0$. If $q_{i,i+1}^{(1)} > 0$ then

$$\begin{aligned}\sum_{\substack{j=1\\ j\neq i+1}}^{n} \delta_{i,j}^{(1)}(1) &= -\sum_{j=1}^{i} \sum_{\substack{1\leq k\leq j-1\\ i+1\leq k\leq n}} q_{i,k}^{(1)} a_{k,j} - \sum_{j=i+2}^{n} \sum_{k=i+1}^{j-1} q_{i,k}^{(1)} a_{k,j} \\ &\geq -\sum_{j=1}^{i} q_{i,i+1}^{(1)} a_{i+1,j} - \sum_{j=i+2}^{n} q_{i,i+1}^{(1)} a_{i+1,j} = -q_{i,i+1}^{(1)} \sum_{\substack{j=1\\ j\neq i+1}}^{n} a_{i+1,j} > 0,\end{aligned}$$

which implies that there exists $j \in \{1, \cdots, n\} \setminus \{i+1\}$ such that $\delta_{i,j}^{(1)}(1) > 0$. This proves (ii_6).

Similarly, we have $\sum_{j=2}^{n} a_{1,j} < 0$. If $q_{n,1}^{(1)} > 0$ then

$$\sum_{j=2}^{n} \delta_{n,j}^{(1)}(1) = -\sum_{j=2}^{n} \sum_{k=1}^{j-1} q_{n,k}^{(1)} a_{k,j} \geq -\sum_{j=2}^{n} q_{n,1}^{(1)} a_{1,j} = -q_{n,1}^{(1)} \sum_{j=2}^{n} a_{1,j} > 0,$$

which implies that there exists $j \in \{2, \cdots, n\}$ such that $\delta_{n,j}^{(1)}(1) > 0$. This proves (ii_7).

At last, assume that $a_{n,1} > 0$ and $a_{k,k+1} > 0$, $k = 1, \cdots, n-1$.

For $i = 1, \cdots, n-1$, we have

$$\delta_{i,1}^{(1)}(1) = -\sum_{k=i+1}^{n} q_{i,k}^{(1)} a_{k,1} \geq -q_{i,n}^{(1)} a_{n,1}.$$

Similarly, we have that for $i = 1, \cdots, n-1$, $j = i+2, \cdots, n$,

$$\delta_{i,j}^{(1)}(1) = -\sum_{k=i+1}^{j-1} q_{i,k}^{(1)} a_{k,j} \geq -q_{i,j-1}^{(1)} a_{j-1,j},$$

and for $i = 2, \cdots, n$, $j = 2, \cdots, i$,

$$\delta_{i,j}^{(1)}(1) = -\sum_{\substack{1\leq k\leq j-1\\ i+1\leq k\leq n}} q_{i,k}^{(1)} a_{k,j} \geq -q_{i,j-1}^{(1)} a_{j-1,j}.$$

Hence it gets that

$$\delta_{i,j}^{(1)}(1) \geq -q_{i,j-1}^{(1)} a_{j-1,j},\ \ i = 1, \cdots, n, j = 2, \cdots, n, j \neq i+1.$$

Denote $\eta = \min\{-a_{n,1};\ -a_{k,k+1} : k = 1, \cdots, n-1\}$. Then $\eta > 0$. Now, we obtain

$$\begin{aligned}\sum_{i,j=1}^{n} \delta_{i,j}^{(1)}(1) &= \sum_{i=1}^{n} \sum_{j=2}^{n} \delta_{i,j}^{(1)}(1) + \sum_{i=1}^{n} \delta_{i,1}^{(1)}(1) \geq \sum_{i=1}^{n} \sum_{\substack{j=2\\ j\neq i+1}}^{n} \delta_{i,j}^{(1)}(1) + \sum_{i=1}^{n-1} \delta_{i,1}^{(1)}(1) \\ &\geq -\sum_{i=1}^{n} \sum_{\substack{j=2\\ j\neq i+1}}^{n} q_{i,j-1}^{(1)} a_{j-1,j} - \sum_{i=1}^{n-1} q_{i,n}^{(1)} a_{n,1} \geq \eta \sum_{i=1}^{n} \sum_{\substack{j=2\\ j\neq i+1}}^{n} q_{i,j-1}^{(1)} + \eta \sum_{i=1}^{n-1} q_{i,n}^{(1)} \\ &= \eta \left(\sum_{i=1}^{n} \sum_{\substack{j=1\\ j\neq i}}^{n-1} q_{i,j}^{(1)} + \sum_{i=1}^{n-1} q_{i,n}^{(1)} \right) = \eta \sum_{\substack{i,j=1\\ i\neq j}}^{n} q_{i,j}^{(1)} > 0,\end{aligned}$$

which implies that there exist $i, j \in \{1, \cdots, n\}$ such that $\delta_{i,j}^{(1)}(1) > 0$, i.e., (ii_8) holds. $\square$

We first give the condition for the establishment of the Stein-Rosenberg Type Theorem I.

Theorem 3.4. *Suppose that P_1A is an L-matrix. Then Theorem A is valid for $\nu = 1$.*

Proof. Denote $\rho = \rho(\mathscr{L}_{\gamma,\omega})$.

Since A is an L-matrix, then it gets that $E_i \geq 0$, $F_i \geq 0$, $i = 1, 2$, $M^{-1}_{\gamma,\omega} > 0$ and

$$\mathscr{L}_{\gamma,\omega} = M^{-1}_{\gamma,\omega}N_{\gamma,\omega} = (1-\omega)I + \omega(I-\gamma L)^{-1}[(1-\gamma)L + U] \geq 0,$$

which shows that the splitting (4) is weak regular. By Lemma 2.2, ρ is an eigenvalue of $\mathscr{L}_{\gamma,\omega}$ with associated eigenvector $x > 0$.

Similarly, the AOR splitting $P_1A = M^{(1)}_{\gamma,\omega} - N^{(1)}_{\gamma,\omega}$ is weak regular, i.e., $[M^{(1)}_{\gamma,\omega}]^{-1} > 0$ and $\mathscr{L}^{(1)}_{\gamma,\omega} \geq 0$.

By Lemma 3.2 we obtain

$$\mathscr{L}^{(1)}_{\gamma,\omega}x - \rho x = (\rho - 1)[M^{(1)}_{\gamma,\omega}]^{-1}[E_1 + E_2 + \gamma(F_1 + F_2) + (1-\gamma)Q^{(l)} + \omega Q^{(u)}M_{\gamma,\omega}]x. \tag{12}$$

By Lemma 3.1 we just need to consider the case when $\gamma \leq \omega$. In this case $N_{\gamma,\omega} \geq 0$.

When $\rho \geq 1$, then $M_{\gamma,\omega}x = N_{\gamma,\omega}x/\rho \geq 0$. Since $[M^{(1)}_{\gamma,\omega}]^{-1} > 0$, $Q^{(l)} \geq 0$ and $Q^{(u)} \geq 0$, then from (12) it derives that $\mathscr{L}^{(1)}_{\gamma,\omega}x \geq \rho x$. It follows by Lemma 2.3 that $\rho(\mathscr{L}^{(1)}_{\gamma,\omega}) \geq \rho$.

Assume that $\rho \leq 1$.

When A is irreducible, by Lemma 2.9, $\rho > 0$ and we can choose $x \gg 0$. Since $M_{\gamma,\omega}x = N_{\gamma,\omega}x/\rho \geq 0$, then from (12) it derives that $\mathscr{L}^{(1)}_{\gamma,\omega}x \leq \rho x$. It follows by Lemma 2.3 that $\rho(\mathscr{L}^{(1)}_{\gamma,\omega}) \leq \rho$.

If A is reducible, then definite $\check{A} = (\check{a}_{i,j})$ with

$$\check{a}_{i,j} = \begin{cases} 0, & \text{if} \quad a_{i,j} \neq 0, \\ 1, & \text{if} \quad a_{i,j} = 0, \end{cases} \qquad i, j = 1, \cdots, n.$$

Let $A(\epsilon) = A - \epsilon\check{A}$ with $\epsilon > 0$. Then $A(\epsilon)$ is an irreducible L-matrix. From $P_1A(\epsilon) = P_1A - \epsilon P_1\check{A}$, it is easy to see that $P_1A(\epsilon)$ is an L-matrix for sufficient small ϵ, since the matrix P_1A is an L-matrix and $P_1\check{A} \geq 0$. Denote the AOR iteration matrices corresponding to $A(\epsilon)$ and $P_1A(\epsilon)$ by $\mathscr{L}_{\gamma,\omega}(\epsilon)$ and $\mathscr{L}^{(1)}_{\gamma,\omega}(\epsilon)$, respectively. By the proof above we have $\rho(\mathscr{L}^{(1)}_{\gamma,\omega}(\epsilon)) \leq \rho(\mathscr{L}_{\gamma,\omega}(\epsilon))$, so that

$$\rho(\mathscr{L}^{(1)}_{\gamma,\omega}) = \lim_{\epsilon\to 0^+} \rho(\mathscr{L}^{(1)}_{\gamma,\omega}(\epsilon)) \leq \lim_{\epsilon\to 0^+} \rho(\mathscr{L}_{\gamma,\omega}(\epsilon)) = \rho.$$

Now, we have proved that either $\rho(\mathscr{L}^{(\nu)}_{\gamma,\omega}) \leq \rho(\mathscr{L}_{\gamma,\omega}) \leq 1$ or $\rho(\mathscr{L}^{(\nu)}_{\gamma,\omega}) \geq \rho(\mathscr{L}_{\gamma,\omega}) \geq 1$, which implies that one of the three mutually exclusive relations (*i*), (*ii*) and (*iii*) of Theorem A holds.

The proof is completed. □

This result is consistent with [94, Theorem 2.6].

Theorem 3.5. *Suppose that P_1A is a Z-matrix. Then Theorem B is valid for $\nu = 1$.*

Proof. Since A is a nonsingular M-matrix, then the splitting (4) is weak regular. By Lemma 2.6, the AOR method is convergent, i.e., $\rho(\mathscr{L}_{\gamma,\omega}) < 1$.

On the other hand, by Lemma 2.11, P_1A is a nonsingular M-matrix so that it is an L-matrix.

Now, it follows by Theorem 3.4 that Theorem B is valid. □

Next, we give the Stein-Rosenberg Type Theorem II.

Theorem 3.6. *Suppose that P_1A is an L-matrix. Then Theorem C is valid for $\nu = 1$, provided one of the following conditions is satisfied:*

(i) $0 \leq \gamma < 1$ *and P_1A is irreducible.*

(ii) $\gamma = 1$ *and P_1A is irreducible. One of the following conditions holds:*

(ii_1) There exist $i, j \in \{1, \cdots, n\}$ such that $\delta^{(1)}_{i,j}(1) > 0$.

(ii_2) $q^{(1)}_{n,1} > 0$.

(ii_3) There exists $k \in \{1, \cdots, n-1\}$ such that $q^{(1)}_{k,k+1} > 0$.

(ii_4) There exist $i, j \in \{1, \cdots, n\}$ such that $q^{(1)}_{i,j} a_{j,i} < 0$.

(ii_5) There exist $i \in \{1, \cdots, n-1\}$ and $j \in \{1, \cdots, i\}$ such that $q^{(1)}_{i,n} a_{n,j} < 0$.

(ii_6) There exist $i \in \{1, \cdots, n-1\}$ and $j \in \{i+1, \cdots, n\}$ such that $q^{(1)}_{i,j} a_{j,1} < 0$.

(ii_7) There exist $i \in \{1, \cdots, n\}$ and $j \in \{1, \cdots, n-1\}$ such that $q^{(1)}_{i,j} a_{j,j+1} < 0$.

(ii_8) There exist $i \in \{2, \cdots, n\}$ and $j \in \{2, \cdots, i\}$ such that $q^{(1)}_{i,1} a_{1,j} < 0$.

(ii_9) $a_{n,1} < 0$, $a_{k,k+1} < 0$, $k = 1, \cdots, n-1$.

(iii) $0 \le \gamma < 1$ and for each $i \in \{1, \cdots, n-1\}$ there exists $j(i) \in \{1, \cdots, n\}$ such that $q^{(1)}_{i,j(i)} > 0$.

(iv) $\gamma = 1$ and for each $i \in \{2, \cdots, n-1\}$ one of the following conditions holds:

(iv_1) There exists $j(i) \in \{1, \cdots, n\}$ such that $\delta^{(1)}_{i,j(i)}(1) > 0$.

(iv_2) $q^{(1)}_{i,i+1} > 0$.

(iv_3) There exists $j_i \in \{1, \cdots, n\}$ such that $q^{(1)}_{i,j_i} a_{j_i,i} < 0$.

(iv_4) There exists $j_i \in \{1, \cdots, i\}$ such that $q^{(1)}_{i,n} a_{n,j_i} < 0$.

(iv_5) There exists $j_i \in \{i+1, \cdots, n\}$ such that $q^{(1)}_{i,j_i} a_{j_i,1} < 0$.

(iv_6) There exists $j_i \in \{1, \cdots, n-1\}$ such that $q^{(1)}_{i,j_i} a_{j_i,j_i+1} < 0$.

(iv_7) There exists $j_i \in \{2, \cdots, i\}$ such that $q^{(1)}_{i,1} a_{1,j_i} < 0$.

At the same time, one of the following conditions also holds:

(iv^a) There exist $j \in \{2, \cdots, n\}$ and $k \in \{1, \cdots, j-1\}$ such that $q^{(1)}_{n,k} a_{k,j} < 0$.

(iv^b) $q^{(1)}_{n,1} > 0$.

(iv^c) There exists $j \in \{2, \cdots, n-1\}$ such that

$$a_{n,j} + q^{(1)}_{n,j} + \sum_{\substack{k=1 \\ k \neq j}}^{n-1} q^{(1)}_{n,k} a_{k,j} < 0. \tag{13}$$

(iv^d) One of the conditions (iv_1)-(iv_6) holds for $i = 1$ and

$$a_{n,1} + q^{(1)}_{n,1} + \sum_{k=2}^{n-1} q^{(1)}_{n,k} a_{k,1} < 0. \tag{14}$$

(iv^e) One of the conditions (iv_1)-(iv_6) holds for $i = 1$ and $a_{n,1} < 0$.

Proof. Denote $\rho = \rho(\mathscr{L}_{\gamma,\omega})$. Assume that $x > 0$ is its associated eigenvector.

Consider two splittings of P_1A given by

$$P_1A = M^{(1)}_{\gamma,\omega} - N^{(1)}_{\gamma,\omega} = P_1M_{\gamma,\omega} - P_1N_{\gamma,\omega}. \tag{15}$$

Since A and P_1A are L-matrices, then the splittings are respectively weak regular and nonnegative, so that $[M^{(1)}_{\gamma,\omega}]^{-1} > 0$.

By Lemma 3.1 we just need to consider the case when $\gamma \le \omega$. By Lemma 2.9 it follows that $\rho > 0$ and $x \gg 0$.

Furthermore, we have $1/(\omega - \gamma + \gamma\rho) > 0$. From $N_{\gamma,\omega}x = \rho M_{\gamma,\omega}x$, it gets that

$$Lx = \frac{1}{\omega - \gamma + \gamma\rho}[(\omega + \rho - 1)I - \omega U]x,$$

so that

$$M_{\gamma,\omega}x = \frac{1}{\omega - \gamma + \gamma\rho}[(1 - \gamma)I + \gamma U]x.$$

By Lemma 3.2, we obtain

$$\begin{aligned}(P_1N_{\gamma,\omega} - N^{(1)}_{\gamma,\omega})x &= \frac{1}{\omega}\left\{E_1 + E_2 + \gamma(F_1 + F_2) + (1-\gamma)Q^{(l)} + \frac{\omega}{\omega - \gamma + \gamma\rho}Q^{(u)}[(1-\gamma)I + \gamma U]\right\}x \\ &= \Phi(\gamma,\omega)x,\end{aligned} \tag{16}$$

where

$$\Phi(\gamma,\omega) = \frac{1}{\omega}(E_1 + E_2 + \gamma F_1 + \gamma F_2) = +\frac{\gamma}{\omega - \gamma + \gamma\rho}Q^{(u)}U + (1-\gamma)\left[\frac{1}{\omega}Q^{(l)} + \frac{1}{\omega - \gamma + \gamma\rho}Q^{(u)}\right].$$

Clearly, $\Phi(\gamma,\omega) \ge 0$, $\Delta^{(1)}(\gamma) \ge 0$, $\Delta_{1k} \ge 0$, $k = 1,2,3,4$, and the positions of the positive elements of the both matrices $\Phi(\gamma,\omega)$ and $\Delta^{(1)}(\gamma)$ are completely same, since $\omega > 0$ and $\omega - \gamma + \gamma\rho > 0$.

Since P_1A is an irreducible L-matrix, then, by Lemma 2.9, we can obtain $\rho(\mathscr{L}^{(1)}_{\gamma,\omega}) > 0$ and $y^T(D_1 - \gamma L_1)^{-1} \gg 0$ whenever y satisfies $y > 0$ and $y^T\mathscr{L}^{(1)}_{\gamma,\omega} = \rho(\mathscr{L}^{(1)}_{\gamma,\omega})y^T$.

We first prove (*i*).

In this case by (i_2) in Lemma 3.3 it follows that $\Delta^{(1)}(\gamma) > 0$ so that $\Phi(\gamma,\omega) > 0$. From (16), we can get $(P_1N_{\gamma,\omega} - N^{(1)}_{\gamma,\omega})x > 0$. This shows that the condition (*ii*) of Lemma 2.12 is satisfied. The required result follows by Lemma 2.12 directly.

We prove (*ii*).

Since $\gamma = 1$, then $\omega = 1$. In this case, the AOR method reduces to the Gauss-Seidel method. The equality (16) reduces to $P_1N_{1,1} - N^{(1)}_{1,1} = E_1 + E_2 + F_1 + F_2 + Q^{(u)}U/\rho = \Phi(1,1)$.

If one of the conditions (ii_2)-(ii_9) is satisfied, then by (*ii*) of Lemma 3.3 it is easy to prove that there exist $i, j \in \{1, \cdots, n\}$ such that $\delta^{(1)}_{i,j}(1) > 0$, which shows that ($ii_1$) is satisfied.

If (ii_1) is satisfied, then $\Phi(1,1) > 0$, so that $(P_1N_{1,1} - N^{(1)}_{1,1})x > 0$. This shows that the condition (*ii*) of Lemma 2.12 is satisfied.

We prove (*iii*). Let

$$M^{(1)}_{\gamma,\omega} = (m^{(1)}_{i,j}) = \begin{pmatrix} \bar{M}_{1,1} & \bar{m}_{1,2} \\ \bar{m}_{2,1} & m^{(1)}_{n,n}\end{pmatrix},\ \bar{M}_{1,1} \in \mathscr{R}^{(n-1)\times(n-1)}.$$

Then $\bar{m}_{1,2} = 0$, $\bar{m}_{2,1} = (m^{(1)}_{n,1} \cdots m^{(1)}_{n,n-1})$ and for $j = 1, \cdots, n-1$, $m^{(1)}_{j,j} > 0$,

$$m^{(1)}_{n,j} = a_{n,j} + q^{(1)}_{n,j} + \sum_{\substack{k=1\\k\ne j}}^{n-1} q^{(1)}_{n,k}a_{k,j} \le 0. \tag{17}$$

Furthermore, we have

$$[M^{(1)}_{\gamma,\omega}]^{-1} = (\hat{m}_{i,j}) = \begin{pmatrix} \bar{M}^{-1}_{1,1} & 0 \\ -\frac{1}{m^{(1)}_{n,n}}\bar{m}_{2,1}\bar{M}^{-1}_{1,1} & \frac{1}{m^{(1)}_{n,n}} \end{pmatrix} > 0,$$

where $\hat{m}_{k,k} > 0$, $\hat{m}_{n,k} \geq -m^{(1)}_{n,k}\hat{m}_{k,k}/m^{(1)}_{n,n}$, $k = 1, \cdots, n-1$.

By (i_1) in Lemma 3.3, for each $i \in \{1, \cdots, n-1\}$, $\delta^{(1)}_{i,j(i)}(\gamma) > 0$. Hence, in this case, the every row of $\Phi(\gamma,\omega)$ has positive elements except the last row, so that the first to $(n-1)$th elements of $\Phi(\gamma,\omega)x$ are positive. Since $[M^{(1)}_{\gamma,\omega}]^{-1} > 0$ and $\hat{m}_{k,k} > 0$ for $k = 1, \cdots, n-1$, then the first to $(n-1)$th elements of $[M^{(1)}_{\gamma,\omega}]^{-1}\Phi(\gamma,\omega)x$ are also positive.

Since A is irreducible, then there exists $j_n \in \{1, \cdots, n-1\}$ such that $a_{n,j_n} < 0$.

If $q^{(1)}_{n,j_n} > 0$ then $\delta^{(1)}_{n,j_n}(\gamma) \geq (1-\gamma)q^{(1)}_{n,j_n} > 0$. This shows that the last row of $\Phi(\gamma,\omega)$ has positive elements, so that the last element of $\Phi(\gamma,\omega)x$ is positive. From (16) we have proved that $(P_1N_{\gamma,\omega} - N^{(1)}_{\gamma,\omega})x \gg 0$ and, therefore,

$$[M^{(1)}_{\gamma,\omega}]^{-1}(P_1N_{\gamma,\omega} - N^{(1)}_{\gamma,\omega})x \gg 0. \tag{18}$$

When $q^{(1)}_{n,j_n} = 0$ then from (17) $m^{(1)}_{n,j_n} \leq a_{n,j_n} < 0$, so that $\hat{m}_{n,j_n} > 0$. Hence the last element of $[M^{(1)}_{\gamma,\omega}]^{-1}\Phi(\gamma,\omega)x$ is positive, which shows that (18) holds.

Now, we have proved that the condition (i) of Lemma 2.12 is satisfied for the splittings given in (15). By Lemma 2.12 we can prove that Theorem C is valid.

At last, we prove (iv).

In this case, the AOR method reduces to the Gauss-Seidel method.

If one of (iv_2)-(iv_7) holds, then it follows by (ii_1)-(ii_6) in Lemma 3.3 that (iv_1) is satisfied.

When (iv_1) holds, then the every row of $\Phi(1,1)$ has positive elements except the first and last rows, so that the second to $(n-1)$th elements of $\Phi(1,1)x$ and $[M^{(1)}_{1,1}]^{-1}\Phi(1,1)x$ are positive.

If (iv^a) holds then $\delta^{(1)}_{n,j}(1) \geq -q^{(1)}_{n,k}a_{k,j} > 0$. And if (iv^b) is satisfied, then by (ii_7) in Lemma 3.3 there exists $j \in \{2, \cdots, n\}$ such that $\delta^{(1)}_{n,j}(1) > 0$. Hence, for these two cases the nth row of $\Phi(1,1)$ has positive elements. This has proved that the second to nth rows of $\Phi(1,1)$ has positive elements, so that the second to nth elements of $\Phi(1,1)x$ and $[M^{(1)}_{1,1}]^{-1}\Phi(1,1)x$ are all positive.

If (iv^c) holds, then $m^{(1)}_{n,j} < 0$, so that $\hat{m}_{n,j} > 0$. Hence the last element of $[M^{(1)}_{\gamma,\omega}]^{-1}\Phi(1,1)x$ is positive, which shows that its second to nth elements are all positive.

When (iv^e) holds, if $q^{(1)}_{n,1} > 0$ then the proof is given above. If $q^{(1)}_{n,1} = 0$ then

$$a_{n,1} + q^{(1)}_{n,1} + \sum_{k=2}^{n-1} q^{(1)}_{n,k}a_{k,1} \leq a_{n,1} < 0,$$

which implies that (iv^d) is satisfied.

Now, we consider that (iv^d) holds. Just as the proof above, the first row of $\Phi(1,1)$ has positive elements, so that the first to $(n-1)$th elements of $\Phi(1,1)x$ and $[M^{(1)}_{1,1}]^{-1}\Phi(1,1)x$ are positive. the inequality (14) shows that $m^{(1)}_{n,1} < 0$, so that $\hat{m}_{n,1} > 0$. Hence the last element of $[M^{(1)}_{1,1}]^{-1}\Phi(1,1)x$ is positive, and therefore, its second to nth elements are all positive.

We have proved that if one of (iv_1)-(iv_7) and one of (iv^a)-(iv^e) hold at the same time, then the condition of Lemma 2.13 is satisfied. By Lemma 2.13 we can prove that Theorem C is valid.

The proof is completed. □

Theorem 3.7. *Suppose that P_1A is a Z-matrix. Then Theorem D is valid for $\nu = 1$, provided one of the following conditions is satisfied:*

(i) One of the conditions (i)-(iv) of Theorem 3.6 holds.

(ii) For $i = 2, \cdots, n$, $j = 1, \cdots, i-1$, $a_{i,j} \geq a_{i,j}^{(1)}$. And one of the following conditions holds:

(i_1) There exists $i_0 \in \{1, \cdots, n\}$ such that $a_{i_0,i_0}^{(1)} < 1$.

(ii_2) $\gamma > 0$ and there exist $i_0 \in \{2, \cdots, n\}$, $j_0 \in \{1, \cdots, i_0 - 1\}$ such that $a_{i_0,j_0} > a_{i_0,j_0}^{(1)}$.

Proof. By Lemma 2.11, PA ia a nonsingular M-matrix. Hence both AOR splittings $A = M_{\gamma,\omega} - N_{\gamma,\omega}$ and $P_1A = M_{\gamma,\omega}^{(1)} - N_{\gamma,\omega}^{(1)}$ are weak regular.

Denote $\rho = \rho(\mathscr{L}_{\gamma,\omega})$ and $x > 0$ being its associated eigenvector. By Theorem 3.5, we have $\rho(\mathscr{L}_{\gamma,\omega}^{(1)}) \leq \rho(\mathscr{L}_{\gamma,\omega}) < 1$.

When (*i*) holds, the proof is completely same as the proof of Theorem 3.5, by Theorem 3.6 we can prove the required result.

Now, we prove (*ii*).

Since A is an irreducible nonsingular M-matrix, then, by Lemma 2.5, $A^{-1} \gg 0$. While by Lemma 2.10, $(P_1A)^{-1} > 0$. From $A^{-1} - (P_1A)^{-1} = (P_1A)^{-1}(P_1 - I) = (P_1A)^{-1}Q_1 > 0$, it gets that $A^{-1} > (P_1A)^{-1}$.

By Lemma 3.1 we just need to consider the case when $\gamma \leq \omega$. Then $N_{\gamma,\omega} > 0$ and $N_{\gamma,\omega}^{(1)} \geq 0$. By Lemma 2.9, $\rho > 0$ and $x \gg 0$. Then it is easy to prove that $M_{\gamma,\omega} > M_{\gamma,\omega}^{(1)}$, so that $M_{\gamma,\omega}x > M_{\gamma,\omega}^{(1)}x$. Noticing that $Ax = (1/\rho - 1)N_{\gamma,\omega}x > 0$, we have $M_{\gamma,\omega}^{(1)}x = P_1Ax + N_{\gamma,\omega}^{(1)}x > 0$. Now we have proved that the condition of Lemma 2.14 is satisfied. By Lemma 2.14 it follows that $\rho(\mathscr{L}_{\gamma,\omega}^{(1)}) < \rho(\mathscr{L}_{\gamma,\omega}) < 1$. □

By the definitions of L-matrix and Z-matrix, the following two corollaries can be derived from Theorems 3.4 and 3.5 directly.

Corollary 3.8. *Suppose that $a_{i,i}^{(1)} > 0$, $a_{i,j}^{(1)} \leq 0$, $i, j = 1, \cdots, n$, $i \neq j$. Then Theorem A is valid for $\nu = 1$.*

Corollary 3.9. *Suppose that $a_{i,j}^{(1)} \leq 0$, $i, j = 1, \cdots, n$, $i \neq j$. Then Theorem B is valid for $\nu = 1$.*

In all of the following, for the case when A is irreducible, the symbol "$\lesssim$" ("$\gtrsim$") indicates "$\leq$" ("$\geq$") if $A^{(\nu)}$ is irreducible even when it appears "=", otherwise it is "<" (">").

Corollary 3.10. *Suppose that $a_{i,i}^{(1)} > 0$, $a_{i,j}^{(1)} \leq 0$, $i, j = 1, \cdots, n$, $i \neq j$. Then Theorem C is valid for $\nu = 1$, provided one of the following conditions is satisfied:*

(i) For $i, j = 1, \cdots, n$, $i \neq j$, $a_{i,j}^{(1)} \lesssim 0$ whenever $a_{i,j} < 0$. One of the conditions $0 \leq \gamma < 1$ and (ii_1)-(ii_9) whenever $\gamma = 1$ in Theorem 3.6 holds.

(ii) One of the conditions (iii) and (iv) of Theorem 3.6 holds.

Proof. Clearly, the matrix P_1A is an L-matrix. The condition $a_{i,j}^{(1)} \lesssim 0$ whenever $a_{i,j} < 0$ ensures that P_1A is irreducible, since A is irreducible. This shows that the condition of Theorem 3.6 is satisfied, so that Theorem C is valid. □

Similarly, the following corollary can be derived from Theorem 3.7 directly.

Corollary 3.11. *Suppose that $a_{i,j}^{(1)} \leq 0$, $i, j = 1, \cdots, n$, $i \neq j$. Then Theorem D is valid for $\nu = 1$, provided one of the conditions (ii) of Theorem 3.7, (i) and (ii) of Corollary 3.10 is satisfied.*

Furthermore, noticing (10), from Corollaries 3.8-3.11, we give the following corollaries.

Corollary 3.12. *Suppose that* $q_{i,j}^{(1)} \leq -a_{i,j}$, $i, j = 1, \cdots, n$, $i \neq j$, *and*

$$1 + \sum_{\substack{k=1 \\ k\neq i}}^{n} q_{i,k}^{(1)} a_{k,i} > 0, \; i = 1, \cdots, n. \tag{19}$$

Then Theorem A is valid for $\nu = 1$.

Proof. For $i, j = 1, \cdots, n$, we have

$$a_{i,j}^{(1)} = a_{i,j} + q_{i,j}^{(1)} + \sum_{\substack{k=1 \\ k\neq i,j}}^{n} q_{i,k}^{(1)} a_{k,j} \leq a_{i,j} + q_{i,j}^{(1)} \leq 0, \; i \neq j,$$

and

$$a_{i,i}^{(1)} = 1 + \sum_{\substack{k=1 \\ k\neq i}}^{n} q_{i,k}^{(1)} a_{k,i} > 0, \; i = j.$$

This shows that the condition of Corollary 3.8 is satisfied. Therefore Theorem A is valid. □

By Corollary 3.9 and the proof of Corollary 3.12 we obtain the following corollary directly.

Corollary 3.13. *Suppose that* $q_{i,j}^{(1)} \leq -a_{i,j}$, $i, j = 1, \cdots, n$, $i \neq j$. *Then Theorem B is valid for* $\nu = 1$.

Corollary 3.14. *Suppose that the condition of Corollary 3.12 is satisfied. Then Theorem C is valid for* $\nu = 1$, *provided one of the following conditions is satisfied:*

(i) For $i, j = 1, \cdots, n$, $i \neq j$, $q_{i,j}^{(1)} \lessapprox -a_{i,j}$ *whenever* $a_{i,j} < 0$. *One of the conditions* $0 \leq \gamma < 1$ *and* (ii_1)-(ii_9) *whenever* $\gamma = 1$ *in Theorem 3.6 holds.*

(ii) One of the conditions (iii) and (iv) of Theorem 3.6 holds, where the inequality (14) can be replaced by $q_{n,j}^{(1)} < -a_{n,j}$.

Proof. Since A is irreducible, then there exists $j \in \{1, \cdots, n-1\}$ such that $a_{n,j} < 0$. So we can choose Q_1 such that $q_{n,j}^{(1)} < -a_{n,j}$ in (*ii*).

From the proof of Corollary 3.12 it is easy to prove that the condition of Corollary 3.10 is satisfied. Therefore Theorem C is valid. □

Similarly, by Corollary 3.11 we can prove the following corollary directly.

Corollary 3.15. *Suppose that* $q_{i,j}^{(1)} \leq -a_{i,j}$, $i, j = 1, \cdots, n$, $i \neq j$. *Then Theorem D is valid for* $\nu = 1$, *provided one of the conditions (ii) of Theorem 3.7, (i) and (ii) of Corollary 3.14 is satisfied.*

As a special case, in [94] we propose $q_{i,j}^{(2)} = -\alpha_{i,j} a_{i,j}$ and get

$$Q_2 = (-\alpha_{i,j} a_{i,j})$$

with

$$\alpha_{i,i} = 0, \; \alpha_{i,j} \geq 0, \; i, j = 1, \cdots, n, \; i \neq j, \text{ and } \sum_{\substack{i,j=1 \\ i\neq j}}^{n} \alpha_{i,j} a_{i,j} \neq 0.$$

Of course, when $a_{i,j} = 0$, the choice of $\alpha_{i,j}$ is meaningless.

In [20], two special preconditioners are proposed for the preconditioned Gauss-Seidel method, where one is $\alpha_{i,j} = 1$, the other is $\alpha_{i,j} = 1 + \alpha$ for $\alpha \geq 0$, $i, j = 1, \cdots, n$, $i \neq j$. In [58, 90], for the preconditioned

Gauss-Seidel and AOR methods respectively, the authors consider the case when $\alpha_{i,j} = \alpha \geq 0$ for $i > j$, $\alpha_{i,j} = \beta \geq 0$ for $i < j$, $i, j = 1, \cdots, n$, with $\alpha + \beta \neq 0$.

Denote

$$\delta^{(2)}_{i,j}(\gamma) = \begin{cases} \sum\limits_{\substack{k=1\\k\neq i}}^{n} \alpha_{i,k}a_{i,k}a_{k,i}, & i = j = 1, \cdots, n; \\ (\gamma-1)\alpha_{i,j}a_{i,j} + \gamma \sum\limits_{k=i+1}^{j-1} \alpha_{i,k}a_{i,k}a_{k,j}, & i = 1, \cdots, n-1, j = i+1, \cdots, n; \\ (\gamma-1)\alpha_{i,j}a_{i,j} + \gamma \sum\limits_{\substack{1\leq k\leq j-1\\i+1\leq k\leq n}} \alpha_{i,k}a_{i,k}a_{k,j}, & i = 2, \cdots, n, j = 1, \cdots, i-1 \end{cases}$$

and

$$\delta^{(2)}_{i,j}(1) = \begin{cases} \sum\limits_{\substack{1\leq k\leq j-1\\i+1\leq k\leq n}} \alpha_{i,k}a_{i,k}a_{k,j}; & i = 1, \cdots, n, j = 1, \cdots, i, \ (i,j) \neq (n,1); \\ \sum\limits_{k=i+1}^{j-1} \alpha_{i,k}a_{i,k}a_{k,j}, & i = 1, \cdots, n-2, j = i+2, \cdots, n; \\ 0, & i = 1, \cdots, n-1, j = i+1; \\ 0, & i = n, j = 1. \end{cases}$$

Using Corollaries 3.8-3.11, we prove corresponding comparison theorems.

Theorem 3.16. *Suppose that* $\sum_{k=1,k\neq i}^{n} \alpha_{i,k}a_{i,k}a_{k,i} < 1$, $i = 1, \cdots, n$, *and*

$$(1-\alpha_{i,j})a_{i,j} - \sum_{\substack{k=1\\k\neq i,j}}^{n} \alpha_{i,k}a_{i,k}a_{k,j} \leq 0, \ i, j = 1, \cdots, n, \ i \neq j. \tag{20}$$

Then Theorem A is valid for $\nu = 2$.

Proof. The inequality (20) shows that $a^{(2)}_{i,j} \leq 0$, $i, j = 1, \cdots, n$, $i \neq j$, and the inequality $\sum_{k=1,k\neq i}^{n} \alpha_{i,k}a_{i,k}a_{k,i} < 1$ shows that $a^{(2)}_{i,i} > 0$, $i = 1, \cdots, n$.

It has proved that the condition of Corollary 3.8 is satisfied so that Theorem A is valid. □

Theorem 3.17. *Suppose that (20) holds. Then Theorem B is valid for* $\nu = 2$.

Proof. From the proof of Theorem 3.16 it can prove that the condition of Corollary 3.9 is satisfied. Therefore Theorem B is valid. □

Theorem 3.18. *Suppose that* $\sum_{k=1,k\neq i}^{n} \alpha_{i,k}a_{i,k}a_{k,i} < 1$, $i = 1, \cdots, n$. *Then Theorem C is valid for* $\nu = 2$, *provided one of the following conditions is satisfied:*

(i) $0 \leq \gamma < 1$ *and*

$$(1-\alpha_{i,j})a_{i,j} - \sum_{\substack{k=1\\k\neq i,j}}^{n} \alpha_{i,k}a_{i,k}a_{k,j} \lneqq 0 \ \text{ whenever } \ a_{i,j} < 0, \ i, j = 1, \cdots, n, \ i \neq j. \tag{21}$$

(ii) $\gamma = 1$, *the inequality (21) holds and one of the following conditions holds:*

- *(ii$_1$) There exist* $i, j \in \{1, \cdots, n\}$ *such that* $\delta^{(2)}_{i,j}(1) > 0$.
- *(ii$_2$)* $a_{n,1} < 0$ *and* $\alpha_{n,1} > 0$.
- *(ii$_3$) There exists* $k \in \{1, \cdots, n-1\}$ *such that* $a_{k,k+1} < 0$ *and* $\alpha_{k,k+1} > 0$.

(ii_4) There exist $i, j \in \{1, \cdots, n\}$ such that $\alpha_{i,j}a_{i,j}a_{j,i} > 0$.

(ii_5) There exist $i \in \{1, \cdots, n-1\}$ and $j \in \{1, \cdots, i\}$ such that $\alpha_{i,n}a_{i,n}a_{n,j} > 0$.

(ii_6) There exist $i \in \{1, \cdots, n-1\}$ and $j \in \{i+1, \cdots, n\}$ such that $\alpha_{i,j}a_{i,j}a_{j,1} > 0$.

(ii_7) There exist $i \in \{1, \cdots, n\}$ and $j \in \{1, \cdots, n-1\}$ such that $\alpha_{i,j}a_{i,j}a_{j,j+1} > 0$.

(ii_8) There exist $i \in \{2, \cdots, n\}$ and $j \in \{2, \cdots, i\}$ such that $\alpha_{i,1}a_{i,1}a_{1,j} > 0$.

(ii_9) $a_{n,1} < 0$ and $a_{k,k+1} < 0$, $k = 1, \cdots, n-1$.

(iii) $0 \le \gamma < 1$, the inequality (20) holds and for each $i \in \{1, \cdots, n-1\}$ there exists $j(i) \in \{1, \cdots, n\}$ such that $\alpha_{i,j(i)}a_{i,j(i)} < 0$.

(iv) $\gamma = 1$, the inequality (20) holds and for each $i \in \{2, \cdots, n-1\}$ one of the following conditions holds:

(iv_1) There exists $j(i) \in \{1, \cdots, n\}$ such that $\delta^{(2)}_{i,j(i)}(1) > 0$.

(iv_2) $a_{i,i+1} < 0$ and $\alpha_{i,i+1} > 0$.

(iv_3) There exists $j_i \in \{1, \cdots, n\}$ such that $\alpha_{i,j_i}a_{i,j_i}a_{j_i,i} > 0$.

(iv_4) There exists $j_i \in \{1, \cdots, i\}$ such that $\alpha_{i,n}a_{i,n}a_{n,j_i} > 0$.

(iv_5) There exists $j_i \in \{i+1, \cdots, n\}$ such that $\alpha_{i,j_i}a_{i,j_i}a_{j_i,1} > 0$.

(iv_6) There exists $j_i \in \{1, \cdots, n-1\}$ such that $\alpha_{i,j_i}a_{i,j_i}a_{j_i,j_i+1} > 0$.

(iv_7) There exists $j_i \in \{2, \cdots, i\}$ such that $\alpha_{i,1}a_{i,1}a_{1,j_i} > 0$.

At the same time, one of the following conditions also holds:

(iv^a) There exist $j \in \{2, \cdots, n\}$ and $k \in \{1, \cdots, j-1\}$ such that $\alpha_{n,k}a_{n,k}a_{k,j} > 0$.

(iv^b) $a_{n,1} < 0$ and $\alpha_{n,1} > 0$.

(iv^c) There exists $j \in \{2, \cdots, n-1\}$ such that

$$(1-\alpha_{n,j})a_{n,j} - \sum_{\substack{k=1\\k\neq j}}^{n-1} \alpha_{n,k}a_{n,k}a_{k,j} < 0. \tag{22}$$

(iv^d) One of the conditions (iv_1)-(iv_6) holds for $i = 1$ and

$$(1-\alpha_{n,1})a_{n,1} - \sum_{k=2}^{n-1} \alpha_{n,k}a_{n,k}a_{k,1} < 0. \tag{23}$$

(iv^e) One of the conditions (iv_1)-(iv_6) holds for $i = 1$ and $a_{n,1} < 0$.

Proof. The inequality $\sum_{k=1,k\neq i}^{n} \alpha_{i,k}a_{i,k}a_{k,i} < 1$ shows that $a^{(2)}_{i,i} > 0$, $i = 1, \cdots, n$. Now, $\delta^{(1)}_{i,j}(\gamma)$ reduces to $\delta^{(2)}_{i,j}(\gamma)$, (13) and (14) reduce to (22) and (23), respectively.

For $i \neq j$, if $a_{i,j} = 0$ then

$$a^{(2)}_{i,j} = -\sum_{\substack{k=1\\k\neq i,j}}^{n} \alpha_{i,k}a_{i,k}a_{k,j} \le 0.$$

When $a_{i,j} < 0$, the inequality (21) implies $a^{(2)}_{i,j} \lneqq 0$.

Now, we have proved that the condition of Corollary 3.10 is satisfied. Hence Theorem C is valid. □

By Corollary 3.11 it is easy to prove the following theorem.

Theorem 3.19. *Theorem D is valid for $\nu = 2$, provided one of the following conditions is satisfied:*

(i) One of the conditions (i)-(iv) of Theorem 3.18 holds.

(ii) The inequality (20) holds. For $i = 2, \cdots, n$, $j = 1, \cdots, i-1$,

$$\alpha_{i,j} a_{i,j} + \sum_{\substack{k=1 \\ k \neq i,j}}^{n} \alpha_{i,k} a_{i,k} a_{k,j} \geq 0.$$

And one of the following conditions holds:

(ii_1) There exists $i_0 \in \{1, \cdots, n\}$ such that

$$\sum_{\substack{k=1 \\ k \neq i_0}}^{n} \alpha_{i_0,k} a_{i_0,k} a_{k,i_0} > 0$$

(ii_2) $\gamma > 0$ and there exist $i_0 \in \{2, \cdots, n\}$, $j_0 \in \{1, \cdots, i_0 - 1\}$ such that

$$\alpha_{i_0,j_0} a_{i_0,j_0} + \sum_{\substack{k=1 \\ k \neq i_0, j_0}}^{n} \alpha_{i_0,k} a_{i_0,k} a_{k,j_0} > 0.$$

Since

$$(1 - \alpha_{i,j}) a_{i,j} - \sum_{\substack{k=1 \\ k \neq i,j}}^{n} \alpha_{i,k} a_{i,k} a_{k,j} \leq (1 - \alpha_{i,j}) a_{i,j}, \ i \neq j,$$

then from Theorems 3.16-3.19, we can prove the following corollaries, directly.

Corollary 3.20. *Suppose that $0 \leq \alpha_{i,j} \leq 1$, $i, j = 1, \cdots, n$, $i \neq j$, and $\sum_{k=1, k \neq i}^{n} \alpha_{i,k} a_{i,k} a_{k,i} < 1$, $i = 1, \cdots, n$. Then Theorem A is valid for $\nu = 2$.*

For the special case when $\alpha_{i,j} = \alpha \geq 0$ for $i > j$ and $\alpha_{i,j} = \beta \geq 0$ for $i < j$, $i = 1, \cdots, n$, the result is better than the corresponding ones given by [90, Theorem 3.1, Corollaries 3.2, 3.3], where the assumption that A is irreducible is redundant.

Corollary 3.21. *Suppose that $0 \leq \alpha_{i,j} \leq 1$, $i, j = 1, \cdots, n$, $i \neq j$. Then Theorem B is valid for $\nu = 2$.*

The result is consistent with [94, Theorem 2.7] and it is better than the corresponding one given by [90, Theorem 2.2], where there are problems in the expression.

Corollary 3.22. *Suppose that $\sum_{k=1, k \neq i}^{n} \alpha_{i,k} a_{i,k} a_{k,i} < 1$, $i = 1, \cdots, n$. Then Theorem C is valid for $\nu = 2$, provided one of the following conditions is satisfied:*

(i) For $i, j = 1, \cdots, n$, $i \neq j$, $0 \leq \alpha_{i,j} \lesssim 1$. One of the conditions $0 \leq \gamma < 1$ and (ii_1)-(ii_9) whenever $\gamma = 1$ in Theorem 3.18 holds.

(ii) One of the conditions (iii) and (iv) of Theorem 3.18 holds, where the inequality (20) is replaced by $0 \leq \alpha_{i,j} \leq 1$, $i, j = 1, \cdots, n$, $i \neq j$.

Corollary 3.23. *Theorem D is valid for $\nu = 2$, provided one of the conditions (i), (ii) of Corollary 3.22 and (ii) of Theorem 3.19 is satisfied, where the inequality (20) is replaced by $0 \leq \alpha_{i,j} \leq 1$, $i, j = 1, \cdots, n$, $i \neq j$.*

3.2. Lower triangular preconditioners

Let

$$\alpha_{i,j} = 0,\ i = 1, \cdots, n, j \ge i.$$

Then Q_2 reduces to

$$Q_3 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ -\alpha_{2,1}a_{2,1} & 0 & \cdots & 0 & 0 \\ -\alpha_{3,1}a_{3,1} & -\alpha_{3,2}a_{3,2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\alpha_{n,1}a_{n,1} & -\alpha_{n,2}a_{n,2} & \cdots & -\alpha_{n,n-1}a_{n,n-1} & 0 \end{pmatrix}$$

with $\alpha_{i,j} \ge 0$, $i = 2, \cdots, n$, $j < i$, and

$$\sum_{i=2}^{n}\sum_{j=1}^{i-1} \alpha_{i,j}a_{i,j} \neq 0.$$

Theorem 3.24. *Suppose that $\sum_{k=1}^{i-1} \alpha_{i,k}a_{i,k}a_{k,i} < 1$, $i = 2, \cdots, n$, and*

$$(1 - \alpha_{i,j})a_{i,j} - \sum_{\substack{k=1 \\ k\neq j}}^{i-1} \alpha_{i,k}a_{i,k}a_{k,j} \le 0,\ i = 2, \cdots, n, j < i. \tag{24}$$

Then Theorem A is valid for $\nu = 3$.

Proof. For $i, j = 1, \cdots, n$, we have

$$\sum_{\substack{k=1 \\ k\neq i,j}}^{n} \alpha_{i,k}a_{i,k}a_{k,j} = \sum_{\substack{k=1 \\ k\neq j}}^{i-1} \alpha_{i,k}a_{i,k}a_{k,j}.$$

If $j > i$ then

$$(1 - \alpha_{i,j})a_{i,j} - \sum_{\substack{k=1 \\ k\neq i,j}}^{n} \alpha_{i,k}a_{i,k}a_{k,j} = a_{i,j} - \sum_{k=1}^{i-1} \alpha_{i,k}a_{i,k}a_{k,j} \le a_{i,j} \le 0.$$

This proves that the condition of Theorem 3.16 is satisfied, so that Theorem A is valid. □

Similarly, by Theorem 3.17 we can prove the following theorem.

Theorem 3.25. *Suppose that (24) holds. Then Theorem B is valid for $\nu = 3$.*

In this case, since $\alpha_{i,j} = 0$ for $i \le j$, then $\delta_{1,j}^{(2)}(\gamma) = 0$, $j = 1, \cdots, n$, so that the conditions (ii_3), (ii_5), (ii_6), (iii), (iv_2), (iv_4), (iv_5), (iv^d) and (iv^e) in Theorem 3.18 can be not satisfied.

Now, $\delta_{i,j}^{(2)}(1)$ reduces to

$$\delta_{i,j}^{(3)}(1) = \begin{cases} \sum\limits_{k=1}^{j-1} \alpha_{i,k}a_{i,k}a_{k,j}, & i = 2, \cdots, n, j = 2, \cdots, i; \\ 0, & \textit{otherwise}. \end{cases}$$

Theorem 3.26. *Suppose that $\sum_{k=1}^{i-1} \alpha_{i,k}a_{i,k}a_{k,i} < 1$, $i = 2, \cdots, n$. Then Theorem C is valid for $\nu = 3$, provided one of the following conditions is satisfied:*

(i) $0 \le \gamma < 1$ *and*

$$(1-\alpha_{i,j})a_{i,j} - \sum_{\substack{k=1\\k\neq j}}^{i-1} \alpha_{i,k}a_{i,k}a_{k,j} \lesssim 0 \ \text{ whenever } \ a_{i,j} < 0,\ i = 2,\cdots,n,\ j < i. \tag{25}$$

(ii) $\gamma = 1$, *(25) holds and one of the following conditions holds:*

- *(ii_1) There exist $i \in \{2,\cdots,n\}$, $j \in \{2,\cdots,i\}$ and $k \in \{1,\cdots,j-1\}$ such that $\alpha_{i,k}a_{i,k}a_{k,j} > 0$.*
- *(ii_2) $a_{n,1} < 0$ and $\alpha_{n,1} > 0$.*
- *(ii_3) $a_{k,k+1} < 0$, $k = 1,\cdots,n-1$.*

(iii) $\gamma = 1$, *the inequality (24) holds. For each $i \in \{2,\cdots,n-1\}$ there exist $j(i) \in \{2,\cdots,i\}$ and $k(i) \in \{1,\cdots,j(i)-1\}$ such that $\alpha_{i,k(i)}a_{i,k(i)}a_{k(i),j(i)} > 0$. And one of the following conditions holds:*

- *(iii_1) There exist $j \in \{2,\cdots,n\}$ and $k \in \{1,\cdots,j-1\}$ such that $\alpha_{n,k}a_{n,k}a_{k,j} > 0$.*
- *(iii_2) $a_{n,1} < 0$ and $\alpha_{n,1} > 0$.*
- *(iii_3) There exists $j \in \{2,\cdots,n-1\}$ such that*

$$(1-\alpha_{n,j})a_{n,j} - \sum_{\substack{k=1\\k\neq j}}^{n-1} \alpha_{n,k}a_{n,k}a_{k,j} < 0.$$

Proof. By Theorem 3.18, (i) and (ii_2) are obvious.

When (ii_3) holds, we have $\max\limits_{1\le k\le n-1}\{a_{k,k+1}\} < 0$ and so that

$$\sum_{i=2}^{n}\sum_{j=2}^{i}\sum_{k=1}^{j-1} \alpha_{i,k}a_{i,k}a_{k,j} \ge \sum_{i=2}^{n}\sum_{j=2}^{i} \alpha_{i,j-1}a_{i,j-1}a_{j-1,j} \ge \max_{1\le k\le n-1}\{a_{k,k+1}\} \sum_{i=2}^{n}\sum_{j=1}^{i-1} \alpha_{i,j}a_{i,j} > 0,$$

which implies that (ii_1) holds.

When (ii_1) holds we have that $\delta^{(3)}_{i,j}(1) \ge \alpha_{i,k}a_{i,k}a_{k,j} > 0$, i.e., the condition (ii_1) in Theorem 3.18 holds, so that Theorem C is valid.

Now, we prove (iii). Clearly, $\delta^{(3)}_{i,j(i)}(1) \ge \alpha_{i,k(i)}a_{i,k(i)}a_{k(i),j(i)} > 0$, which implies that the condition (iv_1) in Theorem 3.18 holds. The required result follows by (iv^a), (iv^b) and (iv^c) in Theorem 3.18, immediately. □

The later part of the condition (ii_1) is equivalent to that there exist positive elements in lower triangular part of the matrix Q_3U except the first column.

For (ii_1) we can choose some special $\{i,j,k\}$ to construct Q_3, e.g., $j = i$, $k = 1$, $k = j-1$, etc.

Similarly, by Theorem 3.19 we can prove the following result immediately.

Theorem 3.27. *Theorem D is valid for $\nu = 3$, provided one of the conditions (i), (ii) and (iii) of Theorem 3.26 is satisfied.*

Similar to Corollaries 3.20-3.23, from Theorems 3.24-3.27 we have the following corollaries, immediately.

Corollary 3.28. *Suppose that $0 \le \alpha_{i,j} \le 1$ and $\sum_{k=1}^{i-1} \alpha_{i,k}a_{i,k}a_{k,i} < 1$, $i = 2,\cdots,n$, $j < i$. Then Theorem A is valid for $\nu = 3$.*

Corollary 3.29. *Suppose that $0 \le \alpha_{i,j} \le 1$, $i = 2,\cdots,n$, $j < i$. Then Theorem B is valid for $\nu = 3$.*

Corollary 3.30. *Suppose that $\sum_{k=1}^{i-1} \alpha_{i,k}a_{i,k}a_{k,i} < 1$, $i = 2,\cdots,n$. Then Theorem C is valid for $\nu = 3$, provided one of the following conditions is satisfied:*

(i) *For* $i = 2, \cdots, n$, $j < i$, $0 \le \alpha_{i,j} \lesssim 1$. *One of the conditions* $0 \le \gamma < 1$ *and* (ii_1), (ii_2), (ii_3) *whenever* $\gamma = 1$ *in Theorem 3.26 holds.*

(ii) *The condition (iii) of Theorem 3.26 holds, where the inequality (24) is replaced by* $0 \le \alpha_{i,j} \le 1$, $i = 2, \cdots, n$, $j < i$.

Corollary 3.31. *Theorem D is valid for* $\nu = 3$, *provided one of the conditions (i) and (ii) of Corollary 3.30 is satisfied.*

Many known corresponding results about the preconditioned AOR method proposed in the references are the special cases of Theorems 3.24-3.27 and Corollaries 3.28-3.31, i.e., they can be derived from these theorems, immediately.

As a special case of Q_3, let

$$\alpha_{i,j} = \alpha, \ i = 2, \cdots, n, \ j < i,$$

with $\alpha > 0$. Then in [90] Q is defined as

$$Q_4 = \alpha L,$$

which is studied in [108]. When $\alpha = 1$ it is given in [58] for the preconditioned Gauss-Seidel method.

From Theorems 3.24-3.27 and Corollaries 3.28-3.31, we have the following comparison results.

Theorem 3.32. *Suppose that* $\alpha \sum_{k=1}^{i-1} a_{i,k} a_{k,i} < 1$, $i = 2, \cdots, n$, *and*

$$(1 - \alpha) a_{i,j} - \alpha \sum_{\substack{k=1 \\ k \ne j}}^{i-1} a_{i,k} a_{k,j} \le 0, \ i = 2, \cdots, n, j < i. \tag{26}$$

Then Theorem A is valid for $\nu = 4$.

Theorem 3.33. *Suppose that (26) holds. Then Theorem B is valid for* $\nu = 4$.

This theorem is better than the corresponding one given by [90, Theorem 2.1], where there are problems in the expression.

Theorem 3.34. *Suppose that* $\alpha \sum_{k=1}^{i-1} a_{i,k} a_{k,i} < 1$, $i = 2, \cdots, n$. *Then Theorem C is valid for* $\nu = 4$, *provided one of the following conditions is satisfied:*

(i) $0 \le \gamma < 1$ *and*

$$(1 - \alpha) a_{i,j} - \alpha \sum_{\substack{k=1 \\ k \ne j}}^{i-1} a_{i,k} a_{k,j} \lesssim 0 \ \text{ whenever } \ a_{i,j} < 0, \ i = 2, \cdots, n, \ j < i. \tag{27}$$

(ii) $\gamma = 1$, *(27) holds and one of the following conditions holds:*

- (ii_1) *There exist* $i \in \{2, \cdots, n\}$, $j \in \{2, \cdots, i\}$ *and* $k \in \{1, \cdots, j-1\}$ *such that* $a_{i,k} a_{k,j} > 0$.
- (ii_2) $a_{n,1} < 0$.
- (ii_3) $a_{1,2} < 0$.

(iii) $\gamma = 1$ *and (26) holds. For each* $i \in \{2, \cdots, n-1\}$ *there exist* $j(i) \in \{2, \cdots, i\}$ *and* $k(i) \in \{1, \cdots, j(i) - 1\}$ *such that* $a_{i,k(i)} a_{k(i),j(i)} > 0$. *And one of the following conditions holds:*

- (iii_1) *There exist* $j \in \{2, \cdots, n\}$ *and* $k \in \{1, \cdots, j-1\}$ *such that* $a_{n,k} a_{k,j} > 0$.
- (iii_2) $a_{n,1} < 0$.

(iii_3) There exists $j \in \{2, \cdots, n-1\}$ such that

$$(1-\alpha)a_{n,j} - \alpha \sum_{\substack{k=1 \\ k\neq j}}^{n-1} a_{n,k}a_{k,j} < 0.$$

Proof. By Theorem 3.26 we just need to prove (ii_3). In fact, if $a_{n,1} < 0$ then (ii_2) holds. If $a_{n,1} = 0$ then, from the irreducibility of A, $\sum_{i=2}^{n-1} a_{i,1} = \sum_{i=2}^{n} a_{i,1} < 0$ so that there exists $i_0 \in \{2, \cdots, n-1\}$ such that $a_{i_0,1} < 0$. Hence $a_{i_0,1}a_{1,2} > 0$. This shows that (ii_1) holds for $i = i_0$, $j = 2$ and $k = 1$. □

Theorem 3.35. *Theorem D is valid for $\nu = 4$, provided one of the conditions (i), (ii) and (iii) of Theorem 3.34 is satisfied.*

From Theorems 3.32-3.35, the following results are directly.

Corollary 3.36. *Suppose that $0 < \alpha \leq 1$ and $\alpha \sum_{k=1}^{i-1} a_{i,k}a_{k,i} < 1$, $i = 2, \cdots, n$. Then Theorem A is valid for $\nu = 4$.*

Corollary 3.37. *Suppose that $0 < \alpha \leq 1$. Then Theorem B is valid for $\nu = 4$.*

Corollary 3.38. *Suppose that $\alpha \sum_{k=1}^{i-1} a_{i,k}a_{k,i} < 1$, $i = 2, \cdots, n$. Then Theorem C is valid for $\nu = 4$, provided one of the following conditions is satisfied:*

(i) $0 < \alpha \lesssim 1$. One of the conditions $0 \leq \gamma < 1$ and (ii_1), (ii_2), (ii_3) whenever $\gamma = 1$ in Theorem 3.34 holds.

(ii) The condition (iii) of Theorem 3.34 holds, where the inequality (26) is replaced by $0 < \alpha \leq 1$.

The result when (i) holds is better than the corresponding one given by [108, Theorem 4.2].

If $\sum_{k=1}^{i-1} a_{i,k}a_{k,i} > 0$, $i = 2, \cdots, n$, then for each $i \in \{2, \cdots, n\}$, there exists $k(i) \in \{1, \cdots, i-1\}$ such that $a_{i,k(i)}a_{k(i),i} > 0$, which implies that (iii_1) in Theorem 3.34 holds. Hence, Corollary 3.38 when (ii) holds is better than the corresponding one given by [108, Theorem 4.1].

Corollary 3.39. *Theorem D is valid for $\nu = 4$, provided one of the conditions (i) and (ii) of Corollary 3.38 is satisfied.*

Specially, for some r, $2 \leq r \leq n$, $\alpha_{r,j} = \alpha_j \geq 0$, $j = 1, \cdots, r-1$, and $\alpha_{i,j} = 0$ otherwise, in [92] the matrix Q is defined as

$$Q_5 = \begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ -\alpha_1 a_{r,1} & \cdots & -\alpha_{r-1}a_{r,r-1} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}$$

with

$$\sum_{k=1}^{r-1} \alpha_k a_{r,k} \neq 0.$$

When $r = n$, it is proposed in [72] for the preconditioned Gauss-Seidel method.

In this case, for $i = 2, \cdots, n$, if $i \neq r$, then

$$\sum_{k=1}^{i-1} \alpha_{i,k}a_{i,k}a_{k,i} = 0$$

and

$$(1-\alpha_{i,j})a_{i,j} - \sum_{\substack{k=1\\k\neq i,j}}^{i-1} \alpha_{i,k}a_{i,k}a_{k,j} = a_{i,j}, \quad \text{for} \quad j < i.$$

Now, $\delta^{(3)}_{i,j}(1)$ reduces to

$$\delta^{(5)}_{i,j}(1) = \begin{cases} \sum\limits_{k=1}^{j-1} \alpha_k a_{r,k}a_{k,j}, & i = r, j = 2, \cdots, r; \\ 0, & \textit{otherwise}. \end{cases}$$

Hence, from Theorems 3.24-3.27 and Corollaries 3.28-3.31, we can obtain the following comparison results, directly.

Theorem 3.40. *Suppose that $\sum_{k=1}^{r-1} \alpha_k a_{r,k}a_{k,r} < 1$ and*

$$(1-\alpha_j)a_{r,j} - \sum_{\substack{k=1\\k\neq j}}^{r-1} \alpha_k a_{r,k}a_{k,j} \leq 0, \ j = 1, \cdots, r-1. \tag{28}$$

Then Theorem A is valid for $\nu = 5$.

Theorem 3.41. *Suppose that (28) holds. Then Theorem B is valid for $\nu = 5$.*

When $\alpha_j = 1$, $j = 1, \cdots, r-1$, the inequality (28) is trivial. Hence the result is better than [72, Theorem 2.9], where the convergence hypothesis of two Gauss-Seidel methods is unnecessary and the proof is insufficient, which is pointed out by [59]. While the condition $\rho(\mathscr{L}) > 0$ in [59, Theorem 3.2] is unnecessary.

Theorem 3.42. *Suppose that $\sum_{k=1}^{r-1} \alpha_k a_{r,k}a_{k,r} < 1$ and*

$$(1-\alpha_j)a_{r,j} - \sum_{\substack{k=1\\k\neq j}}^{r-1} \alpha_k a_{r,k}a_{k,j} \lesssim 0 \ \ \textit{whenever} \ \ a_{r,j} < 0, \ j = 1, \cdots, r-1. \tag{29}$$

Then Theorem C is valid for $\nu = 5$, provided one of the following conditions is satisfied:

(i) $0 \leq \gamma < 1$.

(ii) $\gamma = 1$ *and one of the following conditions holds:*

- *(ii_1) There exist $j \in \{2, \cdots, r\}$ and $k \in \{1, \cdots, j-1\}$ such that $\alpha_k a_{r,k}a_{k,j} > 0$.*
- *(ii_2) $a_{k,k+1} < 0$, $k = 1, \cdots, r-1$.*
- *(ii_3) $a_{k,r} < 0$, $k = 1, \cdots, r-1$.*
- *(ii_4) $r = n$, $a_{n,1} < 0$ and $\alpha_1 > 0$.*

Proof. By (i) and (ii) of Theorem 3.26, (i), (ii_1) and (ii_4) are derived directly.
By the definition of Q_5, there exists $k_0 \in \{1, \cdots, r-1\}$ such that $\alpha_{k_0}a_{r,k_0} < 0$.
If (ii_2) holds then $\alpha_{k_0}a_{r,k_0}a_{k_0,k_0+1} > 0$, which shows that ($ii_1$) holds for $j = k_0+1$ and $k = k_0$.
Similarly, if (ii_3) holds then $\alpha_{k_0}a_{r,k_0}a_{k_0,r} > 0$, which shows that ($ii_1$) holds for $j = r$ and $k = k_0$. □

Theorem 3.43. *Suppose that (29) holds. Then Theorem D is valid for $\nu = 5$, provided one of the conditions (i) and (ii) of Theorem 3.42 is satisfied.*

Corollary 3.44. *Suppose that $0 \leq \alpha_k \leq 1$, $k = 1, \cdots, r-1$, and $\sum_{k=1}^{r-1} \alpha_k a_{r,k}a_{k,r} < 1$. Then Theorem A is valid for $\nu = 5$.*

Corollary 3.45. *Suppose that* $0 \le \alpha_k \le 1$, $k = 1, \cdots, r-1$. *Then Theorem B is valid for* $\nu = 5$.

The result includes the corresponding one given in [92, Corollary 2.3].

Corollary 3.46. *Suppose that* $0 \le \alpha_j \lesssim 1$, $j = 1, \cdots, r-1$, *and* $\sum_{k=1}^{r-1} \alpha_k a_{r,k} a_{k,r} < 1$. *Then Theorem C is valid for* $\nu = 5$, *provided one of the conditions (i) and (ii) of Theorem 3.42 is satisfied.*

Corollary 3.47. *Suppose that* $0 \le \alpha_k \lesssim 1$, $j = 1, \cdots, r-1$. *Then Theorem D is valid for* $\nu = 5$, *provided one of the conditions (i) and (ii) of Theorem 3.42 is satisfied.*

Similarly, for some r, $2 \le r \le n$, $\alpha_{i,r-1} = \alpha_i \ge 0$, $i = r, \cdots, n$, and $\alpha_{i,j} = 0$ otherwise, the matrix Q_3 reduces to

$$Q_6 = \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -\alpha_r a_{r,r-1} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & -\alpha_n a_{n,r-1} & 0 & \cdots & 0 \end{pmatrix}$$

with

$$\sum_{k=r}^{n} \alpha_k a_{k,r-1} \ne 0.$$

When $r = 2$, it is investigated in [55, 60, 104], in [22] for the preconditioned SOR method and in [17] for the preconditioned Gauss-Seidel and Jacobi methods, respectively. When $r = 2$ and $\alpha_i = 1$, $i = 2, \cdots, n$, it is a special case in [66] for the preconditioned Gauss-Seidel and Jacobi methods, and it is used to the preconditioned AOR method in [52].

In this case, $\delta^{(3)}_{i,j}(1)$ reduces to

$$\delta^{(6)}_{i,j}(1) = \begin{cases} \alpha_i a_{i,r-1} a_{r-1,j}, & i = r, \cdots, n,\ j = r, \cdots, i; \\ 0, & \textit{otherwise}. \end{cases}$$

Theorem 3.48. *Suppose that* $0 \le \alpha_k \le 1$ *and* $\alpha_k a_{k,r-1} a_{r-1,k} < 1$, $k = r, \cdots, n$. *Then Theorem A is valid for* $\nu = 6$.

Proof. It is easy to prove that the condition of Corollary 3.28 is satisfied, so that Theorem A is valid. □

The result includes the corresponding one given by [55, Theorem 2.2-(a)]. The result for $\omega = \gamma$ includes the corresponding one given by [102, Theorem 3.3], where the condition is too strong.

Similarly, by Corollary 3.29 we can prove the following theorem.

Theorem 3.49. *Suppose that* $0 \le \alpha_k \le 1$, $k = r, \cdots, n$. *Then Theorem B is valid for* $\nu = 6$.

In order to give the Stein-Rosenberg Type Theorem II, we prove a lemma.

Lemma 3.50. *Let A be an irreducible Z-matrix. Assume that* $r = 2$, $0 < \alpha_k \le 1$, $k = 2, \cdots, n$ *and* $A^{(6)}$ *has the block form*

$$A^{(6)} = \begin{pmatrix} 1 & \bar{a}_{1,2} \\ \bar{a}^{(6)}_{2,1} & A^{(6)}_{2,2} \end{pmatrix},\ A^{(6)}_{2,2} \in \mathscr{R}^{(n-1)\times(n-1)}.$$

Then

(i) $A^{(6)}_{2,2}$ *is an irreducible Z-matrix.*

(ii) $A^{(6)}$ *is an irreducible Z-matrix if and only if there exists* $i_0 \in \{2, \cdots, n\}$ *such that* $(1 - \alpha_{i_0}) a_{i_0,1} \ne 0$.

Proof. Since

$$a_{i,j}^{(6)} = \begin{cases} a_{1,j}, & i = 1, j = 1, \cdots n, \\ (1-\alpha_i)a_{i,1}, & i = 2, \cdots, n, j = 1, \\ a_{i,j} - \alpha_i a_{i,1} a_{1,j}, & i, j = 2, \cdots, n, \end{cases}$$

then it is clearly that $a_{1,k}^{(6)} = a_{1,k} \le 0$, $a_{k,1}^{(6)} = (1-\alpha_k)a_{k,1} \le 0$ for $k = 2, \cdots, n$ and

$$a_{i,j}^{(6)} = a_{i,j} - \alpha_i a_{i,1} a_{1,j} \le a_{i,j} \le 0, \ i, j = 2, \cdots, n, \ i \ne j. \tag{30}$$

Hence, $A^{(6)}$ and $A_{2,2}^{(6)}$ are Z-matrices.

For any $i, j \in \{2, \cdots, n\}$, $i \ne j$, since A is irreducible, then there exists a path $\sigma_{i,j} = (j_0, j_1, \cdots, j_{l+1}) \in G(A)$ with $i = j_0$ and $j = j_{l+1}$.

If $j_k \in \{2, \cdots, n\}$ for $k = 1, \cdots, l$, then, by (30), it gets that $a_{j_k,j_{k+1}}^{(6)} \le a_{j_k,j_{k+1}} < 0$ and $a_{j_0,j_1}^{(6)} \le a_{j_0,j_1} < 0$, so that $\sigma_{i,j} \in G(A_{2,2}^{(6)})$.

For the case when there exists $s \in \{1, \cdots, l\}$ such that $j_s = 1$, we have $j_{s-1} > 1$, $j_{s+1} > 1$, $a_{j_{s-1},1} < 0$ and $a_{1,j_{s+1}} < 0$. By (30), it gets that $a_{j_{s-1},j_{s+1}}^{(6)} = a_{j_{s-1},j_{s+1}} - \alpha_{j_{s-1}} a_{j_{s-1},1} a_{1,j_{s+1}} \le -\alpha_{j_{s-1}} a_{j_{s-1},1} a_{1,j_{s+1}} < 0$. It follows that $\tilde{\sigma}_{i,j} = (j_0, \cdots, j_{s-1}, j_{s+1}, \cdots, j_{l+1}) \in G(A_{2,2}^{(6)})$.

We have proved (*i*).

The necessity of (*ii*) is obvious. Now we prove the sufficiency.

For any $i, j \in \{1, \cdots, n\}$, $i \ne j$, if $i, j \in \{2, \cdots, n\}$ then, by (*i*), there exists a path $\sigma_{i,j}$ such that $\sigma_{i,j} \in G(A_{2,2}^{(6)}) \subseteq G(A^{(6)})$.

For the case when $i = 1$, since A is an irreducible Z-matrix, then there exists $j_0 \in \{2, \cdots, n\}$ such that $a_{1,j_0}^{(6)} = a_{1,j_0} < 0$. By (*i*), there exists a path $\sigma_{j_0,j}$ such that $\sigma_{j_0,j} \in G(A_{2,2}^{(6)})$ so that $(1, \sigma_{j_0,j}) \in G(A^{(6)})$.

For the case when $j = 1$, there exists a path σ_{i,i_0} such that $\sigma_{i,i_0} \in G(A_{2,2}^{(6)})$. Since $a_{i_0,1}^{(6)} = (1-\alpha_{i_0})a_{i_0,1} \ne 0$, it follows that $(\sigma_{i,i_0}, 1) \in G(A^{(6)})$.

We have proved (*ii*). □

This lemma improves [11, Theorem 3].

Theorem 3.51. *Suppose that $\alpha_k a_{k,r-1} a_{r-1,k} < 1$, $k = r, \cdots, n$. Then Theorem C is valid for $\nu = 6$, provided one of the following conditions is satisfied:*

(i) $0 \le \gamma < 1$ *and* $0 \le \alpha_k \lesssim 1$, $k = r, \cdots, n$.

(ii) $\gamma = 1$ *and* $0 \le \alpha_k \lesssim 1$, $k = r, \cdots, n$. *And one of the following conditions holds:*

- *(ii_1) There exist $i \in \{r, \cdots, n\}$ and $j \in \{r, \cdots, i\}$ such that $\alpha_i a_{i,r-1} a_{r-1,j} > 0$.*
- *(ii_2)* $a_{r-1,r} < 0$.
- *(ii_3)* $r = 2$, $a_{n,1} < 0$ *and* $\alpha_n > 0$.

(iii) $r = 2$, $0 \le \gamma < 1$ *and* $0 < \alpha_k \le 1$, $k = 2, \cdots, n$.

(iv) $r = 2$, $\gamma = 1$ *and* $0 < \alpha_k \le 1$, $k = 2, \cdots, n$. *And one of the following conditions holds:*

- *(iv_1) There exist $i \in \{2, \cdots, n\}$ and $j \in \{2, \cdots, i\}$ such that $a_{i,1} a_{1,j} > 0$.*
- *(iv_2)* $a_{1,2} < 0$.
- *(iv_3)* $a_{n,1} < 0$.

Proof. By Corollary 3.30, (i), (ii_1) and (ii_3) are obvious.

Assume that (ii_2) holds. By the definition of Q_6, there exists $k_0 \in \{r, \cdots, n\}$ such that $\alpha_{k_0} a_{k_0,r-1} < 0$, so that $\alpha_{k_0} a_{k_0,r-1} a_{r-1,r} > 0$, which implies that (ii_1) holds for $i = k_0$ and $j = r$.

We now prove (iii) and (iv).

If there exists $i_0 \in \{2, \cdots, n\}$ such that $(1 - \alpha_{i_0}) a_{i_0,1} \neq 0$, then, by Lemma 3.50, $A^{(6)}$ is irreducible. By Corollary 3.22 and (i), (ii_8), (ii_2) in Theorem 3.18 we can derive (iii), (iv_1) and (iv_3), directly. When (iv_2) holds, then from the irreducibility of A, there exists $i_1 \in \{2, \cdots, n\}$ such that $a_{i_1,1} < 0$, so that $a_{i_1,1} a_{1,2} > 0$, which implies that (iv_1) holds for $i = i_1$ and $j = 2$.

For the case when $(1 - \alpha_k) a_{k,1} = 0$, $k = 2, \cdots, n$, then it gets that $\alpha_k = 1$ whenever $a_{k,1} < 0$. In this case the matrix $A^{(6)}$ can be partitioned as

$$A^{(6)} = \begin{pmatrix} 1 & \bar{a}_{1,2} \\ 0 & A_{2,2}^{(6)} \end{pmatrix},$$

where, by Lemma 3.50, $A_{2,2}^{(6)} \in \mathscr{R}^{(n-1)\times(n-1)}$ is an irreducible Z-matrix, so that it is also an L-matrix, since $a_{k,k}^{(6)} = 1 - \alpha_k a_{k,1} a_{1,k} > 0$ for $k = 2, \cdots, n$.

Denote $\rho = \rho(\mathscr{L}_{\gamma,\omega})$. Let $x > 0$ be its associated eigenvector.

By Lemma 3.1 we just need to consider the case when $\gamma \leq \omega$. By Lemma 2.9 it follows that $\rho > 0$ and $x \gg 0$.

Let

$$A_{2,2}^{(6)} = \bar{M}_{\gamma,\omega} - \bar{N}_{\gamma,\omega}$$

be the AOR splitting of $A_{2,2}^{(6)}$. Then

$$M_{\gamma,\omega}^{(6)} = \begin{pmatrix} \frac{1}{\omega} & 0 \\ 0 & \bar{M}_{\gamma,\omega} \end{pmatrix}, \; N_{\gamma,\omega}^{(6)} = \begin{pmatrix} \frac{1-\omega}{\omega} & -\bar{a}_{1,2} \\ 0 & \bar{N}_{\gamma,\omega} \end{pmatrix}$$

and

$$[M_{\gamma,\omega}^{(6)}]^{-1} = \begin{pmatrix} \omega & 0 \\ 0 & \bar{M}_{\gamma,\omega}^{-1} \end{pmatrix}, \; \mathscr{L}_{\gamma,\omega}^{(6)} = \begin{pmatrix} 1-\omega & -\omega\bar{a}_{1,2} \\ 0 & \bar{M}_{\gamma,\omega}^{-1}\bar{N}_{\gamma,\omega} \end{pmatrix}.$$

Let $\bar{E}_1$ and $\bar{F}_1$ be diagonal part and strictly lower triangular part of $Q_6 U$ with block forms

$$\bar{E}_1 = \begin{pmatrix} e_{1,1} & 0 \\ 0 & E_{2,2} \end{pmatrix}, \; \bar{F}_1 = \begin{pmatrix} f_{1,1} & 0 \\ f_{1,2} & F_{2,2} \end{pmatrix}, \; E_{2,2}, F_{2,2} \in \mathscr{R}^{(n-1)\times(n-1)}.$$

Then $e_{1,1} = f_{1,1} = 0$ and $f_{1,2} = 0$. Let

$$Q_6 = \begin{pmatrix} q_{1,1}^{(6)} & \bar{q}_{1,2} \\ \bar{q}_{2,1} & Q_{2,2} \end{pmatrix}, \; Q_{2,2} \in \mathscr{R}^{(n-1)\times(n-1)}, \; x = \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix}, \; \bar{x}_1 \in \mathfrak{R}, \; \bar{x}_2 \in \mathscr{R}^{n-1}.$$

Then $q_{1,1}^{(6)} = 0$, $\bar{q}_{1,2} = 0$, $\bar{q}_{2,1} > 0$, $Q_{2,2} = 0$, $\bar{x}_1 > 0$ and $\bar{x}_2 \gg 0$. Now, by Lemma 3.2, we obtain

$$\begin{aligned}
\mathscr{L}_{\gamma,\omega}^{(6)} x - \rho x &= \begin{pmatrix} (1-\omega)\bar{x}_1 - \omega\bar{a}_{1,2}\bar{x}_2 - \rho\bar{x}_1 \\ \bar{M}_{\gamma,\omega}^{-1}\bar{N}_{\gamma,\omega}\bar{x}_2 - \rho\bar{x}_2 \end{pmatrix} = (\rho - 1)[\bar{E}_1 + \gamma\bar{F}_1 + (1-\gamma)Q_6]x \\
&= (\rho - 1)\begin{pmatrix} \omega & 0 \\ 0 & \bar{M}_{\gamma,\omega}^{-1} \end{pmatrix}\left[\begin{pmatrix} 0 & 0 \\ 0 & E_{2,2} \end{pmatrix} + \gamma\begin{pmatrix} 0 & 0 \\ 0 & F_{2,2} \end{pmatrix} + (1-\gamma)\begin{pmatrix} 0 & 0 \\ \bar{q}_{2,1} & 0 \end{pmatrix}\right]\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix} \\
&= (\rho - 1)\begin{pmatrix} 0 \\ \bar{M}_{\gamma,\omega}^{-1}[(E_{2,2} + \gamma F_{2,2})\bar{x}_2 + (1-\gamma)\bar{q}_{2,1}\bar{x}_1] \end{pmatrix}.
\end{aligned}$$

Hence, we have

$$(1-\omega)\bar{x}_1 - \rho\bar{x}_1 = \omega\bar{a}_{1,2}\bar{x}_2 \tag{31}$$

and

$$\bar{M}_{\gamma,\omega}^{-1}\bar{N}_{\gamma,\omega}\bar{x}_2 - \rho\bar{x}_2 = (\rho-1)\bar{M}_{\gamma,\omega}^{-1}[(E_{2,2}+\gamma F_{2,2})\bar{x}_2 + (1-\gamma)\bar{q}_{2,1}\bar{x}_1]. \tag{32}$$

Since A is an irreducible L-matrix, then $\bar{a}_{1,2} < 0$ so that $\omega\bar{a}_{1,2}\bar{x}_2 < 0$. From (31), it gets that $1-\omega < \rho$.

When (*iii*) holds, i.e., $\gamma < 1$, since $\bar{A}_{2,2}$ is an irreducible L-matrix, then it can derive that $\bar{M}_{\gamma,\omega}^{-1}\bar{N}_{\gamma,\omega} \geq 0$ is irreducible. Furthermore, $\bar{M}_{\gamma,\omega}^{-1}[(E_{2,2}+\gamma F_{2,2})\bar{x}_2 + (1-\gamma)\bar{q}_{2,1}\bar{x}_1] \geq (1-\gamma)\bar{M}_{\gamma,\omega}^{-1}\bar{q}_{2,1}\bar{x}_1 > 0$, since $\bar{q}_{2,1} > 0$ and $\bar{x}_1 > 0$.

Now, Theorem 3.48 has shown that $\rho = 1$ if and only if $\rho(\mathscr{L}_{\gamma,\omega}^{(6)}) = 1$. If $\rho < 1$ then, from (32), it gets that $\bar{M}_{\gamma,\omega}^{-1}\bar{N}_{\gamma,\omega}\bar{x}_2 < \rho\bar{x}_2$ so that $\rho(\bar{M}_{\gamma,\omega}^{-1}\bar{N}_{\gamma,\omega}) < \rho$. Hence, we derive $\rho(\mathscr{L}_{\gamma,\omega}^{(6)}) = \max\{1-\omega,\ \rho(\bar{M}_{\gamma,\omega}^{-1}\bar{N}_{\gamma,\omega})\} < \rho$. If $\rho > 1$ then, from (32), it gets that $\bar{M}_{\gamma,\omega}^{-1}\bar{N}_{\gamma,\omega}\bar{x}_2 > \rho\bar{x}_2$ so that $\rho(\bar{M}_{\gamma,\omega}^{-1}\bar{N}_{\gamma,\omega}) > \rho$. Hence, we derive $\rho(\mathscr{L}_{\gamma,\omega}^{(6)}) = \max\{1-\omega,\ \rho(\bar{M}_{\gamma,\omega}^{-1}\bar{N}_{\gamma,\omega})\} = \rho(\bar{M}_{\gamma,\omega}^{-1}\bar{N}_{\gamma,\omega}) > \rho$.

When (*iv*) holds, then $\omega = 1$ and the AOR method reduces to the Gauss-Seidel method. In this case, we have

$$\mathscr{L}^{(6)} = \begin{pmatrix} 0 & -\bar{a}_{1,2} \\ 0 & \bar{M}_{1,1}^{-1}\bar{N}_{1,1} \end{pmatrix}$$

and therefore $\rho(\mathscr{L}^{(6)}) = \rho(\bar{M}_{1,1}^{-1}\bar{N}_{1,1})$.

Above we have proved that $\bar{A}_{2,2}^{(6)}$ is an irreducible L-matrix. By Lemma 2.9 it gets that $\bar{y}^T\bar{M}_{1,1}^{-1} \gg 0$ whenever $\bar{y}$ satisfies $\bar{y} > 0$ and $\bar{y}^T\bar{M}_{1,1}^{-1}\bar{N}_{1,1} = \rho(\bar{M}_{1,1}^{-1}\bar{N}_{1,1})\bar{y}^T$. Multiply $\bar{y}^T$ on the left side of (32), we can derive

$$\rho(\bar{M}_{1,1}^{-1}\bar{N}_{1,1})\bar{y}^T\bar{x}_2 - \rho\bar{y}^T\bar{x}_2 = (\rho-1)\bar{y}^T\bar{M}_{\gamma,\omega}^{-1}(E_{2,2}+F_{2,2})\bar{x}_2.$$

When (*iv*$_1$) holds, i.e., there exist $i \in \{2,\cdots,n\}$ and $j \in \{2,\cdots,i\}$ such that $a_{i,1}a_{1,j} > 0$, it is easy to prove that $E_{2,2} > 0$ or $F_{2,2} > 0$ so that $(E_{2,2}+F_{2,2})\bar{x}_2 > 0$ and $\bar{y}^T\bar{M}_{\gamma,\omega}^{-1}(E_{2,2}+F_{2,2})\bar{x}_2 > 0$. Since $\bar{y}^T\bar{x}_2 > 0$, then we can get

$$\rho(\mathscr{L}^{(6)}) = \rho(\bar{M}_{1,1}^{-1}\bar{N}_{1,1}) \begin{cases} < \rho, & \text{if} \quad \rho < 1 \\ = \rho, & \text{if} \quad \rho = 1 \\ > \rho, & \text{if} \quad \rho > 1. \end{cases}$$

When (*iv*$_2$) holds, the irreducibility of A or the definition of Q_6 ensures that there exists $i \in \{2,\cdots,n\}$ such that $a_{i,1} < 0$, we have $a_{i,1}a_{1,2} > 0$, which implies that (*iv*$_1$) holds.

Similarly, when (*iv*$_3$) holds, the irreducibility of A ensures that there exists $j \in \{2,\cdots,n\}$ such that $a_{1,j} < 0$, we have $a_{n,1}a_{1,j} > 0$, which also implies that (*iv*$_1$) holds. □

The result for the case when $r = 2$ is better than the corresponding ones given by [52, Theorem 1, Corollary 1], [60, Theorems 3.3, 3.4, 3.5] and [104, Theorems 3.11, 3.13, 3.14 and 3.15]. The proof of [52, Theorem 1] is insufficient, which is pointed out by [107] and [11]. When $r = 2$ and $\gamma = \omega$, the result is better than [22, Theorems 2.1 and 2.2], where the condition $a_{k,k+1}a_{k+1,k} > 0$, $k = 1,\cdots,n-1$, implies that A is irreducible. While the proofs in [22] are insufficient, which is pointed out by [102]. The comparison result [55, Theorem 2.2-(b)] is problematic, because [55, Lemma 2.1] is wrong, which has been shown by [107, Example 3.1].

From Theorem 3.51, we can prove the following theorem.

Theorem 3.52. *Theorem D is valid for $\nu = 6$, provided one of the conditions (i)-(iv) of Theorem 3.51 is satisfied.*

When $r = 2$, $\alpha_k = 1$, $k = 2,\cdots,n$, the results given in Theorems 3.49 and 3.52 are better than [74, Theorem 3.4] and [76, Theorem 3.4].

As a special case of $Q_6(Q_5)$, for some $r > s$ with $a_{r,s} < 0$ and $\alpha > 0$, the matrix $Q_6(Q_5)$ reduces to

$$Q_7 = \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \cdots & \vdots \\ \vdots & \cdots & 0 & \cdots & 0 \\ \vdots & -\frac{a_{r,s}}{\alpha} & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix},$$

which is proposed in [23, 53, 104] for $r = n$ and $s = 1$. And for $r = n$, $s = 1$ and $\alpha = 1$ in [14]. It is given in [109] to replace $-a_{r,s}/\alpha$ with a constant β.

Now, $\delta^{(6)}_{i,j}(1)$ reduces to

$$\delta^{(7)}_{i,j}(1) = \begin{cases} \frac{1}{\alpha} a_{r,s} a_{s,j}, & i = r, j = s + 1, \cdots, r; \\ 0, & otherwise. \end{cases}$$

From Theorems 3.48 and 3.49, we can obtain the following comparison results, directly.

Theorem 3.53. *Suppose that $\alpha \geq 1$ and $\alpha > a_{r,s} a_{s,r}$. Then Theorem A is valid for $\nu = 7$.*

This result is better than that given by [104, Theorem 3.7], where A is assumed to be irreducible.

Theorem 3.54. *Suppose that $\alpha \geq 1$. Then Theorem B is valid for $\nu = 7$.*

This result includes [105, Theorem 3.4] and the corresponding one given in [45, Theorem 3.1].

In order to give the Stein-Rosenberg Type Theorem II, we prove a lemma.

Lemma 3.55. *Let A be an irreducible Z-matrix. Assume that $r = n$, $\alpha \geq 1$ and $A^{(7)}$ has the block form*

$$A^{(7)} = \begin{pmatrix} 1 & \bar{a}_{1,2} \\ \bar{a}^{(7)}_{2,1} & A^{(7)}_{2,2} \end{pmatrix}, \ A^{(7)}_{2,2} \in \mathscr{R}^{(n-1)\times(n-1)}.$$

Then one of the following two mutually exclusive relations holds:

(i) *$A^{(7)}$ is an irreducible Z-matrix.*

(ii) *$A^{(7)}$ is a reducible Z-matrix, but $A^{(7)}_{2,2}$ is an irreducible Z-matrix and $a_{k,1} = a^{(7)}_{k,1} = a^{(7)}_{n,1} = 0$, $k = 2, \cdots, n-1$.*

Proof. Since

$$a^{(7)}_{i,j} = \begin{cases} a_{i,j}, & i = 1, \cdots n-1, j = 1, \cdots n, \\ (1 - \frac{1}{\alpha}) a_{n,1}, & i = n, j = 1, \\ a_{n,j} - \frac{1}{\alpha} a_{n,1} a_{1,j}, & i = n, j = 2, \cdots, n, \end{cases} \tag{33}$$

then it is clearly that, $a^{(7)}_{i,j} = a_{i,j} \leq 0$ for $i = 1, \cdots, n-1$, $j = 1, \cdots, n$, $i \neq j$, $a^{(7)}_{n,1} = (1 - 1/\alpha) a_{n,1} \leq 0$ and

$$a^{(7)}_{n,j} = a_{n,j} - \frac{1}{\alpha} a_{n,1} a_{1,j} \leq a_{n,j} \leq 0, \ j = 2, \cdots, n. \tag{34}$$

Hence, $A^{(7)}$ and $A^{(7)}_{2,2}$ are Z-matrices.

Let

$$A = \begin{pmatrix} \hat{a}_{1,1} & \hat{A}_{1,2} \\ a_{n,1} & \hat{a}_{2,2} \end{pmatrix}, \ \hat{A} = \begin{pmatrix} \hat{a}_{1,1} & \hat{A}_{1,2} \\ 0 & \hat{a}_{2,2} \end{pmatrix}, \ \hat{A}_{1,2} \in \mathscr{R}^{(n-1)\times(n-1)}.$$

Clearly, if $\hat{A}$ is irreducible then $A^{(7)}$ is irreducible, since A is irreducible. When $\alpha > 1$ then $A^{(7)}$ is also irreducible.

Assume that $A^{(7)}$ is reducible. Then $\hat{A}$ is reducible and $\alpha = 1$. The latter implies $a^{(7)}_{n,1} = 0$. In this case, there must be $i^*, j^* \in \{1, \cdots, n\}$ such that there is no path from i^* to j^* in $G(A^{(7)})$.

It is easy to see that, for any $i, j \in \{1, \cdots, n\}$, $\sigma_{i,j} = (j_0, j_1, \cdots, j_{l+1}) \notin G(A^{(7)})$ with $i = j_0$ and $j = j_{l+1}$ if and only if there is $s \in \{0, \cdots, l\}$ such that $i_s = n$ and $i_{s+1} = 1$.

Since A is irreducible, then there exists $\sigma_{i^*,j^*} = (i_0, i_1, \cdots, i_{t+1}) \in G(A)$ with $i^* = i_0$ and $j^* = i_{t+1}$.

If $j^* > 1$ then there is $\mu \in \{0, \cdots, t-1\}$ such that $i_\mu = n$ and $i_{\mu+1} = 1$, which implies $a_{1,i_{\mu+2}} < 0$. Hence, by (34), we have $a^{(7)}_{n,i_{\mu+2}} = a_{n,i_{\mu+2}} - a_{n,1}a_{1,i_{\mu+2}} \le -a_{n,1}a_{1,i_{\mu+2}} < 0$, so that $\sigma = (i_0, \cdots, i_{\mu-1}, n, i_{\mu+2}, \cdots, i_{t+1}) \in G(A^{(7)})$. This is a contradiction. Therefore, $j^* = 1$. This shows that, for any $k \in \{2, \cdots, n-1\}$, there exists $\sigma_{i^*,k} \in G(A^{(7)})$. If $a^{(7)}_{k,1} < 0$ then $\tilde{\sigma}_{i^*,j^*} = (\sigma_{i^*,k}, 1) \in G(A^{(7)})$. This is also a contradiction. Therefore, $a^{(7)}_{k,1} = a_{k,1} = 0$.

Now, let us prove the irreducibility of $A^{(7)}_{2,2}$. For any $i, j \in \{2, \cdots, n\}$, $i \ne j$, since A is irreducible, then there exists a path $\sigma_{i,j} = (\tau_0, \tau_1, \cdots, \tau_{v+1}) \in G(A)$ with $i = \tau_0$ and $j = \tau_{v+1}$.

If $\tau_k \in \{2, \cdots, n\}$ for $k = 1, \cdots, v$, then it gets that $\sigma_{i,j} \in G(A^{(7)}_{2,2})$.

For the case when there exists $s \in \{1, \cdots, v\}$ such that $\tau_s = 1$, we have $a_{\tau_{s-1},1} < 0$ and $a_{1,\tau_{s+1}} < 0$. By (33), τ_{s-1} must be n, since $a_{k,1} = a^{(7)}_{k,1} = 0$ for $k = 2, \cdots, n-1$. By (34), it gets that $a^{(7)}_{n,\tau_{s+1}} = a_{n,\tau_{s+1}} - a_{n,1}a_{1,\tau_{s+1}} \le -a_{n,1}a_{1,\tau_{s+1}} < 0$. This shows that $\tilde{\sigma}_{i,j} = (\tau_0, \cdots, \tau_{s-2}, n, \tau_{s+1}, \cdots, \tau_{v+1}) \in G(A^{(7)}_{2,2})$.

This has proved that $G(A^{(7)}_{2,2})$ is irreducible.

We have proved (*ii*). □

Using this lemma, completely similar to the proof of Theorem 3.51, we can prove the following theorem.

Theorem 3.56. *Suppose that $\alpha > a_{r,s}a_{s,r}$. Then Theorem C is valid for $\nu = 7$, provided one of the following conditions is satisfied:*

(i) $\alpha \gtrsim 1$. *And one of the following conditions holds:*

- *(i_1)* $0 \le \gamma < 1$.
- *(i_2)* $\gamma = 1$ *and there exists $k \in \{s+1, \cdots, r\}$ such that $a_{s,k} < 0$.*

(ii) $r = n$, $s = 1$ *and* $\alpha \ge 1$.

The result when (*ii*) holds includes [104, Theorems 3.8 and 3.9] and [53, Theorem 1], where the proof is insufficient, which is pointed out by [103]. We also have to point out that there exist some mistakes in [23, Theorems 4 and 5]. For $\alpha = 1$, the result is better than the corresponding ones given by [54, Theorem 2.1, Corollaries 2.1, 2.2], where the condition $a_{k,k+1}a_{k+1,k} > 0$, $k = 1, \cdots, n-1$, implies that A is irreducible and so that the condition $a_{1,n}a_{n,1} > 0$ is unnecessary.

From Theorem 3.56, we can prove the following theorem.

Theorem 3.57. *Theorem D is valid for $\nu = 7$, provided one of the conditions (i) and (ii) of Theorem 3.56 is satisfied.*

For the case when (*ii*) in Theorem 3.56 is satisfied, Theorem 3.57 is better than [14, Theorem 2.2] and the corresponding ones given in [45, Theorem 3.1, Corollaries 3.1, 3.2, 3.3]. In [14, Theorem 2.2], the condition $a_{k,k+1}a_{k+1,k} > 0$, $k = 1, \cdots, n-1$, implies that A is irreducible and $\rho(\mathscr{L}_{\gamma,\omega}) < 1$ implies that A is a nonsingular M-matrix. The condition $a_{n,1}a_{1,n} > 0$ is unnecessary.

In [61], for the preconditioned Gauss-Seidel method, a special case of the matrix Q_3 is proposed as

$$Q_8 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ -\alpha_1 a_{2,1} & 0 & \cdots & 0 & 0 \\ 0 & -\alpha_2 a_{3,2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\alpha_{n-1}a_{n,n-1} & 0 \end{pmatrix}$$

with $\alpha_k \ge 0$, $k = 1, \cdots, n-1$, and

$$\sum_{k=1}^{n-1} \alpha_k a_{k+1,k} \ne 0.$$

It is used to the preconditioned AOR method in [19, 39].

In this case, for $i = 2, \cdots, n$,

$$\sum_{k=1}^{i-1} \alpha_{i,k} a_{i,k} a_{k,i} = \alpha_{i-1} a_{i,i-1} a_{i-1,i}.$$

Now, $\delta^{(3)}_{i,j}(1)$ reduces to

$$\delta^{(8)}_{i,j}(1) = \begin{cases} \alpha_{i-1} a_{i,i-1} a_{i-1,i}, & i = j = 2, \cdots, n; \\ 0, & \textit{otherwise}. \end{cases}$$

When $n \ge 3$, the conditions (ii_2) and (iii_2) in Theorem 3.26 can be not satisfied. By Corollaries 3.28 and 3.29, it is easy to prove the following comparison results.

Theorem 3.58. *Suppose that* $0 \le \alpha_k \le 1$ *and* $\alpha_k a_{k,k+1} a_{k+1,k} < 1$, $k = 1, \cdots, n-1$. *Then Theorem A is valid for* $\nu = 8$.

Theorem 3.59. *Suppose that* $0 \le \alpha_k \le 1$, $k = 1, \cdots, n-1$. *Then Theorem B is valid for* $\nu = 8$.

In order to give the Stein-Rosenberg Type Theorem II, we prove a lemma.

Lemma 3.60. *Let A be a Z-matrix. Then A and* $A^{(8)}$ *are irreducible Z-matrices, provided one of the following conditions is satisfied:*

(i) $a_{k,k+1} a_{k+1,k} > 0$, $0 < \alpha_k \le 1$, $k = 1, \cdots, n-1$.

(ii) $n \ge 3$, $a_{n,1} < 0$, $a_{k,k+1} < 0$, $0 \le \alpha_k \le 1$, $k = 1, \cdots, n-1$.

Proof. The condition (i) implies $a_{k,k+1} < 0$ and $a_{k+1,k} > 0$, $k = 1, \cdots, n-1$. Hence, if one of (i) and (ii) holds, then it is easy to prove that A is irreducible.

Since

$$a^{(8)}_{i,j} = \begin{cases} a_{1,j} \le 0, & i = 1, j = 2, \cdots n, \\ (1 - \alpha_{i-1}) a_{i,i-1} \le 0, & i = 2, \cdots, n, j = i-1, \\ a_{i,j} - \alpha_{i-1} a_{i,i-1} a_{i-1,j} \le a_{i,j} \le 0, & i = 2, \cdots, n, j = 1, \cdots, n, j \ne i, i-1, \end{cases} \tag{35}$$

then $A^{(8)}$ is a Z-matrix.

When (i) holds, for any $i, j \in \{1, \cdots, n\}$, $i \ne j$, since A is irreducible, then there exists a path $\sigma_{i,j} = (j_{(0)}, j_{(1)}, \cdots, j_{(l+1)}) \in G(A)$ with $i = j_{(0)}$ and $j = j_{(l+1)}$.

By (35), it is obviously that either if there is no $j_{(k+1)} = j_{(k)} - 1$, $k \in \{0, 1, \cdots, l\}$, or if there exists some $s \in \{0, 1, \cdots, l\}$ such that $j_{(s+1)} = j_{(s)} - 1$ but $\alpha_{j_{(s)}-1} < 1$, then $\sigma_{i,j} \in G(A^{(8)})$.

For the case when $j_{(s+1)} = j_{(s)} - 1$ and $\alpha_{j_{(s)}-1} = 1$ for some $s \in \{0, 1, \cdots, l\}$, then $\sigma_{i,j} \notin G(A^{(8)})$, since $a^{(8)}_{j_{(s)},j_{(s+1)}} = a^{(8)}_{j_{(s)},j_{(s)}-1} = 0$. If $j_{(s)} < n$, then it gets that $a^{(8)}_{j_{(s)},j_{(s)}+1} = a_{j_{(s)},j_{(s)}+1} - a_{j_{(s)},j_{(s)}-1} a_{j_{(s)}-1,j_{(s)}+1} \le a_{j_{(s)},j_{(s)}+1} < 0$ and $a^{(8)}_{j_{(s)}+1,j_{(s)}-1} = a_{j_{(s)}+1,j_{(s)}-1} - \alpha_{j_{(s)}} a_{j_{(s)}+1,j_{(s)}} a_{j_{(s)},j_{(s)}-1} \le -\alpha_{j_{(s)}} a_{j_{(s)}+1,j_{(s)}} a_{j_{(s)},j_{(s)}-1} < 0$. While, if $j_{(s)} = n$, then $j_{(s+1)} = n-1$. In this case it gets that $a^{(8)}_{n,n-2} = a_{n,n-2} - a_{n,n-1} a_{n-1,n-2} \le -a_{n,n-1} a_{n-1,n-2} < 0$ and $a^{(8)}_{n-2,n-1} = a_{n-2,n-1} - \alpha_{n-3} a_{n-2,n-3} a_{n-3,n-1} \le a_{n-2,n-1} < 0$. Now we can construct a path $\sigma^{(1)}_{i,j} = (j_{(0)}, \cdots, j_{(s)}, j_{(s)} + 1, j_{(s+1)}, \cdots, j_{(l+1)})$ whenever $j_{(s)} < n$ or $\sigma^{(1)}_{i,j} = (j_{(0)}, \cdots, j_{(s-1)}, n, n-2, n-1, j_{(s+2)}, \cdots, j_{(l+1)})$ whenever $j_s = n$. To continue this process, we can eventually construct a path $\sigma^{(t)}_{i,j}$, $t \le l$, such that $\sigma^{(t)}_{i,j} \in G(A^{(8)})$. We have proved that $A^{(8)}$ is irreducible.

When (*ii*) holds, then by (35) we can obtain $a^{(8)}_{1,2} = a_{1,2} < 0$, $a^{(8)}_{k,k+1} \le a_{k,k+1} - \alpha_{k-1}a_{k,k-1}a_{k-1,k+1} \le a_{k,k+1} < 0$, $k = 2, \cdots, n-1$, $a^{(8)}_{n,1} \le a_{n,1} - \alpha_{n-1}a_{n,n-1}a_{n-1,1} \le a_{n,1} < 0$. From this it is easy to see that $A^{(8)}$ is irreducible. □

Theorem 3.61. *Suppose that $\alpha_k a_{k,k+1}a_{k+1,k} < 1$, $k = 1, \cdots, n-1$. Then Theorem C is valid for $\nu = 8$, provided one of the following conditions is satisfied:*

(i) $0 \le \alpha_k \lesssim 1$, $k = 1, \cdots, n-1$. *And one of the following conditions holds:*

- *(i_1)* $0 \le \gamma < 1$.
- *(i_2)* $\gamma = 1$ *and there exists* $k \in \{1, \cdots, n-1\}$ *such that* $\alpha_k a_{k,k+1}a_{k+1,k} > 0$.
- *(i_3)* $\gamma = 1$ *and* $a_{k,k+1} < 0$, $k = 1, \cdots, n-1$.

(ii) $a_{k,k+1}a_{k+1,k} > 0$ *and* $0 < \alpha_k \le 1$, $k = 1, \cdots, n-1$.

(iii) $n \ge 3$, $a_{n,1} < 0$, $a_{k,k+1} < 0$, $0 \le \alpha_k \le 1$, $k = 1, \cdots, n-1$.

Proof. By (*i*) of Corollary 3.30, (*i*), (*ii*$_1$) and (*ii*$_3$) in Theorem 3.26, (*i*) follows directly.

If (*ii*) holds, then by Lemma 3.60 $A^{(8)}$ is an irreducible L-matrix. Now, for $0 \le \gamma < 1$ the condition (*i*) of Theorem 3.6 is satisfied. By the definition of Q_8, there exists some $k_0 \in \{1, \cdots, n-1\}$ such that $\alpha_{k_0} > 0$. Hence, for $\gamma = 1$, we can prove that (*ii*$_4$) in Theorem 3.6 holds, since $\alpha_{k_0}a_{k_0+1,k_0}a_{k_0,k_0+1} > 0$.

When (*iii*) holds, the proof is completely same. □

Obviously, from Lemma 3.60, if (*ii*) or (*iii*) holds, then the assumption that A is irreducible is redundant.

This theorem when (*ii*) holds is better than the results in [39]. The corresponding result in [19, Theorem 3.2] is problematic, because [19, Lemma 3.1] is wrong. In fact, Let

$$A = \begin{pmatrix} 1 & -0.5 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

Then it is easy to prove that A is an irreducible L-matrix and it satisfies the assumption of [19, Lemma 3.1]. But the iteration matrices of the preconditioned AOR methods are reducible when we choose $\alpha_3 = 1$.

The following result is easy to prove.

Theorem 3.62. *Theorem D is valid for $\nu = 8$, provided one of the conditions (i), (ii) and (iii) of Theorem 3.61 is satisfied.*

Different from Q_5, a special Q is proposed as

$$Q_9 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ -\alpha_1 a_{n,1} + \beta_1 & -\alpha_2 a_{n,2} + \beta_2 & \cdots & -\alpha_{n-1}a_{n,n-1} + \beta_{n-1} & 0 \end{pmatrix}$$

with $\alpha_k \ge 0$, $-\alpha_k a_{n,k} + \beta_k \ge 0$, $k = 1, \cdots, n-1$, and $\sum_{k=1}^{n-1}(-\alpha_k a_{n,k} + \beta_k) \ne 0$. For $\alpha_k = \alpha \ge 0$, $\beta_k = \beta \ge 0$ and $\alpha + \beta \ne 0$, it is given in [18].

In this case, we have that

$$a^{(9)}_{i,j} = a_{i,j},\ i = 1, \cdots, n-1,\ j = 1, \cdots, n,$$

$$a^{(9)}_{n,j} = (1 - \alpha_j)a_{n,j} + \beta_j + \sum_{\substack{k=1\\k\ne j}}^{n-1}(-\alpha_k a_{n,k} + \beta_k)a_{k,j},\ j = 1, \cdots, n-1,$$

$$a^{(9)}_{n,n} = 1 + \sum_{k=1}^{n-1}(-\alpha_k a_{n,k} + \beta_k)a_{k,n},$$

so that

$$q_{i,j}^{(9)} = q_{n,n}^{(9)} = 0,\ i = 1, \cdots, n-1,\ j = 1, \cdots, n,$$
$$q_{n,j}^{(9)} = -\alpha_j a_{n,j} + \beta_j,\ j = 1, \cdots, n-1$$

and

$$\sum_{\substack{k=1\\k\neq i}}^{n} q_{i,k}^{(9)} a_{k,i} = \begin{cases} 0, & i = 1, \cdots, n-1,\\ \sum\limits_{k=1}^{n-1} (-\alpha_k a_{n,k} + \beta_k) a_{k,n}, & i = n. \end{cases}$$

Now, $\delta_{i,j}^{(1)}(1)$ reduces to

$$\delta_{i,j}^{(9)}(1) = \begin{cases} \sum\limits_{k=1}^{j-1} (\alpha_k a_{n,k} - \beta_k) a_{k,j}, & i = n, j = 2, \cdots, n;\\ 0, & \text{otherwise}. \end{cases}$$

Hence, by Corollaries 3.8-3.11, we can prove the following comparison theorems directly, where the proof of Theorem 3.65 is similar to that of Theorem 3.42.

Theorem 3.63. *Suppose that* $\sum_{k=1}^{n-1} (\alpha_k a_{n,k} - \beta_k) a_{k,n} < 1$ *and*

$$(1-\alpha_j) a_{n,j} + \beta_j + \sum_{\substack{k=1\\k\neq j}}^{n-1} (-\alpha_k a_{n,k} + \beta_k) a_{k,j} \leq 0,\ j = 1, \cdots, n-1. \tag{36}$$

Then Theorem A is valid for $\nu = 9$.

Theorem 3.64. *Suppose that (36) holds. Then Theorem B is valid for* $\nu = 9$.

The results given by Theorems 3.63 and 3.64 include the corresponding ones given in [18, Theorem 2.3], where $1 \leq j \leq n$ should be $1 \leq j \leq n-1$.

Theorem 3.65. *Suppose that* $\sum_{k=1}^{n-1} (\alpha_k a_{n,k} - \beta_k) a_{k,n} < 1$ *and*

$$(1-\alpha_j) a_{n,j} + \beta_j + \sum_{\substack{k=1\\k\neq j}}^{n-1} (-\alpha_k a_{n,k} + \beta_k) a_{k,j} \begin{cases} \leq 0, \\ \lneqq 0 \text{ whenever } a_{n,j} < 0, \end{cases} \quad j = 1, \cdots, n-1. \tag{37}$$

Then Theorem C is valid for $\nu = 9$, *provided one of the following conditions is satisfied:*

(i) $0 \leq \gamma < 1$.

(ii) $\gamma = 1$ *and one of the following conditions holds:*

- *(ii_1)* *There exist* $i \in \{2, \cdots, n\}$ *and* $j \in \{1, \cdots, i-1\}$ *such that* $(\alpha_j a_{n,j} - \beta_j) a_{j,i} > 0$.
- *(ii_2)* $\alpha_1 a_{n,1} - \beta_1 < 0$.
- *(ii_3)* $a_{k,k+1} < 0, k = 1, \cdots, n-1$.
- *(ii_4)* $a_{k,n} < 0, k = 1, \cdots, n-1$.

Theorem 3.66. *Suppose that (37) holds. Then Theorem D is valid for* $\nu = 9$, *provided one of the conditions (i) and (ii) of Theorem 3.65 is satisfied.*

From Theorems 3.63-3.66, the following results are directly.

Corollary 3.67. *Suppose that* $\sum_{k=1}^{n-1}(\alpha_k a_{n,k} - \beta_k)a_{k,n} < 1$ *and* $(1-\alpha_j)a_{n,j} + \beta_j \le 0$, $j = 1, \cdots, n-1$. *Then Theorem A is valid for* $\nu = 9$.

Corollary 3.68. *Suppose that* $(1-\alpha_k)a_{n,k} + \beta_k \le 0$, $k = 1, \cdots, n-1$. *Then Theorem B is valid for* $\nu = 9$.

Corollary 3.69. *Suppose that* $\sum_{k=1}^{n-1}(\alpha_k a_{n,k} - \beta_k)a_{k,n} < 1$ *and*

$$(1-\alpha_j)a_{n,j} + \beta_j \begin{cases} \le 0, \\ \lesssim 0 \text{ whenever } a_{n,j} < 0, \end{cases} \quad j = 1, \cdots, n-1. \tag{38}$$

Then Theorem C is valid for $\nu = 9$, *provided one of the conditions (i) and (ii) of Theorem 3.65 is satisfied.*

Corollary 3.70. *Suppose that (38) holds. Then Theorem D is valid for* $\nu = 9$, *provided one of the conditions (i) and (ii) of Theorem 3.65 is satisfied.*

Similarly, different from Q_6, a special Q is proposed in [56] as

$$Q_{10} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ -\alpha_2 a_{2,1} + \beta_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\alpha_n a_{n,1} + \beta_n & 0 & \cdots & 0 \end{pmatrix}$$

with $\alpha_k \ge 0$, $-\alpha_k a_{k,1} + \beta_k \ge 0$, $k = 2, \cdots, n$, and $\sum_{k=2}^{n}(-\alpha_k a_{k,1} + \beta_k) \ne 0$.

It is given in [9] for $\alpha_k = 1$, $k = 2, \cdots, n$, in [18] for $\alpha_k = \alpha \ge 0$, $\beta_k = \beta \ge 0$, $k = 2, \cdots, n$ with $\alpha + \beta \ne 0$, and in [8] for the preconditioned SOR method, where $\alpha_k = 1$, $k = 2, \cdots, n$.

In this case, we have that

$$a_{i,j}^{(10)} = \begin{cases} a_{1,j}, & i = 1, j = 1, \cdots, n, \\ (1-\alpha_i)a_{i,1} + \beta_i, & i = 2, \cdots, n, j = 1, \\ a_{i,j} + (-\alpha_i a_{i,1} + \beta_i)a_{1,j}, & i, j = 2, \cdots, n, \end{cases}$$

so that

$$q_{i,1}^{(10)} = -\alpha_i a_{i,1} + \beta_i, \; i = 2, \cdots, n,$$
$$q_{i,j}^{(10)} = q_{1,1}^{(10)} = 0, \; i = 1, \cdots, n, \; j = 2, \cdots, n$$

and

$$\sum_{\substack{k=1 \\ k \ne i}}^{n} q_{i,k}^{(10)} a_{k,i} = \begin{cases} 0, & i = 1, \\ (-\alpha_i a_{i,1} + \beta_i)a_{1,i}, & i = 2, \cdots, n. \end{cases}$$

Now, $\delta_{i,j}^{(1)}(1)$ reduces to

$$\delta_{i,j}^{(10)}(1) = \begin{cases} (\alpha_i a_{i,1} - \beta_i)a_{1,j}, & i = 2, \cdots, n, j = 2, \cdots, i; \\ 0, & \textit{otherwise}. \end{cases}$$

Hence, by Corollaries 3.12-3.15, we can prove the following comparison theorems directly.

Theorem 3.71. *Suppose that* $(1-\alpha_k)a_{k,1} + \beta_k \le 0$ *and* $(\alpha_k a_{k,1} - \beta_k)a_{1,k} < 1$, $k = 2, \cdots, n$. *Then Theorem A is valid for* $\nu = 10$.

Theorem 3.72. *Suppose that* $(1-\alpha_k)a_{k,1} + \beta_k \le 0$, $k = 2, \cdots, n$. *Then Theorem B is valid for* $\nu = 10$.

The results given by Theorems 3.71 and 3.72 include the corresponding ones given in [18, Theorem 3.3].

The result given by 3.72 is better than the corresponding one given in [56, Theorem 4.2], where the condition that A is irreducible is unnecessary.

In order to give the Stein-Rosenberg Type Theorem II, we give a lemma, whose proof is completely same as that of Lemma 3.50.

Lemma 3.73. *Let A be an irreducible Z-matrix. Assume that $0 < -\alpha_k a_{k,1} + \beta_k \le -a_{k,1}$, $k = 2, \cdots, n$ and $A^{(10)}$ has the block form*

$$A^{(10)} = \begin{pmatrix} 1 & \bar{a}_{1,2} \\ \bar{a}_{2,1}^{(10)} & A_{2,2}^{(10)} \end{pmatrix}, \ A_{2,2}^{(10)} \in \mathscr{R}^{(n-1)\times(n-1)}.$$

Then

(i) $A_{2,2}^{(10)}$ *is an irreducible Z-matrix.*

(ii) $A^{(10)}$ *is an irreducible Z-matrix if and only if there exists $k_0 \in \{2, \cdots, n\}$ such that $(1 - \alpha_{k_0})a_{k_0,1} + \beta_{k_0} \neq 0$.*

Using this lemma, similar to the proof of Theorem 3.51 we prove the following theorem.

Theorem 3.74. *Suppose that $(1 - \alpha_k)a_{k,1} + \beta_k \le 0$ and $(\alpha_k a_{k,1} - \beta_k)a_{1,k} < 1$, $k = 2, \cdots, n$. Then Theorem C is valid for $\nu = 10$, provided one of the following conditions is satisfied:*

(i) $0 \le \gamma < 1$ *and* $(1 - \alpha_k)a_{k,1} + \beta_k \lesssim 0$ *whenever* $a_{k,1} < 0$, $k = 2, \cdots, n$.

(ii) $\gamma = 1$ *and* $(1 - \alpha_k)a_{k,1} + \beta_k \lesssim 0$ *whenever* $a_{k,1} < 0$, $k = 2, \cdots, n$. *And one of the following conditions holds:*

- *(ii_1) There exist $i \in \{2, \cdots, n\}$ and $j \in \{2, \cdots, i\}$ such that $(\alpha_i a_{i,1} - \beta_i)a_{1,j} > 0$.*
- *(ii_2)* $-\alpha_n a_{n,1} + \beta_n > 0$.
- *(ii_3)* $a_{1,2} < 0$.

(iii) $-\alpha_k a_{k,1} + \beta_k > 0$, $k = 2, \cdots, n$.

Proof. By Theorem 3.6, we just need to prove (ii_3) and (*iii*).

For (ii_3), by the definition of Q_{10}, there exists $i_0 \in \{2, \cdots, n\}$ such that $-\alpha_{i_0} a_{i_0,1} + \beta_{i_0} > 0$, so that $(\alpha_{i_0} a_{i_0,1} - \beta_{i_0})a_{1,2} > 0$, which implies that ($ii_1$) holds for $i = i_0$ and $j = 2$.

For (*iii*), since A is irreducible, then there exists $j \in \{2, \cdots, n\}$ such that $a_{1,j} < 0$, so that $\delta_{n,j}^{(10)}(1) = (\alpha_n a_{n,1} - \beta_n)a_{1,j} > 0$. Using Lemma 3.73, the rest of the proof is completely similar to that of (*iii*) and (iv_1) in Theorem 3.51. □

When (*iii*) holds it includes [9, Theorem 3.1]. The result for $\gamma = \omega$ is better that the corresponding ones given by [8, Theorem 3.1, Corollary 3.1].

Theorem 3.75. *Suppose that $(1 - \alpha_k)a_{k,1} + \beta_k \le 0$, $k = 2, \cdots, n$. Then Theorem D is valid for $\nu = 10$, provided one of the conditions (i), (ii) and (iii) of Theorem 3.74 is satisfied.*

As a special case of Q_9 and Q_{10}, Q is proposed in [91] as

$$Q_{11} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ -\frac{1}{\alpha}a_{n,1} - \beta & 0 & \cdots & 0 \end{pmatrix}$$

with $\alpha > 0$ and $a_{n,1}/\alpha + \beta < 0$. It is discussed in [51] for $\alpha = 1$.

Now, $\delta_{i,j}^{(10)}(1)$ reduces to

$$\delta_{i,j}^{(11)}(1) = \begin{cases} (\frac{1}{\alpha}a_{n,1} + \beta)a_{1,j}, & i = n, j = 2, \cdots, n; \\ 0, & otherwise. \end{cases}$$

By Theorems 3.71 and 3.72, the following comparison results are obtained, directly.

Theorem 3.76. *Suppose that $\beta \geq (1 - 1/\alpha)a_{n,1}$ and $(a_{n,1}/\alpha + \beta)a_{1,n} < 1$. Then Theorem A is valid for $\nu = 11$.*

Theorem 3.77. *Suppose that $\beta \geq (1 - 1/\alpha)a_{n,1}$. Then Theorem B is valid for $\nu = 11$.*

In order to give the Stein-Rosenberg Type Theorem II, completely similar to Lemma 3.55, we can prove the following lemma.

Lemma 3.78. *Let A be an irreducible Z-matrix. Assume that $\beta \geq (1 - 1/\alpha)a_{n,1}$ and $A^{(11)}$ has the block form*

$$A^{(11)} = \begin{pmatrix} 1 & \bar{a}_{1,2} \\ \bar{a}_{2,1}^{(11)} & A_{2,2}^{(11)} \end{pmatrix}, \ A_{2,2}^{(11)} \in \mathscr{R}^{(n-1)\times(n-1)}.$$

Then one of the following two mutually exclusive relations holds:

(i) $A^{(11)}$ is an irreducible Z-matrix.

(ii) $A^{(11)}$ is a reducible Z-matrix, but $A_{2,2}^{(11)}$ is an irreducible Z-matrix and $a_{k,1} = a_{k,1}^{(11)} = a_{n,1}^{(11)} = 0$, $k = 2, \cdots, n-1$.

Using this lemma, similar to the proof of Theorem 3.56, we prove the following theorem.

Theorem 3.79. *Suppose that $\beta \geq (1 - 1/\alpha)a_{n,1}$ and $(a_{n,1}/\alpha + \beta)a_{1,n} < 1$. Then Theorem C is valid for $\nu = 11$.*

Proof. Since $(a_{n,1}/\alpha + \beta)a_{1,n} < 1$, then, by Lemma 3.78, $A^{(11)}$ is an L-matrix.

If $A^{(11)}$ is irreducible then it follows by (i) and (ii_2) in Theorem 3.6 that Theorem C is valid, since $q_{n,1}^{(11)} = -a_{n,1}/\alpha - \beta > 0$.

For the case when $A^{(11)}$ is reducible, then the irreducibility of A ensures that there exists $j \in \{2, \cdots, n\}$ such that $a_{1,j} < 0$ so that $-(a_{n,1}/\alpha + \beta)a_{1,j} > 0$. Now, using Lemma 3.78, the rest of the proof is completely similar to that of (iii) and (iv_1) in Theorem 3.51. □

The result is better than [91, Theorem 1], where the condition $0 < a_{1,n}a_{n,1} < \alpha(\alpha > 1)$ is unnecessary. For $\alpha = 1$, it also better than the corresponding ones given by [51, Theorem 8, Corollaries 10, 11].

Theorem 3.80. *Suppose that $\beta \geq (1 - 1/\alpha)a_{n,1}$. Then Theorem D is valid for $\nu = 11$.*

By the definition of Q_{11}, $a_{n,1}/\alpha + \beta < 0$. While in the comparison theorems above we need the condition $\beta \geq (1 - 1/\alpha)a_{n,1}$. Hence it implies $a_{n,1} < 0$.

3.3. Upper triangular preconditioners

Corresponding to Q_3, we let

$$\alpha_{i,j} = 0,\ i = 1, \cdots, n, j \le i.$$

Then Q_2 reduces to

$$Q_{12} = \begin{pmatrix} 0 & -\alpha_{1,2}a_{1,2} & -\alpha_{1,3}a_{1,3} & \cdots & -\alpha_{1,n}a_{1,n} \\ 0 & 0 & -\alpha_{2,3}a_{2,3} & \cdots & -\alpha_{2,n}a_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\alpha_{n-1,n}a_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

with $\alpha_{i,j} \ge 0$, $i = 1, \cdots, n-1$, $j > i$, and

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \alpha_{i,j}a_{i,j} \ne 0.$$

In [37] it is proposed for the preconditioned Gauss-Seidel method, where $\alpha_{i,j} = \alpha_i \ge 0$, $i = 1, \cdots, n-1$.

In this case, for $i, j = 1, \cdots, n$, we have that

$$\sum_{\substack{k=1 \\ k \ne i,j}}^{n} \alpha_{i,k}a_{i,k}a_{k,j} = \sum_{\substack{k=i+1 \\ k \ne j}}^{n} \alpha_{i,k}a_{i,k}a_{k,j},$$

so that if $j < i$, then

$$(1 - \alpha_{i,j})a_{i,j} - \sum_{\substack{k=1 \\ k \ne i,j}}^{n} \alpha_{i,k}a_{i,k}a_{k,j} = a_{i,j} - \sum_{k=i+1}^{n} \alpha_{i,k}a_{i,k}a_{k,j} \le a_{i,j}.$$

Since $\alpha_{n,j} = 0$ for $j = 1, \cdots, n$, then $\delta_{i,j}^{(2)}(\gamma)$ and $\delta_{i,j}^{(2)}(1)$ reduce respectively to

$$\delta_{i,j}^{(12)}(\gamma) = \begin{cases} \sum\limits_{k=i+1}^{n} \alpha_{i,k}a_{i,k}a_{k,i}, & i = j = 1, \cdots, n-1; \\ (\gamma - 1)\alpha_{i,j}a_{i,j} + \gamma \sum\limits_{k=i+1}^{j-1} \alpha_{i,k}a_{i,k}a_{k,j}, & i = 1, \cdots, n-1, j = i+1, \cdots, n; \\ \gamma \sum\limits_{k=i+1}^{n} \alpha_{i,k}a_{i,k}a_{k,j}, & i = 2, \cdots, n-1, j = 1, \cdots, i-1; \\ 0, & i = n, j = 1, \cdots, n \end{cases}$$

and

$$\delta_{i,j}^{(12)}(1) = \begin{cases} \sum\limits_{k=i+1}^{n} \alpha_{i,k}a_{i,k}a_{k,j}, & i = 1, \cdots, n-1, j = 1, \cdots, i; \\ \sum\limits_{k=i+1}^{j-1} \alpha_{i,k}a_{i,k}a_{k,j}, & i = 1, \cdots, n-1, j = i+2, \cdots, n; \\ 0, & i = 1, \cdots, n-1, j = i+1; \\ 0, & i = n, j = 1, \cdots, n. \end{cases}$$

In this case, the conditions (ii_2), (ii_8), (iv_7), (iv^a) and (iv^b) in Theorem 3.18 can be not satisfied. While the inequality (22) or (23) is trivial because that A is irreducible and

$$(1 - \alpha_{n,j})a_{n,j} - \sum_{\substack{k=1 \\ k \ne j}}^{n-1} \alpha_{n,k}a_{n,k}a_{k,j} = a_{n,j}.$$

Completely similar to Theorems 3.24-3.27 and Corollaries 3.28-3.31, by Theorems 3.16-3.19, we can prove the following comparison results, immediately.

Theorem 3.81. *Suppose that* $\sum_{k=i+1}^{n} \alpha_{i,k}a_{i,k}a_{k,i} < 1$, $i = 1, \cdots, n-1$, *and*

$$(1-\alpha_{i,j})a_{i,j} - \sum_{\substack{k=i+1 \\ k\neq j}}^{n} \alpha_{i,k}a_{i,k}a_{k,j} \leq 0,\ i = 1, \cdots, n-1, j > i. \tag{39}$$

Then Theorem A is valid for $\nu = 12$.

Theorem 3.82. *Suppose that (39) holds. Then Theorem B is valid for* $\nu = 12$.

Theorem 3.83. *Suppose that* $\sum_{k=i+1}^{n} \alpha_{i,k}a_{i,k}a_{k,i} < 1$, $i = 1, \cdots, n-1$. *Then Theorem C is valid for* $\nu = 12$, *provided one of the following conditions is satisfied:*

(i) $0 \leq \gamma < 1$ *and*

$$(1-\alpha_{i,j})a_{i,j} - \sum_{\substack{k=i+1 \\ k\neq j}}^{n} \alpha_{i,k}a_{i,k}a_{k,j} \lessapprox 0 \ \textit{ whenever } \ a_{i,j} < 0,\ i = 1, \cdots, n-1,\ j > i. \tag{40}$$

(ii) $\gamma = 1$, *(40) holds and one of the following conditions holds:*

- *(ii₁) There exist* $i \in \{1, \cdots, n-1\}$, $j \in \{1, \cdots, i\}$ *and* $k \in \{i+1, \cdots, n\}$ *such that* $\alpha_{i,k}a_{i,k}a_{k,j} > 0$.
- *(ii₂) There exist* $i \in \{1, \cdots, n-1\}$, $j \in \{i+2, \cdots, n\}$ *and* $k \in \{i+1, \cdots, j-1\}$ *such that* $\alpha_{i,k}a_{i,k}a_{k,j} > 0$.
- *(ii₃) There exists* $k \in \{1, \cdots, n-1\}$ *such that* $a_{k,k+1} < 0$ *and* $\alpha_{k,k+1} > 0$.
- *(ii₄)* $a_{n,1} < 0$ *and* $a_{k,k+1} < 0$, $k = 1, \cdots, n-1$.
- *(ii₅)* $a_{n,1} < 0$ *and* $a_{k,n} < 0$, $k = 2, \cdots, n-1$.
- *(ii₆)* $a_{k,1} < 0$, $k = 2, \cdots, n$.

(iii) $0 \leq \gamma < 1$ *and (39) holds. For each* $i \in \{1, \cdots, n-1\}$, *there exists* $j(i) \in \{i+1, \cdots, n\}$ *such that* $\alpha_{i,j(i)}a_{i,j(i)} < 0$.

(iv) $\gamma = 1$ *and (39) holds. For each* $i \in \{1, \cdots, n-1\}$, *one of the following conditions holds:*

- *(iv₁) There exist* $j(i) \in \{1, \cdots, i\}$ *and* $k(i) \in \{i+1, \cdots, n\}$ *such that* $\alpha_{i,k(i)}a_{i,k(i)}a_{k(i),j(i)} > 0$.
- *(iv₂) There exist* $j(i) \in \{i+2, \cdots, n\}$ *and* $k(i) \in \{i+1, \cdots, j-1\}$ *such that* $\alpha_{i,k(i)}a_{i,k(i)}a_{k(i),j(i)} > 0$.
- *(iv₃)* $a_{i,i+1} < 0$ *and* $\alpha_{i,i+1} > 0$.

Proof. By Theorem 3.18, we just need to prove (ii_5), (ii_6) and (iv).

By the definition of Q_{12}, there exist $i_0 \in \{1, \cdots, n-1\}$ and $j_0 \in \{i+1, \cdots, n\}$ such that $\alpha_{i_0,j_0}a_{i_0,j_0} < 0$.

When (ii_5) holds, if $j_0 < n$ then $\alpha_{i_0,j_0}a_{i_0,j_0}a_{j_0,n} > 0$, which implies that (ii_2) holds for $i = i_0$, $j = n$ and $k = j_0$. If $j_0 = n$ then $\alpha_{i_0,n}a_{i_0,n}a_{n,1} > 0$, which implies that (ii_1) holds for $i = i_0$, $j = 1$ and $k = n$.

When (ii_6) holds, it gets that $\alpha_{i_0,j_0}a_{i_0,j_0}a_{j_0,1} > 0$, which implies that (ii_1) holds for $i = i_0$, $j = 1$ and $k = j_0$.

By the irreducibility of A, there exists $j \in \{1, \cdots, n-1\}$ such that $a_{n,j} < 0$, which implies that (iv^c) or (iv^d) in Theorem 3.18 holds because $\alpha_{n,k} = 0, k = 1, \cdots, n-1$. By (iv_1) and (iv_2) in Theorem 3.18 we derive (iv). □

Theorem 3.84. *Theorem D is valid for* $\nu = 12$, *provided one of the following conditions is satisfied:*

(i) One of the conditions (i)-(iv) of Theorem 3.83 holds.

(ii) The inequality (39) holds and one of the following conditions holds:

- *(ii₁) There exist* $i \in \{1, \cdots, n-1\}$ *and* $j \in \{i+1, \cdots, n\}$ *such that* $\alpha_{i,j}a_{i,j}a_{j,i} > 0$.

(ii_2) $\gamma > 0$. *There exist* $i \in \{2, \cdots, n-1\}$, $j \in \{1, \cdots, i-1\}$ *and* $k \in \{i+1, \cdots, n\}$ *such that* $\alpha_{i,k} a_{i,k} a_{k,j} > 0$.

Proof. We just need to prove (*ii*). In fact, since $\alpha_{i,j} = 0$ for $j \le i$, then the result follows by (*ii*) of Theorem 3.19 directly. □

Corollary 3.85. *Suppose that* $0 \le \alpha_{i,j} \le 1$ *and* $\sum_{k=i+1}^{n} \alpha_{i,k} a_{i,k} a_{k,i} < 1$, $i = 1, \cdots, n-1$, $j > i$. *Then Theorem A is valid for* $\nu = 12$.

Corollary 3.86. *Suppose that* $0 \le \alpha_{i,j} \le 1$, $i = 1, \cdots, n-1$, $j > i$. *Then Theorem B is valid for* $\nu = 12$.

Corollary 3.87. *Suppose that* $\sum_{k=i+1}^{n} \alpha_{i,k} a_{i,k} a_{k,i} < 1$, $i = 1, \cdots, n-1$. *Then Theorem C is valid for* $\nu = 12$, *provided one of the following conditions is satisfied:*

(i) For $i = 1, \cdots, n-1$, $j > i$, $0 \le \alpha_{i,j} \lessapprox 1$. *One of the conditions* $0 \le \gamma < 1$ *and* (ii_1)-(ii_6) *whenever* $\gamma = 1$ *in Theorem 3.83 holds.*

(ii) One of the conditions (iii) and (iv) of Theorem 3.83 holds, where the inequality (39) is replaced by $0 \le \alpha_{i,j} \le 1$, $i = 1, \cdots, n-1$, $j > i$.

Corollary 3.88. *Theorem D is valid for* $\nu = 12$, *provided one of the following conditions is satisfied:*

(i) One of the conditions (i) and (ii) of Corollary 3.87 holds.

(ii) For $i = 1, \cdots, n-1$, $j > i$, $0 \le \alpha_{i,j} \le 1$. *One of the conditions* (ii_1) *and* (ii_2) *in Theorem 3.84 holds.*

Many known corresponding results about the preconditioned AOR method proposed in the references are the special cases of Theorems 3.81-3.84 and Corollaries 3.85-3.88, i.e., they can be derived from these theorems, immediately.

When $\alpha_{i,j} = \alpha > 0$, $i = 1, \cdots, n-1$, $j > i$, the matrix Q_{12} reduces to

$$Q_{13} = \alpha U,$$

which is proposed in [35] for the preconditioned Gauss-Seidel method. It is investigated in [97, 108] for the preconditioned AOR method. For $\alpha = 1$ it is proposed in [85] for the preconditioned Gauss-Seidel method and in [63, 84] for the preconditioned SOR method.

Denote

$$\delta_{i,j}^{(13)}(\gamma) = \begin{cases} \sum\limits_{k=i+1}^{n} a_{i,k} a_{k,i}, & i = j = 1, \cdots, n-1; \\ (\gamma - 1) a_{i,j} + \gamma \sum\limits_{k=i+1}^{j-1} a_{i,k} a_{k,j}, & i = 1, \cdots, n-1, j = i+1, \cdots, n; \\ \gamma \sum\limits_{k=i+1}^{n} a_{i,k} a_{k,j}, & i = 2, \cdots, n-1, j = 1, \cdots, i-1; \\ 0, & i = n, j = 1, \cdots, n \end{cases}$$

and

$$\delta_{i,j}^{(13)}(1) = \begin{cases} \sum\limits_{k=i+1}^{n} a_{i,k} a_{k,j}, & i = 1, \cdots, n-1, j = 1, \cdots, i; \\ \sum\limits_{k=i+1}^{j-1} a_{i,k} a_{k,j}, & i = 1, \cdots, n-1, j = i+2, \cdots, n; \\ 0, & i = 1, \cdots, n-1, j = i+1; \\ 0, & i = n, j = 1, \cdots, n. \end{cases}$$

By Theorems 3.81-3.84 and Corollaries 3.85-3.88, the following results are obtained.

Theorem 3.89. *Suppose that* $\alpha\sum_{k=i+1}^{n}a_{i,k}a_{k,i}<1$, $i=1,\cdots,n-1$, *and*

$$(1-\alpha)a_{i,j}-\alpha\sum_{\substack{k=i+1\\k\neq j}}^{n}a_{i,k}a_{k,j}\leq 0,\ i=1,\cdots,n-1,j>i. \tag{41}$$

Then Theorem A is valid for $\nu=13$.

Theorem 3.90. *Suppose that (41) holds. Then Theorem B is valid for* $\nu=13$.

Theorem 3.91. *Suppose that* $\alpha\sum_{k=i+1}^{n}a_{i,k}a_{k,i}<1$, $i=1,\cdots,n-1$. *Then Theorem C is valid for* $\nu=13$, *provided one of the following conditions is satisfied:*

(i) $0\leq\gamma<1$ *and*

$$(1-\alpha)a_{i,j}-\alpha\sum_{\substack{k=i+1\\k\neq j}}^{n}a_{i,k}a_{k,j}\lesssim 0\ \text{ whenever } a_{i,j}<0,\ i=1,\cdots,n-1,j>i. \tag{42}$$

(ii) $\gamma=1$. *The inequality (42) holds and one of the following conditions holds:*

- *(ii_1)* *There exist* $i\in\{1,\cdots,n-1\}$, $j\in\{1,\cdots,i\}$ *and* $k\in\{i+1,\cdots,n\}$ *such that* $a_{i,k}a_{k,j}>0$.
- *(ii_2)* *There exist* $i\in\{1,\cdots,n-1\}$, $j\in\{i+2,\cdots,n\}$ *and* $k\in\{i+1,\cdots,j-1\}$ *such that* $a_{i,k}a_{k,j}>0$.
- *(ii_3)* *There exists* $k\in\{1,\cdots,n-1\}$ *such that* $a_{k,k+1}<0$.
- *(ii_4)* $a_{n,1}<0$.

(iii) $0\leq\gamma<1$ *and the inequality (41) holds. For each* $i\in\{1,\cdots,n-1\}$, *there exists* $j(i)\in\{i+1,\cdots,n\}$ *such that* $a_{i,j(i)}<0$.

(iv) $\gamma=1$ *and the inequality (41) holds. For each* $i\in\{1,\cdots,n-1\}$ *one of the following conditions holds:*

- *(iv_1)* *There exist* $j(i)\in\{1,\cdots,i\}$ *and* $k(i)\in\{i+1,\cdots,n\}$ *such that* $a_{i,k(i)}a_{k(i),j(i)}>0$.
- *(iv_2)* *There exist* $j(i)\in\{i+2,\cdots,n\}$ *and* $k(i)\in\{i+1,\cdots,j-1\}$ *such that* $a_{i,k(i)}a_{k(i),j(i)}>0$.
- *(iv_3)* $a_{i,i+1}<0$.

Proof. We just need to prove (ii_4). From the irreducibility of A, there exists $i_0\in\{1,\cdots,n-1\}$ such that $a_{i_0,n}<0$ so that $a_{i_0,n}a_{n,1}>0$, which implies that (ii_1) holds for $i=i_0$, $j=1$ and $k=n$. □

Theorem 3.92. *Theorem D is valid for* $\nu=13$, *provided one of the following conditions is satisfied:*

(i) One of the conditions (i)-(iv) of Theorem 3.91 holds.

(ii) The inequality (41) holds and one of the following conditions holds:

- *(ii_1)* *There exist* $i\in\{1,\cdots,n-1\}$ *and* $j\in\{i+1,\cdots,n\}$ *such that* $a_{i,j}a_{j,i}>0$.
- *(ii_1)* $\gamma>0$. *There exist* $i\in\{2,\cdots,n-1\}$, $j\in\{1,\cdots,i-1\}$ *and* $k\in\{i+1,\cdots,n\}$ *such that* $a_{i,k}a_{k,j}>0$.

Corollary 3.93. *Suppose that* $0<\alpha\leq 1$ *and* $\alpha\sum_{k=i+1}^{n}a_{i,k}a_{k,i}<1$, $i=1,\cdots,n-1$. *Then Theorem A is valid for* $\nu=13$.

The result improves the corresponding one given by [97, Theorem 3.6].

Corollary 3.94. *Suppose that* $0<\alpha\leq 1$. *Then Theorem B is valid for* $\nu=13$.

Corollary 3.95. *Suppose that* $\alpha \sum_{k=i+1}^{n} a_{i,k}a_{k,i} < 1$, $i = 1, \cdots, n-1$. *Then Theorem C is valid for* $\nu = 13$, *provided one of the following conditions is satisfied:*

(i) $0 \le \gamma < 1$ *and* $0 < \alpha \lesssim 1$.

(ii) $\gamma = 1$ *and* $0 < \alpha \lesssim 1$. *One of the conditions (ii*$_1$*)-(ii*$_4$*) in Theorem 3.91 is satisfied.*

(iii) $0 \le \gamma < 1$, $0 < \alpha \le 1$ *and for each* $i \in \{1, \cdots, n-1\}$, *there exists* $j(i) \in \{i+1, \cdots, n\}$ *such that* $a_{i,j(i)} < 0$.

(iv) $\gamma = 1$, $0 < \alpha \le 1$ *and for each* $i \in \{1, \cdots, n-1\}$ *one of the conditions (iv*$_1$*)-(iiv*$_3$*) in Theorem 3.91 holds.*

The result when (*i*) holds is better than the corresponding one given by [108, Theorem 3.3].

If $\sum_{k=i+1}^{n} a_{i,k}a_{k,i} > 0$, $i = 1, \cdots, n-1$, then for each $i \in \{1, \cdots, n-1\}$, there exists $k(i) \in \{i+1, \cdots, n\}$ such that $a_{i,k(i)}a_{k(i),i} > 0$ and $a_{i,k(i)} < 0$, which implies that (*iii*) of Theorem 3.91 holds for $j(i) = k(i)$ and (iv_1) in Theorem 3.91 holds for $j(i) = i$. Hence, Corollary 3.95 when (*iii*) and (*iv*) hold is better than [108, Theorems 3.1, 3.2, 3.4].

When $\alpha = 1$, [63] studies the preconditioned SOR method. The main result [63, Theorem 3.1] presents a Stein-Rosenberg type comparison theorem. But it is incorrect. Where the authors assume that A is strictly diagonally dominant. Under this condition, by [4, Theorem 6-2.3], A is a nonsingular M-matrix. Then, by [4, Theorem 7-5.24], it gets that $\rho(\mathscr{L}_\omega) < 1$. Hence, with our sign, [63, Theorem 3.1] should be corrected as follows: *"If A is a strictly diagonally dominant Z-matrix such that* $0 < a_{k,k+1}a_{k+1,k} < 1$, $k = 1, \cdots, n-1$ *and* $0 < \omega < 1$, *then* $\rho(\mathscr{L}_\omega^{(13)}) < \rho(\mathscr{L}_\omega) < 1$*"*. While this result can be also derived directly from Corollary 3.96 below. In fact, the condition $a_{k,k+1}a_{k+1,k} > 0$, $k = 1, \cdots, n-1$, implies that A is irreducible and $a_{k,k+1} < 0$, $k = 1, \cdots, n-1$. Therefore, the conditions (*iii*) and (iv_2) in Theorem 3.91 are satisfied. By (*i*) of Corollary 3.96, it follows that $\rho(\mathscr{L}_\omega^{(13)}) < \rho(\mathscr{L}_\omega) < 1$ holds for $0 < \omega \le 1$.

Corollary 3.96. *Theorem D is valid for* $\nu = 13$, *provided one of the following conditions is satisfied:*

(i) One of the conditions (i)-(iv) of Corollary 3.95 holds.

(ii) $0 \le \alpha \le 1$ *and one of the conditions (ii*$_1$*) and (ii*$_2$*) in Theorem 3.92 holds.*

Corresponding to Q_6, for $r = 2, \cdots, n$, in [92] the matrix Q is defined as

$$Q_{14} = \begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & -\alpha_r a_{r-1,r} & \cdots & -\alpha_n a_{r-1,n} \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}$$

with $\alpha_k \ge 0$, $k = r, \cdots, n$, and

$$\sum_{k=r}^{n} \alpha_k a_{r-1,k} \ne 0.$$

In this case, $\delta_{i,j}^{(12)}(\gamma)$ and $\delta_{i,j}^{(12)}(1)$ reduce respectively to

$$\delta_{i,j}^{(14)}(\gamma) = \begin{cases} \sum\limits_{k=r}^{n} \alpha_k a_{r-1,k}a_{k,r-1}, & i = j = r-1; \\ (\gamma - 1)\alpha_r a_{r-1,j} + \gamma \sum\limits_{k=r}^{j-1} \alpha_k a_{r-1,k}a_{k,j}, & i = r-1, j = r, \cdots, n; \\ \gamma \sum\limits_{k=r}^{n} \alpha_k a_{r-1,k}a_{k,j}, \text{ whenever } r \ge 3, & i = r-1, j = 1, \cdots, r-2; \\ 0, & \textit{otherwise} \end{cases}$$

and

$$\delta^{(14)}_{i,j}(1) = \begin{cases} \sum\limits_{k=r}^{n} \alpha_k a_{r-1,k} a_{k,j}, & i = r-1, j = 1, \cdots, r-1; \\ \sum\limits_{k=r}^{j-1} \alpha_k a_{r-1,k} a_{k,j}, & i = r-1, j = r+1, \cdots, n; \\ 0, & otherwise. \end{cases}$$

Clearly, the conditions (*iii*) and (*iv*) of Theorem 3.83 can be not satisfied.

From Theorems 3.81-3.84 and Corollaries 3.85-3.88, we have the following comparison results.

Theorem 3.97. *Suppose that* $\sum_{k=r}^{n} \alpha_k a_{r-1,k} a_{k,r-1} < 1$ *and*

$$(1-\alpha_j) a_{r-1,j} - \sum_{\substack{k=r \\ k \neq j}}^{n} \alpha_k a_{r-1,k} a_{k,j} \leq 0, \ \ j = r, \cdots, n. \tag{43}$$

Then Theorem A is valid for $\nu = 14$.

Theorem 3.98. *Suppose that (43) holds. Then Theorem B is valid for* $\nu = 14$.

Theorem 3.99. *Suppose that* $\sum_{k=r}^{n} \alpha_k a_{r-1,k} a_{k,r-1} < 1$ *and*

$$(1-\alpha_j) a_{r-1,j} - \sum_{\substack{k=r \\ k \neq j}}^{n} \alpha_k a_{r-1,k} a_{k,j} \lesssim 0 \text{ whenever } \ a_{r-1,j} < 0, \ \ j = r, \cdots, n. \tag{44}$$

Then Theorem C is valid for $\nu = 14$*, provided one of the following conditions is satisfied:*

- *(i)* $0 \leq \gamma < 1$.
- *(ii)* $\gamma = 1$ *and one of the following conditions holds:*
  - *(ii*$_1$*) There exist* $i \in \{1, \cdots, r-1\}$ *and* $j \in \{r, \cdots, n\}$ *such that* $\alpha_j a_{r-1,j} a_{j,i} > 0$.
  - *(ii*$_2$*) There exist* $i \in \{r+1, \cdots, n\}$ *and* $j \in \{r, \cdots, i-1\}$ *such that* $\alpha_j a_{r-1,j} a_{j,i} > 0$.
  - *(ii*$_3$*)* $a_{r-1,r} < 0$ *and* $\alpha_r > 0$.
  - *(ii*$_4$*)* $a_{k,1} < 0, k = r, \cdots, n$.
  - *(ii*$_5$*)* $a_{k,r-1} < 0, k = r, \cdots, n$.
  - *(ii*$_6$*)* $a_{n,1} < 0$ *and* $a_{k,n} < 0, k = r, \cdots, n-1$.
  - *(ii*$_7$*)* $a_{n,1} < 0$ *and* $a_{k,k+1} < 0, k = r, \cdots, n-1$.

Proof. By Theorem 3.83 we just need to prove (*ii*$_4$)-(*ii*$_7$).

From the definition of Q_{14}, there exists $k_0 \in \{r, \cdots, n\}$ such that $\alpha_{k_0} a_{r-1,k_0} < 0$.

If (*ii*$_4$) holds then $\alpha_{k_0} a_{r-1,k_0} a_{k_0,1} > 0$, which implies that (*ii*$_1$) holds for $i = 1$ and $j = k_0$.

Similarly, if (*ii*$_5$) holds then we can prove that (*ii*$_1$) holds for $i = r-1$ and $j = k_0$.

When (*ii*$_6$) holds, if $k_0 < n$ then $\alpha_{k_0} a_{r-1,k_0} a_{k_0,n} > 0$, which implies that (*ii*$_2$) holds for $i = n$ and $j = k_0$. If $k_0 = n$ then $\alpha_n a_{r-1,n} a_{n,1} > 0$, which implies that (*ii*$_1$) holds for $i = 1$ and $j = n$.

Similarly, when (*ii*$_7$) holds we can prove that (*ii*$_2$) holds. □

Theorem 3.100. *Theorem D is valid for* $\nu = 14$*, provided one of the following conditions is satisfied:*

- *(i) The inequality (44) and one of the conditions (i) and (ii) of Theorem 3.99 holds.*
- *(ii) The inequality (43) holds and one of the following conditions holds:*
  - *(ii*$_1$*) There exists* $k \in \{r, \cdots, n\}$ *such that* $\alpha_k a_{r-1,k} a_{k,r-1} > 0$.

(ii_2) $\gamma > 0$ and there exist $k \in \{r, \cdots, n\}$ and $j \in \{1, \cdots, r-2\}$ such that $\alpha_k a_{r-1,k} a_{k,j} > 0$.

Corollary 3.101. *Suppose that $0 \le \alpha_k \le 1$, $k = r, \cdots, n$, and $\sum_{k=r}^{n} \alpha_k a_{r-1,k} a_{k,r-1} < 1$. Then Theorem A is valid for $\nu = 14$.*

Corollary 3.102. *Suppose that $0 \le \alpha_k \le 1$, $k = r, \cdots, n$. Then Theorem B is valid for $\nu = 14$.*

The result includes the corresponding one given in [92, Corollary 2.3].

Corollary 3.103. *Suppose that $0 \le \alpha_k \lesssim 1$, $k = r, \cdots, n$, and $\sum_{k=r}^{n} \alpha_k a_{r-1,k} a_{k,r-1} < 1$. Then Theorem C is valid for $\nu = 14$, provided one of the conditions (i) and (ii) of Theorem 3.99 is satisfied.*

Corollary 3.104. *Theorem D is valid for $\nu = 14$, provided one of the following conditions is satisfied:*

(i) $0 \le \alpha_k \lesssim 1$, $k = r, \cdots, n$, and one of the conditions (i) and (ii) of Theorem 3.99 holds.

(ii) $0 \le \alpha_k \le 1$, $k = r, \cdots, n$, and one of the conditions (ii_1) and (ii_2) in Theorem 3.100 holds.

Similar to Q_{14}, for $r = 2, \cdots, n$, the matrix Q is chosen as

$$Q_{15} = \begin{pmatrix} 0 & \cdots & 0 & -\alpha_1 a_{1,r} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & -\alpha_{r-1} a_{r-1,r} & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}$$

with $\alpha_k \ge 0$, $k = 1, \cdots, r-1$, and

$$\sum_{k=1}^{r-1} \alpha_k a_{k,r} \neq 0.$$

It is proposed in [104] for $r = n$. When $r = n$ and $\alpha_i = 1$, $i = 2, \cdots, n$, it is a special case in [66] for the preconditioned Gauss-Seidel and Jacobi methods.

In this case, $\delta_{i,j}^{(12)}(\gamma)$ and $\delta_{i,j}^{(12)}(1)$ reduce respectively to

$$\delta_{i,j}^{(15)}(\gamma) = \begin{cases} \alpha_i a_{i,r} a_{r,i}, & i = j = 1, \cdots, r-1; \\ (\gamma - 1)\alpha_i a_{i,r}, & i = 1, \cdots, r-1, j = r; \\ \gamma \alpha_i a_{i,r} a_{r,j}, & i = 1, \cdots, r-1, j \in \{1, \cdots, i-1\} \cup \{r+1, \cdots, n\}; \\ 0, & \text{otherwise} \end{cases}$$

and

$$\delta_{i,j}^{(15)}(1) = \begin{cases} \alpha_i a_{i,r} a_{r,j}, & i = 1, \cdots, r-1, 1 \le j \le i, r+1 \le j \le n; \\ 0, & \text{otherwise.} \end{cases}$$

From Corollaries 3.85 and 3.86, we have the following comparison results.

Theorem 3.105. *Suppose that $0 \le \alpha_k \le 1$ and $\alpha_k a_{k,r} a_{r,k} < 1$, $k = 1, \cdots, r-1$. Then Theorem A is valid for $\nu = 15$.*

Theorem 3.106. *Suppose that $0 \le \alpha_k \le 1$, $k = 1, \cdots, r-1$. Then Theorem B is valid for $\nu = 15$.*

In order to give the Stein-Rosenberg Type Theorem II, completely similar to Lemma 3.50, we can prove the following lemma.

Lemma 3.107. *Let A be an irreducible Z-matrix. Assume that $r = n$, $0 < \alpha_k \leq 1$, $k = 1, \cdots, n-1$ and $A^{(15)}$ has the block form*

$$A^{(15)} = \begin{pmatrix} A_{1,1}^{(15)} & \bar{a}_{1,2}^{(15)} \\ \bar{a}_{2,1} & 1 \end{pmatrix}, \; A_{1,1}^{(15)} \in \mathscr{R}^{(n-1)\times(n-1)}.$$

Then

(i) $A_{1,1}^{(15)}$ *is an irreducible Z-matrix.*

(ii) $A^{(15)}$ *is an irreducible Z-matrix if and only if there exists $j_0 \in \{1, \cdots, n-1\}$ such that $(1 - \alpha_{j_0})a_{j_0,n} \neq 0$.*

Theorem 3.108. *Suppose that $\alpha_k a_{k,r} a_{r,k} < 1$, $k = 1, \cdots, r-1$. Then Theorem C is valid for $\nu = 15$, provided one of the following conditions is satisfied:*

(i) $0 \leq \gamma < 1$ *and* $0 \leq \alpha_k \lesssim 1$, $k = 1, \cdots, r-1$.

(ii) $\gamma = 1$, $0 \leq \alpha_k \lesssim 1$, $k = 1, \cdots, r-1$. *And one of the following conditions holds:*

- *(ii_1) There exist $i \in \{1, \cdots, r-1\}$ and $j \in \{1, \cdots, i\} \cup \{r+1, \cdots, n\}$ such that $\alpha_i a_{i,r} a_{r,j} > 0$.*
- *(ii_2)* $a_{r-1,r} < 0$ *and* $\alpha_{r-1} > 0$.
- *(ii_3)* $a_{r,1} < 0$.
- *(ii_4) There exists $k \in \{r+1, \cdots, n\}$ such that $a_{r,k} < 0$.*

(iii) $r = n$, $0 \leq \gamma < 1$ *and* $0 < \alpha_k \leq 1$, $k = 1, \cdots, n-1$.

(iv) $r = n$, $\gamma = 1$, $0 < \alpha_k \leq 1$, $k = 1, \cdots, n-1$. *And one of the following conditions holds:*

- *(iv_1) There exist $i \in \{1, \cdots, n-1\}$ and $j \in \{1, \cdots, i\}$ such that $a_{i,n} a_{n,j} > 0$.*
- *(iv_2)* $a_{n-1,n} < 0$.
- *(iv_3)* $a_{n,1} < 0$.

Proof. We just need to prove (ii_3), (ii_4) and (iii) and (iv).

From the definition of Q_{15}, there exists $i_0 \in \{1, \cdots, r-1\}$ such that $\alpha_{i_0} a_{i_0,r} < 0$. If ($ii_3$) holds then $\alpha_{i_0} a_{i_0,r} a_{r,1} > 0$, which implies that ($ii_1$) holds for $i = i_0$ and $j = 1$. If (ii_4) holds then we can prove that (ii_1) holds for $i = i_0$ and $j = k$.

For (iii) and (iv), if $A^{(15)}$ is irreducible then the result is obvious.

Now, we consider the case when $A^{(15)}$ is reducible. Suppose that $\rho = \rho(\mathscr{L}_{\gamma,\omega})$ and $x > 0$ is its associated eigenvector. By Theorem 3.105, $\rho = 1$ if and only if $\rho(\mathscr{L}_{\gamma,\omega}^{(15)}) = 1$.

By Lemma 3.1 we just need to consider the case when $\gamma \leq \omega$. Then, by Lemma 2.9 it follows that $\rho > 0$ and $x \gg 0$.

Let A have the block form

$$A = \begin{pmatrix} A_{1,1} & \bar{a}_{1,2} \\ \bar{a}_{2,1} & 1 \end{pmatrix}, \; A_{1,1} \in \mathscr{R}^{(n-1)\times(n-1)}.$$

Then, by Lemma 3.107, $A^{(15)}$ has the block form

$$A^{(15)} = \begin{pmatrix} A_{1,1}^{(15)} & 0 \\ \bar{a}_{2,1} & 1 \end{pmatrix}, \; A_{1,1}^{(15)} \in \mathscr{R}^{(n-1)\times(n-1)},$$

where $A_{1,1}^{(15)}$ is an irreducible L-matrix, since $a_{k,k}^{(15)} = 1 - \alpha_k a_{k,n} a_{n,k} > 0$, $k = 1, \cdots, n-1$. Let $A_{1,1} = \hat{M}_{\gamma,\omega} - \hat{N}_{\gamma,\omega}$ and $A_{1,1}^{(15)} = \bar{M}_{\gamma,\omega} - \bar{N}_{\gamma,\omega}$ be the AOR splittings of $A_{1,1}$ and $A_{1,1}^{(15)}$, respectively. Then they are regular splittings and

$$
\begin{aligned}
&M_{\gamma,\omega} = \begin{pmatrix} \hat{M}_{\gamma,\omega} & 0 \\ \frac{\gamma}{\omega}\bar{a}_{2,1} & \frac{1}{\omega} \end{pmatrix},\ N_{\gamma,\omega} = \begin{pmatrix} \hat{N}_{\gamma,\omega} & -\bar{a}_{1,2} \\ \frac{\gamma-\omega}{\omega}\bar{a}_{2,1} & \frac{1-\omega}{\omega} \end{pmatrix},\\
&M_{\gamma,\omega}^{(15)} = \begin{pmatrix} \bar{M}_{\gamma,\omega} & 0 \\ \frac{\gamma}{\omega}\bar{a}_{2,1} & \frac{1}{\omega} \end{pmatrix},\ N_{\gamma,\omega}^{(15)} = \begin{pmatrix} \bar{N}_{\gamma,\omega} & 0 \\ \frac{\gamma-\omega}{\omega}\bar{a}_{2,1} & \frac{1-\omega}{\omega} \end{pmatrix},\ [M_{\gamma,\omega}^{(15)}]^{-1} = \begin{pmatrix} \bar{M}_{\gamma,\omega}^{-1} & 0 \\ -\gamma\bar{a}_{2,1}\bar{M}_{\gamma,\omega}^{-1} & \omega \end{pmatrix},\\
&\mathscr{L}_{\gamma,\omega}^{(15)} = \begin{pmatrix} \bar{M}_{\gamma,\omega}^{-1}\bar{N}_{\gamma,\omega} & 0 \\ \bar{a}_{2,1}[(\gamma-\omega)I - \gamma\bar{M}_{\gamma,\omega}^{-1}\bar{N}_{\gamma,\omega}] & 1-\omega \end{pmatrix}.
\end{aligned}
$$

Let $\bar{E}_2$ and $\bar{F}_2$ be diagonal part and strictly lower triangular part of $Q_{15}L$ with block forms

$$
\bar{E}_2 = \begin{pmatrix} E_{1,1} & 0 \\ 0 & e_{2,2} \end{pmatrix},\ \bar{F}_2 = \begin{pmatrix} F_{1,1} & 0 \\ f_{1,2} & f_{2,2} \end{pmatrix},\ E_{1,1}, F_{1,1} \in \mathscr{R}^{(n-1)\times(n-1)}.
$$

Then $e_{2,2} = f_{2,2} = 0$ and $f_{1,2} = 0$. Let

$$
Q_{15} = \begin{pmatrix} Q_{1,1} & \bar{q}_{1,2} \\ \bar{q}_{2,1} & q_{2,2}^{(15)} \end{pmatrix},\ Q_{1,1} \in \mathscr{R}^{(n-1)\times(n-1)}
$$

and

$$
x = \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix},\ \bar{x}_1 \in \mathscr{R}^{n-1},\ \bar{x}_2 \in \mathscr{R}.
$$

Then $Q_{1,1} = 0$, $\bar{q}_{2,1} = 0$, $q_{2,2}^{(15)} = 0$, $\bar{q}_{1,2} > 0$, $\bar{x}_1 \gg 0$ and $\bar{x}_2 > 0$. Now, by Lemma 3.2, we obtain

$$
\begin{aligned}
&\mathscr{L}_{\gamma,\omega}^{(15)}x - \rho x\\
&= \begin{pmatrix} \bar{M}_{\gamma,\omega}^{-1}\bar{N}_{\gamma,\omega}\bar{x}_1 - \rho\bar{x}_1 \\ \bar{a}_{2,1}[(\gamma-\omega)I - \gamma\bar{M}_{\gamma,\omega}^{-1}\bar{N}_{\gamma,\omega}]\bar{x}_1 + (1-\omega)\bar{x}_2 - \rho\bar{x}_2 \end{pmatrix}\\
&= (\rho-1)[M_{\gamma,\omega}^{(15)}]^{-1}[\bar{E}_2 + \gamma\bar{F}_2 + \frac{\omega}{\rho}Q_{15}N_{\gamma,\omega}]x\\
&= (\rho-1)\begin{pmatrix} \bar{M}_{\gamma,\omega}^{-1} & 0 \\ -\gamma\bar{a}_{2,1}\bar{M}_{\gamma,\omega}^{-1} & \omega \end{pmatrix}\left[\begin{pmatrix} E_{1,1} & 0 \\ 0 & 0 \end{pmatrix} + \gamma\begin{pmatrix} F_{1,1} & 0 \\ 0 & 0 \end{pmatrix} + \frac{\omega}{\rho}\begin{pmatrix} 0 & \bar{q}_{1,2} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \hat{N}_{\gamma,\omega} & -\bar{a}_{1,2} \\ \frac{\gamma-\omega}{\omega}\bar{a}_{2,1} & \frac{1-\omega}{\omega} \end{pmatrix}\right]\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix}\\
&= (\rho-1)\begin{pmatrix} \bar{M}_{\gamma,\omega}^{-1}[(E_{1,1} + \gamma F_{1,1} + \frac{\gamma-\omega}{\rho}\bar{q}_{1,2}\bar{a}_{2,1})\bar{x}_1 + \frac{1-\omega}{\rho}\bar{q}_{1,2}\bar{x}_2] \\ -\gamma\bar{a}_{2,1}\bar{M}_{\gamma,\omega}^{-1}[(E_{1,1} + \gamma F_{1,1} + \frac{\gamma-\omega}{\rho}\bar{q}_{1,2}\bar{a}_{2,1})\bar{x}_1 + \frac{1-\omega}{\rho}\bar{q}_{1,2}\bar{x}_2] \end{pmatrix}.
\end{aligned}
$$

Hence, we have

$$
\bar{M}_{\gamma,\omega}^{-1}\bar{N}_{\gamma,\omega}\bar{x}_1 - \rho\bar{x}_1 = (\rho-1)\bar{M}_{\gamma,\omega}^{-1}[(E_{1,1} + \gamma F_{1,1} + \frac{\gamma-\omega}{\rho}\bar{q}_{1,2}\bar{a}_{2,1})\bar{x}_1 + \frac{1-\omega}{\rho}\bar{q}_{1,2}\bar{x}_2] \tag{45}
$$

and

$$
\begin{aligned}
&(1-\omega)\bar{x}_2 - \rho\bar{x}_2 \qquad (46)\\
&= \bar{a}_{2,1}[(\omega-\gamma)I + \gamma\bar{M}_{\gamma,\omega}^{-1}\bar{N}_{\gamma,\omega}]\bar{x}_1 - (\rho-1)\gamma\bar{a}_{2,1}\bar{M}_{\gamma,\omega}^{-1}[(E_{1,1} + \gamma F_{1,1} + \frac{\gamma-\omega}{\rho}\bar{q}_{1,2}\bar{a}_{2,1})\bar{x}_1 + \frac{1-\omega}{\rho}\bar{q}_{1,2}\bar{x}_2].
\end{aligned}
$$

For the case when (iii) holds, i.e., $\gamma < 1$, let $A_{1,1}^{(15)} = \check{D} - \check{L} - \check{U}$, where $\check{D}$, $\check{L}$ and $\check{U}$ are respectively diagonal, strictly lower and upper triangular matrices. Since $A_{1,1}^{(15)}$ is an irreducible L-matrix and

$$
\bar{M}_{\gamma,\omega}^{-1}\bar{N}_{\gamma,\omega} = \check{D}^{-1}[(1-\omega)\check{D} + \omega(1-\gamma)\check{L} + \omega\check{U}] + T \geq 0
$$

with

$$T = \omega\gamma\check{D}^{-1}\check{L}(\check{D} - \gamma\check{L})^{-1}[(1-\gamma)\check{L} + \check{U}] \geq 0,$$

then it follows that $\bar{M}_{\gamma,\omega}^{-1}\bar{N}_{\gamma,\omega}$ is irreducible, so that $\bar{M}_{\gamma,\omega}^{-1}\bar{N}_{\gamma,\omega}\bar{x}_1 \gg 0$. The irreducibility of A ensures that $\bar{a}_{2,1} < 0$. It is easy to see that

$$\frac{\gamma-\omega}{\rho}\bar{q}_{1,2}\bar{a}_{2,1}\bar{x}_1 + \frac{1-\omega}{\rho}\bar{q}_{1,2}\bar{x}_2 > 0,\ \bar{a}_{2,1}[(\omega-\gamma)I + \gamma\bar{M}_{\gamma,\omega}^{-1}\bar{N}_{\gamma,\omega}]\bar{x}_1 < 0.$$

When $\rho < 1$, from (45) and (46) it gets that

$$\bar{M}_{\gamma,\omega}^{-1}\bar{N}_{\gamma,\omega}\bar{x}_1 - \rho\bar{x}_1 < 0,\ (1-\omega)\bar{x}_2 - \rho\bar{x}_2 < 0,$$

which shows that $\rho(\bar{M}_{\gamma,\omega}^{-1}\bar{N}_{\gamma,\omega}) < \rho$ and $1-\omega < \rho$. Therefore we have $\rho(\mathscr{L}_{\gamma,\omega}^{(15)}) = \max\{1-\omega,\ \rho(\bar{M}_{\gamma,\omega}^{-1}\bar{N}_{\gamma,\omega})\} < \rho$. When $\rho > 1$ then, from (45) it gets that $\bar{M}_{\gamma,\omega}^{-1}\bar{N}_{\gamma,\omega}\bar{x}_1 - \rho\bar{x}_1 > 0$, which shows that $\rho(\bar{M}_{\gamma,\omega}^{-1}\bar{N}_{\gamma,\omega}) > \rho > 1$ and hence, $\rho(\mathscr{L}_{\gamma,\omega}^{(15)}) = \max\{1-\omega,\ \rho(\bar{M}_{\gamma,\omega}^{-1}\bar{N}_{\gamma,\omega})\} = \rho(\bar{M}_{\gamma,\omega}^{-1}\bar{N}_{\gamma,\omega}) > \rho$.

When (iv) holds, i.e., $\gamma = 1$, then $\omega = 1$ and the AOR method reduces to the Gauss-Seidel method. In this case, we have

$$\mathscr{L}^{(15)} = \begin{pmatrix} \bar{M}_{1,1}^{-1}\bar{N}_{1,1} & 0 \\ -\bar{a}_{2,1}\bar{M}_{1,1}^{-1}\bar{N}_{1,1} & 0 \end{pmatrix}$$

and therefore $\rho(\mathscr{L}^{(15)}) = \rho(\bar{M}_{1,1}^{-1}\bar{N}_{1,1})$. Now, (45) reduces to

$$\bar{M}_{1,1}^{-1}\bar{N}_{1,1}\bar{x}_1 - \rho\bar{x}_1 = (\rho-1)\bar{M}_{1,1}^{-1}(E_{1,1} + F_{1,1})\bar{x}_1.$$

Since $A_{1,1}^{(15)}$ is irreducible, then the rest of the proof is completely similar to that of Theorem 3.51. □

The result when (*iii*) and (*iv*) hold is better than [104, Theorem 3.16, Corollary 3.17], where the condition that $a_{k,n} \neq 0$ for $k = 1, \cdots, n-1$ is redundant. Again, the result when (*i*) and (*ii*) hold includes [104, Theorems 3.18, 3.19 and 3.20].

Theorem 3.109. *Theorem D is valid for $\nu = 15$, provided one of the following conditions is satisfied:*

(i) One of the conditions (i)-(iv) of Theorem 3.108 holds.

(ii) $0 \leq \alpha_k \leq 1$, $k = 1, \cdots, r-1$, one of the following conditions holds:

(ii_1) There exists $i \in \{1, \cdots, r-1\}$ such that $\alpha_i a_{i,r} a_{r,i} > 0$.

(ii_2) $\gamma > 0$ and there exist $i \in \{2, \cdots, r-1\}$ and $j \in \{1, \cdots, i-1\}$ such that $\alpha_i a_{i,r} a_{r,j} > 0$.

As a special case, for some $r < s$ with $a_{r,s} < 0$ and $\alpha > 0$, the matrix Q_{15} reduces to

$$Q_{16} = \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & -\frac{a_{r,s}}{\alpha} & \vdots \\ 0 & \cdots & 0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix},$$

which is given in [53] for $r = 1$ and $s = n$. And for $r = 1$, $s = n$ and $\alpha = 1$ it is proposed in [14]. When $r = 1$ and $s = n$, it is given in [96] for the preconditioned Gauss-Seidel method. When $\alpha = 1$ and $s = r + 1$, it is proposed in [13] for the preconditioned MSOR method. It is given in [109] to replace $-a_{r,s}/\alpha$ with a constant β.

In this case, $\delta_{i,j}^{(15)}(\gamma)$ and $\delta_{i,j}^{(15)}(1)$ reduce respectively to

$$\delta_{i,j}^{(16)}(\gamma) = \begin{cases} \frac{1}{\alpha} a_{r,s} a_{s,r}, & i = j = r; \\ \frac{\gamma - 1}{\alpha} a_{r,s}, & i = r, j = s; \\ \frac{\gamma}{\alpha} a_{r,s} a_{s,j}, & i = r, 1 \le j \le r-1, s+1 \le j \le n; \\ 0, & \textit{otherwise} \end{cases}$$

and

$$\delta_{i,j}^{(16)}(1) = \begin{cases} \frac{1}{\alpha} a_{r,s} a_{s,j}, & i = r, 1 \le j \le r, s+1 \le j \le n; \\ 0, & \textit{otherwise}. \end{cases}$$

Clearly, the conditions (*iii*) and (*iv*) of Theorem 3.108 can be not satisfied.
From Theorems 3.105 and 3.106, the following comparison results are immediately.

Theorem 3.110. *Suppose that $\alpha \ge 1$ and $\alpha > a_{r,s}a_{s,r}$. Then Theorem A is valid for $\nu = 16$.*

Theorem 3.111. *Suppose that $\alpha \ge 1$. Then Theorem B is valid for $\nu = 16$.*

In order to give the Stein-Rosenberg Type Theorem II, completely similar to Lemma 3.55, we can prove the following lemma.

Lemma 3.112. *Let A be an irreducible Z-matrix. Assume that $r = 1$ and $s = n$, $\alpha \ge 1$ and $A^{(16)}$ has the block form*

$$A^{(16)} = \begin{pmatrix} A_{2,2}^{(16)} & \bar{a}_{1,2}^{(16)} \\ \bar{a}_{2,1} & 1 \end{pmatrix}, \; A_{1,1}^{(16)} \in \mathscr{R}^{(n-1)\times(n-1)}.$$

Then one of the following two mutually exclusive relations holds:

(i) *$A^{(16)}$ is an irreducible Z-matrix.*

(ii) *$A^{(16)}$ is a reducible Z-matrix, but $A_{1,1}^{(16)}$ is an irreducible Z-matrix and $a_{1,k} = a_{1,k}^{(16)} = a_{1,n}^{(16)} = 0$, $k = 2, \cdots, n-1$.*

Using this lemma, completely similar to the proof of Theorem 3.108, we can prove the following theorem.

Theorem 3.113. *Suppose that $\alpha > a_{r,s}a_{s,r}$. Then Theorem C is valid for $\nu = 16$, provided one of the following conditions is satisfied:*

(i) *$\alpha \gtrsim 1$. And one of the following conditions holds:*

- *(i_1)* *$0 \le \gamma < 1$.*
- *(i_2)* *$\gamma = 1$ and there exists $k \in \{1, \cdots, r\} \cup \{s+1, \cdots, n\}$ such that $a_{s,k} < 0$.*
- *(i_3)* *$\gamma = 1$ and $s = r + 1$.*

(ii) *$r = 1$, $s = n$ and $\alpha \ge 1$.*

The result when (*ii*) holds is better than [53, Theorem 2]. For $\alpha = 1$, it is also better than the corresponding ones given by [54, Theorem 2.2, Corollaries 2.1, 2.2], where the condition $a_{k,k+1}a_{k+1,k} > 0$, $k = 1, \cdots, n-1$, implies that A is irreducible and so that the condition $a_{1,n}a_{n,1} > 0$ is unnecessary.

Theorem 3.114. *Theorem D is valid for $\nu = 16$, provided one of the following conditions is satisfied:*

(i) *One of the conditions (i) and (ii) of Theorem 3.113 holds.*

(ii) *$\alpha \ge 1$ and one of the following conditions holds:*

- *(ii_1)* *$a_{s,r} < 0$.*

(ii_2) $\gamma > 0$, $r \geq 2$ *and there exists* $k \in \{1, \cdots, r-1\}$ *such that* $a_{s,k} < 0$.

The result when (ii_1) holds for $(r, s) = (1, n)$ and $\alpha = 1$, is better than [14, Theorem 2.3].

In [30], for the preconditioned Gauss-Seidel method, Q is chosen as

$$Q_{17} = \begin{pmatrix} 0 & -\alpha_1 a_{1,2} & 0 & \cdots & 0 \\ 0 & 0 & -\alpha_2 a_{2,3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\alpha_{n-1} a_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

with $\alpha_k \geq 0$, $k = 1, \cdots, n-1$, and

$$\sum_{k=1}^{n-1} \alpha_k a_{k,k+1} \neq 0,$$

which is used to the preconditioned AOR method in [97, 104], to the preconditioned SOR method in [82] and to the preconditioned Gauss-Seidel and Jacobi methods in [17]. For $\alpha_k = 1$, $k = 1, \cdots, n-1$, it is proposed in [15] for the preconditioned Gauss-Seidel method, in [40] for the preconditioned SOR method, and in [12, 52] for the preconditioned AOR method.

In this case, $\delta_{i,j}^{(12)}(\gamma)$ and $\delta_{i,j}^{(12)}(1)$ reduce respectively to

$$\delta_{i,j}^{(17)}(\gamma) = \begin{cases} \alpha_i a_{i,i+1} a_{i+1,i}, & i = j = 1, \cdots, n-1; \\ (\gamma - 1)\alpha_i a_{i,i+1}, & i = 1, \cdots, n-1, j = i+1; \\ \gamma \alpha_i a_{i,i+1} a_{i+1,j}, & i = 1, \cdots, n-1, j = 1, \cdots, n, j \neq i, i+1; \\ 0, & i = n, j = 1, \cdots, n \end{cases}$$

and

$$\delta_{i,j}^{(17)}(1) = \begin{cases} \alpha_i a_{i,i+1} a_{i+1,j}, & i = 1, \cdots, n-1, j = 1, \cdots, n, j \neq i+1; \\ 0, & \textit{otherwise}. \end{cases}$$

From Corollaries 3.85 and 3.86 we can obtain the following comparison result, directly.

Theorem 3.115. *Suppose that* $0 \leq \alpha_k \leq 1$ *and* $\alpha_k a_{k,k+1} a_{k+1,k} < 1$, $k = 1, \cdots, n-1$. *Then Theorem A is valid for* $\nu = 17$.

The result improves the corresponding one given by [97, Theorem 2.1] and includes [50, Theorem 4.1] for the preconditioned Gauss-Seidel method. It is also better than [104, Theorem 3.6], where it is assumed that A is irreducible.

Theorem 3.116. *Suppose that* $0 \leq \alpha_k \leq 1$, $k = 1, \cdots, n-1$. *Then Theorem B is valid for* $\nu = 17$.

The result includes the corresponding one given by [97, Corollary 2.3]. For the Gauss-Seidel method it is better than [31, Theorem 2] where the condition that the Gauss-Seidel methods are convergent is redundant, [58, Theorem 28] where the assumption that A is irreducible is redundant, [34, Theorem 3.5] and [69, Theorem 2.4] since an irreducibly diagonally dominant Z-matrix is a nonsingular M-matrix.

Completely similar to Lemma 3.60 we can prove the following lemma.

Lemma 3.117. *Let A be a Z-matrix. Assume that* $n \geq 3$, $a_{1,n} < 0$, $a_{k+1,k} < 0$, $0 \leq \alpha_k \leq 1$, $k = 1, \cdots, n-1$. *Then A and $A^{(17)}$ are irreducible Z-matrices.*

Theorem 3.118. *Suppose that* $\alpha_k a_{k,k+1} a_{k+1,k} < 1$, $k = 1, \cdots, n-1$. *Then Theorem C is valid for* $\nu = 17$, *provided one of the following conditions is satisfied:*

(i) $0 \leq \alpha_k \lesssim 1$, $k = 1, \cdots, n-1$.

(ii) $0 < \alpha_k \le 1$ *and* $a_{k,k+1} < 0$, $k = 1, \cdots, n-1$.

(iii) $n \ge 3$, $0 \le \alpha_k \le 1$, $a_{1,n} < 0$, $a_{k+1,k} < 0$, $k = 1, \cdots, n-1$.

Proof. By the definition of Q_{17}, there exists $k_0 \in \{1, \cdots, n-1\}$ such that $a_{k_0,k_0+1} < 0$

When (*i*) holds, the conditions (*i*) and (*ii*$_3$) in Theorem 3.83 are satisfied, so that (*i*) of Corollary 3.87 holds.

Similarly, when (*ii*) holds, the conditions (*iii*) and (*iv*$_3$) in Theorem 3.83 are satisfied for $j(i) = i+1$, so that (*ii*) of Corollary 3.87 holds.

When (*iii*) holds, then, by Lemma 3.117, $A^{(17)}$ is an irreducible L-matrix. From (*i*) we can prove (*iii*).

The proof is complete. □

Obviously, from Lemma 3.117, if (*iii*) holds, then the assumption that A is irreducible is redundant.

The result when the condition (*i*) holds is better than [104, Theorems 3.4, 3.5] and [99, Theorem 2], where $\gamma < 1$.

The result when the condition (*ii*) holds includes [104, Theorem 3.1, Corollaries 3.2, 3.3] and [50, Theorem 4.2]. It is better than [12, Theorem 3.1, Corollary 3.1], [15, Theorem 4.1] and [40, Theorem 3], since the condition $a_{k,k+1}a_{k+1,k} > 0$, $k = 1, \cdots, n-1$, implies that A is irreducible and $a_{k,k+1} < 0$, $k = 1, \cdots, n-1$.

The corresponding result given in [52, Theorem 2] is incorrect, which is pointed out by [107]. But the corresponding one given in [107, Theorem 3.5] is also incorrect, which is pointed out by [99].

Similarly, from Corollary 3.88, we can obtain the following comparison result, directly.

Theorem 3.119. *Theorem D is valid for* $\nu = 17$, *provided one of the following conditions is satisfied:*

(i) One of the conditions (i), (ii) and (iii) of Theorem 3.118 holds.

(ii) $0 \le \alpha_k \le 1$, $k = 1, \cdots, n-1$ *and one of the following conditions holds:*

(ii$_1$) There exists $i \in \{1, \cdots, n-1\}$ *such that* $\alpha_i a_{i,i+1}a_{i+1,i} > 0$.

(ii$_2$) $\gamma > 0$ *and there exist* $i \in \{2, \cdots, n-1\}$ *and* $j \in \{1, \cdots, i-1\}$ *such that* $\alpha_i a_{i,i+1}a_{i+1,j} > 0$.

Different from Q_{15} and Q_{17}, we define $Q_{18} = (q_{i,j}^{(18)})$ as

$$q_{i,j}^{(18)} = \begin{cases} -\alpha_i a_{i,s_i}, & i = 1, \cdots, n-1, j = s_i, \\ 0, & \textit{otherwise}, \end{cases}$$

where

$$s_i = \min\{s : s \in \{k : |a_{i,k}| \text{ is maximal for } i+1 \le k \le n\}\}$$

and $\sum_{i=1}^{n-1} \alpha_i a_{i,s_i} \neq 0$, which is proposed in [34] for the preconditioned Gauss-Seidel method and $\alpha_k = 1$, $k = 1, \cdots, n-1$. In [33] its convergence for H-matrix is discussed.

In this case, $\delta_{i,j}^{(12)}(\gamma)$ and $\delta_{i,j}^{(12)}(1)$ reduce respectively to

$$\delta_{i,j}^{(18)}(\gamma) = \begin{cases} \alpha_i a_{i,s_i}a_{s_i,i}, & i = j = 1, \cdots, n-1; \\ (\gamma-1)\alpha_i a_{i,s_i}, & i = 1, \cdots, n-1, j = s_i; \\ \gamma\alpha_i a_{i,s_i}a_{s_i,j}, & i = 1, \cdots, n-1, j \in \{1, \cdots, i-1\} \cup \{s_i+1, \cdots, n\}; \\ 0, & \textit{otherwise} \end{cases}$$

and

$$\delta_{i,j}^{(18)}(1) = \begin{cases} \alpha_i a_{i,s_i}a_{s_i,j}, & i = 1, \cdots, n-1, \\ & j \in \{1, \cdots, i\} \cup \{s_i+1, \cdots, n\}; \\ 0, & \textit{otherwise}. \end{cases}$$

From Corollaries 3.85-3.88 we can obtain the following comparison result, directly.

Theorem 3.120. *Suppose that $0 \le \alpha_k \le 1$ and $\alpha_k a_{k,s_k} a_{s_k,k} < 1$, $k = 1, \cdots, n-1$. Then Theorem A is valid for $\nu = 18$.*

Theorem 3.121. *Suppose that $0 \le \alpha_k \le 1$, $k = 1, \cdots, n-1$. Then Theorem B is valid for $\nu = 18$.*

The result for the case when $\omega = \gamma$ and $\alpha_k = 1$, $k = 1, \cdots, n-1$, reduces to [47, Theorem 4.2].

Theorem 3.122. *Suppose that $\alpha_k a_{k,s_k} a_{s_k,k} < 1$, $k = 1, \cdots, n-1$. Then Theorem C is valid for $\nu = 18$, provided one of the following conditions is satisfied:*

(i) $0 \le \gamma < 1$ *and* $0 \le \alpha_k \lesssim 1$, $k = 1, \cdots, n-1$.

(ii) $\gamma = 1$, $0 \le \alpha_k \lesssim 1$, $k = 1, \cdots, n-1$. *And one of the following conditions holds:*

- *(ii₁)* *There exist* $i \in \{1, \cdots, n-1\}$, $j \in \{1, \cdots, i\} \cup \{s_i + 1, \cdots, n\}$ *such that* $\alpha_i a_{i,s_i} a_{s_i,j} > 0$.
- *(ii₂)* $a_{n,1} < 0$ *and* $a_{k,k+1} < 0$, $k = 1, \cdots, n-1$.
- *(ii₃)* $a_{n,1} < 0$ *and* $a_{k,n} < 0$, $k = 2, \cdots, n-1$.
- *(ii₄)* $a_{k,1} < 0$, $k = 2, \cdots, n$.

(iii) $0 < \alpha_k \le 1$ *and* $a_{k,s_k} < 0$, $k = 1, \cdots, n-1$. *For each* $i \in \{1, \cdots, n-1\}$ *one of the following conditions holds:*

- *(iii₁)* $0 \le \gamma < 1$.
- *(iii₂)* $\gamma = 1$ *and there exists* $j(i) \in \{1, \cdots, i\} \cup \{s_i + 1, \cdots, n\}$ *such that* $a_{s_i,j(i)} > 0$.

The result when (iii_2) holds for $\alpha_k = 1$, $k = 1, \cdots, n-1$, is better than [47, Theorem 4.3], since its condition insures that A is irreducible.

Theorem 3.123. *Theorem D is valid for $\nu = 18$, provided one of the following conditions is satisfied:*

(i) *One of the conditions (i), (ii) and (iii) of Theorem 3.122 holds.*

(ii) $0 \le \alpha_k \le 1$, $k = 1, \cdots, n-1$ *and one of the following conditions holds:*

- *(ii₁)* *There exists* $i \in \{1, \cdots, n-1\}$ *such that* $\alpha_i a_{i,s_i} a_{s_i,i} > 0$.
- *(ii₂)* $\gamma > 0$ *and there exist* $i \in \{2, \cdots, n-1\}$ *and* $j \in \{1, \cdots, i-1\}$ *such that* $\alpha_i a_{i,s_i} a_{s_i,j} > 0$.

Similar to Q_{10}, Q can be defined as

$$Q_{19} = \begin{pmatrix} 0 & \cdots & 0 & -\alpha_1 a_{1,n} + \beta_1 \\ 0 & \cdots & 0 & -\alpha_2 a_{2,n} + \beta_2 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & -\alpha_{n-1} a_{n-1,n} + \beta_{n-1} \\ 0 & \cdots & 0 & 0 \end{pmatrix}$$

with $\alpha_k \ge 0$, $-\alpha_k a_{k,n} + \beta_k \ge 0$, $k = 1, \cdots, n-1$, and

$$\sum_{k=1}^{n-1} (-\alpha_k a_{k,n} + \beta_k) \ne 0.$$

For $\alpha_k = 1$, it is given in [9] for the preconditioned AOR method, and in [8] for the preconditioned SOR method.

In this case, $\delta^{(1)}_{i,j}(\gamma)$ and $\delta^{(1)}_{i,j}(1)$ reduce respectively to

$$\delta^{(19)}_{i,j}(\gamma) = \begin{cases} (\alpha_i a_{i,n} - \beta_i) a_{n,i}, & i = j = 1, \cdots, n-1; \\ (\gamma - 1)(\alpha_i a_{i,n} - \beta_i), & i = 1, \cdots, n-1, j = n; \\ \gamma(\alpha_i a_{i,n} - \beta_i) a_{n,j}, & i = 1, \cdots, n-1, j = 1, \cdots, i-1; \\ 0, & \textit{otherwise} \end{cases}$$

and

$$\delta_{i,j}^{(19)}(1) = \begin{cases} (\alpha_i a_{i,n} - \beta_i) a_{n,j}, & i = 1, \cdots, n-1, j = 1, \cdots, i; \\ 0, & \text{otherwise}. \end{cases}$$

Similar to Lemma 3.107, we have the following lemma.

Lemma 3.124. *Let A be an irreducible Z-matrix. Assume that $0 < -\alpha_k a_{k,n} + \beta_k \leq -a_{k,n}$, $k = 1, \cdots, n-1$ and $A^{(19)}$ has the block form*

$$A^{(19)} = \begin{pmatrix} A_{1,1}^{(19)} & \bar{a}_{1,2}^{(19)} \\ \bar{a}_{2,1} & 1 \end{pmatrix}, \ A_{1,1}^{(19)} \in \mathscr{R}^{(n-1)\times(n-1)}.$$

Then

(i) $A_{1,1}^{(19)}$ *is an irreducible Z-matrix.*

(ii) $A^{(19)}$ *is an irreducible Z-matrix if and only if there exists $j_0 \in \{1, \cdots, n-1\}$ such that $(1 - \alpha_{j_0}) a_{j_0,n} + \beta_{j_0} \neq 0$.*

Similar to Theorems 3.105-3.109, we can prove the following comparison theorems.

Theorem 3.125. *Suppose that $0 \leq -\alpha_k a_{k,n} + \beta_k \leq -a_{k,n}$ and $(\alpha_k a_{k,n} - \beta_k) a_{n,k} < 1$, $k = 1, \cdots, n-1$. Then Theorem A is valid for $\nu = 19$.*

Theorem 3.126. *Suppose that $0 \leq -\alpha_k a_{k,n} + \beta_k \leq -a_{k,n}$, $k = 1, \cdots, n-1$. Then Theorem B is valid for $\nu = 19$.*

Theorem 3.127. *Suppose that $(1 - \alpha_k) a_{k,n} + \beta_k \leq 0$ and $(\alpha_k a_{k,n} - \beta_k) a_{n,k} < 1$, $k = 1, \cdots, n-1$. Then Theorem C is valid for $\nu = 19$, provided one of the following conditions is satisfied:*

(i) $0 \leq \gamma < 1$ *and* $(1 - \alpha_k) a_{k,n} + \beta_k \lesssim 0$ *whenever* $a_{k,n} < 0$, $k = 1, \cdots, n-1$.

(ii) $\gamma = 1$ *and* $(1 - \alpha_k) a_{k,n} + \beta_k \lesssim 0$ *whenever* $a_{k,n} < 0$, $k = 1, \cdots, n-1$. *And one of the following conditions holds:*

- *(ii_1) There exist $i \in \{1, \cdots, n-1\}$ and $j \in \{1, \cdots, i\}$ such that $(\alpha_i a_{i,n} - \beta_i) a_{n,j} > 0$.*
- *(ii_2)* $-\alpha_{n-1} a_{n-1,n} + \beta_{n-1} > 0$.
- *(ii_3)* $a_{n,1} < 0$.

(iii) $-\alpha_k a_{k,n} + \beta_k > 0$, $k = 1, \cdots, n-1$.

Proof. By Theorem 3.6 and referring to the proof of Theorem 3.108, we just need to prove the case when $\gamma = 1$ in (*iii*).

In fact, using Lemma 3.124, it is easy to prove that a sufficient condition similar with (*iv*) of Theorem 3.108 is that there exist $i \in \{1, \cdots, n-1\}$ and $j \in \{1, \cdots, i\}$ such that $(\alpha_i a_{i,n} - \beta_i) a_{n,j} > 0$, which is equivalent to that there exists $j \in \{1, \cdots, n-1\}$ such that $a_{n,j} < 0$, since $-\alpha_i a_{i,n} + \beta_i > 0$. While, the irreducibility of A ensures that it is true. □

When (*iii*) holds, the result is better that the corresponding ones given by [8, Theorem 3.3, Corollary 3.3] and [9, Theorem 3.2].

Theorem 3.128. *Theorem D is valid for $\nu = 19$, provided one of the following conditions is satisfied:*

(i) One of the conditions (i), (ii) and (iii) of Theorem 3.127 holds.

(ii) $0 \leq \alpha_k \leq 1$, $k = 1, \cdots, n-1$ *and one of the following conditions holds:*

- *(ii_1) There exists $i \in \{1, \cdots, n-1\}$ such that $(\alpha_i a_{i,n} - \beta_i) a_{n,i} > 0$.*
- *(ii_2) $\gamma > 0$ and there exist $i \in \{2, \cdots, n-1\}$ and $j \in \{1, \cdots, i-1\}$ such that $(\alpha_i a_{i,n} - \beta_i) a_{n,j} > 0$.*

As a special of Q_{19}, Q is proposed in [38] as

$$Q_{20} = \begin{pmatrix} 0 & \cdots & 0 & -\frac{a_{1,n}}{\alpha} - \beta \\ 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix}$$

with $\alpha > 0$ and $a_{1,n}/\alpha + \beta < 0$, which is given in [51] for $\alpha = 1$.

In this case, similar to $\delta_{i,j}^{(16)}(1)$, we can derive $\delta_{i,j}^{(20)}(1)$, whose all elements are zero except $\delta_{1,1}^{(20)}(1) = (a_{1,n}/\alpha + \beta)a_{n,1}$.

Theorem 3.129. *Suppose that $\beta \geq (1 - 1/\alpha)a_{1,n}$ and $(a_{1,n}/\alpha + \beta)a_{n,1} < 1$. Then Theorem A is valid for $\nu = 20$.*

Proof. Since

$$\left(1 - \frac{1}{\alpha}\right)a_{1,n} \leq \beta \text{ iff } -\frac{a_{1,n}}{\alpha} - \beta \leq -a_{1,n},$$

and the inequality (19) reduces to $(a_{1,n}/\alpha + \beta)a_{n,1} < 1$, then the condition of Corollary 3.12 is satisfied so that Theorem A is valid. □

It is easy to prove the following theorem.

Theorem 3.130. *Suppose that $\beta \geq (1 - 1/\alpha)a_{1,n}$. Then Theorem B is valid for $\nu = 20$.*

In order to give the Stein-Rosenberg Type Theorem II, completely similar to Lemma 3.112, we can prove the following lemma.

Lemma 3.131. *Let A be an irreducible Z-matrix. Assume that $\beta \geq (1 - 1/\alpha)a_{1,n}$ and $A^{(20)}$ has the block form*

$$A^{(20)} = \begin{pmatrix} A_{1,1}^{(20)} & \bar{a}_{1,2}^{(20)} \\ \bar{a}_{2,1} & 1 \end{pmatrix}, \ A_{1,1}^{(20)} \in \mathscr{R}^{(n-1)\times(n-1)}.$$

Then one of the following two mutually exclusive relations holds:

(i) *$A^{(20)}$ is an irreducible Z-matrix.*

(ii) *$A^{(20)}$ is a reducible Z-matrix, but $A_{1,1}^{(20)}$ is an irreducible Z-matrix and $a_{1,k} = a_{1,k}^{(20)} = a_{1,n}^{(20)} = 0$, $k = 2, \cdots, n-1$.*

Using this lemma, completely similar to (*ii*) of Theorem 3.113, we can prove the following theorem.

Theorem 3.132. *Suppose that $\beta \geq (1 - 1/\alpha)a_{1,n}$ and $(a_{1,n}/\alpha + \beta)a_{n,1} < 1$. Then Theorem C is valid for $\nu = 20$.*

The result is better than [41, Theorem 6], where the condition $0 < a_{1,n}a_{n,1} < \alpha(\alpha > 1)$ is unnecessary.

Theorem 3.133. *Suppose that $\beta \geq (1 - 1/\alpha)a_{1,n}$. Then Theorem D is valid for $\nu = 20$.*

By the definition of Q_{20}, $a_{1,n}/\alpha + \beta < 0$. While in the comparison theorems above we need the condition $\beta \geq (1 - 1/\alpha)a_{1,n}$. Hence it implies $a_{1,n} < 0$.

For $\alpha = 1$, the result is better than the corresponding ones given by [51, Theorem 9, Corollaries 10, 11], where the condition $a_{1,n}a_{n,1} > 0$ is unnecessary.

3.4. Combination preconditioners

In this subsection, when the matrix Q is composed of two different combinations of Q_i and Q_j, we always assume $Q_i > 0$ and $Q_j > 0$. Otherwise, the corresponding situation has been discussed in the above two subsections.

First, the matrix Q is chosen as

$$Q_{21} = Q_5 + Q_{12},$$

i.e.,

$$Q_{21} = \begin{pmatrix} 0 & -\beta_{1,2}a_{1,2} & \cdots & -\beta_{1,n-1}a_{1,n-1} & -\beta_{1,n}a_{1,n} \\ 0 & 0 & \cdots & -\beta_{2,n-1}a_{2,n-1} & -\beta_{2,n}a_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -\beta_{n-1,n}a_{n-1,n} \\ -\alpha_1 a_{n,1} & -\alpha_2 a_{n,2} & \cdots & -\alpha_{n-1}a_{n,n-1} & 0 \end{pmatrix}$$

with $\alpha_i \geq 0$, $\beta_{i,j} \geq 0$, $i = 1, \cdots, n-1$, $j = i+1, \cdots, n$, and

$$\sum_{k=1}^{n-1} \alpha_k a_{n,k} \neq 0 \ \text{ and } \ \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \beta_{i,j}a_{i,j} \neq 0,$$

where for simplicity we set $r = n$ for Q_5.

When $\alpha_i = \beta_{i,j} = 1$, $i = 1, \cdots, n-1$, $j = i+1, \cdots, n$, it is proposed in [100] for the preconditioned Gauss-Seidel method.

By Corollaries 3.20 and 3.21, the following two comparison theorems are directly.

Theorem 3.134. *Suppose that $0 \leq \alpha_i \leq 1$, $0 \leq \beta_{i,j} \leq 1$, $i = 1, \cdots, n-1$, $j = i+1, \cdots, n$ and*

$$\sum_{k=1}^{n-1} \alpha_k a_{n,k}a_{k,n} < 1, \ \sum_{k=i+1}^{n} \beta_{i,k}a_{i,k}a_{k,i} < 1, \ i = 1, \cdots, n-1. \tag{47}$$

Then Theorem A is valid for $\nu = 21$.

Theorem 3.135. *Suppose that $0 \leq \alpha_i \leq 1$, $0 \leq \beta_{i,j} \leq 1$, $i = 1, \cdots, n-1$, $j = i+1, \cdots, n$. Then Theorem B is valid for $\nu = 21$.*

In this case, $\delta_{i,j}^{(2)}(\gamma)$ and $\delta_{i,j}^{(2)}(1)$ reduce respectively to

$$\delta_{i,j}^{(21)}(\gamma) = \begin{cases} \sum\limits_{k=i+1}^{n} \beta_{i,k}a_{i,k}a_{k,i}, & i = j = 1, \cdots, n-1; \\ \sum\limits_{k=1}^{n-1} \alpha_k a_{n,k}a_{k,n}, & i = j = n; \\ (\gamma - 1)\beta_{i,j}a_{i,j} + \gamma \sum\limits_{k=i+1}^{j-1} \beta_{i,k}a_{i,k}a_{k,j}, & i = 1, \cdots, n-1, j = i+1, \cdots, n; \\ \gamma \sum\limits_{k=i+1}^{n} \beta_{i,k}a_{i,k}a_{k,j}, & i = 2, \cdots, n-1, j = 1, \cdots, i-1; \\ (\gamma - 1)\alpha_j a_{n,j} + \gamma \sum\limits_{k=1}^{j-1} \alpha_k a_{n,k}a_{k,j}, & i = n, j = 1, \cdots, n-1 \end{cases}$$

and

$$\delta_{i,j}^{(21)}(1)=\begin{cases}\sum\limits_{k=i+1}^{n}\beta_{i,k}a_{i,k}a_{k,j}, & i=1,\cdots,n-1,\ j=1,\cdots,i;\\ \sum\limits_{k=i+1}^{j-1}\beta_{i,k}a_{i,k}a_{k,j}, & i=1,\cdots,n-2,\ j=i+2,\cdots,n;\\ \sum\limits_{k=1}^{j-1}\alpha_k a_{n,k}a_{k,j}, & i=n,\ j=2,\cdots,n;\\ 0, & \textit{otherwise}.\end{cases}$$

Using Corollary 3.22, we prove Stein-Rosenberg type comparison theorem.

Theorem 3.136. *Suppose that (47) holds. Then Theorem C is valid for $\nu = 21$, provided one of the following conditions is satisfied:*

- *(i)* $0 \le \gamma < 1$ *and* $0 \le \alpha_i \lesssim 1$, $0 \le \beta_{i,j} \lesssim 1$, $i = 1, \cdots, n-1$, $j = i+1, \cdots, n$.
- *(ii)* $\gamma = 1$ *and* $0 \le \alpha_i \lesssim 1$, $0 \le \beta_{i,j} \lesssim 1$, $i = 1, \cdots, n-1$, $j = i+1, \cdots, n$. *And one of the following conditions holds:*
  - *(ii_1)* *There exist* $i \in \{1, \cdots, n-1\}$, $j \in \{1, \cdots, i\}$ *and* $k \in \{i+1, \cdots, n\}$ *such that* $\beta_{i,k}a_{i,k}a_{k,j} > 0$.
  - *(ii_2)* *There exist* $i \in \{1, \cdots, n-2\}$, $j \in \{i+2, \cdots, n\}$ *and* $k \in \{i+1, \cdots, j-1\}$ *such that* $\beta_{i,k}a_{i,k}a_{k,j} > 0$.
  - *(ii_3)* *There exist* $j \in \{2, \cdots, n\}$ *and* $k \in \{1, \cdots, j-1\}$ *such that* $\alpha_k a_{n,k}a_{k,j} > 0$.
  - *(ii_4)* *There exists* $k \in \{1, \cdots, n-1\}$ *such that* $a_{k,k+1} < 0$ *and* $\beta_{k,k+1} > 0$.
  - *(ii_5)* $a_{n,1} < 0$ *and* $\alpha_1 > 0$.
  - *(ii_6)* $a_{k,1} < 0$, $k = 2, \cdots, n$.
  - *(ii_7)* $a_{n,1} < 0$ *and* $a_{k,n} < 0$, $k = 2, \cdots, n-1$.
  - *(ii_8)* $a_{k,n} < 0$, $k = 1, \cdots, n-1$.
  - *(ii_9)* $a_{k,k+1} < 0$, $k = 1, \cdots, n-1$.
- *(iii)* $0 \le \gamma < 1$ *and* $0 \le \alpha_i \le 1$, $0 \le \beta_{i,j} \le 1$, $i = 1, \cdots, n-1$, $j = i+1, \cdots, n$. *And for each* $i \in \{1, \cdots, n-1\}$, *there exists* $j(i) \in \{i+1, \cdots, n\}$ *such that* $\beta_{i,j(i)}a_{i,j(i)} < 0$.
- *(iv)* $\gamma = 1$ *and* $0 \le \alpha_i \le 1$, $0 \le \beta_{i,j} \le 1$, $i = 1, \cdots, n-1$, $j = i+1, \cdots, n$. *For each* $i \in \{2, \cdots, n-1\}$, *one of the following conditions holds:*
  - *(iv_1)* *There exist* $j(i) \in \{1, \cdots, i\}$ *and* $k(i) \in \{i+1, \cdots, n\}$ *such that* $\beta_{i,k(i)}a_{i,k(i)}a_{k(i),j(i)} > 0$.
  - *(iv_2)* *There exist* $j(i) \in \{i+2, \cdots, n\}$ *and* $k(i) \in \{i+1, \cdots, j-1\}$ *such that* $\beta_{i,k(i)}a_{i,k(i)}a_{k(i),j(i)} > 0$.
  - *(iv_3)* $a_{i,i+1} < 0$ *and* $\beta_{i,i+1} > 0$.

  At the same time, one of the following conditions also holds:

  - *(iv^a)* *There exist* $j \in \{2, \cdots, n\}$ *and* $k \in \{1, \cdots, j-1\}$ *such that* $\alpha_k a_{n,k}a_{k,j} > 0$.
  - *(iv^b)* $a_{n,1} < 0$ *and* $\alpha_1 > 0$.
  - *(iv^c)* *There exists* $j \in \{2, \cdots, n-1\}$ *such that*

    $$(1-\alpha_j)a_{n,j} - \sum_{\substack{k=1\\k\neq j}}^{n-1}\alpha_k a_{n,k}a_{k,j} < 0.$$

  - *(iv^d)* *One of the conditions (iv_1)-(iv_3) holds for* $i = 1$ *and*

    $$(1-\alpha_1)a_{n,1} - \sum_{k=2}^{n-1}\alpha_k a_{n,k}a_{k,1} < 0.$$

(iv^e) One of the conditions (iv_1)-(iv_3) holds for $i = 1$ and $a_{n,1} < 0$.

Proof. When (i) or (iii) holds, then the condition (i) or (iii) of Theorem 3.18 is satisfied.

When one of (ii_1), (ii_2) and (ii_3) holds, then $\delta^{(21)}_{i,j}(1) > 0$ or $\delta^{(21)}_{n,j}(1) > 0$, which implies that ($ii_1$) in Theorem 3.18 is satisfied.

When (ii_4) or (ii_5) holds, then (ii_3) or (ii_2) in Theorem 3.18 is satisfied.

When one of (ii_6) and (ii_7) holds, then by the proof of Theorem 3.83 it follows that (ii_1) is satisfied.

Similarly, when one of (ii_8) and (ii_9) holds, then by the proof of Theorem 3.42 it follows that (ii_3) is satisfied.

Exactly the same, we can prove that if (iv_4) holds, then the condition (iv_4) in Theorem 3.18 is satisfied.

By Corollary 3.22 the proof is complete. □

Theorem 3.137. *Theorem D is valid for $\nu = 21$, provided one of the following conditions is satisfied:*

(i) One of the conditions (i)-(iv) of Theorem 3.136 holds.

(ii) $0 \leq \alpha_i \leq 1$, $0 \leq \beta_{i,j} \leq 1$, $i = 1, \cdots, n-1$, $j = i+1, \cdots, n$. For $j = 1, \cdots, n-1$,

$$\alpha_j a_{n,j} + \sum_{\substack{k=1\\k\neq j}}^{n-1} \alpha_k a_{n,k} a_{k,j} \geq 0.$$

And one of the following conditions holds:

(ii_1) There exists $i_0 \in \{1, \cdots, n-1\}$ such that $\sum\limits_{k=i_0+1}^{n} \beta_{i_0,k} a_{i_0,k} a_{k,i_0} > 0$.

(ii_2) $\sum_{k=1}^{n-1} \alpha_k a_{n,k} a_{k,n} > 0$.

(ii_3) $\gamma > 0$ and there exist $i_0 \in \{2, \cdots, n-1\}$, $j_0 \in \{1, \cdots, i_0 - 1\}$ such that $\sum_{k=i_0+1}^{n} \beta_{i_0,k} a_{i_0,k} a_{k,j_0} > 0$.

(ii_4) $\gamma > 0$ and there exists $j_0 \in \{1, \cdots, n-1\}$ such that $\alpha_{j_0} a_{n,j_0} + \sum_{k=1,k\neq j_0}^{n-1} \alpha_k a_{n,k} a_{k,j_0} > 0$.

Proof. We just need to prove (ii). It is easy to obtain that

$$a^{(21)}_{i,j} = \begin{cases} 1 - \sum\limits_{k=i+1}^{n} \alpha_{i,k} a_{i,k} a_{k,i}, & i = j = 1, \cdots, n-1, \\ 1 - \sum\limits_{k=1}^{n-1} \alpha_k a_{n,k} a_{k,n}, & i = j = n, \\ a_{i,j} - \sum\limits_{k=i+1}^{n} \beta_k a_{i,k} a_{k,j}, & i = 2, \cdots, n-1, j = 1, \cdots, i-1, \\ (1-\alpha_j) a_{n,j} - \sum\limits_{k=1}^{n-1} \alpha_k a_{n,k} a_{k,j}, & i = n, j = 1, \cdots, n-1. \end{cases}$$

By Corollary 3.23 we can derive (ii_1)-(ii_4). □

As a special case of Q_{21} for $\alpha_k = 0$, $k = 2, \cdots, n-1$, it gets that

$$Q_{22} = Q_7 + Q_{12},$$

i.e.,

$$Q_{22} = \begin{pmatrix} 0 & -\alpha_{1,2} a_{1,2} & -\alpha_{1,3} a_{1,3} & \cdots & -\alpha_{1,n} a_{1,n} \\ 0 & 0 & -\alpha_{2,3} a_{2,3} & \cdots & -\alpha_{2,n} a_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\alpha_{n-1,n} a_{n-1,n} \\ -\alpha a_{n,1} & 0 & 0 & \cdots & 0 \end{pmatrix}$$

with $a_{n,1} < 0$, $\alpha > 0$, $\beta_{i,j} \geq 0$, $i = 1, \cdots, n-1$, $j = i+1, \cdots, n$, and $\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \beta_{i,j} a_{i,j} \neq 0$, where we set $r = n$ and $s = 1$ for Q_7.

In this case, (ii_5) and (iv^b) in Theorem 3.136 are satisfied. While, (ii) of Theorem 3.137 can be not satisfied. Hence, from Theorems 3.134-3.137, we can derive the following theorems, directly.

Theorem 3.138. *Suppose that* $0 < \alpha \leq 1$, $0 \leq \beta_{i,j} \leq 1$, $i = 1, \cdots, n-1$, $j = i+1, \cdots, n$, *and*

$$\alpha a_{n,1} a_{1,n} < 1, \ \sum_{k=i+1}^{n} \beta_{i,k} a_{i,k} a_{k,i} < 1, \ i = 1, \cdots, n-1. \tag{48}$$

Then Theorem A is valid for $\nu = 22$.

Theorem 3.139. *Suppose that* $0 < \alpha \leq 1$, $0 \leq \beta_{i,j} \leq 1$, $i = 1, \cdots, n-1$, $j = i+1, \cdots, n$. *Then Theorem B is valid for* $\nu = 22$.

Theorem 3.140. *Suppose that (48) holds. Then Theorem C is valid for* $\nu = 22$, *provided one of the following conditions is satisfied:*

(i) $0 < \alpha \lesssim 1$, $0 \leq \beta_{i,j} \lesssim 1$, $i = 1, \cdots, n-1$, $j = i+1, \cdots, n$.

(ii) $0 \leq \gamma < 1$ *and* $0 < \alpha \leq 1$, $0 \leq \beta_{i,j} \leq 1$, $i = 1, \cdots, n-1$, $j = i+1, \cdots, n$. *And for each* $i \in \{1, \cdots, n-1\}$, *there exists* $j(i) \in \{i+1, \cdots, n\}$ *such that* $\beta_{i,j(i)} a_{i,j(i)} < 0$.

(iii) $\gamma = 1$ *and* $0 < \alpha \leq 1$, $0 \leq \beta_{i,j} \leq 1$, $i = 1, \cdots, n-1$, $j = i+1, \cdots, n$. *For each* $i \in \{2, \cdots, n-1\}$, *one of the following conditions holds:*

- *(iii_1) There exist* $j(i) \in \{1, \cdots, i\}$ *and* $k(i) \in \{i+1, \cdots, n\}$ *such that* $\beta_{i,k(i)} a_{i,k(i)} a_{k(i),j(i)} > 0$.
- *(iii_2) There exist* $j(i) \in \{i+2, \cdots, n\}$ *and* $k(i) \in \{i+1, \cdots, j-1\}$ *such that* $\beta_{i,k(i)} a_{i,k(i)} a_{k(i),j(i)} > 0$.
- *(iii_3)* $a_{i,i+1} < 0$ *and* $\beta_{i,i+1} > 0$.

Theorem 3.141. *Theorem D is valid for* $\nu = 22$, *provided one of the conditions (i), (ii) and (iii) of Theorem 3.140 is satisfied.*

Similar to Q_{21}, the matrix Q is chosen as

$$Q_{23} = Q_6 + Q_{12},$$

i.e.,

$$Q_{23} = \begin{pmatrix} 0 & -\beta_{1,2} a_{1,2} & \cdots & -\beta_{1,n-1} a_{1,n-1} & -\beta_{1,n} a_{1,n} \\ -\alpha_2 a_{2,1} & 0 & \cdots & -\beta_{2,n-1} a_{2,n-1} & -\beta_{2,n} a_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\alpha_{n-1} a_{n-1,1} & 0 & \cdots & 0 & -\beta_{n-1,n} a_{n-1,n} \\ -\alpha_n a_{n,1} & 0 & \cdots & 0 & 0 \end{pmatrix}$$

with $\alpha_{i+1} \geq 0$, $\beta_{i,j} \geq 0$, $i = 1, \cdots, n-1$, $j = i+1, \cdots, n$, and

$$\sum_{k=2}^{n} \alpha_k a_{k,1} \neq 0 \ \text{ and } \ \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \beta_{i,j} a_{i,j} \neq 0,$$

where we set $r = 2$ for Q_6.

Similar to the Theorems 3.134 and 3.135, we have the following two comparison theorems.

Theorem 3.142. *Suppose that* $0 \le \alpha_{i+1} \le 1$, $0 \le \beta_{i,j} \le 1$, $i = 1, \cdots, n-1$, $j = i+1, \cdots, n$ *and*

$$\alpha_n a_{n,1} a_{1,n} < 1,\ \sum_{k=2}^{n} \beta_{1,k} a_{1,k} a_{k,1} < 1,\ \alpha_i a_{i,1} a_{1,i} + \sum_{k=i+1}^{n} \beta_{i,k} a_{i,k} a_{k,i} < 1,\ i = 2, \cdots, n-1. \tag{49}$$

Then Theorem A is valid for $\nu = 23$.

Theorem 3.143. *Suppose that* $0 \le \alpha_{i+1} \le 1$, $0 \le \beta_{i,j} \le 1$, $i = 1, \cdots, n-1$, $j = i+1, \cdots, n$. *Then Theorem B is valid for* $\nu = 23$.

In this case, $\delta^{(2)}_{i,j}(\gamma)$ and $\delta^{(2)}_{i,j}(1)$ reduce respectively to

$$\delta^{(23)}_{i,j}(\gamma) = \begin{cases} \sum\limits_{k=i+1}^{n} \beta_{1,k} a_{1,k} a_{k,1}, & i = j = 1; \\ \alpha_i a_{i,1} a_{1,i} + \sum\limits_{k=i+1}^{n} \beta_{i,k} a_{i,k} a_{k,i}, & i = j = 2, \cdots, n-1; \\ \alpha_n a_{n,1} a_{1,n}, & i = j = n; \\ (\gamma - 1)\beta_{i,i+1} a_{i,i+1}, & i = 1, \cdots, n-1, j = i+1; \\ (\gamma - 1)\beta_{i,j} a_{i,j} + \gamma \sum\limits_{k=i+1}^{j-1} \beta_{i,k} a_{i,k} a_{k,j}, & i = 1, \cdots, n-1, j = i+2, \cdots, n; \\ (\gamma - 1)\alpha_i a_{i,1} + \gamma \sum\limits_{k=i+1}^{n} \beta_{i,k} a_{i,k} a_{k,1}, & i = 2, \cdots, n-1, j = 1; \\ \gamma \alpha_i a_{i,1} a_{1,j} + \gamma \sum\limits_{k=i+1}^{n} \beta_{i,k} a_{i,k} a_{k,j}, & i = 3, \cdots, n-1, j = 2, \cdots, i-1; \\ (\gamma - 1)\alpha_n a_{n,1}, & i = n, j = 1; \\ \gamma \alpha_n a_{n,1} a_{1,j}, & i = n, j = 2, \cdots, n-1 \end{cases}$$

and

$$\delta^{(23)}_{i,j}(1) = \begin{cases} \sum\limits_{k=i+1}^{n} \beta_{i,k} a_{i,k} a_{k,1}, & i = 1, \cdots, n-1, j = 1; \\ \alpha_i a_{i,1} a_{1,j} + \sum\limits_{k=i+1}^{n} \beta_{i,k} a_{i,k} a_{k,j}, & i = 2, \cdots, n-1, j = 2, \cdots, i; \\ \sum\limits_{k=i+1}^{j-1} \beta_{i,k} a_{i,k} a_{k,j}, & i = 1, \cdots, n-1, j = i+2, \cdots, n; \\ \alpha_n a_{n,1} a_{1,j}, & i = n, j = 2, \cdots, n; \\ 0, & \text{otherwise}. \end{cases}$$

Using Corollary 3.22, we prove Stein-Rosenberg type comparison theorem.

Theorem 3.144. *Suppose that (49) holds. Then Theorem C is valid for* $\nu = 23$, *provided one of the following conditions is satisfied:*

(i) $0 \le \gamma < 1$ *and* $0 \le \alpha_{i+1} \lesssim 1$, $0 \le \beta_{i,j} \lesssim 1$, $i = 1, \cdots, n-1$, $j = i+1, \cdots, n$.

(ii) $\gamma = 1$ *and* $0 \le \alpha_{i+1} \lesssim 1$, $0 \le \beta_{i,j} \lesssim 1$, $i = 1, \cdots, n-1$, $j = i+1, \cdots, n$. *And one of the following conditions holds:*

- *(ii$_1$) There exist* $i \in \{1, \cdots, n-1\}$, $j \in \{1, \cdots, i\}$ *and* $k \in \{i+1, \cdots, n\}$ *such that* $\beta_{i,k} a_{i,k} a_{k,j} > 0$.
- *(ii$_2$) There exist* $i \in \{1, \cdots, n-2\}$, $j \in \{i+2, \cdots, n\}$ *and* $k \in \{i+1, \cdots, j-1\}$ *such that* $\beta_{i,k} a_{i,k} a_{k,j} > 0$.
- *(ii$_3$) There exist* $i \in \{2, \cdots, n\}$ *and* $j \in \{2, \cdots, i\}$ *such that* $\alpha_i a_{i,1} a_{1,j} > 0$.
- *(ii$_4$) There exists* $k \in \{2, \cdots, n-1\}$ *such that* $a_{k,k+1} < 0$ *and* $\beta_{k,k+1} > 0$.
- *(ii$_5$)* $a_{n,1} < 0$ *and* $\alpha_n > 0$.
- *(ii$_6$)* $a_{1,2} < 0$.

(iii) $0 \le \gamma < 1$ *and* $0 \le \alpha_{i+1} \le 1$, $0 \le \beta_{i,j} \le 1$, $i = 1, \cdots, n-1$, $j = i+1, \cdots, n$. *And for each* $i \in \{1, \cdots, n-1\}$, $\alpha_i a_{i,1} < 0$ *(for* $i \ge 2$*) or there exists* $j(i) \in \{i+1, \cdots, n\}$ *such that* $\beta_{i,j(i)} a_{i,j(i)} < 0$.

(iv) $\gamma = 1$ *and* $0 \le \alpha_{i+1} \le 1$, $0 \le \beta_{i,j} \le 1$, $i = 1, \cdots, n-1$, $j = i+1, \cdots, n$. *For each* $i \in \{2, \cdots, n-1\}$, *one of the following conditions holds:*

(iv_1) *There exist* $j(i) \in \{1, \cdots, i\}$ *and* $k(i) \in \{i+1, \cdots, n\}$ *such that* $\beta_{i,k(i)} a_{i,k(i)} a_{k(i),j(i)} > 0$.

(iv_2) *There exist* $j(i) \in \{i+2, \cdots, n\}$ *and* $k(i) \in \{i+1, \cdots, j-1\}$ *such that* $\beta_{i,k(i)} a_{i,k(i)} a_{k(i),j(i)} > 0$.

(iv_3) *There exists* $j(i) \in \{2, \cdots, i\}$ *such that* $\alpha_i a_{i,1} a_{1,j(i)} > 0$.

(iv_4) $a_{i,i+1} < 0$ *and* $\beta_{i,i+1} > 0$.

At the same time, one of the following conditions also holds:

(iv^a) $a_{n,1} < 0$ *and* $\alpha_n > 0$.

(iv^b) *There exists* $j \in \{2, \cdots, n-1\}$ *such that* $a_{n,j} - \alpha_n a_{n,1} a_{1,j} < 0$.

(iv^c) *One of the conditions (iv_1), (iv_2) and (iv_4) holds for* $i = 1$ *and* $a_{n,1} < 0$.

Proof. We only prove the case when (ii_6) holds. The proof for all other cases is similar to that in Theorem 3.136.

Now, by the definition of Q_{23} there exists $k \in \{2, \cdots, n\}$ such that $\alpha_k a_{k,1} < 0$. If $a_{1,2} < 0$ then $\alpha_k a_{k,1} a_{1,2} > 0$, so that $\delta_{k,2}^{(23)}(1) > 0$. This shows that the condition (ii_1) of Theorem 3.18 is satisfied. By Corollary 3.22 the proof is complete. □

Similar to the proof of Theorem 3.137 we can prove the following theorem.

Theorem 3.145. *Theorem D is valid for* $\nu = 23$, *provided one of the following conditions is satisfied:*

(i) *One of the conditions (i)-(iv) of Theorem 3.144 holds.*

(ii) $0 \le \alpha_{i+1} \le 1$, $0 \le \beta_{i,j} \le 1$, $i = 1, \cdots, n-1$, $j = i+1, \cdots, n$. $\alpha_n a_{n,1} = 0$ *and* $\alpha_i a_{i,1} + \sum_{k=i+1}^n \beta_{i,k} a_{i,k} a_{k,1} \ge 0$ *for* $i = 2, \cdots, n-1$. *And one of the following conditions holds:*

(ii_1) *There exists* $i_0 \in \{2, \cdots, n-1\}$ *such that* $\alpha_{i_0} a_{i_0,1} a_{1,i_0} + \sum_{k=i_0+1}^n \beta_{i_0,k} a_{i_0,k} a_{k,i_0} > 0$.

(ii_2) $\sum_{k=2}^n \beta_{1,k} a_{1,k} a_{k,1} > 0$.

(ii_3) $\gamma > 0$ *and there exist* $i_0 \in \{3, \cdots, n-1\}$, $j_0 \in \{2, \cdots, i_0-1\}$ *such that* $\alpha_{i_0} a_{i_0,1} a_{1,i_0} + \sum_{k=i_0+1}^n \beta_{i_0,k} a_{i_0,k} a_{k,j_0} > 0$

(ii_4) $\gamma > 0$ *and there exists* $i_0 \in \{2, \cdots, n-1\}$ *such that* $\alpha_{i_0} a_{i_0,1} + \sum_{k=i_0+1}^n \beta_{i_0,k} a_{i_0,k} a_{k,1} > 0$.

As a special case of Q_{23} for $\beta_{i,j} = 0$, $i = 1, \cdots, n-2$, $j = i+2, \cdots, n$, except $\beta_{1,n} \neq 0$, it gets that

$$Q_{24} = Q_6 + Q_{16} + Q_{17},$$

i.e.,

$$Q_{24} = \begin{pmatrix} 0 & -\beta_1 a_{1,2} 0 & 0 & \cdots & 0 & -\beta_n a_{1,n} \\ -\alpha_2 a_{2,1} & 0 & -\beta_2 a_{2,3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\alpha_{n-1} a_{n-1,1} & 0 & 0 & \cdots & 0 & -\beta_{n-1} a_{n-1,n} \\ -\alpha_n a_{n,1} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

with $\alpha_i \ge 0$, $i = 2, \cdots, n$, $\beta_j \ge 0$, $j = 1, \cdots, n$, and

$$\sum_{k=2}^n \alpha_k a_{k,1} \neq 0 \ \text{ and } \ \beta_n a_{1,n} + \sum_{k=1}^{n-1} \beta_k a_{k,k+1} \neq 0,$$

where we set $r = 2$ for Q_6, $r = 1$ and $s = n$ for Q_{16}.

It is proposed in [1], where either $\alpha_k = \beta_k = \beta_n = 1, k = 1, \cdots, n-1$ or $\alpha_k = \beta_n = 1, k = 1, \cdots, n-1$.

In this case, $\delta_{i,j}^{(23)}(\gamma)$ and $\delta_{i,j}^{(23)}(1)$ reduce respectively to

$$\delta_{i,j}^{(24)}(\gamma) = \begin{cases} \beta_1 a_{1,2}a_{2,1} + \beta_1 a_{1,n}a_{n,1}, & i = j = 1; \\ \alpha_i a_{i,1}a_{1,i} + \beta_i a_{i,i+1}a_{i+1,i}, & i = j = 2, \cdots, n-1; \\ \alpha_n a_{n,1}a_{1,n}, & i = j = n; \\ (\gamma - 1)\alpha_i a_{i,1} + \gamma\beta_i a_{i,i+1}a_{i+1,1}, & i = 2, \cdots, n-1, j = 1; \\ \gamma\alpha_i a_{i,1}a_{1,j} + \gamma\beta_i a_{i,i+1}a_{i+1,j}, & i = 3, \cdots, n-1, j = 2, \cdots, i-1; \\ (\gamma - 1)\alpha_n a_{n,1}, & i = n, j = 1; \\ \gamma\alpha_n a_{n,1}a_{1,j}, & i = n, j = 2, \cdots, n-1; \\ (\gamma - 1)\beta_i a_{i,i+1}, & i = 1, \cdots, n-1, j = i+1; \\ \gamma\beta_i a_{i,i+1}a_{i+1,j}, & i = 1, \cdots, n-1, j = i+2, \cdots, n \end{cases}$$

and

$$\delta_{i,j}^{(24)}(1) = \begin{cases} \beta_1 a_{1,2}a_{2,1} + \beta_1 a_{1,n}a_{n,1}, & i = j = 1; \\ \beta_i a_{i,i+1}a_{i+1,1}, & i = 2, \cdots, n-1, j = 1; \\ \alpha_i a_{i,1}a_{1,j} + \beta_i a_{i,i+1}a_{i+1,j}, & i = 2, \cdots, n-1, j = 2, \cdots, i; \\ \beta_i a_{i,i+1}a_{i+1,j}, & i = 1, \cdots, n-1, j = i+2, \cdots, n; \\ \alpha_n a_{n,1}a_{1,j}, & i = n, j = 2, \cdots, n; \\ 0, & otherwise. \end{cases}$$

Form Theorems 3.142-3.145, It is easy to prove the following theorem.

Theorem 3.146. *Suppose that $0 \le \beta_1 \le 1$, $0 \le \alpha_k, \beta_k \le 1$, $k = 2, \cdots, n$ and*

$$\alpha_n a_{n,1}a_{1,n} < 1,\ \beta_1 a_{1,2}a_{2,1} + \beta_n a_{1,n}a_{n,1} < 1,\ \alpha_i a_{i,1}a_{1,i} + \beta_i a_{i,i+1}a_{i+1,i} < 1,\ i = 2, \cdots, n-1. \tag{50}$$

Then Theorem A is valid for $\nu = 24$.

Theorem 3.147. *Suppose that $0 \le \beta_1 \le 1$, $0 \le \alpha_k, \beta_k \le 1$, $k = 2, \cdots, n$. Then Theorem B is valid for $\nu = 24$.*

Theorem 3.148. *Suppose that (50) holds. Then Theorem C is valid for $\nu = 24$, provided one of the following conditions is satisfied:*

(i) $0 \le \gamma < 1$ *and* $0 \le \beta_1 \lesssim 1$, $0 \le \alpha_k, \beta_k \lesssim 1$, $k = 2, \cdots, n$.

(ii) $\gamma = 1$ *and* $0 \le \beta_1 \lesssim 1$, $0 \le \alpha_k, \beta_k \lesssim 1$, $k = 2, \cdots, n$. *And one of the following conditions holds:*

- *(ii_1) There exists $k \in \{2, \cdots, n-1\}$ such that $a_{k,k+1} < 0$ and $\beta_k > 0$.*
- *(ii_2) There exist $i \in \{2, \cdots, n\}$ and $j \in \{2, \cdots, i\}$ such that $\alpha_i a_{i,1}a_{1,j} > 0$.*
- *(ii_3) $\beta_n a_{1,n}a_{n,1} > 0$.*
- *(ii_4) $a_{n,1} < 0$ and $\alpha_n > 0$.*
- *(ii_5) $a_{1,2} < 0$.*

(iii) $0 \le \gamma < 1$ *and* $0 \le \beta_1 \le 1$, $0 \le \alpha_i, \beta_i \le 1$, $i = 2, \cdots, n$. $\beta_1 a_{1,2} + \beta_n a_{1,n} < 0$ *and* $\alpha_i a_{i,1} + \beta_i a_{i,i+1} < 0$, $i = 2, \cdots, n-1$.

(iv) $\gamma = 1$ *and* $0 \le \beta_1 \le 1$, $0 \le \alpha_i, \beta_i \le 1$, $i = 2, \cdots, n$. *For each $i \in \{2, \cdots, n-1\}$, either $\beta_i a_{i,i+1} < 0$ or there exists $j(i) \in \{2, \cdots, i\}$ such that $\alpha_i a_{i,1}a_{1,j(i)} > 0$.*

At the same time, one of the following conditions also holds:

- *(iv^a) $a_{n,1} < 0$ and $\alpha_n > 0$.*
- *(iv^b) There exists $j \in \{2, \cdots, n-1\}$ such that $a_{n,j} - \alpha_n a_{n,1}a_{1,j} < 0$.*

(iv^c) $\beta_1 a_{1,2}a_{n,1} + \beta_n a_{1,n}a_{n,1} > 0$.

Clearly, the condition $\beta_n a_{1,n}a_{n,1} > 0$ implies that $\beta_n a_{1,n} < 0$ so that $\beta_1 a_{1,2} + \beta_n a_{1,n} < 0$. The condition $\alpha_i a_{i,1}a_{1,i} + \beta_i a_{i,i+1}a_{i+1,i} > 0$ implies that either $\alpha_i a_{i,1} < 0$ or $\beta_i a_{i,i+1} < 0$ so that $\alpha_i a_{i,1} + \beta_i a_{i,i+1} < 0$. This shows that if $\beta_n a_{1,n}a_{n,1} > 0$ and $\alpha_i a_{i,1}a_{1,i} + \beta_i a_{i,i+1}a_{i+1,i} > 0$ for $i = 2, \cdots, n-1$, then the condition (iii) in Theorem 3.148 is satisfied. Hence, the result here is far better than the corresponding ones given by [1, Theorems 2 and 4].

Theorem 3.149. *Theorem D is valid for $\nu = 24$, provided one of the following conditions is satisfied:*

(i) One of the conditions (i)-(iv) of Theorem 3.148 holds.

(ii) $0 \le \beta_1 \le 1$, $0 \le \alpha_k, \beta_k \le 1$, $k = 2, \cdots, n$. $\alpha_n a_{n,1} = 0$ and $\alpha_k a_{k,1} + \beta_k a_{k,k+1}a_{k+1,1} \ge 0$ for $k = 2, \cdots, n-1$. And one of the following conditions holds:

(ii_1) There exists $i_0 \in \{2, \cdots, n-1\}$ such that $\alpha_{i_0} a_{i_0,1}a_{1,i_0} + \beta_{i_0} a_{i_0,i_0+1}a_{i_0+1,i_0} > 0$.

(ii_2) $\beta_1 a_{1,2}a_{2,1} + \beta_n a_{1,n}a_{n,1} > 0$.

(ii_3) $\gamma > 0$ and there exist $i_0 \in \{3, \cdots, n-1\}$, $j_0 \in \{2, \cdots, i_0 - 1\}$ such that $\alpha_{i_0} a_{i_0,1}a_{1,i_0} + \beta_{i_0} a_{i_0,i_0+1}a_{i_0+1,j_0} > 0$

(ii_4) $\gamma > 0$ and there exists $i_0 \in \{2, \cdots, n-1\}$ such that $\alpha_{i_0} a_{i_0,1} + \beta_{i_0} a_{i_0,i_0+1}a_{i_0+1,1} > 0$.

Relative to Q_{21}, the matrix Q is chosen as

$$Q_{25} = Q_3 + Q_{17},$$

i.e.,

$$Q_{25} = \begin{pmatrix} 0 & -\beta_1 a_{1,2} & \cdots & 0 & 0 \\ -\alpha_{2,1}a_{2,1} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\alpha_{n-1,1}a_{n-1,1} & -\alpha_{n-1,2}a_{n-1,2} & \cdots & 0 & -\beta_{n-1}a_{n-1,n} \\ -\alpha_{n,1}a_{n,1} & -\alpha_{n,2}a_{n,2} & \cdots & -\alpha_{n,n-1}a_{n,n-1} & 0 \end{pmatrix}$$

with $\alpha_{i,j} \ge 0$, $i = 2, \cdots, n$, $j < i$, and $\beta_i \ge 0$, $i = 1, \cdots, n-1$, and

$$\sum_{i=2}^{n} \sum_{j=1}^{i-1} \alpha_{i,j} a_{i,j} \neq 0 \quad \text{and} \quad \sum_{i=1}^{n-1} \beta_i a_{i,i+1} \neq 0.$$

By Corollaries 3.20 and 3.21, the following two comparison theorems are directly.

Theorem 3.150. *Suppose that $0 \le \alpha_{i+1,j} \le 1$, $0 \le \beta_i \le 1$, $i = 1, \cdots, n-1$, $j = 1, \cdots, i$, and $\beta_i a_{i,i+1}a_{i+1,i} + \sum_{k=1}^{i-1} \alpha_{i,k} a_{i,k} a_{k,i} < 1$, $i = 1, \cdots, n$. Then Theorem A is valid for $\nu = 25$.*

Theorem 3.151. *Suppose that $0 \le \alpha_{i+1,j} \le 1$, $0 \le \beta_i \le 1$, $i = 1, \cdots, n-1$, $j = 1, \cdots, i$. Then Theorem B is valid for $\nu = 25$.*

In this case, $\delta_{i,j}^{(2)}(\gamma)$ and $\delta_{i,j}^{(2)}(1)$ reduce respectively to

$$\delta_{i,j}^{(25)}(\gamma) = \begin{cases} \sum\limits_{k=1}^{i-1} \alpha_{i,k} a_{i,k} a_{k,i} + \beta_i a_{i,i+1} a_{i+1,i}, & i = j = 1, \cdots, n; \\ (\gamma - 1)\beta_i a_{i,i+1}, & i = 1, \cdots, n-1, j = i+1; \\ \gamma \beta_i a_{i,i+1} a_{i+1,j}, & i = 1, \cdots, n-1, j = i+2, \cdots, n; \\ (\gamma - 1)\alpha_{i,j} a_{i,j} + \gamma \sum\limits_{k=1}^{j-1} \alpha_{i,k} a_{i,k} a_{k,j} + \gamma \beta_i a_{i,i+1} a_{i+1,j}, & i = 2, \cdots, n, j = 1, \cdots, i-1 \end{cases}$$

and

$$\delta_{i,j}^{(25)}(1) = \begin{cases} \sum\limits_{k=1}^{j-1} \alpha_{i,k}a_{i,k}a_{k,j} + \beta_i a_{i,i+1}a_{i+1,j}, & i = 1, \cdots, n,\ j = 1, \cdots, i; \\ \beta_i a_{i,i+1}a_{i+1,j}, & i = 1, \cdots, n-1,\ j = i+2, \cdots, n; \\ 0, & \textit{otherwise}. \end{cases}$$

Using Corollary 3.22, we prove Stein-Rosenberg type comparison theorem.

Theorem 3.152. *Suppose that $\beta_i a_{i,i+1}a_{i+1,i} + \sum_{k=1}^{i-1} \alpha_{i,k}a_{i,k}a_{k,i} < 1$, $i = 1, \cdots, n$. Then Theorem C is valid for $\nu = 25$, provided one of the following conditions is satisfied:*

(i) *For $i = 1, \cdots, n-1$, $j = 1, \cdots, i$, $0 \le \alpha_{i+1,j} \lnsim 1$, $0 \le \beta_i \lnsim 1$.*

(ii) *$0 \le \gamma < 1$ and $0 \le \alpha_{i+1,j} \le 1$, $0 \le \beta_i \le 1$, $i = 1, \cdots, n-1$, $j = 1, \cdots, i$. For each $i \in \{1, \cdots, n-1\}$, $\beta_i a_{i,i+1} < 0$ or there exists $j(i) \in \{1, \cdots, i-1\}$ such that $\alpha_{i,j(i)}a_{i,j(i)} < 0$.*

(iii) *$\gamma = 1$ and $0 \le \alpha_{i+1,j} \le 1$, $0 \le \beta_i \le 1$, $i = 1, \cdots, n-1$, $j = 1, \cdots, i$. For each $i \in \{2, \cdots, n-1\}$, $\beta_i a_{i,i+1} < 0$ or there exist $j(i) \in \{1, \cdots, i\}$ and $k(i) \in \{1, \cdots, j(i)-1\}$ such that $\alpha_{i,k(i)}a_{i,k(i)}a_{k(i),j(i)} > 0$. At the same time, one of the following conditions holds:*

(iii_1) *There exist $j \in \{2, \cdots, n\}$ and $k \in \{1, \cdots, j-1\}$ such that $\alpha_{n,k}a_{n,k}a_{k,j} > 0$.*

(iii_2) *$a_{n,1} < 0$ and $\alpha_{n,1} > 0$.*

(iii_3) *There exists $j \in \{2, \cdots, n-1\}$ such that*

$$(1 - \alpha_{n,j})a_{n,j} - \sum_{\substack{k=1 \\ k \ne j}}^{n-1} \alpha_{n,k}a_{n,k}a_{k,j} < 0.$$

(iii_4) *$a_{1,2} < 0$, $\beta_1 > 0$ and*

$$(1 - \alpha_{n,1})a_{n,1} - \sum_{k=2}^{n-1} \alpha_{n,k}a_{n,k}a_{k,1} < 0.$$

(iii_5) *$a_{1,2} < 0$, $a_{n,1} < 0$ and $\beta_1 > 0$.*

Proof. Since $\sum_{k=1}^{n-1} \beta_k a_{k,k+1} \ne 0$, then there exists $k \in \{1, \cdots, n-1\}$ such that $a_{k,k+1} < 0$ and $\beta_k > 0$, which shows that the condition (ii_3) in Theorem 3.18 is satisfied, so that the condition (*i*) of Corollary 3.22 is satisfied. This shows (*i*).

When (*ii*) holds, then the condition (*iii*) of Theorem 3.18 is satisfied, so that the condition (*ii*) of Corollary 3.22 is satisfied.

When (*iii*) holds, it gets that $\delta_{i,j(i)}^{(25)}(1) > 0$. In this case $\delta_{1,j}^{(25)}(1) = \beta_1 a_{1,2}a_{2,j}$, $j = 1, 3, \cdots, n$. Since A is irreducible, then there exists $j \in \{1\} \cup \{3, \cdots, n\}$ such that $a_{2,j} < 0$, so that $\delta_{1,j}^{(25)}(1) = \beta_1 a_{1,2}a_{2,j} > 0$ whenever $\beta_1 > 0$ and $a_{1,2} < 0$. Hence, the conditions (iv^a)-(iv^e) in Theorem 3.18 reduce to (iii_1)-(iii_5), respectively. By (*ii*) of Corollary 3.22 the proof of (*iii*) is complete. □

Theorem 3.153. *Theorem D is valid for $\nu = 25$, provided one of the following conditions is satisfied:*

(i) *One of the conditions (i), (ii) and (iii) of Theorem 3.152 holds.*

(ii) *$0 \le \alpha_{i+1,j} \le 1$, $0 \le \beta_i \le 1$, $i = 1, \cdots, n-1$, $j = 1, \cdots, i$. For $i = 2, \cdots, n$, $j = 1, \cdots, i-1$,*

$$\alpha_{i,j}a_{i,j} + \sum_{\substack{k=1 \\ k \ne j}}^{i-1} \alpha_{i,k}a_{i,k}a_{k,j} + \beta_i a_{i,i+1}a_{i+1,j} \ge 0.$$

And one of the following conditions holds:

(ii_1) There exists $i_0 \in \{1, \cdots, n-1\}$ such that $\beta_{i_0} a_{i_0,i_0+1} a_{i_0+1,i_0} > 0$.

(ii_2) There exist $i_0 \in \{2, \cdots, n\}$ and $j_0 \in \{1, \cdots, i_0 - 1\}$ such that $\alpha_{i_0,j_0} a_{i_0,j_0} a_{j_0,i_0} > 0$.

(ii_3) $\gamma > 0$ and $\alpha_{2,1} a_{2,1} + \beta_2 a_{2,3} a_{3,1} > 0$.

(ii_4) $\gamma > 0$ and there exist $i_0 \in \{3, \cdots, n\}$ and $j_0 \in \{1, \cdots, i_0 - 1\}$ such that

$$\alpha_{i_0,j_0} a_{i_0,j_0} + \sum_{\substack{k=1 \\ k \neq j_0}}^{i_0-1} \alpha_{i_0,k} a_{i_0,k} a_{k,j_0} + \beta_{i_0} a_{i_0,i_0+1} a_{i_0+1,j_0} > 0.$$

Proof. We just need to prove (ii). It is easy to obtain

$$a_{i,j}^{(25)} = \begin{cases} 1 - \sum\limits_{k=1}^{i-1} \alpha_{i,k} a_{i,k} a_{k,i} - \beta_i a_{i,i+1} a_{i+1,i}, & i = j = 1, \cdots, n, \\ (1 - \alpha_{i,j}) a_{i,j} - \sum\limits_{k=1}^{i-1} \alpha_{i,k} a_{i,k} a_{k,j} - \beta_i a_{i,i+1} a_{i+1,j}, & i = 2, \cdots, n, j = 1, \cdots, i-1. \end{cases}$$

By Corollary 3.23 we can derive (ii_1)-(ii_4). $\square$

As a special case, Q_{21} and Q_{25} reduce to

$$Q_{26} = Q_5 + Q_{17} = \begin{pmatrix} 0 & -\beta_1 a_{1,2} & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -\beta_{n-1} a_{n-1,n} \\ -\alpha_1 a_{n,1} & -\alpha_2 a_{n,2} & \cdots & -\alpha_{n-1} a_{n,n-1} & 0 \end{pmatrix}$$

with $\alpha_k \geq 0$, $\beta_k \geq 0$, $k = 1, \cdots, n-1$, and

$$\sum_{k=1}^{n-1} \alpha_k a_{n,k} \neq 0 \quad \text{and} \quad \sum_{k=1}^{n-1} \beta_k a_{k,k+1} \neq 0,$$

where for simplicity we set $r = n$ for Q_5.

It is proposed in [69] for the preconditioned Gauss-Seidel method, where $\alpha_k = \beta_k = 1$, $k = 1, \cdots, n-1$.

In this case, $\delta_{i,j}^{(25)}(\gamma)$ and $\delta_{i,j}^{(25)}(1)$ reduce respectively to

$$\delta_{i,j}^{(26)}(\gamma) = \begin{cases} \beta_i a_{i,i+1} a_{i+1,i}, & i = j = 1, \cdots, n-1; \\ \sum\limits_{k=1}^{n-1} \alpha_k a_{n,k} a_{k,n}, & i = j = n; \\ (\gamma - 1) \beta_i a_{i,i+1}, & i = 1, \cdots, n-1, j = i+1; \\ \gamma \beta_i a_{i,i+1} a_{i+1,j}, & i = 1, \cdots, n-1, j = 1, \cdots, n, j \neq i, i+1; \\ (\gamma - 1) \alpha_j a_{n,j} + \gamma \sum\limits_{k=1}^{j-1} \alpha_k a_{n,k} a_{k,j}, & i = n, j = 1, \cdots, n-1 \end{cases}$$

and

$$\delta_{i,j}^{(26)}(1) = \begin{cases} \beta_i a_{i,i+1} a_{i+1,j}, & i = 1, \cdots, n-1, j = 1, \cdots, n, j \neq i+1; \\ \sum\limits_{k=1}^{j-1} \alpha_k a_{n,k} a_{k,j}, & i = n, j = 1, \cdots, n. \end{cases}$$

From Theorems 3.150-3.153 the following comparison results are immediately.

Theorem 3.154. *Suppose that* $0 \leq \alpha_k, \beta_k \leq 1$, $k = 1, \cdots, n-1$, *and*

$$\sum_{k=1}^{n-1} \alpha_k a_{n,k} a_{k,n} < 1,\ \beta_i a_{i,i+1} a_{i+1,i} < 1,\ i = 1, \cdots, n-1. \tag{51}$$

Then Theorem A is valid for $\nu = 26$.

Theorem 3.155. *Suppose that* $0 \leq \alpha_k, \beta_k \leq 1$, $k = 1, \cdots, n-1$. *Then Theorem B is valid for* $\nu = 26$.

Theorem 3.156. *Suppose that (51) holds. Then Theorem C is valid for* $\nu = 26$, *provided one of the following conditions is satisfied:*

(i) *For* $k = 1, \cdots, n-1$, $0 \leq \alpha_k, \beta_k \lesssim 1$.

(ii) $0 \leq \gamma < 1$, $a_{k,k+1} < 0$, $0 \leq \alpha_k \leq 1$ *and* $0 < \beta_k \leq 1$, $k = 1, \cdots, n-1$.

(iii) $\gamma = 1$, $a_{k,k+1} < 0$, $0 \leq \alpha_k \leq 1$ *and* $0 < \beta_k \leq 1$, $k = 1, \cdots, n-1$. *One of the following conditions holds:*

(iii_1) *There exist* $j \in \{2, \cdots, n\}$ *and* $k \in \{1, \cdots, j-1\}$ *such that* $\alpha_k a_{n,k} a_{k,j} > 0$.

(iii_2) $a_{n,1} < 0$ *and* $\alpha_1 > 0$.

(iii_3) *There exists* $j \in \{2, \cdots, n-1\}$ *such that*

$$(1-\alpha_j) a_{n,j} - \sum_{\substack{k=1\\k\neq j}}^{n-1} \alpha_k a_{n,k} a_{k,j} < 0.$$

(iii_4) $a_{1,2} < 0$, $\beta_1 > 0$ *and*

$$(1-\alpha_1) a_{n,1} - \sum_{k=2}^{n-1} \alpha_k a_{n,k} a_{k,1} < 0.$$

(iii_5) $a_{1,2} < 0$, $a_{n,1} < 0$ *and* $\beta_1 > 0$.

Theorem 3.157. *Theorem D is valid for* $\nu = 26$, *provided one of the following conditions is satisfied:*

(i) *One of the conditions (i), (ii) and (iii) of Theorem 3.156 holds.*

(ii) *For* $j = 1, \cdots, n-1$, $0 \leq \alpha_j, \beta_j \leq 1$ *and* $\alpha_j a_{n,j} + \sum_{k=1,k\neq j}^{n-1} \alpha_k a_{n,k} a_{k,j} \geq 0$. *One of the following conditions holds:*

(ii_1) *There exists* $i_0 \in \{1, \cdots, n-1\}$ *such that* $\beta_{i_0} a_{i_0,i_0+1} a_{i_0+1,i_0} > 0$.

(ii_2) *There exists* $i_0 \in \{1, \cdots, n-1\}$ *such that* $\alpha_{i_0} a_{n,i_0} a_{i_0,n} > 0$.

(ii_3) $\gamma > 0$ *and there exist* $i_0 \in \{2, \cdots, n-1\}$ *and* $j_0 \in \{1, \cdots, i_0-1\}$ *such that* $\beta_{i_0} a_{i_0,i_0+1} a_{i_0+1,j_0} > 0$.

(ii_4) $\gamma > 0$ *and there exists* $j_0 \in \{1, \cdots, n-1\}$ *such that*

$$\alpha_{j_0} a_{n,j_0} + \sum_{\substack{k=1\\k\neq j_0}}^{n-1} \alpha_k a_{n,k} a_{k,j_0} > 0.$$

Similarly, as a special case of Q_{24} and Q_{25}, Q is proposed in [79] as

$$Q_{27} = Q_6 + Q_{17} = \begin{pmatrix} 0 & -\beta_1 a_{1,2} & 0 & \cdots & 0 \\ -\alpha_2 a_{2,1} & 0 & -\beta_2 a_{2,3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\alpha_{n-1} a_{n-1,1} & 0 & 0 & \cdots & -\beta_{n-1} a_{n-1,n} \\ -\alpha_n a_{n,1} & 0 & 0 & \cdots & 0 \end{pmatrix}$$

with $\alpha_{k+1} \geq 0$, $\beta_k \geq 0$, $k = 1, \cdots, n-1$, and

$$\sum_{k=2}^{n} \alpha_k a_{k,1} \neq 0 \text{ and } \sum_{k=1}^{n-1} \beta_k a_{k,k+1} \neq 0,$$

where for simplicity we set $r = 2$ for Q_6.

It is proposed in [68] for the preconditioned SOR method, where $\alpha_{k+1} = \beta_k = 1$, $k = 1, \cdots, n-1$.

In this case, $\delta_{i,j}^{(25)}(\gamma)$ and $\delta_{i,j}^{(25)}(1)$ reduce respectively to

$$\delta_{i,j}^{(27)}(\gamma) = \begin{cases} \beta_1 a_{1,2} a_{2,1}, & i = j = 1; \\ \alpha_i a_{i,1} a_{1,i} + \beta_i a_{i,i+1} a_{i+1,i}, & i = j = 2, \cdots, n-1; \\ \alpha_n a_{n,1} a_{1,n}, & i = j = n; \\ (\gamma - 1)\alpha_i a_{i,1} + \gamma \beta_i a_{i,i+1} a_{i+1,1}, & i = 2, \cdots, n-1, j = 1; \\ \gamma \alpha_i a_{i,1} a_{1,j} + \gamma \beta_i a_{i,i+1} a_{i+1,j}, & i = 3, \cdots, n-1, j = 2, \cdots, i-1; \\ (\gamma - 1)\alpha_n a_{n,1}, & i = n, j = 1; \\ \gamma \alpha_n a_{n,1} a_{1,j}, & i = n, j = 2, \cdots, n-1; \\ (\gamma - 1)\beta_i a_{i,i+1}, & i = 1, \cdots, n-1, j = i+1; \\ \gamma \beta_i a_{i,i+1} a_{i+1,j}, & i = 1, \cdots, n-1, j = i+2, \cdots, n \end{cases}$$

and

$$\delta_{i,j}^{(27)}(1) = \begin{cases} \beta_i a_{i,i+1} a_{i+1,j}, & i = 1, \cdots, n-1, j \in \{1\} \cup \{i+2, \cdots, n\}; \\ \alpha_i a_{i,1} a_{1,j} + \beta_i a_{i,i+1} a_{i+1,j}, & i = 2, \cdots, n-1, j = 2, \cdots, i; \\ \alpha_n a_{n,1} a_{1,j}, & i = n, j = 2, \cdots, n; \\ 0, & \textit{otherwise}. \end{cases}$$

From Theorems 3.150-3.153, the following comparison results are directly.

Theorem 3.158. *Suppose that $0 \leq \alpha_{k+1}, \beta_k \leq 1$, $k = 1, \cdots, n-1$, and $\beta_1 a_{1,2} a_{2,1} < 1$, $\alpha_n a_{n,1} a_{1,n} < 1$, $\alpha_k a_{k,1} a_{1,k} + \beta_k a_{k,k+1} a_{k+1,k} < 1$, $k = 2, \cdots, n-1$. Then Theorem A is valid for $\nu = 27$.*

Theorem 3.159. *Suppose that $0 \leq \alpha_{k+1}, \beta_k \leq 1$, $k = 1, \cdots, n-1$. Then Theorem B is valid for $\nu = 27$.*

Theorem 3.160. *Suppose that $\beta_1 a_{1,2} a_{2,1} < 1$, $\alpha_n a_{n,1} a_{1,n} < 1$, $\alpha_k a_{k,1} a_{1,k} + \beta_k a_{k,k+1} a_{k+1,k} < 1$, $k = 2, \cdots, n-1$. Then Theorem C is valid for $\nu = 27$, provided one of the following conditions is satisfied:*

(i) *For $k = 1, \cdots, n-1$, $0 \leq \alpha_{k+1}, \beta_k \lesssim 1$.*

(ii) *$0 \leq \gamma < 1$, $0 \leq \alpha_{k+1}, \beta_k \leq 1$, $k = 1, \cdots, n-1$. $\beta_1 > 0$, $a_{1,2} < 0$ and for each $i \in \{2, \cdots, n-1\}$, $\beta_i a_{i,i+1} < 0$ or $\alpha_i a_{i,1} < 0$.*

(iii) *$\gamma = 1$, $0 \leq \alpha_{k+1}, \beta_k \leq 1$, $k = 1, \cdots, n-1$. For each $i \in \{2, \cdots, n-1\}$, $\beta_i a_{i,i+1} < 0$ or there exists $j(i) \in \{2, \cdots, i\}$ such that $\alpha_i a_{i,1} a_{1,j(i)} > 0$. At the same time, one of the following conditions holds:*

(iii_1) *$a_{n,1} < 0$ and $\alpha_n > 0$.*

(iii_2) *There exists $j \in \{2, \cdots, n-1\}$ such that $a_{n,j} - \alpha_n a_{n,1} a_{1,j} < 0$.*

(iii_3) *$a_{1,2} < 0$, $a_{n,1} < 0$ and $\beta_1 > 0$.*

It can be proved that if $0 < a_{1,2} a_{2,1} < 1$, $0 < a_{n,1} a_{1,n} < 1$ and $0 < a_{k,1} a_{1,k} + a_{k,k+1} a_{k+1,k} < 1$, $k = 2, \cdots, n-1$, then A is irreducible. Hence, for the Gauss-Seidel method the result when *(iii)* holds is better than [79, Theorems 2,3,4], where in Theorem 4 it should be that $\alpha_{k+1} > 0$ and $\beta_k > 0$, $k = 1, \cdots, n-1$.

All the corresponding results given in [68] are problematic, because [68, Theorem 3.1] is wrong. In fact, let

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -0.5 \\ -0.5 & 1 & -0.5 & 0 & 0 & 0 \\ 0 & -0.5 & 1 & 0 & 0 & 0 \\ 0 & -0.5 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -0.5 \\ -0.5 & -0.5 & 0 & -0.5 & -0.5 & 1 \end{pmatrix}. \tag{52}$$

Then it is easy to prove that A is an irreducible L-matrix and the assumption of [68, Theorem 3.1] is satisfied. But the iteration matrix of the preconditioned SOR method is reducible.

Theorem 3.161. *Theorem D is valid for $\nu = 27$, provided one of the conditions (i), (ii) and (iii) of Theorem 3.160 is satisfied.*

As a special case of Q_{26} and Q_{27}, Q is defined in [98] as

$$Q_{28} = Q_7 + Q_{17} = \begin{pmatrix} 0 & -\beta_1 a_{1,2} & 0 & \cdots & 0 \\ 0 & 0 & -\beta_2 a_{2,3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\beta_{n-1} a_{n-1,n} \\ -\alpha a_{n,1} & 0 & 0 & \cdots & 0 \end{pmatrix}$$

with $a_{n,1} < 0$, $\alpha > 0$, $\beta_k \ge 0$, $k = 1, \cdots, n-1$, and $\sum_{k=1}^{n-1} \beta_k a_{k,k+1} \ne 0$, where we set $r = n$ and $s = 1$ for Q_7.

It is continued to study in [26] and it is proposed in [100, 105] for the preconditioned Gauss-Seidel method, where $\alpha_1 = \beta_k = 1$, $k = 1, \cdots, n-1$ and $\alpha_k = 0$, $k = 2, \cdots, n-1$.

In this case, $\delta_{i,j}^{(25)}(\gamma)$ and $\delta_{i,j}^{(25)}(1)$ reduce respectively to

$$\delta_{i,j}^{(28)}(\gamma) = \begin{cases} \beta_i a_{i,i+1} a_{i+1,i}, & i = j = 1, \cdots, n-1; \\ \alpha a_{n,1} a_{1,n}, & i = j = n; \\ (\gamma - 1)\beta_i a_{i,i+1}, & i = 1, \cdots, n-1, j = i+1; \\ \gamma \beta_i a_{i,i+1} a_{i+1,j}, & i = 1, \cdots, n-1, j = 1, \cdots, n, j \ne i, i+1; \\ \gamma \alpha a_{n,1} a_{1,j}, & i = n, j = 2, \cdots, n-1; \\ (\gamma - 1)\alpha a_{n,1} & i = n, j = 1 \end{cases}$$

and

$$\delta_{i,j}^{(28)}(1) = \begin{cases} \beta_i a_{i,i+1} a_{i+1,j}, & i = 1, \cdots, n-1, j = 1, \cdots, n, j \ne i+1; \\ \alpha a_{n,1} a_{1,j}, & i = n, j = 2, \cdots, n; \\ 0, & \text{otherwise}. \end{cases}$$

From Theorems 3.154 and 3.155 the following comparison results are immediately.

Theorem 3.162. *Suppose that $\alpha \le 1$, $0 \le \beta_k \le 1$, $\alpha a_{n,1} a_{1,n} < 1$, $\beta_k a_{k,k+1} a_{k+1,k} < 1$, $k = 1, \cdots, n-1$. Then Theorem A is valid for $\nu = 28$.*

Theorem 3.163. *Suppose that $\alpha \le 1$, $0 \le \beta_k \le 1$, $k = 1, \cdots, n-1$. Then Theorem B is valid for $\nu = 28$.*

Completely similar to Lemma 3.117 we can prove the following lemma.

Lemma 3.164. *Let A be a Z-matrix. Assume that $n \ge 3$, $a_{1,n} < 0$, $a_{k+1,k} < 0$, $\alpha \le 1$, $0 \le \beta_k \le 1$, $k = 1, \cdots, n-1$. Then $A^{(28)}$ is an irreducible Z-matrix.*

Theorem 3.165. *Suppose that $\alpha a_{n,1} a_{1,n} < 1$, $\beta_k a_{k,k+1} a_{k+1,k} < 1$, $k = 1, \cdots, n-1$. Then Theorem C is valid for $\nu = 28$, provided one of the following conditions is satisfied:*

(i) $\alpha \lesssim 1$ *and* $0 \le \beta_k \lesssim 1$, $k = 1, \cdots, n-1$.

(ii) $\alpha \le 1$, $0 < \beta_k \le 1$, $a_{k,k+1} < 0$, $k = 1, \cdots, n-1$.

(iii) $n \ge 3$, $\alpha \le 1$, $0 \le \beta_k \le 1$, $a_{1,n} < 0$, $a_{k+1,k} < 0$, $k = 1, \cdots, n-1$.

Proof. By Theorem 3.156, we just need to prove (*ii*) and (*iii*).
When (*ii*) holds, since $a_{n,1} < 0$ and $\alpha > 0$, the condition (iii_2) in Theorem 3.156 is satisfied.
When (*iii*) holds, by Lemma 3.164, $A^{(28)}$ is an irreducible Z-matrix. From (*i*) we can prove (*iii*). □

For (*ii*) and (*iii*) the assumption that A is irreducible is redundant. Hence, [105, Theorem 3.1] can be derived, directly.

All the corresponding results given in [98] are problematic, because [98, Lemmas 4.1, 4.3] are wrong. In fact, let A be defined by (52). Then A is an irreducible L-matrix and it is easy to prove that the assumptions of [98, Lemmas 4.1, 4.3] are satisfied. But the iteration matrices of the preconditioned AOR methods are reducible when we choose $\beta_2 = 1$.

In this case, (*ii*) of Theorem 3.157 can be not satisfied. Hence, from Theorems 3.157, the following theorem is derived, directly.

Theorem 3.166. *Theorem D is valid for* $\nu = 28$, *provided one of the conditions (i), (ii) and (iii) of Theorem 3.165 is satisfied.*

Corresponding to Q_{28}, in [26] Q is defined as

$$Q_{29} = Q_8 + Q_{16} = \begin{pmatrix} 0 & 0 & \cdots & 0 & -\beta a_{1,n} \\ -\alpha_1 a_{2,1} & 0 & \cdots & 0 & 0 \\ 0 & -\alpha_2 a_{3,2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\alpha_{n-1} a_{n,n-1} & 0 \end{pmatrix}$$

with $a_{1,n} < 0$, $\beta > 0$, $\alpha_k \ge 0$, $k = 1, \cdots, n-1$, and

$$\sum_{k=1}^{n-1} \alpha_k a_{k+1,k} \ne 0,$$

where we set $r = 1$ and $s = n$ for Q_{16}

In this case, $\delta_{i,j}^{(2)}(\gamma)$ and $\delta_{i,j}^{(2)}(1)$ reduce respectively to

$$\delta_{i,j}^{(29)}(\gamma) = \begin{cases} \beta a_{1,n} a_{n,1}, & i = j = 1; \\ \alpha_{i-1} a_{i,i-1} a_{i-1,i}, & i = j = 2, \cdots, n; \\ (\gamma - 1)\beta a_{1,n}, & i = 1, j = n; \\ (\gamma - 1)\alpha_{i-1} a_{i,i-1}, & i = 2, \cdots, n, j = i-1; \\ 0, & \text{otherwise} \end{cases}$$

and

$$\delta_{i,j}^{(29)}(\gamma) = \begin{cases} \beta a_{1,n} a_{n,1}, & i = j = 1; \\ \alpha_{i-1} a_{i,i-1} a_{i-1,i}, & i = j = 2, \cdots, n; \\ 0, & \text{otherwise.} \end{cases}$$

By Corollaries 3.20 and 3.21, the following comparison theorems are directly.

Theorem 3.167. *Suppose that* $\beta \le 1$, $0 \le \alpha_k \le 1$, $\beta a_{1,n} a_{n,1} < 1$, $\alpha_k a_{k+1,k} a_{k,k+1} < 1$, $k = 1, \cdots, n-1$. *Then Theorem A is valid for* $\nu = 29$.

Theorem 3.168. *Suppose that* $\beta \le 1$, $0 \le \alpha_k \le 1$, $k = 1, \cdots, n-1$. *Then Theorem B is valid for* $\nu = 29$.

Completely similar to Lemma 3.164 we can prove the following lemma.

Lemma 3.169. *Let A be a Z-matrix. Assume that $n \geq 3$, $a_{n,1} < 0$, $\beta \leq 1$, $a_{k,k+1} < 0$, $0 \leq \alpha_k \leq 1$, $k = 1, \cdots, n-1$. Then $A^{(29)}$ is an irreducible Z-matrix.*

Theorem 3.170. *Suppose that $\beta a_{1,n} a_{n,1} < 1$, $\alpha_k a_{k+1,k} a_{k,k+1} < 1$, $k = 1, \cdots, n-1$. Then Theorem C is valid for $\nu = 29$, provided one of the following conditions is satisfied:*

(i) $0 \leq \gamma < 1$, $\beta \lesssim 1$ *and* $0 \leq \alpha_k \lesssim 1$, $k = 1, \cdots, n-1$.

(ii) $\gamma = 1$, $\beta \lesssim 1$ *and* $0 \leq \alpha_k \lesssim 1$, $k = 1, \cdots, n-1$. *One of the following conditions holds:*

- *(ii_1)* $a_{n,1} < 0$.
- *(ii_2) There exists* $k \in \{1, \cdots, n-1\}$ *such that* $\alpha_k a_{k+1,k} a_{k,k+1} > 0$.
- *(ii_3)* $a_{k,k+1} < 0$, $k = 1, \cdots, n-1$.

(iii) $0 \leq \gamma < 1$, $\beta \leq 1$, $0 \geq \alpha_{n-1} \leq 1$, $0 < \alpha_k \leq 1$ *and* $a_{k+1,k} < 0$, $k = 1, \cdots, n-2$.

(iv) $\gamma = 1$, $\beta \leq 1$, $0 \leq \alpha_{n-1} \leq 1$, $0 < \alpha_k \leq 1$ *and* $a_{k+1,k} a_{k,k+1} > 0$, $k = 1, \cdots, n-2$. *One of the following conditions holds:*

- *(iv_1)* $a_{n,1} < 0$.
- *(iv_2)* $\alpha_{n-1} > 0$ *and* $a_{n,n-1} a_{n-1,n} > 0$.

(v) $n \geq 3$, $a_{n,1} < 0$, $a_{k,k+1} < 0$, $\beta \leq 1$, $0 \leq \alpha_k \leq 1$, $k = 1, \cdots, n-1$.

Proof. (i), (ii_1) and (ii_2) satisfy respectively (i), (ii_1) in Theorem 3.18. Hence they satisfy the condition (i) of Corollaries 3.22.

By the definition of Q_{29}, there exists $k_0 \in \{1, \cdots, n-1\}$ such that $\alpha_{k_0} a_{k_0+1,k_0} < 0$ so that $\alpha_{k_0} a_{k_0+1,k_0} a_{k_0,k_0+1} > 0$, which implies that ($ii_2$) holds for $k = k_0$.

(iii) can be derived by (ii) of Corollaries 3.22.

(iv) satisfies the conditions (iv_1), (iv^e) and (iv^a) in Theorem 3.18, so that it satisfies condition (ii) of Corollaries 3.22.

When (v) holds, by Lemma 3.169, $A^{(29)}$ is an irreducible L-matrix. From (i) and (ii) we can prove (v), where (ii_1) is satisfied. □

Obviously, for (v) the assumption that A is irreducible is redundant.

The following result is easy to prove.

Theorem 3.171. *Theorem D is valid for $\nu = 29$, provided one of the conditions (i)-(v) of Theorem 3.165 is satisfied.*

As a special case of Q_{25}, Q is defined in [42] for the preconditioned Gauss-Seidel method as

$$Q_{30} = Q_8 + Q_{17} = \begin{pmatrix} 0 & -\beta_1 a_{1,2} & \cdots & 0 & 0 \\ -\alpha_1 a_{2,1} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -\beta_{n-1} a_{n-1,n} \\ 0 & 0 & \cdots & -\alpha_{n-1} a_{n,n-1} & 0 \end{pmatrix}$$

with $\beta_k, \alpha_k \geq 0$, $k = 1, \cdots, n-1$, and

$$\sum_{k=1}^{n-1} \beta_k a_{k,k+1} \neq 0 \quad \text{and} \quad \sum_{k=1}^{n-1} \alpha_k a_{k+1,k} \neq 0.$$

It is continued to be studied in [43] for the preconditioned Gauss-Seidel and Jacobi methods. It is given in [87, 89] for the preconditioned GAOR method. In [88], it is used to preconditioned parallel multisplitting USAOR method.

In this case, $\delta_{i,j}^{(25)}(\gamma)$ and $\delta_{i,j}^{(25)}(1)$ reduce respectively to

$$\delta_{i,j}^{(30)}(\gamma) = \begin{cases} \alpha_{i-1}a_{i,i-1}a_{i-1,i} + \beta_i a_{i,i+1}a_{i+1,i}, & i = j = 1, \cdots, n; \\ (\gamma - 1)\alpha_{i-1}a_{i,i-1} + \gamma\beta_i a_{i,i+1}a_{i+1,j}, & i = 2, \cdots, n, j = i - 1; \\ (\gamma - 1)\beta_i a_{i,i+1}, & i = 1, \cdots, n - 1, j = i + 1; \\ \gamma\beta_i a_{i,i+1}a_{i+1,j}, & i = 1, \cdots, n - 1, j = 1, \cdots, n, j \neq i - 1, i, i + 1; \\ 0, & i = n, j = 1, \cdots, n - 2 \end{cases}$$

and

$$\delta_{i,j}^{(30)}(1) = \begin{cases} \beta_1 a_{1,2}a_{2,1} < 1, & i = j = 1; \\ \alpha_{i-1}a_{i,i-1}a_{i-1,i} + \beta_i a_{i,i+1}a_{i+1,i}, & i = j = 2, \cdots, n - 1; \\ \alpha_{n-1}a_{n,n-1}a_{n-1,n} < 1, & i = j = n; \\ \beta_i a_{i,i+1}a_{i+1,j}, & i = 1, \cdots, n - 1, \\ & j = 1, \cdots, n, j \neq i, i + 1; \\ 0, & \textit{otherwise}. \end{cases}$$

From Theorems 3.150 and 3.151, the following two comparison results are immediately.

Theorem 3.172. *Suppose that* $0 \leq \alpha_k, \beta_k \leq 1$, $k = 1, \cdots, n - 1$, *and* $\beta_1 a_{1,2}a_{2,1} < 1$, $\alpha_{n-1}a_{n,n-1}a_{n-1,n} < 1$, $\alpha_{k-1}a_{k,k-1}a_{k-1,k} + \beta_k a_{k,k+1}a_{k+1,k} < 1$, $k = 2, \cdots, n - 1$. *Then Theorem A is valid for* $\nu = 30$.

Theorem 3.173. *Suppose that* $0 \leq \alpha_k, \beta_k \leq 1$, $k = 1, \cdots, n - 1$. *Then Theorem B is valid for* $\nu = 30$.

Theorem 3.174. *Suppose that* $\beta_1 a_{1,2}a_{2,1} < 1$, $\alpha_{n-1}a_{n,n-1}a_{n-1,n} < 1$, $\alpha_{k-1}a_{k,k-1}a_{k-1,k} + \beta_k a_{k,k+1}a_{k+1,k} < 1$, $k = 2, \cdots, n-1$. *Then Theorem C is valid for* $\nu = 30$, *provided one of the following conditions is satisfied:*

(i) *For* $k = 1, \cdots, n - 1$, $0 \leq \alpha_k, \beta_k \lesssim 1$.

(ii) $0 \leq \gamma < 1$, $0 \leq \alpha_k, \beta_k \leq 1$, $k = 1, \cdots, n - 1$. $a_{1,2} < 0$, $\beta_1 > 0$ *and for each* $i \in \{2, \cdots, n - 1\}$, $\beta_i a_{i,i+1} < 0$ *or* $\alpha_{i-1}a_{i,i-1} < 0$.

(iii) $\gamma = 1$, $0 \leq \alpha_k, \beta_k \leq 1$, $k = 1, \cdots, n - 1$. *For each* $i \in \{2, \cdots, n - 1\}$, $\beta_i a_{i,i+1} < 0$ *or* $\alpha_{i-1}a_{i,i-1}a_{i-1,i} > 0$. *At the same time, one of the following conditions holds:*

- *(iii_1)* $\alpha_{n-1}a_{n,n-1}a_{n-1,n} > 0$.
- *(iii_2)* $(1 - \alpha_{n-1})a_{n,n-1} < 0$.
- *(iii_3)* *There exists* $j \in \{2, \cdots, n - 2\}$ *such that* $a_{n,j} - \alpha_{n-1}a_{n,n-1}a_{n-1,j} < 0$.
- *(iii_4)* $a_{1,2} < 0$, $\beta_1 > 0$ *and* $a_{n,1} - \alpha_{n-1}a_{n,n-1}a_{n-1,1} < 0$.
- *(iii_5)* $a_{1,2} < 0$, $a_{n,1} < 0$ *and* $\beta_1 > 0$.

Proof. We just need to prove (*iii*).

When $\alpha_{i-1}a_{i,i-1}a_{i-1,i} > 0$, the inequality $\alpha_{i,k(i)}a_{i,k(i)}a_{k(i),j(i)} > 0$ holds for $j(i) = i$ and $k(i) = j(i) - 1$.

The conditions (iii_1), (iii_4) and (iii_5) can be derived from corresponding ones in Theorem 3.152. While, the conditions (iii_2) and (iii_3) can be derived by (iii_3) in Theorem 3.152, directly. □

The result for the Gauss-Seidel method is better than [42, Theorems 3.2-3.5], where in Theorem 3.5 it should be that $\alpha_k > 0$ and $\beta_k > 0$, $k = 1, \cdots, n - 1$.

Theorem 3.175. *Theorem D is valid for* $\nu = 30$, *provided one of the conditions (i), (ii) and (iii) of Theorem 3.174 is satisfied.*

Another combination is given as

$$Q_{31} = Q_5 + Q_{15} = \begin{pmatrix} 0 & \cdots & 0 & -\beta_1 a_{1,n} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & -\beta_{n-1} a_{n-1,n} \\ -\alpha_1 a_{n,1} & \cdots & -\alpha_{n-1} a_{n,n-1} & 0 \end{pmatrix}$$

with $\alpha_k \geq 0, \beta_k \geq 0, k = 1, \cdots, n-1$, and

$$\sum_{k=1}^{n-1} \alpha_k a_{n,k} \neq 0 \ \text{ and } \ \sum_{k=1}^{n-1} \beta_k a_{k,n} \neq 0,$$

where for simplicity we set $r = n$ for Q_5 and Q_{15}.

In this case, $\delta_{i,j}^{(2)}(\gamma)$ and $\delta_{i,j}^{(2)}(1)$ reduce respectively to

$$\delta_{i,j}^{(31)}(\gamma) = \begin{cases} \beta_i a_{i,n} a_{n,i}, & i = j = 1, \cdots, n-1; \\ \sum\limits_{k=1}^{n-1} \alpha_k a_{n,k} a_{k,n}, & i = j = n; \\ (\gamma - 1)\beta_i a_{i,n}, & i = 1, \cdots, n-1, j = n; \\ \gamma \beta_i a_{i,n} a_{n,j}, & i = 2, \cdots, n-1, j = 1, \cdots, i-1; \\ (\gamma - 1)\alpha_j a_{n,j} + \gamma \sum\limits_{k=1}^{j-1} \alpha_k a_{n,k} a_{k,j}, & i = n, j = 1, \cdots, n-1; \\ 0, & \textit{otherwise} \end{cases}$$

and

$$\delta_{i,j}^{(31)}(1) = \begin{cases} \beta_i a_{i,n} a_{n,j}, & i = 1, \cdots, n-1, j = 1, \cdots, i; \\ \sum\limits_{k=1}^{j-1} \alpha_k a_{n,k} a_{k,j}, & i = n, j = 2, \cdots, n; \\ 0, & \textit{otherwise.} \end{cases}$$

Using Corollaries 3.20 and 3.21, we can prove the following theorems, directly.

Theorem 3.176. *Suppose that* $0 \leq \alpha_k, \beta_k \leq 1$, $k = 1, \cdots, n-1$, *and*

$$\sum_{k=1}^{n-1} \alpha_k a_{n,k} a_{k,n} < 1, \ \beta_i a_{i,n} a_{n,i} < 1, i = 1, \cdots, n-1. \tag{53}$$

Then Theorem A is valid for $\nu = 31$.

Theorem 3.177. *Suppose that* $0 \leq \alpha_k, \beta_k \leq 1$, $k = 1, \cdots, n-1$. *Then Theorem B is valid for* $\nu = 31$.

Theorem 3.178. *Suppose that (53) holds. Then Theorem C is valid for* $\nu = 31$, *provided one of the following conditions is satisfied:*

(i) $0 \leq \gamma < 1$. *For* $k = 1, \cdots, n-1$, $0 \leq \alpha_k, \beta_k \lesssim 1$.

(ii) $\gamma = 1$. *For* $k = 1, \cdots, n-1$, $0 \leq \alpha_k, \beta_k \lesssim 1$. *And one of the following conditions holds:*

- *(ii_1) There exist* $i \in \{1, \cdots, n-1\}$ *and* $j \in \{1, \cdots, i\}$ *such that* $\beta_i a_{i,n} a_{n,j} > 0$.
- *(ii_2) There exist* $j \in \{2, \cdots, n\}$ *and* $k \in \{1, \cdots, j-1\}$ *such that* $\alpha_k a_{n,k} a_{k,j} > 0$.
- *(ii_3)* $a_{n-1,n} < 0$ *and* $\beta_{n-1} > 0$.
- *(ii_4)* $a_{n,1} < 0$.
- *(ii_5)* $a_{k,n} < 0$, $k = 1, \cdots, n-1$.

(ii_6) $a_{k,k+1} < 0, k = 1, \cdots, n-1$.

(iii) $a_{k,n} < 0, 0 \le \alpha_k \le 1, 0 < \beta_k \le 1, k = 1, \cdots, n-1$. *And one of the following conditions holds:*

(iii_1) $0 \le \gamma < 1$

(iii_2) $\gamma = 1$ *and for each* $i \in \{1, \cdots, n-1\}$ *there exists* $j(i) \in \{1, \cdots, i\}$ *such that* $a_{n,j(i)} < 0$.

Proof. Since A is irreducible, then there exists $j_0 \in \{1, \cdots, n-1\}$ such that $a_{n,j_0} < 0$.

By (*i*) of Corollary 3.22, (*i*) is obvious.

If (ii_1) holds, then $\delta_{i,j}^{(31)}(1) > 0$ for $i = 1, \cdots, n-1$ and $j = 1, \cdots, i$. While if (ii_2) holds, then $\delta_{n,j}^{(31)}(1) \ge \alpha_k a_{n,k} a_{k,j} > 0$ for $j \in \{2, \cdots, n\}$. This shows that (ii_1) in Theorem 3.18 holds, so that the required result follows by (*i*) of Corollary 3.22, directly.

If (ii_3) holds, then $\beta_{n-1} a_{n-1,n} a_{n,j_0} > 0$, which implies that ($ii_1$) holds for $i = n-1$ and $j = j_0$.

If (ii_4) holds, by the definition of Q_{31}, there exists $i_0 \in \{1, \cdots, n-1\}$ such that $\beta_{i_0} a_{i_0,n} < 0$, so that $\beta_{i_0} a_{i_0,n} a_{n,1} > 0$, which implies that ($ii_1$) holds for $i = i_0$ and $j = 1$.

If (ii_5) holds, by the definition of Q_{31}, there exists $k_0 \in \{1, \cdots, n-1\}$ such that $\alpha_{k_0} a_{n,k_0} < 0$, so that $\alpha_{k_0} a_{n,k_0} a_{k_0,n} > 0$, which implies that ($ii_2$) holds for $k = k_0$ and $j = n$. While when (ii_6) holds, it is easy to see that (ii_2) holds for $k = k_0$ and $j = k_0 + 1$.

When (*iii*) holds, for each $i \in \{1, \cdots, n-1\}$, if $\gamma < 1$ then $\beta_i a_{i,n} < 0$, i.e., (*iii*) of Theorem 3.18 is satisfied.

While, if $\gamma = 1$ then $\delta_{i,j(i)}^{(31)}(1) = \beta_i a_{i,n} a_{n,j(i)} > 0$, i.e., ($iv_1$) in Theorem 3.18 holds. If $\alpha_{j_0} = 0$ then $(1-\alpha_{j_0}) a_{n,j_0} = a_{n,j_0} < 0$, which implies that (22) or (23) holds, so that (iv^c) or (iv^d) in Theorem 3.18 is satisfied. If $\alpha_{j_0} > 0$ then $\alpha_{j_0} a_{n,j_0} a_{j_0,n} > 0$, which implies that ($iv^a$) in Theorem 3.18 is satisfied for $k = j_0$ and $j = n$. This has proved that the condition (*iv*) of Theorem 3.18 is satisfied.

By Corollary 3.22, (*iii*) is proved. □

Similarly, by Corollary 3.23, we can prove the following result.

Theorem 3.179. *Theorem D is valid for* $\nu = 31$, *provided one of the following conditions is satisfied:*

(i) One of the conditions (i), (ii) and (iii) of Theorem 3.178 holds.

(ii) For $j = 1, \cdots, n-1, 0 \le \alpha_j, \beta_j \le 1$ *and*

$$\alpha_j a_{n,j} + \sum_{\substack{k=1\\k\neq j}}^{n-1} \alpha_k a_{n,k} a_{k,j} > 0.$$

One of the following conditions holds:

(ii_1) There exists $i_0 \in \{1, \cdots, n-1\}$ *such that* $\beta_{i_0} a_{i_0,n} a_{n,i_0} > 0$.

(ii_2) There exists $i_0 \in \{1, \cdots, n-1\}$ *such that* $\alpha_{i_0} a_{n,i_0} a_{i_0,n} > 0$.

(ii_3) $\gamma > 0$. *There exist* $i_0 \in \{2, \cdots, n-1\}$, $j_0 \in \{1, \cdots, i_0 - 1\}$ *such that* $\beta_{i_0} a_{i_0,n} a_{n,j_0} > 0$.

(ii_4) $\gamma > 0$. *There exists* $j_0 \in \{1, \cdots, n-1\}$ *such that* $\alpha_{j_0} a_{n,j_0} + \sum_{k=1,k\neq j_0}^{n-1} \alpha_k a_{n,k} a_{k,j_0} > 0$.

Similar to Q_{31}, we give a new combination preconditioner as

$$Q_{32} = Q_6 + Q_{14} = \begin{pmatrix} 0 & -\beta_2 a_{1,2} & \cdots & -\beta_n a_{1,n} \\ -\alpha_2 a_{2,1} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\alpha_n a_{n,1} & 0 & \cdots & 0 \end{pmatrix}$$

with $\alpha_k \geq 0$, $\beta_k \geq 0$, $k = 2, \cdots, n$, and

$$\sum_{k=2}^{n} \alpha_k a_{k,1} \neq 0 \text{ and } \sum_{k=2}^{n} \beta_k a_{1,k} \neq 0,$$

where for simplicity we set $r = 2$ for Q_6 and Q_{14}.

It is proposed in [87] for the preconditioned GAOR method for weighted linear least squares problems.

In this case, $\delta_{i,j}^{(2)}(\gamma)$ and $\delta_{i,j}^{(2)}(1)$ reduce respectively to

$$\delta_{i,j}^{(32)}(\gamma) = \begin{cases} \sum\limits_{k=2}^{n} \beta_k a_{1,k} a_{k,1}, & i = j = 1; \\ \alpha_i a_{i,1} a_{1,i}, & i = j = 2, \cdots, n; \\ (\gamma - 1)\beta_j a_{1,j} + \gamma \sum\limits_{k=2}^{j-1} \beta_k a_{1,k} a_{k,j}, & i = 1, j = 2, \cdots, n; \\ (\gamma - 1)\alpha_i a_{i,1}, & i = 2, \cdots, n, j = 1; \\ \gamma \alpha_i a_{i,1} a_{1,j}, & i = 3, \cdots, n, j = 2, \cdots, i-1; \\ 0, & otherwise \end{cases}$$

and

$$\delta_{i,j}^{(32)}(1) = \begin{cases} \sum\limits_{k=2}^{n} \beta_k a_{1,k} a_{k,1}, & i = j = 1; \\ \sum\limits_{k=2}^{j-1} \beta_k a_{1,k} a_{k,j}, & i = 1, j = 3, \cdots, n; \\ \alpha_i a_{i,1} a_{1,j}, & i = 2, \cdots, n, j = 2, \cdots, i; \\ 0, & otherwise. \end{cases}$$

Using Corollaries 3.20 and 3.21, we can prove the following theorems.

Theorem 3.180. *Suppose that $0 \leq \alpha_i, \beta_i \leq 1$, $\alpha_i a_{i,1} a_{1,i} < 1$, $i = 2, \cdots, n$, $\sum_{k=2}^{n} \beta_k a_{1,k} a_{k,1} < 1$. Then Theorem A is valid for $\nu = 32$.*

Theorem 3.181. *Suppose that $0 \leq \alpha_k, \beta_k \leq 1$, $k = 2, \cdots, n$. Then Theorem B is valid for $\nu = 32$.*

Theorem 3.182. *Suppose that $\sum_{k=2}^{n} \beta_k a_{1,k} a_{k,1} < 1$, $\alpha_i a_{i,1} a_{1,i} < 1$, $i = 2, \cdots, n$. Then Theorem C is valid for $\nu = 32$, provided one of the following conditions is satisfied:*

(i) $0 \leq \gamma < 1$. *For* $k = 2, \cdots, n$, $0 \leq \alpha_k, \beta_k \lesssim 1$.

(ii) $\gamma = 1$. *For* $k = 2, \cdots, n$, $0 \leq \alpha_k, \beta_k \lesssim 1$. *One of the following conditions holds:*

- *(ii_1) There exists $k \in \{2, \cdots, n\}$ such that $\beta_k a_{1,k} a_{k,1} > 0$.*
- *(ii_2) There exist $j \in \{3, \cdots, n\}$ and $k \in \{2, \cdots, j-1\}$ such that $\beta_k a_{1,k} a_{k,j} > 0$.*
- *(ii_3) There exist $i \in \{2, \cdots, n\}$ and $j \in \{2, \cdots, i\}$ such that $\alpha_i a_{i,1} a_{1,j} > 0$.*
- *(ii_4) $a_{1,2} < 0$.*
- *(ii_5) $a_{n,1} < 0$ and $\alpha_n > 0$.*
- *(ii_6) $a_{k,1} < 0$, $k = 2, \cdots, n$.*
- *(ii_7) $a_{n,1} < 0$ and $a_{k,n} < 0$, $k = 2, \cdots, n-1$.*
- *(ii_8) $a_{n,1} < 0$ and $a_{k,k+1} < 0$, $k = 2, \cdots, n-1$.*

(iii) $0 \leq \gamma < 1$. *For* $k = 2, \cdots, n-1$, $a_{k,1} < 0$, $0 < \alpha_k \leq 1$, $0 \leq \alpha_n \leq 1$, *and for* $k = 2, \cdots, n$, $0 \leq \beta_k \leq 1$.

(iv) $\gamma = 1$. $0 < \alpha_k \leq 1$, $0 \leq \beta_k \leq 1$, $k = 2, \cdots, n$. *For each* $i \in \{2, \cdots, n-1\}$, $a_{i,1} < 0$ *and there exists* $j(i) \in \{2, \cdots, i\}$ *such that* $a_{1,j(i)} < 0$.

Proof. From (i) and (ii_2) in Theorem 3.18, we can derive (i) and (ii_5) directly.

If one of (ii_1), (ii_2) and (ii_3) holds, then there exist $i, j \in \{1, \cdots, n\}$ such that $\delta^{(32)}_{i,j}(1) > 0$, which implies that (ii_1) in Theorem 3.18 holds.

If (ii_4) holds, then from the definition of Q_{32}, there exists $i_0 \in \{2, \cdots, n\}$ such that $\alpha_{i_0} a_{i_0,1} < 0$ so that $\alpha_{i_0} a_{i_0,1} a_{1,2} > 0$. This has shown that the condition (ii_3) is satisfied for $i = i_0$ and $j = 2$.

From

$$\sum_{k=2}^{n} \beta_k a_{1,k} a_{k,1} \geq \max_{2\leq k\leq n}\{a_{k,1}\} \sum_{k=2}^{n} \beta_k a_{1,k} > 0,$$

$$\sum_{k=2}^{n-1} \beta_k a_{1,k} a_{k,n} + \beta_n a_{1,n} a_{n,1} \geq \max\{a_{n,1};\ a_{k,n} : k = 2, \cdots, n-1\} \sum_{k=2}^{n} \beta_k a_{1,k} > 0$$

and

$$\sum_{j=3}^{n} \beta_{j-1} a_{1,j-1} a_{j-1,j} + \beta_n a_{1,n} a_{n,1} \geq \max\{a_{n,1};\ a_{k,k+1} : k = 2, \cdots, n-1\} \sum_{j=2}^{n} \beta_j a_{1,j} > 0,$$

it is easy to see that if one of (ii_6), (ii_7) and (ii_8) holds, then (ii_1) or (ii_2) is satisfied.

For (iii), by the definition of Q_{32} again, there exists $j(1) \in \{2, \cdots, n\}$ such that $\beta_{j(1)} a_{1,j(1)} < 0$. For $i \in \{2, \cdots, n-1\}$, $\alpha_i a_{i,1} < 0$. The condition (iii) follows by (iii) of Theorem 3.18, immediately.

If (iv) holds, then $\delta^{(32)}_{i,j(i)}(1) > 0$, which implies that (iv_1) in Theorem 3.18 holds. From the irreducibility of A, there exists $j_0 \in \{1, \cdots, n-1\}$ such that $a_{n,j_0} < 0$. If $j_0 = 1$ then (iv^b) in Theorem 3.18 holds. If $j_0 \geq 2$ then $a_{n,j_0} - \alpha_n a_{n,1} a_{1,j_0} \leq a_{n,j_0} < 0$, which shows that (22) holds for $j = j_0$, i.e., the condition (iv^c) in Theorem 3.18 holds. We have proved (iv). □

Theorem 3.183. *Theorem D is valid for $\nu = 32$, provided one of the conditions (i)-(iv) of Theorem 3.182 is satisfied.*

Unlike Q_{31} and Q_{32}, we give Q as

$$Q_{33} = Q_5 + Q_{14} = \begin{pmatrix} 0 & -\beta_2 a_{1,2} & \cdots & -\beta_{n-1} a_{1,n-1} & -\beta_n a_{1,n} \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ -\alpha_1 a_{n,1} & -\alpha_2 a_{n,2} & \cdots & -\alpha_{n-1} a_{n,n-1} & 0 \end{pmatrix}$$

with $\alpha_k \geq 0$, $\beta_{k=1} \geq 0$, $k = 1, \cdots, n-1$, and

$$\sum_{k=1}^{n-1} \alpha_k a_{n,k} \neq 0 \ \text{ and } \ \sum_{k=2}^{n} \beta_k a_{1,k} \neq 0.$$

It is proposed in [93] with $\alpha_1 = 0$.

In this case, $\delta^{(2)}_{i,j}(\gamma)$ and $\delta^{(2)}_{i,j}(1)$ reduce respectively to

$$\delta^{(33)}_{i,j}(\gamma) = \begin{cases} \sum\limits_{k=2}^{n} \beta_k a_{1,k} a_{k,1}, & i = j = 1; \\ \sum\limits_{k=1}^{n-1} \alpha_k a_{n,k} a_{k,n}, & i = j = n; \\ (\gamma - 1)\beta_j a_{1,j} + \gamma \sum\limits_{k=2}^{j-1} \beta_k a_{1,k} a_{k,j}, & i = 1, j = 2, \cdots, n; \\ (\gamma - 1)\alpha_j a_{n,j} + \gamma \sum\limits_{k=1}^{j-1} \alpha_k a_{n,k} a_{k,j}, & i = n, j = 1, \cdots, n-1; \\ 0, & \textit{otherwise} \end{cases}$$

and

$$\delta_{i,j}^{(33)}(1) = \begin{cases} \sum\limits_{k=2}^{n} \beta_k a_{1,k} a_{k,1}, & i = j = 1; \\ \sum\limits_{k=2}^{j-1} \beta_k a_{1,k} a_{k,j}, & i = 1, j = 3, \cdots, n; \\ \sum\limits_{k=1}^{j-1} \alpha_k a_{n,k} a_{k,j}, & i = n, j = 2, \cdots, n; \\ 0, & otherwise. \end{cases}$$

Using Corollaries 3.20-3.23, similar to Theorems 3.180-3.183, we can prove the following results.

Theorem 3.184. *Suppose that* $0 \le \alpha_k, \beta_{k+1} \le 1$, $k = 1, \cdots, n-1$, *and*

$$\sum_{k=2}^{n} \beta_k a_{1,k} a_{k,1} < 1, \ \sum_{k=1}^{n-1} \alpha_k a_{n,k} a_{k,n} < 1. \tag{54}$$

Then Theorem A is valid for $\nu = 33$.

Theorem 3.185. *Suppose that* $0 \le \alpha_k, \beta_{k+1} \le 1$, $k = 1, \cdots, n-1$. *Then Theorem B is valid for* $\nu = 33$.

The results given in Theorems 3.184 and 3.185 are better than the corresponding ones given in [93, Theorem 2.2].

Theorem 3.186. *Suppose that (54) holds and* $0 \le \alpha_k, \beta_{k+1} \lesssim 1$, $k = 1, \cdots, n-1$. *Then Theorem C is valid for* $\nu = 33$, *provided one of the following conditions is satisfied:*

(i) $0 \le \gamma < 1$.

(ii) $\gamma = 1$ *and one of the following conditions holds:*

- *(ii_1) There exists* $k \in \{2, \cdots, n\}$ *such that* $\beta_k a_{1,k} a_{k,1} > 0$.
- *(ii_2) There exist* $j \in \{3, \cdots, n\}$ *and* $k \in \{2, \cdots, j-1\}$ *such that* $\beta_k a_{1,k} a_{k,j} > 0$.
- *(ii_3) There exist* $j \in \{2, \cdots, n\}$ *and* $k \in \{1, \cdots, j-1\}$ *such that* $\alpha_k a_{n,k} a_{k,j} > 0$.
- *(ii_4)* $a_{n,1} < 0$ *and* $\alpha_n > 0$.
- *(ii_5)* $a_{k,1} < 0$, $k = 2, \cdots, n$.
- *(ii_6)* $a_{k,n} < 0$, $k = 1, \cdots, n-1$.
- *(ii_7)* $a_{n,1} < 0$ *and* $a_{k,n} < 0$, $k = 2, \cdots, n-1$.
- *(ii_8)* $a_{k,k+1} < 0$, $k = 1, \cdots, n-1$.
- *(ii_9)* $a_{n,1} < 0$ *and* $a_{k,k+1} < 0$, $k = 2, \cdots, n-1$.

Theorem 3.187. *Suppose that* $0 \le \alpha_k, \beta_{k+1} \lesssim 1$, $k = 1, \cdots, n-1$. *Then Theorem D is valid for* $\nu = 33$, *provided one of (i) and (ii) of Theorem 3.186 is satisfied.*

A combination is proposed in [92] as $Q_{34} = Q_5 + Q_{14}$, i.e.,

$$Q_{34} = \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ -\alpha_1 a_{r,1} & \cdots & -\alpha_{r-1} a_{r,r-1} & 0 & -\alpha_{r+1} a_{r,r+1} & \cdots & -\alpha_n a_{r,n} \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

with $2 \le r \le n-1$, $\alpha_k \ge 0$, $k = 1, \cdots, n$, $k \ne r$ and

$$\sum_{k=1}^{r-1} \alpha_k a_{r,k} \ne 0 \quad \text{and} \quad \sum_{k=r+1}^{n} \alpha_k a_{r,k} \ne 0.$$

In this case, $\delta_{i,j}^{(2)}(\gamma)$ and $\delta_{i,j}^{(2)}(1)$ reduce respectively to

$$\delta_{i,j}^{(34)}(\gamma) = \begin{cases} (\gamma - 1)\alpha_j a_{r,j} + \gamma \sum\limits_{\substack{1 \le k \le j-1 \\ r+1 \le k \le n}} \alpha_k a_{r,k} a_{k,j}, & i = r, j = 1, \cdots, r-1; \\ \sum\limits_{\substack{k=1 \\ k \ne r}}^{n} \alpha_k a_{r,k} a_{k,r}, & i = j = r; \\ (\gamma - 1)\alpha_j a_{r,j} + \gamma \sum\limits_{k=r+1}^{j-1} \alpha_k a_{r,k} a_{k,j}, & i = r, j = r+1, \cdots, n; \\ 0, & \textit{otherwise} \end{cases}$$

and

$$\delta_{i,j}^{(34)}(1) = \begin{cases} \sum\limits_{\substack{1 \le k \le j-1 \\ r+1 \le k \le n}} \alpha_k a_{r,k} a_{k,j}, & i = r, j = 1, \cdots, r; \\ \sum\limits_{k=r+1}^{j-1} \alpha_k a_{r,k} a_{k,j}, & i = r, j = r+2, \cdots, n; \\ 0, & \textit{otherwise}. \end{cases}$$

Similar to the proof of Theorems 3.42 and 3.99, we can prove corresponding comparison results, directly.

Theorem 3.188. *Suppose that $0 \le \alpha_k \le 1$, $k = 1, \cdots, n$, $k \ne r$, and $\sum_{k=1,k\ne r}^{n} \alpha_k a_{r,k} a_{k,r} < 1$. Then Theorem A is valid for $\nu = 34$.*

Theorem 3.189. *Suppose that $0 \le \alpha_k \le 1$, $k = 1, \cdots, n$, $k \ne r$. Then Theorem B is valid for $\nu = 34$.*

The results given in Theorems 3.188 and 3.189 are better than the corresponding ones given in [92, Theorem 2.2].

Theorem 3.190. *Suppose that $0 \le \alpha_k \lesssim 1$, $k = 1, \cdots, n$, $k \ne r$, and $\sum_{k=1,k\ne r}^{n} \alpha_k a_{r,k} a_{k,r} < 1$. Then Theorem C is valid for $\nu = 34$, provided one of the following conditions is satisfied:*

(i) $0 \le \gamma < 1$.

(ii) $\gamma = 1$ *and one of the following conditions holds:*

- *(ii_1) There exist $i \in \{1, \cdots, r\}$ and $j \in \{1, \cdots, i-1\} \cup \{r+1, \cdots, n\}$ such that $\alpha_j a_{r,j} a_{j,i} > 0$.*
- *(ii_2) There exist $i \in \{r+2, \cdots, n\}$ and $j \in \{r+1, \cdots, i-1\}$ such that $\alpha_j a_{r,j} a_{j,i} > 0$.*
- *(ii_3) $a_{r,r+1} < 0$ and $\alpha_{r+1} > 0$.*
- *(ii_4) $a_{k,k+1} < 0$, $k = 1, \cdots, r-1$.*
- *(ii_5) $a_{n,1} < 0$ and $a_{k,k+1} < 0$, $k = r+1, \cdots, n-1$.*
- *(ii_6) $a_{k,r} < 0$, $k = 1, \cdots, r-1$.*
- *(ii_7) $a_{k,r} < 0$, $k = r+1, \cdots, n$.*
- *(ii_8) $a_{n,1} < 0$ and $a_{k,n} < 0$, $k = r+1, \cdots, n-1$.*
- *(ii_9) $a_{k,1} < 0$, $k = r+1, \cdots, n$.*

Theorem 3.191. *Suppose that $0 \le \alpha_k \lesssim 1$, $k = 1, \cdots, n$, $k \ne r$. Then Theorem D is valid for $\nu = 34$, provided one of the conditions (i) and (ii) of Theorem 3.190 is satisfied.*

Similarly, let $Q_{35} = Q_6 + Q_{15}$. Then we can propose a combination. For simplicity we set

$$Q_{35} = \begin{pmatrix} 0 & \cdots & 0 & -\alpha_1 a_{1,r} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & -\alpha_{r-1} a_{r-1,r} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -\alpha_{r+1} a_{r+1,r} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & -\alpha_n a_{n,r} & 0 & \cdots & 0 \end{pmatrix}$$

with $2 \le r \le n-1$, $\alpha_k \ge 0$, $k = 1, \cdots, n$, $k \ne r$ and

$$\sum_{k=1}^{r-1} \alpha_k a_{k,r} \ne 0 \text{ and } \sum_{k=r+1}^{n} \alpha_k a_{k,r} \ne 0.$$

It is proposed for the preconditioned Jacobi and Gauss-Seidel methods in [66].

In this case, $\delta_{i,j}^{(2)}(\gamma)$ and $\delta_{i,j}^{(2)}(1)$ reduce respectively to

$$\delta_{i,j}^{(35)}(\gamma) = \begin{cases} \alpha_i a_{i,r} a_{r,i}, & i = j \in \{1, \cdots, n\} \setminus \{r\}; \\ (\gamma - 1)\alpha_i a_{i,r}, & i \in \{1, \cdots, n\} \setminus \{r\}, j = r; \\ \gamma \alpha_i a_{i,r} a_{r,j}, & i = 1, \cdots, r-1, j \in \{1, \cdots, i-1\} \cup \{r+1, \cdots, n\}; \\ \gamma \alpha_i a_{i,r} a_{r,j}, & i = r+2, \cdots, n, j = r+1, \cdots, i-1; \\ 0, & \textit{otherwise} \end{cases}$$

and

$$\delta_{i,j}^{(35)}(1) = \begin{cases} \alpha_i a_{i,r} a_{r,j}, & i = 1, \cdots, r-1, j \in \{1, \cdots, i\} \cup \{r+1, \cdots, n\}; \\ \alpha_i a_{i,r} a_{r,j}, & i = r+1, \cdots, n, j = r+1, \cdots, i; \\ 0, & \textit{otherwise.} \end{cases}$$

Since $\delta_{r,j}^{(35)}(\gamma) = 0$, $j = 1, \cdots, n$, then the condition (*ii*) of Theorem 3.18 can be not satisfied.

By Corollaries 3.20-3.23, we can prove the following results, directly.

Theorem 3.192. *Suppose that $0 \le \alpha_k \le 1$ and $\alpha_k a_{k,r} a_{r,k} < 1$, $k = 1, \cdots, n$, $k \ne r$. Then Theorem A is valid for $\nu = 35$.*

Theorem 3.193. *Suppose that $0 \le \alpha_k \le 1$, $k = 1, \cdots, n$, $k \ne r$. Then Theorem B is valid for $\nu = 35$.*

Theorem 3.194. *Suppose that $0 \le \alpha_k \lesssim 1$ and $\alpha_k a_{k,r} a_{r,k} < 1$, $k = 1, \cdots, n$, $k \ne r$. Then Theorem C is valid for $\nu = 35$, provided one of the following conditions is satisfied:*

(i) $0 \le \gamma < 1$.

(ii) $\gamma = 1$ *and one of the following conditions holds:*

- *(ii_1) There exist $i \in \{1, \cdots, r-1\}$ and $j \in \{1, \cdots, i\} \cup \{r+1, \cdots, n\}$ such that $\alpha_i a_{i,r} a_{r,j} > 0$.*
- *(ii_2) There exist $i \in \{r+1, \cdots, n\}$ and $j \in \{r+1, \cdots, i\}$ such that $\alpha_i a_{i,r} a_{r,j} > 0$.*
- *(ii_3) $a_{r-1,r} < 0$ and $\alpha_{r-1} > 0$.*
- *(ii_4) $a_{r,1} < 0$.*
- *(ii_5) There exists $k \in \{r+1, \cdots, n\}$ such that $a_{r,k} < 0$.*

Proof. We just need to prove (ii_3), (ii_4) and (ii_v).

When (ii_3) holds, from the irreducibility of A, there exists $j_0 \in \{1, \cdots, n\} \setminus \{r\}$ such that $a_{r,j_0} < 0$, so that $\alpha_{r-1}a_{r-1,r}a_{r,j_0} > 0$, which implies that (ii_1) holds for $i = r-1$ and $j = j_0$.

If (ii_4) holds, then by the definition of Q_{35} there exists $k_0\{1, \cdots, r-1\}$ such that $\alpha_{k_0}a_{k_0,r} < 0$, so that $\alpha_{k_0}a_{k_0,r}a_{r,1} > 0$, which implies that (ii_1) holds for $i = k_0$ and $j = 1$.

For the case when (ii_5) holds, the proof is completely same. □

Theorem 3.195. *Suppose that $0 \le \alpha_k \lesssim 1$, $k = 1, \cdots, n$, $k \ne r$. Then Theorem D is valid for $\nu = 35$, provided one of the conditions (i) and (ii) of Theorem 3.194 is satisfied.*

Clearly, there exists a permutation matrix V such that $V^TQ_{35}V = Q_6$ with $q^{(6)}_{r,1} = \alpha_1 a_{1,r}$, $q^{(6)}_{k.1} = \alpha_k a_{k,r}$, $k = 2, \cdots, n$, $k \ne r$. It is easy to see that the matrices A and V^TAV have the same irreducibility and $\rho(\mathscr{J}(A)) = \rho(\mathscr{J}(V^TAV))$, $\rho(\mathscr{J}(A^{(35)})) = \rho(\mathscr{J}(V^TP_{35}VV^TAV))$. Hence, by (iii) of Theorem 3.51, Theorems 3.194 and 3.195 for $\gamma = 0$ are valid whenever we set $\alpha_k = 1$, $k = 1, \cdots, n$, $k \ne r$. So from Theorem 3.195 it can derive [66, Theorem 2.2].

At last, in [26], Q is chosen as

$$Q_{36} = Q_{11} + Q_{20} = \begin{pmatrix} 0 & 0 & \cdots & 0 & -\frac{a_{1,n}}{\alpha_2} - \beta_2 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ -\frac{a_{n,1}}{\alpha_1} - \beta_1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

with $a_{1,n} < 0$, $a_{n,1} < 0$, $\alpha_i > 0$, $i = 1, 2$, $a_{n,1}/\alpha_1 + \beta_1 < 0$ and $a_{1,n}/\alpha_2 + \beta_2 < 0$.

It is proposed in [41] for $\alpha_i = \alpha$, $\beta_i = \beta$, $i = 1, 2$.

In this case, for $n \ge 3$, the condition (ii) of Theorem 3.6 can be not satisfied.

By Corollaries 3.12-3.15, it can be prove the following comparison results.

Theorem 3.196. *Suppose that*

$$\beta_1 \ge \left(1 - \frac{1}{\alpha_1}\right)a_{n,1}, \quad \beta_2 \ge \left(1 - \frac{1}{\alpha_2}\right)a_{1,n} \tag{55}$$

and

$$\beta_1 > \frac{1}{a_{1,n}} - \frac{a_{n,1}}{\alpha_1}, \quad \beta_2 > \frac{1}{a_{n,1}} - \frac{a_{1,n}}{\alpha_2}. \tag{56}$$

Then Theorem A is valid for $\nu = 36$.

Theorem 3.197. *Suppose that (55) holds. Then Theorem B is valid for $\nu = 36$.*

Theorem 3.198. *Suppose that (56) holds and*

$$\beta_1 \gtrsim \left(1 - \frac{1}{\alpha_1}\right)a_{n,1}, \quad \beta_2 \gtrsim \left(1 - \frac{1}{\alpha_2}\right)a_{1,n}. \tag{57}$$

Then Theorem C is valid for $\nu = 36$.

Proof. In this case, for $i \ne j$, the condition $q^{(1)}_{i,j} \lesssim -a_{i,j}$ reduces to (57).

On the other hand, since $a_{n,1}/\alpha_1 + \beta_1 < 0$, then the condition (ii_2) in Theorem 3.6 is satisfied. It follows by Corollary 3.14 that Theorem C is valid. □

Theorem 3.199. *Suppose that (57) holds. Then Theorem D is valid for $\nu = 36$.*

4. Conclusions

In this paper, we have investigated the preconditioned AOR method for solving linear systems. We have studied two general preconditioners and proposed some lower triangular, upper triangular and combination preconditioners. For A being an L-matrix, a nonsingular M-matrix, an irreducible L-matrix and an irreducible nonsingular M-matrix, four types of comparison theorems are presented, respectively. They contain a general comparison result, a strict comparison result and two Stein-Rosenberg type comparison results. Our theorems include and are better than almost all known corresponding results. We also pointed out some incorrect known results.

Of course, according to the construction of combination preconditioners, we can define more combination preconditioners.

When (γ, ω) is equal to (ω, ω), (1, 1) and (0, 1), from the results above, we can derive respectively the corresponding comparison results about the preconditioned SOR method, Gauss-Seidel method and Jacobi method directly.

Similar to [2, 7, 10, 12, 13] for the block preconditioned Jacobi, Gauss-Seidel, SOR methods and the block preconditioned AOR method respectively, when A is partitioned by block, then Q can be chosen as a block matrix, so that we can derive the same comparison results for the block preconditioned AOR method.

Similar to [3, 25, 27, 64, 87, 106, 112, 114] for the preconditioned AOR methods for solving linear least squares problems, we can derive the corresponding comparison results as above.

The comparisons between either different preconditioners or different parameters of a same type preconditioner have investigated by many authors in [1–3, 17, 18, 22, 23, 31, 32, 34, 38, 41, 43, 44, 47, 48, 59, 62, 66, 67, 69–72, 74, 76–78, 82, 90–93, 96, 100–102, 105, 108]. This is an important and interesting research subject. Because of the length, this paper does not cover this topic.

In [19, 24, 33, 36, 55, 58, 60, 61, 65, 75, 83, 88, 94, 97, 101] and some related literatures, the preconditioned iterative methods for H-matrix is studied.

Recently, in [5, 6, 21, 28, 49, 57, 73, 95], the preconditioned tensor splitting methods and the preconditioned AOR (SOR) methods for solving multi-linear systems are proposed. These are new subjects to be studied.

References

[1] I. Abdullahi, A. Ndanusa, *A new modified preconditioned accelerated overrelaxation (AOR) iterative method for matrix linear algebraic systems*, Sci. World J. **15** (2020), 45–50.

[2] M. Alanelli, A. Hadjidimos, *Block Gauss elimination followed by a classical iterative method for the solution of linear systems*, J. Comput. Appl. Math. **163** (2004), 381–400.

[3] Y.-Q. Bai, A.-Q. Wang, *An acceleration of preconditioned generalized accelerated over relaxation methods for systems of linear equations*, Int. J. Comput. Math. **92** (2015), 1012–1024.

[4] A. Berman, R.J. Plemmons, *Nonnegative matrices in the mathematics sciences*, Classics in Applied Mathematics, Vol. 9. SIAM, Philadelphia, 1994.

[5] Y. Chen, C. Li, *A new preconditioned AOR method for solving multi-linear systems*, Linear and Multilinear Algebra **72** (2024), 1385–1402.

[6] L.-B. Cui, M.-H. Li, Y. Song, *Preconditioned tensor splitting iterations method for solving multi-linear systems*, Appl. Math. Lett. **96** (2019), 89–94.

[7] M.T. Darvishi, P. Hessari, B.-C. Shin, *Preconditioned modified AOR method for systems of linear equations*, Int. J. Numer. Meth. Biomed. Engng. **27** (2011), 758–769.

[8] M. Dehghan, M. Hajarian, *Improving preconditioned SOR-type iterative methods for L-matrices*, Int. J. Numer. Meth. Biomed. Engng. **27** (2011), 774–784.

[9] M. Dehghan, M. Hajarian, *Modied AOR iterative methods to solve linear systems*, J. Vib. Control **20** (2014), 661–669.

[10] Q.-Y. Dou, J.-F. Yin, *Multi-level preconditioned block accelerated overrelaxation iteration method for Z-matrices*, J. Appl. Math. Comput. **38** (2012), 653–667.

[11] S.A. Edalatpanah, *On the modified methods for irreducible linear systems with L-matrices*, Bull. Comput. Appl. Math. **6** (2018), 119–128.

[12] D.J. Evans, M.M. Martins, *The AOR method for preconditioned linear systems*, Int. J. Comput. Math. **56** (1995), 69–76.

[13] D.J. Evans, M.M. Martins, M.E. Trigo, *On preconditioned MSOR iterations*, Int. J. Comput. Math. **59** (1996), 251–257.

[14] D.J. Evans, M.M. Martins, M.E. Trigo, *The AOR iterative method for new preconditioned linear systems*, J. Comput. Appl. Math. **132** (2001), 461–466.

[15] A.D. Gunawardena, S.K. Jain, L. Snyder, *Modified iteration methods for consistent linear systems*, Linear Algebra Appl. **154-156** (1991), 123–143.

[16] A. Hadjidimos, *Accelerated overrelaxation method*, Math. Comput. **32** (1978), 149–157.

[17] A. Hadjidimos, D. Noutsos, M. Tzoumas, *More on modifications and improvements of classical iterative schemes for M-matrices*, Linear Algebra Appl. **364** (2003), 253–279.

[18] M. Hasani, D.K. Salkuyeh, *Improvements of two preconditioned AOR iterative methods for Z-matrices*, Bull. Iranian Math. Soc. **40** (2014), 357–371.

[19] P. Hessari, M.T. Darvishi, B.-C. Shin, *Convergence analysis of preconditioned AOR iterative method*, Honam Math. J. **32** (2010), 399–412.

[20] H. Hirano, H. Niki, *Application of a preconditioning iterative method to the computation of fluid flow*, Numer. Funct. Anal. Optim. **22** (2001), 405–417.

[21] B. Huang, *Apreconditioned tensor splitting iteration method and associated global correction technique for solving multilinear systems*, Calcolo **60** (2023), 4.

[22] T.-Z. Huang, G.-H. Cheng, X.-Y. Cheng, *Modified SOR-type iterative method for Z-matrices*, Appl. Math. Comput. **175** (2006), 258–268.

[23] T.-Z. Huang, G.-H. Cheng, D.J. Evans, X.-Y. Cheng, AOR type iterations for solving preconditioned linear systems, Int. J. Comput. Math. **82** (2005), 969–976.

[24] T.-Z. Huang, X.-Z. Wang, Y.-D. Fu, *Improving Jacobi methods for nonnegative H-matrices linear systems*, Appl. Math. Comput. **186** (2007), 1542–1550.

[25] Z.-G. Huang, L. Wang, Q. Lu, J.-J. Cui, *Some new preconditioned generalized AOR methods for generalized least-squares problems*, Appl. Math. Comput. **269** (2015), 87–104.

[26] Z. Huang, L. Wang, Z. Xu, J. Cui, *Convergence analysis of some new preconditioned AOR iterative methods for L-matrices*, IAENG Int. J. Appl. Math. **46** (2016), 202–209.

[27] Z.-G. Huang, L.-G. Wang, Z. Xu, J.-J. Cui, *Some new preconditioned generalized AOR methods for solving weighted linear least squares problems*, Comp. Appl. Math. **37** (2018), 415–438.

[28] Z. Jiang, J. Li, *A new preconditioned AOR-type method for $\mathcal{M}-$tensor equation*, Appl. Numer. Math. **189** (2023), 39–52.

[29] B. Karasözen, A.Y. Özban, *Modified iterative methods for linear sustems of equations*, Int. J. Comput. Math. **70** (1998), 179–196.

[30] T. Kohno, H. Kotakemori, H. Niki, M. Usui, *Improving modified iterative methods for Z-matrices*, Linear Algebra Appl. **267** (1997), 113–123.

[31] T. Kohno, H. Niki, *A note on the preconditioner $P_m = (I + S_m)$*, J. Comput. Appl. Math. **225** (2009), 316–319.

[32] T. Kohno, H. Niki, *Letter to the editor: A note on the preconditioned Gauss-Seidel (GS) method for linear systems*, J. Comput. Appl. Math. **233** (2010), 2413–2421.

[33] V. Kostić, L. Cvetković, *Iterative method with the preconditioner $(I + S_{max})$ for H-matrices*, Proc. Appl. Math. Mech. **3** (2003), 547–548.

[34] H. Kotakemori, K. Harada, M. Morimoto, H. Niki, *A comparison theorem for the iterative method with the preconditioner $(I + S_{max})$*, J. Comput. Appl. Math. **145** (2002), 373–378.

[35] H. Kotakemori, H. Niki, N. Okamoto, *Accelerated iterative method for Z-matrices*, J. Comput. Appl. Math. **75** (1996), 87–97.

[36] H. Kotakemori, H. Niki, N. Okamoto, *Convergence of a preconditioned iterative method for H-matrices*, J. Comput. Appl. Math. **83** (1997), 115–118.

[37] H. Kotakemori, H. Niki, N. Okamoto, *A generalization of the adaptive Gauss-Seidel method for Z-matrices*, Int. J. Comput. Math. **64** (1997), 317–326.

[38] A. Li, *Improving AOR iterative methods for irreducible L-matrices*, Eng. Let. **19** (2011), 46–49.

[39] A. Li, *A new preconditioned AOR iterative method and comparison theorems for linear systems*, IAENG Int. J. Appl. Math. **42** (2012), 161–163.

[40] C. Li, D.J. Evans, *Improving the SOR method*, Int. J. Comput. Math. **54** (1994), 207–213.

[41] C.-X. Li, S.-L. Wu, *Some comparison results with new effective preconditioners for L-matrices*, WSEAS Trans. Math. **13** (2014), 314–323.

[42] J. Li, T. Huang, *Preconditioned methods of Z-matrices*, Acta Math. Sci. **25A** (2005), 5–10.

[43] J.-C. Li, Y.-L. Jiang, *Generalized tridiagonal preconditioners for solving linear systems*, Int. J. Comput. Math. **87** (2010), 3297–3310.

[44] W. Li, J. Li, *Comparison results between preconditioned Jacobi and the AOR iterative method*, Numer. Math. J. Chinese Univ. (English Ser.) **16** (2007), 313–319.

[45] W. Li, *Preconditioned AOR iterative methods for linear systems*, Int. J. Comput. Math. **79** (2002), 89–101.

[46] W. Li, *The convergence of the modified Gauss-Seidel methods for consistent linear systems*, J. Comput. Appl. Math. **154** (2003), 97–105.

[47] W. Li, *Comparison results for solving preconditioned linear systems*, J. Comput. Appl. Math. **176** (2005), 319–329.

[48] W. Li, *A note on the preconditioned Gauss-Seidel (GS) method for linear systems*, J. Comput. Appl. Math. **182** (2005), 81–90.

[49] W. Li, D. Liu, S.-W. Vong, *Comparison results for splitting iterations for solving multi-linear systems*, Appl. Numer. Math. **134** (2018), 105–121.

[50] W. Li, W. Sun, *Modified Gauss-Seidel type methods and Jacobi type methods for Z-matrices*, Linear Algebra Appl. **317** (2000), 227–240.

[51] Y. Li, Z. Wang, *A modified AOR iterative method for preconditioned linear systems*, Southeast Asian Bull. Math. **28** (2004), 305–320.

[52] Y.-T. Li, C.-X. Li, S.-L. Wu, *Improving AOR method for consistent linear systems*, Appl. Math. Comput. **186** (2007), 379–388.

[53] Y.-T. Li, C.-X. Li, S.-L. Wu, *Improvements of preconditioned AOR iterative method for L-matrices*, J. Comput. Appl. Math. **206** (2007), 656–665.

[54] Y.-T. Li, Z.-D. Wang, *Notes on the AOR iterative method for preconditioned linear systems*, Chinese J. Engrg. Math. **21** (2004), 685–690.

[55] Y.-T. Li, S. Yang, *A multi-parameters preconditioned AOR iterative method for linear systems*, Appl. Math. Comput. **206** (2008), 465–473.

[56] C. Liu, C. Li, *A new preconditioned generalised AOR method for the linear complementarity problem based on a generalised Hadjidimos preconditioner*, E. Asian J. Appl. Math. **2** (2012), 94–107.

[57] D. Liu, W. Li, S.-W. Vong, *A new preconditioned SOR method for solving multi-linear systems with an $\mathcal{M}-$tensor*, Calcolo **57** (2020), 15.

[58] Q. Liu, *Some preconditioning techniques for linear systems*, WSEAS Trans. Math. **9** (2008), 579–588.

[59] Q. Liu, G. Chen, *Erratum to: "A note on the preconditioned Gauss-Seidel method for M-matrices" [J. Comput. Appl. Math. 219 (1) (2008) 59–71]*, J. Comput. Appl. Math. **228** (2009), 498–502.

[60] Q. Liu, G. Chen, *Convergence analysis of preconditioned AOR iterative method for linear systems*, Math. Probl. Eng. **2010** (2010), ID 341982.

[61] Q. Liu, G. Chen, J. Cai, *Convergence analysis of the preconditioned Gauss-Seidel method for H-matrices*, Comput. Math. Appl. **56** (2008), 2048–2053.
[62] Q. Liu, J. Huang, S. Zeng, *Convergence analysis of the two preconditioned iterative methods for M-matrix linear systems*, J. Comput. Appl. Math. **281** (2015), 49–57.
[63] M.M. Martins, D.J. Evans, W. Yousif, *Further results on the preconditioned SOR method*, Int. J. Comput. Math. **l7** (2001), 603–610.
[64] S.-X. Miao, Y.-H. Luo, G.-B. Wang, *Two new preconditioned GAOR methods for weighted linear least squares problems*, Appl. Math. Comput. **324** (2018), 93–104.
[65] S.-X. Miao, B. Zheng, *On convergence of the modified Gauss-Seidel iterative method for H-matrix linear system*, Commun. Korean Math. Soc. **28** (2013), 603–613.
[66] J.P. Milaszewicz, *Improving Jacobi and Gauss-Seidel iterations*, Linear Algebra Appl. **93** (1987), 161–170.
[67] M. Morimoto, *Study on the preconditioners $(I + S_m)$*, J. Comput. Appl. Math. **234** (2010), 209–214.
[68] A. Ndanusa, K.R. Adeboye, *Preconditioned SOR iterative methods for L-matrices*, American J. Comput. Appl. Math. **2** (2012), 300–305.
[69] H. Niki, K. Harada, M. Morimoto, M. Sakakihara, *The survey of preconditioners used for accelerating the rate of convergence in the Gauss-Seidel method*, J. Comput. Appl. Math. **164-165** (2004), 587–600
[70] H. Niki, T. Kohno, *Letter to the editor: Comment on 'A comparison theorem of the SOR iterative method'*, J. Comput. Appl. Math. **234** (2010), 3507–3510.
[71] H. Niki, T. Kohno, K. Abe, *An extended GS method for dense linear systems*, J. Comput. Appl. Math. **231** (2009), 177–186.
[72] H. Niki, T. Kohno, M. Morimoto, *The preconditioned Gauss-Seidel method faster than the SOR method*, J. Comput. Appl. Math. **219** (2008), 59–71.
[73] M. Qi, X. Shao, *Preconditioned AOR iterative methods for solving multi-linear systems with $\mathcal{M}-$tensor*, J. Appl. Math. & Informatics **39** (2021), 587–600.
[74] H. Saberi Najafi, S.A. Edalatpanah, *Comparison analysis for improving preconditioned SOR-type iterative method*, Numer. Anal. Appl. **6** (2013), 62–70.
[75] H. Saberi Najafi, S.A. Edalatpanah, *A new family of $(I + S)$-type preconditioner with some applications*, Comp. Appl. Math. **34** (2015), 917–931.
[76] H. Saberi Najafi, S.A. Edalatpanah, A.H. Refahi Sheikhani, *Convergence analysis of modified iterative methods to solve linear systems*, Mediterr. J. Math. **11** (2014), 1019–1032.
[77] D.K. Salkuyeh, M. Hasani, F.P.A. Beik, *On the preconditioned AOR iterative method for Z-matrices*, Comp. Appl. Math. **36** (2017), 877–883.
[78] D.K. Salkuyeh, M. Hasani, F.P.A. Beik, J. Song, Y. Song, *Correction to: On the preconditioned AOR iterative method for Z-matrices*, Comp. Appl. Math. **39** (2020), 124.
[79] H. Shen, X. Shao, Z. Huang, C. Li, *Preconditioned Gauss-Seidel iterative method for Z-matrices linear systems*, Bull. Korean Math. Soc. **48** (2011), 303–314.
[80] Y. Song, *Comparisons of nonnegative splittings of matrices*, Linear. Algebra. Appl. **154-156** (1991), 433–455.
[81] Y. Song, *A note on comparison theorems for splittings of bounded operators*, Acta Math. Sinica **40** (1997), 313–318.
[82] L.-Y. Sun, *A comparison theorem for the SOR iterative method*, J. Comput. Appl. Math. **181** (2005), 336–341.
[83] L.-Y. Sun, *Some extensions of the improved modified Gauss-Seidel iterative method for H-matrices*, Numer. Linear Algebra Appl. **13** (2006), 869–876.
[84] M. Usui, T. Kohno, H. Niki, *On the preconditioned SOR method*, Int. J. Comput. Math. **59** (1995), 123–130.
[85] M. Usui, H. Niki, T. Kohno, *Adaptive Gauss-Seidel method for linear systems*, Int. J. Comput. Math. **51** (1994), 119–125.
[86] R.S. Varga, *Matrix iterative analysis*, Springer Series in Computational Mathematics 27, Berlin: Springer-Verlag, 2000.
[87] G. Wang, Y. Du, F. Tan, *Comparison results on preconditioned GAOR methods for weighted linear least squares problems*, J. Appl. Math. **2012** (2012), ID 563586.
[88] G. Wang, D. Sun, *Preconditioned parallel multisplitting USAOR method for H-matrices linear systems*, Appl. Math. Comput. **275** (2016), 156–164.
[89] G. Wang, T. Wang, F. Tan, *Some results on preconditioned GAOR methods*, Appl. Math. Comput. **219** (2013), 5811–5816.
[90] G. Wang, T. Zhang, *New preconditioned AOR iterative method for Z-matrices*, J. Appl. Math. Comput. **37** (2011), 103–117.
[91] H. Wang, Y.-T. Li, *A new preconditioned AOR iterative methods for L-matrices*, J. Comput. Appl. Math. **229** (2009), 47–53.
[92] L. Wang, *On a class of row preconditioners for solving linear systems*, Int. J. Comput. Math. **83** (2006), 939–949.
[93] L. Wang, *Comparison results for AOR iterative method with a new preconditioner*, Int. J. Nonlin. Sci. **2** (2006), 16–28.
[94] L. Wang, Y. Song, *Preconditioned AOR iterative methods for M-matrices*, J. Comput. Appl. Math. **226** (2009), 114–124.
[95] X. Wang, M. Che, Y. Wei, *Preconditioned tensor splitting AOR iterative methods for $\mathcal{H}-$tensor equations*, Numer. Linear Algebra Appl. **27** (2020), e2329.
[96] Z.-D. Wang, T.-Z. Huang, *The upper Jacobi and upper Gauss-Seidel type iterative methods for preconditioned linear systems*, Appl. Math. Lett. **19** (2006), 1029–1036.
[97] M. Wu, L. Wang, Y. Song, *Preconditioned AOR iterative method for linear systems*, Appl. Numer. Math. **57** (2007), 672–685.
[98] S.-L. Wu, T.-Z. Huang, *A modified AOR-type iterative method for L-matrix linear systems*, ANZIAM J. **49** (2007), 281–292.
[99] S. Wu, T. Huang, *Erratum to: "Convergence of the preconditioned AOR method for irreducible L-matrices" [Appl. Math. Comput. 201 (2008) 56–64]*, Appl. Math. Comput. **212** (2009), 551–552.
[100] J.Y. Yuan, D.D. Zontini, *Comparison theorems of preconditioned Gauss-Seidel methods for M-matrices*, Appl. Math. Comput. **219** (2012), 1947–1957.
[101] J.H. Yun, *A note on the improving modified Gauss-Seidel (IMGS) method*, Appl. Math. Comput. **184** (2007), 674–679.
[102] J.H. Yun, *A note on the modied SOR method for Z-matrices*, Appl. Math. Comput. **194** (2007), 572–576.
[103] J.H. Yun, *A note on preconditioned AOR method for L-matrices*, J. Comput. Appl. Math. **220** (2008), 13–16.
[104] J.H. Yun, *Comparison results of the preconditioned AOR methods for L-matrices*, Appl. Math. Comput. **218** (2011), 3399–3413.

[105] J.H. Yun, *Comparison results for the preconditioned Gauss-Seidel methods*, Commun. Korean Math. Soc. **27** (2012), 207–215.

[106] J.H. Yun, *Comparison results on the preconditioned GAOR method for generalized least squares problems*, Int. J. Comput. Math. **89** (2012), 2094–2105.

[107] J.H. Yun, S.W. Kim, *Convergence of the preconditioned AOR method for irreducible L-matrices*, Appl. Math. Comput. **201** (2008), 56–64.

[108] J.H. Yun, H.J. Lim, S.W. Kim, *Performance comparison of preconditioned iterative methods with direct preconditioners*, J. Appl. Math. Inform. **32** (2014), 389–403.

[109] Y. Zhang, T.-Z. Huang, X.-P. Liu, *Modified iterative methods for nonnegative matrices and M-matrices linear systems*, Comput. Math. Appl. **50** (2005), 1587–1602.

[110] Y. Zhang, T.-Z. Huang, X.-P. Liu, *Gauss type preconditioning techniques for linear systems*, Appl. Math. Comput. **188** (2007), 612–633.

[111] Y. Zhang, T.-Z. Huang, X.-P. Liu, T.-X. Gu, *A class of preconditioners based on the $(I + S(\alpha))$-type preconditioning matrices for solving linear systems*, Appl. Math. Comput. **189** (2007), 1737–1748.

[112] J. Zhao, C. Li, F. Wang, Y. Li, *Some new preconditioned generalized AOR methods for generalized least-squares problems*, Int. J. Comput. Math. **91** (2014), 1370–1383.

[113] B. Zheng, S.-X. Miao, *Two new modified Gauss-Seidel methods for linear system with M-matrices*, J. Comput. Appl. Math. **233** (2009), 922–930.

[114] X. Zhou, Y. Song, L. Wang, Q. Liu, *Preconditioned GAOR methods for solving weighted linear least squares problems*, J. Comput. Appl. Math. **224** (2009), 242–249.