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The m-MP weak core inverse and its applications

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Abstract. Based on the m-core-nilpotent decomposition, we introduce a generalized inverse named the m-MP weak core inverse, which unifies the CMP inverse and the MP weak core inverse. Some properties, characterizations and representations of this generalized inverse are shown. Then, the relationship between the m-MP weak core inverse and a nonsingular bordered matrix is established. A variant of the successive squaring computational iterative scheme is given for calculating the m-MP weak core inverse. An equivalent condition for the continuity for the m-MP weak core inverse is also studied. In the final, the m-MP weak core inverse is used in solving a system of linear equations.

1. Introduction

Let $\mathbb{Z}^+$ and $\mathbb{C}_{m\times n}$ denote the sets of all positive integers and all $m \times n$ complex matrices, respectively. The symbols * means the conjugate transpose of a matrix. The notations A^*, $R(A)$, $N(A)$, rank(A) and $\rho(A)$ denote the conjugate transpose, range, null space, rank and spectral radius of $A \in \mathbb{C}_{m\times n}$, respectively. The notation $\|\cdot\|$ stands for the matrix norm. Let $A\{2\} = \{X \in \mathbb{C}_{n\times m} | XAX = X\}$. If $X \in A\{2\}$, we call X an outer inverse of A, which is denoted by $A^{(2)}$. If an outer inverse X of A satisfies $R(X) = \mathcal{T}$ and $N(X) = \mathcal{S}$, then X is called the outer inverse with the prescribed range $\mathcal{T}$ and null space $\mathcal{S}$, which is denoted by $A^{(2)}_{\mathcal{T},\mathcal{S}}$. We define that $A^0 = I_n$, where $A \in \mathbb{C}_{n\times n}$ and I_n is the identity matrix of order n. And, 0 denotes the null matrix of an appropriate size. For an $n \times n$ matrix A, the index of A is the smallest non-negative integer k such that rank(A^{k+1}) = rank(A^k), which is denoted as Ind(A). Notice that Ind$(A) = 0$ if and only if A is invertible. For subspaces $\mathcal{T}$ and $\mathcal{S}$ satisfying their direct sum as $\mathbb{C}_{n\times 1}$, i.e., $\mathcal{S} \oplus \mathcal{T} = \mathbb{C}_{n\times 1}$, the projector onto $\mathcal{T}$ along $\mathcal{S}$ is indicated by $P_{\mathcal{T},\mathcal{S}}$. Particularly, the orthogonal projector onto $\mathcal{T}$ is $P_{\mathcal{T}} = P_{\mathcal{T},\mathcal{T}^\perp}$, where $\mathcal{T}^\perp$ presents the orthogonal complement of $\mathcal{T}$.

The Moore Penrose inverse denoted by $A^\dagger = X \in \mathbb{C}_{n\times m}$ of $A \in \mathbb{C}_{m\times n}$ is the unique matrix satisfying the following matrix equations [21]

$$(1)\ AXA = A,\ \ (2)\ XAX = X,\ \ (3)\ (AX)^* = AX,\ \ (4)\ (XA)^* = XA.$$

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The Drazin inverse denoted by $A^D = X \in \mathbb{C}_{n\times n}$ of $A \in \mathbb{C}_{n\times n}$ with $\mathrm{Ind}(A) = k$ is the unique matrix satisfying the following matrix equations [3]

$$(1)\ A^{k+1}X = A^k,\ \ (2)\ XAX = X,\ \ (3)\ AX = XA.$$

When $k = 1$, A^D is called the group inverse of A and is denoted by $A^\#$.

The CMP inverse denoted by $A^{c,\dagger} = X \in \mathbb{C}_{n\times n}$ of $A \in \mathbb{C}_{n\times n}$ with $\mathrm{Ind}(A) = k$ is the unique matrix satisfying the following matrix equations [15]

$$(1)\ XAX = X,\ \ (2)\ AXA = AA^DA,\ \ (3)\ XA = A^\dagger AA^DA,\ \ (4)\ AX = AA^DAA^\dagger.$$

Notice that $A^{c,\dagger} = A^\dagger AA^DAA^\dagger$.

The core-EP inverse denoted by $A^{\textcircled{\dagger}} = X \in \mathbb{C}_{n\times n}$ of $A \in \mathbb{C}_{n\times n}$ with $\mathrm{Ind}(A) = k$ is the unique matrix satisfying the following matrix equations [14]

$$(1)\ XAX = X,\ \ (2)\ XA^{k+1} = A^k,\ \ (3)\ (AX)^* = AX,\ \ (4)\ R(X) \subseteq R(A^k).$$

Notice that $A^{\textcircled{\dagger}} = A^DA^k\left(A^k\right)^\dagger$.

The weak group inverse denoted by $A^{\textcircled{w}} = X \in \mathbb{C}_{n\times n}$ of $A \in \mathbb{C}_{n\times n}$ with $\mathrm{Ind}(A) = k$ is the unique matrix satisfying the following matrix equations [28]

$$(1)\ AX^2 = X,\ \ (2)\ AX = A^{\textcircled{\dagger}}A.$$

Notice that $A^{\textcircled{w}} = \left(A^{\textcircled{\dagger}}\right)^2 A$.

The MP weak core inverse denoted by $A^{\bigcirc} = X \in \mathbb{C}_{n\times n}$ of $A \in \mathbb{C}_{n\times n}$ with $\mathrm{Ind}(A) = k$ is the unique matrix satisfying the following matrix equations [12]

$$(1)\ XAX = X,\ \ (2)\ AX = AA^{\textcircled{w}}AA^\dagger,\ \ (3)\ XA = A^\dagger AA^{\textcircled{w}}A.$$

Notice that $A^{\bigcirc} = A^\dagger AA^{\textcircled{w}}AA^\dagger$.

The m-weak group inverse denoted by $A^{\textcircled{w}_m} = X \in \mathbb{C}_{n\times n}$ of $A \in \mathbb{C}_{n\times n}$ with $\mathrm{Ind}(A) = k$ is the unique matrix satisfying the following matrix equations [9]

$$(1)\ AX^2 = X,\ \ (2)\ AX = \left(A^{\textcircled{\dagger}}\right)^m A^m,\ \text{where } m \in \mathbb{Z}^+.$$

Notice that $A^{\textcircled{w}_m} = \left(A^{\textcircled{\dagger}}\right)^{m+1} A^m$.

The m-weak group MP inverse denoted by $A^{\textcircled{w}_m,\dagger} = X \in \mathbb{C}_{n\times n}$ of $A \in \mathbb{C}_{n\times n}$ with $\mathrm{Ind}(A) = k$ is the unique matrix satisfying the following matrix equations [8]

$$(1)\ XAX = X,\ \ (2)\ AX = AA^{\textcircled{w}_m}AA^\dagger,\ \ (3)\ XA = A^DAA^{\textcircled{w}_m}A,\ \textit{where } m \in \mathbb{Z}^+.$$

Notice that $A^{\textcircled{w}_m,\dagger} = A^{\textcircled{w}_m}AA^\dagger$.

Let $A, B, C \in \mathbb{C}_{n\times n}$. The (B, C)-inverse of A is the unique matrix $Y \in \mathbb{C}_{n\times n}$ satisfying the following matrix equations [1, 22]

$$(1)\ YAB = B,\ \ (2)\ CAY = C,\ \ (3)\ N(C) \subseteq N(Y),\ \ (4)\ R(Y) \subseteq R(B).$$

Lemma 1.1. *[26] Let $A \in \mathbb{C}_{n\times n}$ with $\mathrm{Ind}(A) = k$. Then*
(1) $R(A^\dagger) = R(A^*)$, $N(A^\dagger) = N(A^*)$;
(2) $AA^\dagger = P_{R(A),N(A^*)}$;
(3) $A^\dagger A = P_{R(A^*),N(A)}$.

Lemma 1.2. *[9] Let $A \in \mathbb{C}_{n\times n}$ with $\operatorname{Ind}(A) = k$ and $m \in \mathbb{Z}^+$. Then*
(1) $A^{\textcircled{w}_m} A A^{\textcircled{w}_m} = A^{\textcircled{w}_m}$;
(2) $A^{\textcircled{w}_m} = A^{(2)}_{R(A^k),N\left((A^k)^* A^m\right)}$;
(3) $AA^{\textcircled{w}_m} = P_{R(A^k),N\left((A^k)^* A^m\right)}$;
(4) $A^{\textcircled{w}_m} A = P_{R(A^k),N\left((A^k)^* A^{m+1}\right)}$.

Lemma 1.3. *[8] Let $A \in \mathbb{C}_{n\times n}$ with $\operatorname{Ind}(A) = k$ and $m \in \mathbb{Z}^+$. Then*
(1) $A^{\textcircled{w}_m,\dagger} = A^{(2)}_{R(A^k),N\left((A^k)^* A^{m+1}A^\dagger\right)}$;
(2) $AA^{\textcircled{w}_m,\dagger} = P_{R(A^k),N\left((A^k)^* A^{m+1}A^\dagger\right)}$;
(3) $A^{\textcircled{w}_m,\dagger} A = P_{R(A^k),N\left((A^k)^* A^{m+1}\right)}$.

Lemma 1.4. *[26] Let $A \in \mathbb{C}_{n\times n}$. Let L and M be complementary subspaces of $\mathbb{C}_n$, i.e., $L \oplus M = \mathbb{C}_n$. Then,*
(1) $P_{L,M}A = A \Leftrightarrow R(A) \subseteq L$;
(2) $A = AP_{L,M} \Leftrightarrow M \subseteq N(A)$.

2. The *m*-MP weak core inverse

In this section, we introduce a generalized inverse named m-MP weak core inverse. In order to do this, we firstly recall the m-core-nilpotent decomposition. Then we consider a system of equations.

Lemma 2.1. *[8] Let $A \in \mathbb{C}_{n\times n}$ with $\operatorname{Ind}(A) = k$ and $m \in \mathbb{Z}^+$. Let $A = A_1 + A_2$ be the m-core-nilpotent decomposition of A. Then*

$$A_1 = AA^{\textcircled{w}_m}A \text{ and } A_2 = A - AA^{\textcircled{w}_m}A.$$

Notice that if $m \geqslant k-1$, then A_1 coincides with $\widetilde{A_1} = AA^D A$ which is the core part in the core-nilpotent decomposition. If $m = 1$, then A_1 coincides with $C = AA^{\textcircled{w}}A$ which is the weak core part of A.

Definition 2.2. *[16] Let $A \in \mathbb{C}_{n\times n}$ with $\operatorname{Ind}(A) = k$ and $m \in \mathbb{Z}^+$. Let A_1 be the form of Lemma 2.1. The following matrix equations*

$$XAX = X,\ AX = A_1A^\dagger,\ XA = A^\dagger A_1 \tag{1}$$

have the unique solution $A^\dagger AA^{\textcircled{w}_m}AA^\dagger$. We term $A^{\textcircled{w}_m,\bigcirc} = A^\dagger AA^{\textcircled{w}_m}AA^\dagger$ as the m-MP weak core inverse (in short, the m-MPWC inverse) of A.

Remark 2.3. *From the definitions of CMP inverse and MP weak core inverse, it is easy to infer that:*
(1) If $m = 1$, then $A^{\textcircled{w}_m,\bigcirc} = A^{\bigcirc}$;
(2) If $m \geqslant k-1$, then $A^{\textcircled{w}_m,\bigcirc} = A^{c,\dagger}$.
Hence, the m-MPWC inverse is actually a generalization of the CMP inverse and the MP weak core inverse. It is also a special case of the weak CMP inverse introduced in [16].

Example 2.4. *Let*

$$A = \begin{pmatrix} I_4 & I_4 \\ 0 & N \end{pmatrix}, \textit{where}\ \ N = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then $\operatorname{Ind}(A) = 4$, *and*

$$A^\dagger = \begin{pmatrix} H_1 & -N^\dagger \\ I - H_1 & N^\dagger \end{pmatrix},\ \ A^D = \begin{pmatrix} I_4 & H_2 \\ 0 & 0 \end{pmatrix},\ \ A^{c,\dagger} = \begin{pmatrix} H_1 & H_5 \\ I_4 - H_1 & H_6 \end{pmatrix},$$

$$A^{\textcircled{\dagger}} = \begin{pmatrix} I_4 & 0 \\ 0 & 0 \end{pmatrix},\ A^{\textcircled{w}} = \begin{pmatrix} I_4 & I_4 \\ 0 & 0 \end{pmatrix},\ A^{\textcircled{w}_2} = \begin{pmatrix} I_4 & H_3 \\ 0 & 0 \end{pmatrix},$$

$$A^{\bigcirc} = \begin{pmatrix} H_1 & H_7 \\ I_4 - H_1 & I_4 - H_1 \end{pmatrix},\ A^{\textcircled{w}_2,\dagger} = \begin{pmatrix} I_4 & H_4 \\ 0 & 0 \end{pmatrix} \text{ and } A^{\textcircled{w}_2,\bigcirc} = \begin{pmatrix} H_1 & H_8 \\ I_4 - H_1 & H_9 \end{pmatrix},$$

where

$$H_1 = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ H_2 = \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ H_3 = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ H_4 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$H_5 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ H_6 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ H_7 = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ H_8 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$H_9 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } N^{\dagger} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

This shows that m-MPWC inverse is different from some known generalized inverses.

3. Characterizations of the *m*-MP weak core inverse

More characterizations for the Moore-Penrose inverse, the CMP inverse, the core-EP inverse, the weak group inverse, and the m-weak group inverse can be found in [5, 10, 11, 17–19, 29, 30], and we now present characterizations for the m-MP weak core inverse.

Lemma 3.1. *[16] Let $A \in \mathbb{C}_{n\times n}$ with $\operatorname{Ind}(A) = k$ and $m \in \mathbb{Z}^+$. Then*
(1) $R(A^{\textcircled{w}_m,\bigcirc}) = R(A^{\dagger}A^k)$;
(2) $N(A^{\textcircled{w}_m,\bigcirc}) = N\left(\left(A^k\right)^* A^{m+1}A^{\dagger}\right)$;
(3) $A^{\textcircled{w}_m,\bigcirc} = A^{(2)}_{R(A^{\dagger}A^k),N\left(\left(A^k\right)^* A^{m+1}A^{\dagger}\right)}$.

Theorem 3.2. *Let $A \in \mathbb{C}_{n\times n}$ with $\operatorname{Ind}(A) = k$ and $m \in \mathbb{Z}^+$. Then*

$$A^{\textcircled{w}_m,\bigcirc} = A^{(2)}_{R\left(A^{\dagger}A^k\left(A^k\right)^* A^{m+1}A^{\dagger}\right),N\left(A^{\dagger}A^k\left(A^k\right)^* A^{m+1}A^{\dagger}\right)}.$$

Proof. From Theorem 5.3 of [8], we get that

$$R(A^k) = R\left(A^k\left(A^k\right)^* A^{m+1}A^{\dagger}\right), \qquad rank(A^k) = rank\left(\left(A^k\right)^* A^{m+1}A^{\dagger}\right).$$

Hence, we can deduce that

$$R(A^{\dagger}A^k) = A^{\dagger}R(A^k) = A^{\dagger}R\left(A^k\left(A^k\right)^* A^{m+1}A^{\dagger}\right) = R\left(A^{\dagger}A^k\left(A^k\right)^* A^{m+1}A^{\dagger}\right),$$

which implies

$$rank(A^{\dagger}A^k) = rank\left(A^{\dagger}A^k\left(A^k\right)^* A^{m+1}A^{\dagger}\right).$$

We can also infer directly that

$$\begin{aligned} rank(A^\dagger A^k) &\leqslant rank(A^k) = rank(A^{k+1}) = rank(AA^k) = rank(AA^\dagger AA^k) \\ &\leqslant rank(A^\dagger A^{k+1}) \leqslant rank(A^\dagger A^k), \end{aligned}$$

which implies

$$rank\left(\left(A^k\right)^* A^{m+1}A^\dagger\right) = rank(A^k) = rank(A^\dagger A^k) = rank\left(A^\dagger A^k\left(A^k\right)^* A^{m+1}A^\dagger\right).$$

Since $N\left(\left(A^k\right)^* A^{m+1}A^\dagger\right) \subseteq N\left(A^\dagger A^k\left(A^k\right)^* A^{m+1}A^\dagger\right)$, it follows that

$$N\left(\left(A^k\right)^* A^{m+1}A^\dagger\right) = N\left(A^\dagger A^k\left(A^k\right)^* A^{m+1}A^\dagger\right).$$

Combine with Lemma 3.1, we get $A^{\textcircled{w}_m,\bigcirc} = A^{(2)}_{R\left(A^\dagger A^k\left(A^k\right)^* A^{m+1}A^\dagger\right),N\left(A^\dagger A^k\left(A^k\right)^* A^{m+1}A^\dagger\right)}$. □

Theorem 3.3. *Let $A \in \mathbb{C}_{n\times n}$ with $\mathrm{Ind}(A) = k$ and $m \in \mathbb{Z}^+$. Let A_1 be as in Lemma 2.1. Then*
(1) $A^{\textcircled{w}_m,\bigcirc}$ is a reflexive g-inverse of A_1;
(2) $A_1A^{\textcircled{w}_m,\bigcirc} = A_1A^\dagger$; $A^{\textcircled{w}_m,\bigcirc}A_1 = A^\dagger A_1$.

Proof. (1). It is easy to see that

$$A^{\textcircled{w}_m,\bigcirc}A_1A^{\textcircled{w}_m,\bigcirc} = A^\dagger AA^{\textcircled{w}_m}AA^\dagger AA^{\textcircled{w}_m}AA^\dagger AA^{\textcircled{w}_m}AA^\dagger = A^\dagger AA^{\textcircled{w}_m}AA^\dagger = A^{\textcircled{w}_m,\bigcirc},$$

$$A_1A^{\textcircled{w}_m,\bigcirc}A_1 = AA^{\textcircled{w}_m}AA^\dagger AA^{\textcircled{w}_m}AA^\dagger AA^{\textcircled{w}_m}A = AA^{\textcircled{w}_m}A = A_1.$$

So, $A^{\textcircled{w}_m,\bigcirc}$ is a reflexive g-inverse of A_1.
(2). It is obvious that

$$A_1A^{\textcircled{w}_m,\bigcirc} = AA^{\textcircled{w}_m}AA^\dagger AA^{\textcircled{w}_m}AA^\dagger = AA^{\textcircled{w}_m}AA^\dagger = A_1A^\dagger,$$

$$A^{\textcircled{w}_m,\bigcirc}A_1 = A^\dagger AA^{\textcircled{w}_m}AA^\dagger AA^{\textcircled{w}_m}A = A^\dagger AA^{\textcircled{w}_m}A = A^\dagger A_1.$$

This completes the proof. □

Theorem 3.4. *Let $A \in \mathbb{C}_{n\times n}$ with $\mathrm{Ind}(A) = k$ and $m \in \mathbb{Z}^+$. Then the following statements are equivalent:*
(1) $X = A^{\textcircled{w}_m,\bigcirc}$;
(2) $N(X) = N\left(\left(A^k\right)^* A^{m+1}A^\dagger\right)$, $XA = A^\dagger A_1$;
(3) $N(X) = N\left(\left(A^k\right)^* A^{m+1}A^\dagger\right)$, $XA^k = A^\dagger A^k$;
(4) $N(X) = N\left(\left(A^k\right)^* A^{m+1}A^\dagger\right)$, $XAA^{\textcircled{w}_m} = A^\dagger AA^{\textcircled{w}_m}$.

Proof. (1) ⇒ (2). It is easily obtained by (1) and Lemma 3.1.
(2) ⇒ (3). Post-multiplying $XA = A^\dagger A_1$ by A^{k-1}, we have

$$XA^k = A^\dagger A_1A^{k-1} = A^\dagger AA^{\textcircled{w}_m}AA^{k-1} = A^\dagger AA^{\textcircled{w}_m}A^k = A^\dagger A^k,$$

where the last equality follows from Lemma 1.2 and Lemma 1.4.
(3) ⇒ (4). It follows from the fact $A\left(A^{\textcircled{w}_m}\right)^2 = A^{\textcircled{w}_m}$ that

$$XAA^{\textcircled{w}_m} = XA^k\left(A^{\textcircled{w}_m}\right)^k = A^\dagger A^k\left(A^{\textcircled{w}_m}\right)^k = A^\dagger AA^{\textcircled{w}_m}.$$

(4) ⇒ (1). From Lemma 1.3 and Lemma 1.4, we have

$$N(X) = N\left(\left(A^k\right)^* A^{m+1}A^\dagger\right) = N(AA^{\textcircled{w}_m,\dagger}) = N(AA^{\textcircled{w}_m}AA^\dagger),$$

which implies

$$X = XAA^{\textcircled{w}_m}AA^{\dagger}.$$

Applying $XAA^{\textcircled{w}_m} = A^{\dagger}AA^{\textcircled{w}_m}$ to $X = XAA^{\textcircled{w}_m}AA^{\dagger}$, we have

$$X = XAA^{\textcircled{w}_m}AA^{\dagger} = A^{\dagger}AA^{\textcircled{w}_m}AA^{\dagger} = A^{\textcircled{w}_m,\bigcirc}.$$

This finishes the proof. □

Theorem 3.5. *Let $A \in \mathbb{C}_{n\times n}$ with $\mathrm{Ind}(A) = k$ and $m \in \mathbb{Z}^+$. Then the following statements are equivalent:*
(1) $X = A^{\textcircled{w}_m,\bigcirc}$;
(2) $XAX = X$, $R(X) = R(A^{\dagger}A^k)$, $N(X) = N\left(\left(A^k\right)^* A^{m+1}A^{\dagger}\right)$;
(3) $XAX = X$, $AX = A_1A^{\dagger}$, $R(XA) = R(A^{\dagger}A^k)$;
(4) $XAX = X$, $AX = A_1A^{\dagger}$, $XA^k = A^{\dagger}A^k$;
(5) $XAX = X$, $XA = A^{\dagger}A_1$, $N(AX) = N\left(\left(A^k\right)^* A^{m+1}A^{\dagger}\right)$.

Proof. The item (1) implies items (2), (3), (5) can be checked directly using (1) and Lemma 3.1.
(1) ⇒ (4). It is obvious by (1) that

$$XAX = X, \quad AX = A_1A^{\dagger}.$$

From Lemma 1.2 and Lemma 1.4, we have

$$R(AA^{\textcircled{w}_m}) = R(A^k) \text{ and } AA^{\textcircled{w}_m}A^k = A^k.$$

As a consequence, we can infer that

$$XA^k = XAA^{k-1} = A^{\dagger}A_1A^{k-1} = A^{\dagger}AA^{\textcircled{w}_m}AA^{k-1} = A^{\dagger}AA^{\textcircled{w}_m}A^k = A^{\dagger}A^k.$$

(2) ⇒ (1). Is is clear by Lemma 3.1 (3).
(3) ⇒ (1). Because $R(A^{\dagger}AA^{\textcircled{w}_m}) = A^{\dagger}R(AA^{\textcircled{w}_m}) = A^{\dagger}R(A^k) = R(A^{\dagger}A^k) = R(XA)$ from Lemma 1.2, we have

$$XAA^{\dagger}AA^{\textcircled{w}_m} = A^{\dagger}AA^{\textcircled{w}_m}, \text{ i.e., } XAA^{\textcircled{w}_m} = A^{\dagger}AA^{\textcircled{w}_m}.$$

Applying $AX = A_1A^{\dagger}$ and $XAA^{\textcircled{w}_m} = A^{\dagger}AA^{\textcircled{w}_m}$ to $XAX = X$, we get that

$$X = XAX = XA_1A^{\dagger} = XAA^{\textcircled{w}_m}AA^{\dagger} = A^{\dagger}AA^{\textcircled{w}_m}AA^{\dagger} = A^{\textcircled{w}_m,\bigcirc}.$$

(4) ⇒ (1). Since $AA^{\textcircled{w}_m}$ is an idempotent, it follows that

$$\begin{aligned} X &= XAX = XA_1A^{\dagger} = XAA^{\textcircled{w}_m}AA^{\dagger} = XA^k\left(A^{\textcircled{w}_m}\right)^k AA^{\dagger} \\ &= A^{\dagger}A^k\left(A^{\textcircled{w}_m}\right)^k AA^{\dagger} = A^{\dagger}AA^{\textcircled{w}_m}AA^{\dagger} = A^{\textcircled{w}_m,\bigcirc}. \end{aligned}$$

(5) ⇒ (1). Since $N(AX) = N\left(\left(A^k\right)^* A^{m+1}A^{\dagger}\right) = N(A^{\textcircled{w}_m,\bigcirc})$, it follows that

$$X = XAX = A^{\dagger}A_1X = A^{\dagger}AA^{\textcircled{w}_m}AX = A^{\dagger}AA^{\textcircled{w}_m}AA^{\dagger}AX = A^{\textcircled{w}_m,\bigcirc}AX = A^{\textcircled{w}_m,\bigcirc}.$$

This finishes the proof. □

Theorem 3.6. *Let $A \in \mathbb{C}_{n\times n}$ with $\mathrm{Ind}(A) = k$ and $m \in \mathbb{Z}^+$. Then the following statements are equivalent:*
(1) $X = A^{\textcircled{w}_m,\bigcirc}$;
(2) $A^{\dagger}A_1X = X$, $AX = A_1A^{\dagger}$, $XA = A^{\dagger}A_1$;
(3) $XA_1X = X$, $A_1X = A_1A^{\dagger}$, $XA_1 = A^{\dagger}A_1$;
(4) $XA_1X = X$, $A^{\textcircled{w}_m}AX = A^{\textcircled{w}_m,\dagger}$, $XAA^{\textcircled{w}_m} = A^{\dagger}AA^{\textcircled{w}_m}$.

Proof. (1) ⇒ (2). From (1), we have that

$$XAX = X, \;\; AX = A_1A^\dagger \;\text{ and }\; XA = A^\dagger A_1.$$

Post-multiplying $XA = A^\dagger A_1$ by X, we have

$$A^\dagger A_1 X = XAX = X.$$

(2) ⇒ (1). Post-multiplying $XA = A^\dagger A_1$ by X, we have

$$XAX = A^\dagger A_1 X = X.$$

Since $AX = A_1A^\dagger$ and $XA = A^\dagger A_1$, we know $X = A^{\textcircled{w}_m,\bigcirc}$ by Definition 2.2.
(1) ⇒ (3). It is obvious by Theorem 3.3.
(3) ⇒ (4). Pre-multiplying $A_1X = A_1A^\dagger$ by $A^{\textcircled{w}_m}$, we have

$$A^{\textcircled{w}_m}A_1X = A^{\textcircled{w}_m}A_1A^\dagger = A^{\textcircled{w}_m}AA^{\textcircled{w}_m}AA^\dagger = A^{\textcircled{w}_m}AA^\dagger = A^{\textcircled{w}_m,\dagger}.$$

Post-multiplying $XA_1 = A^\dagger A_1$ by $A^{\textcircled{w}_m}$, we have

$$XA_1A^{\textcircled{w}_m} = XAA^{\textcircled{w}_m}AA^{\textcircled{w}_m} = XAA^{\textcircled{w}_m} = A^\dagger A_1A^{\textcircled{w}_m} = A^\dagger AA^{\textcircled{w}_m}AA^{\textcircled{w}_m} = A^\dagger AA^{\textcircled{w}_m}.$$

(4) ⇒ (1). Applying $A^{\textcircled{w}_m}AX = A^{\textcircled{w}_m,\dagger}$ and $XAA^{\textcircled{w}_m} = A^\dagger AA^{\textcircled{w}_m}$ to $X = XA_1X$, we get that

$$X = XA_1X = XAA^{\textcircled{w}_m}AX = A^\dagger AA^{\textcircled{w}_m}AX = A^\dagger AA^{\textcircled{w}_m,\dagger} = A^\dagger AA^{\textcircled{w}_m}AA^\dagger = A^{\textcircled{w}_m,\bigcirc}.$$

This finishes the proof. □

Theorem 3.7. *Let $A \in \mathbb{C}_{n\times n}$ and $m \in \mathbb{Z}^+$. Then $A^{\textcircled{w}_m,\bigcirc}$ is the $(A^\dagger A_1A^*, A^*A_1A^\dagger)$-inverse of A, where A_1 has the same form as in Lemma 2.1.*

Proof. It is easy to check that

$$A^{\textcircled{w}_m,\bigcirc}AA^\dagger A_1A^* = A^\dagger AA^{\textcircled{w}_m}AA^\dagger AA^\dagger AA^{\textcircled{w}_m}AA^* = A^\dagger AA^{\textcircled{w}_m}AA^{\textcircled{w}_m}AA^* = A^\dagger AA^{\textcircled{w}_m}AA^* = A^\dagger A_1A^*,$$

$$A^*A_1A^\dagger AA^{\textcircled{w}_m,\bigcirc} = A^*AA^{\textcircled{w}_m}AA^\dagger AA^\dagger AA^{\textcircled{w}_m}AA^\dagger = A^*AA^{\textcircled{w}_m}AA^{\textcircled{w}_m}AA^\dagger = A^*AA^{\textcircled{w}_m}AA^\dagger = A^*A_1A^\dagger.$$

In addition, if $x \in N(A^*A_1A^\dagger)$, then we can infer

$$A^{\textcircled{w}_m,\bigcirc}x = A^\dagger A_1A^\dagger x = A^\dagger AA^\dagger A_1A^\dagger x = A^\dagger \left(A^\dagger\right)^* A^*AA^\dagger AA^{\textcircled{w}_m}AA^\dagger x = A^\dagger \left(A^\dagger\right)^* A^*A_1A^\dagger x = 0,$$

implying $x \in N(A^{\textcircled{w}_m,\bigcirc})$. Hence, we obtain $N(A^*A_1A^\dagger) \subseteq N(A^{\textcircled{w}_m,\bigcirc})$. Also, from

$$A^{\textcircled{w}_m,\bigcirc} = A^\dagger A_1A^\dagger = A^\dagger A_1A^\dagger AA^\dagger = A^\dagger A_1A^\dagger AA^*\left(A^\dagger\right)^* A^\dagger = A^\dagger A_1A^*\left(A^\dagger\right)^* A^\dagger,$$

we know that $R(A^{\textcircled{w}_m,\bigcirc}) \subseteq R(A^\dagger A_1A^*)$. Thus, $A^{\textcircled{w}_m,\bigcirc}$ is the $(A^\dagger A_1A^*, A^*A_1A^\dagger)$-inverse of A by the definition of the (B, C)-inverse. □

4. Two canonical forms of the *m*-MP weak core inverse

Lemma 4.1. *[27] Every matrix $A \in \mathbb{C}_{n\times n}$ with $\mathrm{Ind}(A) = k$ and $\mathrm{rank}(A^k) = r$ has a core-EP decomposition $A = \widehat{A_1} + \widehat{A_2}$, and has the following matrix form*

$$A = U\begin{pmatrix} T & S \\ 0 & N \end{pmatrix}U^*, \tag{2}$$

where $\operatorname{Ind}(\widehat{A_1}) \leqslant 1$, $\widehat{A_2}^k = 0$, $A_1^*A_2 = A_2A_1 = 0$, *and* $U \in \mathbb{C}_{n\times n}$ *is unitary. Furthermore,*

$$\widehat{A_1} = U\begin{pmatrix} T & S \\ 0 & 0 \end{pmatrix}U^*,\ \widehat{A_2} = U\begin{pmatrix} 0 & 0 \\ 0 & N \end{pmatrix}U^*,$$

where $T \in \mathbb{C}_{r\times r}$ *is nonsingular,* $S \in \mathbb{C}_{r\times(n-r)}$, $N \in \mathbb{C}_{(n-r)\times(n-r)}$ *is nilpotent, and* $N^k = 0$.

The Moore Penrose inverse of A is given as [2]:

$$A^\dagger = U\begin{pmatrix} T^*\triangle & -T^*\triangle SN^\dagger \\ (I-N^\dagger N)S^*\triangle & N^\dagger - (I-N^\dagger N)S^*\triangle SN^\dagger \end{pmatrix}U^*, \tag{3}$$

where $\triangle = \left(TT^* + S(I-N^\dagger N)S^*\right)^{-1}$.

From [9], we gather

$$A^{\circledR_m} = U\begin{pmatrix} T^{-1} & \left(T^{m+1}\right)^{-1}T_m \\ 0 & 0 \end{pmatrix}U^*, \tag{4}$$

$$A^k = U\begin{pmatrix} T^k & T_k \\ 0 & 0 \end{pmatrix}U^*,$$

$$A^m = U\begin{pmatrix} T^m & T_m \\ 0 & N^m \end{pmatrix}U^*,$$

$$AA^\dagger = U\begin{pmatrix} I & 0 \\ 0 & NN^\dagger \end{pmatrix}U^*,$$

where $m \in \mathbb{Z}^+$, $T_k = \sum_{j=0}^{k-1} T^jSN^{k-1-j}$ *and* $T_m = \sum_{j=0}^{m-1} T^jSN^{m-1-j}$.

Lemma 4.2. *[7] Every matrix* $A \in \mathbb{C}_{n\times n}$ *with* $\operatorname{rank}(A) = r > 0$ *has a Hartwig-Spindelböck decomposition:*

$$A = U\begin{pmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{pmatrix}U^*, \tag{5}$$

where $U \in \mathbb{C}_{n\times n}$ *is a unitary matrix,* $\Sigma = diag(\sigma_1 I_{k_1}, \sigma_2 I_{k_2}, \ldots, \sigma_t I_{k_t})$ *is a diagonal matrix, the elements on the diagonal* $\sigma_1 > \sigma_2 > \ldots > \sigma_t > 0$ *being the singular values of the matrix A,* $k_1 + k_2 + \cdots + k_t = r = rank(A)$, *and* $K \in \mathbb{C}_{r\times r}$ *and* $L \in \mathbb{C}_{r\times(n-r)}$ *satisfy* $KK^* + LL^* = I_r$.

The MP inverse of A is given as [13]:

$$A^\dagger = U\begin{pmatrix} K^*\Sigma^{-1} & 0 \\ L^*\Sigma^{-1} & 0 \end{pmatrix}U^*. \tag{6}$$

The core-EP inverse of A is given as [4]:

$$A^{\textcircled{\dagger}} = U\begin{pmatrix} (\Sigma K)^{\textcircled{\dagger}} & 0 \\ 0 & 0 \end{pmatrix}U^*. \tag{7}$$

The matrix A^m, *where* $m \in \mathbb{Z}^+$, *is given as:*

$$A^m = U\begin{pmatrix} (\Sigma K)^m & (\Sigma K)^{m-1}\Sigma L \\ 0 & 0 \end{pmatrix}U^*. \tag{8}$$

Theorem 4.3. *Let* $A \in \mathbb{C}_{n\times n}$ *with* $\operatorname{Ind}(A) = k$ *be of the form (2) and* $m \in \mathbb{Z}^+$. *Then*

$$A^{\circledR_m,\bigcirc} = U\begin{pmatrix} T^*\triangle & T^*\triangle T^{-m}T_mNN^\dagger \\ (I-N^\dagger N)S^*\triangle & (I-N^\dagger N)S^*\triangle T^{-m}T_mNN^\dagger \end{pmatrix}U^*,$$

where $\triangle = \left(TT^* + S(I-N^\dagger N)S^*\right)^{-1}$ *and* $T_m = \sum_{j=0}^{m-1} T^jSN^{m-1-j}$.

Proof. From (2), (3), (4), and the fact that $A^{\textcircled{w}_m,\bigcirc} = A^{\dagger}AA^{\textcircled{w}_m}AA^{\dagger}$, we can determine the result directly. □

Theorem 4.4. *Let $A \in \mathbb{C}_{n\times n}$ with $\mathrm{Ind}(A) = k$ be of the form (5) and $m \in \mathbb{Z}^{+}$. Then*

$$A^{\textcircled{w}_m} = U\begin{pmatrix} (\Sigma K)^{\textcircled{w}_m} & \left((\Sigma K)^{\textcircled{\dagger}}\right)^{m+1}(\Sigma K)^{m-1}\Sigma L \\ 0 & 0 \end{pmatrix}U^{*}, \tag{9}$$

and

$$A^{\textcircled{w}_m,\bigcirc} = U\begin{pmatrix} K^{*}K(\Sigma K)^{\textcircled{w}_m} & 0 \\ L^{*}K(\Sigma K)^{\textcircled{w}_m} & 0 \end{pmatrix}U^{*}. \tag{10}$$

Proof. From (7), we have

$$\left(A^{\textcircled{\dagger}}\right)^{m+1} = U\begin{pmatrix} \left((\Sigma K)^{\textcircled{\dagger}}\right)^{m+1} & 0 \\ 0 & 0 \end{pmatrix}U^{*}. \tag{11}$$

From (8), (11), and the fact that

$$A^{\textcircled{w}_m} = \left(A^{\textcircled{\dagger}}\right)^{m+1}A^{m},$$

we get that (9) holds. Using (5), (6), (9), and $A^{\textcircled{w}_m,\bigcirc} = A^{\dagger}AA^{\textcircled{w}_m}AA^{\dagger}$, we obtain (10). □

Theorem 4.5. *Let $A \in \mathbb{C}_{n\times n}$ with $\mathrm{Ind}(A) = k$ and $m \in \mathbb{Z}^{+}$. Then $A^{\textcircled{w}_m,\bigcirc}$ is an EP matrix if and only if the following conditions hold:*
(1) $K^{*} \triangledown K = \left(K(\Sigma K)^{\textcircled{w}_m}\right)^{\dagger} K (\Sigma K)^{\textcircled{w}_m}$;
(2) $\triangledown K = 0$;
(3) $L^{*} \triangledown K = 0$,
where $\triangledown = K(\Sigma K)^{\textcircled{w}_m}\left(K(\Sigma K)^{\textcircled{w}_m}\right)^{\dagger}$.

Proof. Let A be of the form (5). It is easy to check that

$$\left(A^{\textcircled{w}_m,\bigcirc}\right)^{\dagger} = U\begin{pmatrix} \left(K(\Sigma K)^{\textcircled{w}_m}\right)^{\dagger}K & \left(K(\Sigma K)^{\textcircled{w}_m}\right)^{\dagger}L \\ 0 & 0 \end{pmatrix}U^{*}. \tag{12}$$

Using (10), (12), and $KK^{*} + LL^{*} = I_r$, we can get

$$\left(A^{\textcircled{w}_m,\bigcirc}\right)^{\dagger}A^{\textcircled{w}_m,\bigcirc} = U\begin{pmatrix} \left(K(\Sigma K)^{\textcircled{w}_m}\right)^{\dagger}K(\Sigma K)^{\textcircled{w}_m} & 0 \\ 0 & 0 \end{pmatrix}U^{*},$$

and

$$A^{\textcircled{w}_m,\bigcirc}\left(A^{\textcircled{w}_m,\bigcirc}\right)^{\dagger} = U\begin{pmatrix} K^{*}K(\Sigma K)^{\textcircled{w}_m}\left(K(\Sigma K)^{\textcircled{w}_m}\right)^{\dagger}K & K^{*}K(\Sigma K)^{\textcircled{w}_m}\left(K(\Sigma K)^{\textcircled{w}_m}\right)^{\dagger}L \\ L^{*}K(\Sigma K)^{\textcircled{w}_m}\left(K(\Sigma K)^{\textcircled{w}_m}\right)^{\dagger}K & L^{*}K(\Sigma K)^{\textcircled{w}_m}\left(K(\Sigma K)^{\textcircled{w}_m}\right)^{\dagger}L \end{pmatrix}U^{*}.$$

Since $A^{\textcircled{w}_m,\bigcirc}$ is an EP matrix if and only if $\left(A^{\textcircled{w}_m,\bigcirc}\right)^{\dagger}A^{\textcircled{w}_m,\bigcirc} = A^{\textcircled{w}_m,\bigcirc}\left(A^{\textcircled{w}_m,\bigcirc}\right)^{\dagger}$, then we have

$$K^{*}K(\Sigma K)^{\textcircled{w}_m}\left(K(\Sigma K)^{\textcircled{w}_m}\right)^{\dagger}K = \left(K(\Sigma K)^{\textcircled{w}_m}\right)^{\dagger}K(\Sigma K)^{\textcircled{w}_m}; \tag{13}$$

$$K^{*}K(\Sigma K)^{\textcircled{w}_m}\left(K(\Sigma K)^{\textcircled{w}_m}\right)^{\dagger}L = 0; \tag{14}$$

$$L^{*}K(\Sigma K)^{\textcircled{w}_m}\left(K(\Sigma K)^{\textcircled{w}_m}\right)^{\dagger}K = 0; \tag{15}$$

$$L^{*}K(\Sigma K)^{\textcircled{w}_m}\left(K(\Sigma K)^{\textcircled{w}_m}\right)^{\dagger}L = 0. \tag{16}$$

Pre-multiplying (14) by K, pre-multiplying (16) by L, and adding the results, we have

$$K(\Sigma K)^{\textcircled{W}_m}\left(K(\Sigma K)^{\textcircled{W}_m}\right)^{\dagger} L = 0. \tag{17}$$

So, if $A^{\textcircled{W}_m,\bigcirc}$ is an EP matrix, then (13), (15), (17) hold, i.e.,

$$K^* \triangledown K = \left(K(\Sigma K)^{\textcircled{W}_m}\right)^{\dagger} K(\Sigma K)^{\textcircled{W}_m},\ L^* \triangledown K = 0 \text{ and } \triangledown K = 0,$$

where $\triangledown = K(\Sigma K)^{\textcircled{W}_m}\left(K(\Sigma K)^{\textcircled{W}_m}\right)^{\dagger}$. On the contrary, if

$$K^* \triangledown K = \left(K(\Sigma K)^{\textcircled{W}_m}\right)^{\dagger} K(\Sigma K)^{\textcircled{W}_m},\ L^* \triangledown K = 0 \text{ and } \triangledown K = 0,$$

then items (13)-(16) hold. This implies $A^{\textcircled{W}_m,\bigcirc}$ is an EP matrix. □

Theorem 4.6. *Let $A \in \mathbb{C}_{n\times n}$ with $\mathrm{Ind}(A) = k$ and $m \in \mathbb{Z}^+$. If $A^{\textcircled{W}_m,\bigcirc}$ is an EP matrix, then the following statements hold:*
(1) $[LL^*, \triangledown] = 0$;
(2) $[KK^*, \triangledown] = 0$;
(3) $\triangledown K = K\left(K(\Sigma K)^{\textcircled{W}_m}\right)^{\dagger} K(\Sigma K)^{\textcircled{W}_m}$,
where $[A\ B] = AB - BA$ and $\triangledown = K(\Sigma K)^{\textcircled{W}_m}\left(K(\Sigma K)^{\textcircled{W}_m}\right)^{\dagger}$.

Proof. According to Theorem 4.5, if $A^{\textcircled{W}_m,\bigcirc}$ is an EP matrix, we have items (13)-(16) hold. Pre-multiplying by K and post-multiplying by L^* on (14), and pre-multiplying by L and post-multiplying by K^* on (15), we have

$$KK^*K(\Sigma K)^{\textcircled{W}_m}\left(K(\Sigma K)^{\textcircled{W}_m}\right)^{\dagger} LL^* = 0, \text{ and } LL^*K(\Sigma K)^{\textcircled{W}_m}\left(K(\Sigma K)^{\textcircled{W}_m}\right)^{\dagger} KK^* = 0.$$

By $KK^* + LL^* = I_r$, we have

$$(I_r - LL^*)K(\Sigma K)^{\textcircled{W}_m}\left(K(\Sigma K)^{\textcircled{W}_m}\right)^{\dagger} LL^* = 0, \text{ and } (I_r - KK^*)K(\Sigma K)^{\textcircled{W}_m}\left(K(\Sigma K)^{\textcircled{W}_m}\right)^{\dagger} KK^* = 0,$$

which can be also written as

$$\begin{aligned}
&K(\Sigma K)^{\textcircled{W}_m}\left(K(\Sigma K)^{\textcircled{W}_m}\right)^{\dagger} LL^* \\
&\qquad = LL^*K(\Sigma K)^{\textcircled{W}_m}\left(K(\Sigma K)^{\textcircled{W}_m}\right)^{\dagger} LL^* \\
&\qquad = LL^*K(\Sigma K)^{\textcircled{W}_m}\left(K(\Sigma K)^{\textcircled{W}_m}\right)^{\dagger} LL^* + LL^*K(\Sigma K)^{\textcircled{W}_m}\left(K(\Sigma K)^{\textcircled{W}_m}\right)^{\dagger} KK^* \\
&\qquad = LL^*K(\Sigma K)^{\textcircled{W}_m}\left(K(\Sigma K)^{\textcircled{W}_m}\right)^{\dagger}, \\
&K(\Sigma K)^{\textcircled{W}_m}\left(K(\Sigma K)^{\textcircled{W}_m}\right)^{\dagger} KK^* \\
&\qquad = KK^*K(\Sigma K)^{\textcircled{W}_m}\left(K(\Sigma K)^{\textcircled{W}_m}\right)^{\dagger} KK^* \\
&\qquad = KK^*K(\Sigma K)^{\textcircled{W}_m}\left(K(\Sigma K)^{\textcircled{W}_m}\right)^{\dagger} KK^* + KK^*K(\Sigma K)^{\textcircled{W}_m}\left(K(\Sigma K)^{\textcircled{W}_m}\right)^{\dagger} LL^* \\
&\qquad = KK^*K(\Sigma K)^{\textcircled{W}_m}\left(K(\Sigma K)^{\textcircled{W}_m}\right)^{\dagger}.
\end{aligned}$$

So we obtain that

$$\left[LL^*, K(\Sigma K)^{\textcircled{W}_m}\left(K(\Sigma K)^{\textcircled{W}_m}\right)^{\dagger}\right] = 0, \text{ and } \left[KK^*, K(\Sigma K)^{\textcircled{W}_m}\left(K(\Sigma K)^{\textcircled{W}_m}\right)^{\dagger}\right] = 0.$$

Pre-multiplying (13) by K and pre-multiplying (15) by L, and adding the results, then we have

$$K(\Sigma K)^{\textcircled{W}_m}\left(K(\Sigma K)^{\textcircled{W}_m}\right)^{\dagger} K = K\left(K(\Sigma K)^{\textcircled{W}_m}\right)^{\dagger} K(\Sigma K)^{\textcircled{W}_m}.$$

This finishes the proof. □

5. Representations of the *m*-MP weak core inverse

Lemma 5.1. *[31] Let $A \in \mathbb{C}_{m\times n}$, $X \in \mathbb{C}_{n\times p}$ and $Y \in \mathbb{C}_{p\times m}$. If $A^{(2)}_{R(XY),N(XY)}$ exists, then*

$$A^{(2)}_{R(XY),N(XY)} = \lim_{\lambda\to 0} X(\lambda I_p + YAX)^{-1}Y.$$

Lemma 5.2. *[6, 23] Let $A \in \mathbb{C}_{m\times n}$ with $\mathrm{Ind}(A) = k$. Then*
(1) $A^{\dagger} = \int_0^{\infty} A^{*}\exp(-AA^{*}t)\,dt$;
(2) $A^{\dagger} = \lim_{t\to 0}(tI_n + A^{*}A)^{-1}A^{*}$.

Lemma 5.3. *[26] Let $A \in \mathbb{C}_{m\times n}$ with $\mathrm{Ind}(A) = k$. If $A = B_1C_1$ is a full-rank decomposition and $C_iB_i = B_{i+1}C_{i+1}$, $(i = 1, 2, \dots, k-1)$ are also full-rank decompositions. Then*
(1) C_kB_k is invertible;
(2) $A^{\dagger} = C_1^{*}\left(C_1C_1^{*}\right)^{-1}\left(B_1^{*}B_1\right)^{-1}B_1^{*}$.

Lemma 5.4. *[9] Let $A \in \mathbb{C}_{n\times n}$ with $\mathrm{Ind}(A) = k$ and $m \in \mathbb{Z}^{+}$. Let A be the full-rank decomposition as in Lemma 5.3, then*

$$\begin{aligned} A^{\text{Ⓦ}_m} &= B\left(B(C_kB_k)^{m+1}\right)^{\dagger}(B_1C_1)^{m} \\ &= B(C_kB_k)^{-m-1}\left(B^{*}B\right)^{-1}B^{*}(B_1C_1)^{m}, \end{aligned}$$

where $B = B_1B_2 \dots B_k$.

Theorem 5.5. *Let $A \in \mathbb{C}_{n\times n}$ with $\mathrm{Ind}(A) = k$ and $m \in \mathbb{Z}^{+}$. Let A be the full-rank decomposition as in Lemma 5.3, then*

$$A^{\text{Ⓦ}_m,\bigcirc} = C_1^{*}\left(C_1C_1^{*}\right)^{-1}C_1B\left(B(C_kB_k)^{m+1}\right)^{\dagger}(B_1C_1)^{m}B_1\left(B_1^{*}B_1\right)^{-1}B_1^{*} \tag{18}$$

$$= C_1^{*}\left(C_1C_1^{*}\right)^{-1}B_2\dots B_k(C_kB_k)^{-m}\left(B^{*}B\right)^{-1}B^{*}(B_1C_1)^{m}B_1\left(B_1^{*}B_1\right)^{-1}B_1^{*}, \tag{19}$$

where $B = B_1B_2 \dots B_k$.

Proof. Using Lemma 5.3 and Lemma 5.4, we have

$$\begin{aligned} A^{\text{Ⓦ}_m,\bigcirc} &= C_1^{*}\left(C_1C_1^{*}\right)^{-1}\left(B_1^{*}B_1\right)^{-1}B_1^{*}B_1C_1B(C_kB_k)^{-m-1}\left(B^{*}B\right)^{-1}B^{*}(B_1C_1)^{m}B_1C_1C_1^{*}\left(C_1C_1^{*}\right)^{-1}\left(B_1^{*}B_1\right)^{-1}B_1^{*} \\ &= C_1^{*}\left(C_1C_1^{*}\right)^{-1}C_1B(C_kB_k)^{-m-1}\left(B^{*}B\right)^{-1}B^{*}(B_1C_1)^{m}B_1\left(B_1^{*}B_1\right)^{-1}B_1^{*} \\ &= C_1^{*}(C_1C_1^{*})^{-1}B_2\dots B_kC_kB_k(C_kB_k)^{-m-1}\left(B^{*}B\right)^{-1}B^{*}(B_1C_1)^{m}B_1\left(B_1^{*}B_1\right)^{-1}B_1^{*} \\ &= C_1^{*}\left(C_1C_1^{*}\right)^{-1}B_2\dots B_k(C_kB_k)^{-m}\left(B^{*}B\right)^{-1}B^{*}(B_1C_1)^{m}B_1\left(B_1^{*}B_1\right)^{-1}B_1^{*}. \end{aligned}$$

We can get (18) in the same way as above. □

Theorem 5.6. *Let $A \in \mathbb{C}_{n\times n}$ with $\mathrm{Ind}(A) = k$ and $m \in \mathbb{Z}^{+}$. Let A be the full-rank decomposition as in Lemma 5.3, then*

$$A^{\text{Ⓦ}_m,\bigcirc} = C_1^{*}\left(C_1C_1^{*}\right)^{-1}C_1B\int_0^{\infty}\left(B\left(C_kB_k\right)^{m+1}\right)^{*}\exp\left(-B\left(C_kB_k\right)^{m+1}\left(B\left(C_kB_k\right)^{m+1}\right)^{*}v\right)dv(B_1C_1)^{m}B_1\left(B_1^{*}B_1\right)^{-1}B_1^{*}.$$

Proof. Using Lemma 5.2(1), we can see

$$\left(B\left(C_kB_k\right)^{m+1}\right)^{\dagger} = \int_0^{\infty}\left(B\left(C_kB_k\right)^{m+1}\right)^{*}\exp\left(-B\left(C_kB_k\right)^{m+1}\left(B\left(C_kB_k\right)^{m+1}\right)^{*}v\right)dv,$$

The rest can be proved by (18). □

Theorem 5.7. *Let $A \in \mathbb{C}_{n\times n}$, $l \geqslant k = \mathrm{Ind}(A)$ and $m \in \mathbb{Z}^+$. Then*

$$A^{\circledneedle_m,\bigcirc} = \int_0^\infty A^* \exp(-AA^*t)\,dt \int_0^\infty A^{l+1}\left(A^{l+m+1}\right)^* \exp\left(-A^{l+m+1}\left(A^{l+m+1}\right)^* u\right) du \int_0^\infty A^{m+1}A^* \exp(-AA^*t)\,dt.$$

Proof. From Theorem 2.1 of [20], i.e., $A^{\circledneedle_m} = A^l\left(A^{l+m+1}\right)^\dagger A^m$, we gather a new representation of the m-MP weak core inverse

$$A^{\circledneedle_m,\bigcirc} = A^\dagger A^{l+1}\left(A^{l+m+1}\right)^\dagger A^{m+1}A^\dagger.$$

It follows from Lemma 5.2(1) that

$$\left(A^{l+m+1}\right)^\dagger = \int_0^\infty \left(A^{l+m+1}\right)^* \exp(-A^{l+m+1}\left(A^{l+m+1}\right)^* u)\,du.$$

The proof can be completed. □

Theorem 5.8. *Let $A \in \mathbb{C}_{n\times n}$ with $\mathrm{Ind}(A) = k$ and $m \in \mathbb{Z}^+$. Let A be the full-rank decomposition as in Lemma 5.3, then*

$$A^{\circledneedle_m,\bigcirc} = \lim_{\lambda\to 0} C_1^*\left(C_1C_1^*\right)^{-1} C_1B\left(\lambda I_n + \left(B\left(C_kB_k\right)^{m+1}\right)^* B\left(C_kB_k\right)^{m+1}\right)^{-1}\left(B\left(C_kB_k\right)^{m+1}\right)^*\left(B_1C_1\right)^m B_1\left(B_1^*B_1\right)^{-1}B_1^*.$$

Proof. Using Lemma 5.2(2), we have

$$\left(B\left(C_kB_k\right)^{m+1}\right)^\dagger = \lim_{\lambda\to 0}\left(\lambda I_n + \left(B\left(C_kB_k\right)^{m+1}\right)^* B\left(C_kB_k\right)^{m+1}\right)^{-1}\left(B\left(C_kB_k\right)^{m+1}\right)^*.$$

The rest can be demonstrated by (18). □

Theorem 5.9. *Let $A \in \mathbb{C}_{n\times n}$, $l \geqslant k = \mathrm{Ind}(A)$ and $m \in \mathbb{Z}^+$. Then*

$$A^{\circledneedle_m,\bigcirc} = \lim_{t\to 0}\left(tI_n + A^*A\right)^{-1}A^* \lim_{\alpha\to 0} A^{l+1}\left(\alpha I_n + \left(A^{l+m+1}\right)^* A^{l+m+1}\right)^{-1}\left(A^{l+m+1}\right)^* A^{m+1} \lim_{t\to 0}\left(tI_n + A^*A\right)^{-1}A^*.$$

Proof. It follows from Lemma 5.2(2) that

$$\left(A^{l+m+1}\right)^\dagger = \lim_{\alpha\to 0}\left(\alpha I_n + \left(A^{l+m+1}\right)^* A^{l+m+1}\right)^{-1}\left(A^{l+m+1}\right)^*.$$

Since $A^{\circledneedle_m,\bigcirc} = A^\dagger A^{l+1}\left(A^{l+m+1}\right)^\dagger A^{m+1}A^\dagger$, the proof can be finished. □

Theorem 5.10. *Let $A \in \mathbb{C}_{m\times n}$ with $\mathrm{Ind}(A) = k$. Then*

(1) $A^{\circledneedle_m,\bigcirc} = \lim_{\lambda\to 0}\left(\lambda I_n + A^*A\right)^{-1}A^*A^k\left(\lambda I_n + \left(A^k\right)^* A^{m+k}\right)^{-1}\left(A^k\right)^* A^{m+1}\left(\lambda I_n + A^*A\right)^{-1}A^*$;

(2) $A^{\circledneedle_m,\bigcirc} = \lim_{\lambda\to 0}\left(\lambda I_n + A^*A\right)^{-1}A^*A^k\left(A^k\right)^*\left(\lambda I_n + A^{m+k}\left(A^k\right)^*\right)^{-1}A^{m+1}\left(\lambda I_n + A^*A\right)^{-1}A^*$;

(3) $A^{\circledneedle_m,\bigcirc} = \lim_{\lambda\to 0}\left(\lambda I_n + A^*A\right)^{-1}A^*A^k\left(A^k\right)^* A^{m+1}\left(\lambda I_n + \left(\lambda I_n + A^*A\right)^{-1}A^*A^k\left(A^k\right)^* A^{m+1}\right)^{-1}\left(\lambda I_n + A^*A\right)^{-1}A^*$;

(4) $A^{\circledneedle_m,\bigcirc} = \lim_{\lambda\to 0}\left(\lambda I_n + \left(\lambda I_n + A^*A\right)^{-1}A^*A^k\left(A^k\right)^* A^{m+1}\right)^{-1}\left(\lambda I_n + A^*A\right)^{-1}A^*A^k\left(A^k\right)^* A^{m+1}\left(\lambda I_n + A^*A\right)^{-1}A^*$.

Proof. Firstly, we have $A^{\circledneedle_m,\bigcirc} = A^{(2)}_{R\left(A^\dagger A^k\left(A^k\right)^* A^{m+1}A^\dagger\right),N\left(A^\dagger A^k\left(A^k\right)^* A^{m+1}A^\dagger\right)}$ by Theorem 3.2. Combining Lemma 5.1 and Lemma 5.2 (2), we have the following proof.

(1) If $X = A^{\dagger}A^k$, $Y = \left(A^k\right)^* A^{m+1}A^{\dagger}$. Then

$$\begin{aligned}
A^{\textcircled{w}_m,\bigcirc} &= \lim_{\lambda\to 0} A^{\dagger}A^k\left(\lambda I_n + \left(A^k\right)^* A^{m+1}A^{\dagger}AA^{\dagger}A^k\right)^{-1}\left(A^k\right)^* A^{m+1}A^{\dagger}\\
&= \lim_{\lambda\to 0} A^{\dagger}A^k\left(\lambda I_n + \left(A^k\right)^* A^{m+k}\right)^{-1}\left(A^k\right)^* A^{m+1}A^{\dagger}\\
&= \lim_{\lambda\to 0} \left(\lambda I_n + A^*A\right)^{-1}A^*A^k\left(\lambda I_n + \left(A^k\right)^* A^{m+k}\right)^{-1}\left(A^k\right)^* A^{m+1}\left(\lambda I_n + A^*A\right)^{-1}A^*.
\end{aligned}$$

(2) If $X = A^{\dagger}A^k\left(A^k\right)^*$, $Y = A^{m+1}A^{\dagger}$, then we can get the result by similar computation.

(3) If $X = A^{\dagger}A^k\left(A^k\right)^* A^{m+1}$, $Y = A^{\dagger}$, then we can get the result.

(4) If $X = I_n$, $Y = A^{\dagger}A^k\left(A^k\right)^* A^{m+1}A^{\dagger}$, then we can get the result. □

6. Successive matrix squaring algorithm for the *m*-MP weak core inverse

Inspired by the successive matrix squaring (SMS) algorithm for general outer inverses given in [24], we infer SMS algorithm for computing the m-MP weak core inverse in the section.

Since

$$\begin{aligned}
\left(A^k\right)^* A^{m+1}A^{\textcircled{w}_m,\bigcirc} &= \left(A^k\right)^* A^{m+1}A^{\dagger}AA^{\textcircled{w}_m}AA^{\dagger}\\
&= \left(A^k\right)^* A^{m+1}\left(A^{\textcircled{\dagger}}\right)^{m+1}A^{m+1}A^{\dagger}\\
&= \left(A^k\right)^* A^{m+1}\left(A^DA^k\left(A^k\right)^{\dagger}\right)^{m+1}A^{m+1}A^{\dagger}\\
&= \left(A^k\right)^* A^{m+1}\left(A^D\right)^{m+1}A^k\left(A^k\right)^{\dagger}A^{m+1}A^{\dagger}\\
&= \left(A^k\right)^* A^k\left(A^k\right)^{\dagger}A^{m+1}A^{\dagger}\\
&= \left(A^k\right)^* A^{m+1}A^{\dagger},
\end{aligned}$$

we have that

$$\begin{aligned}
A^{\textcircled{w}_m,\bigcirc} &= A^{\textcircled{w}_m,\bigcirc} - \beta\left(\left(A^k\right)^* A^{m+1}A^{\textcircled{w}_m,\bigcirc} - \left(A^k\right)^* A^{m+1}A^{\dagger}\right)\\
&= \left(I - \beta\left(A^k\right)^* A^{m+1}\right)A^{\textcircled{w}_m,\bigcirc} + \beta\left(A^k\right)^* A^{m+1}A^{\dagger}.
\end{aligned}$$

Observe the following matrices

$$P = I - \beta\left(A^k\right)^* A^{m+1},\ Q = \beta\left(A^k\right)^* A^{m+1}A^{\dagger},\ \beta > 0.$$

It is obvious that $A^{\textcircled{w}_m,\bigcirc}$ is the unique solution of $X = PX + Q$. Then an iterative procedure for computing $A^{\textcircled{w}_m,\bigcirc}$ can be defined as follows

$$\begin{cases} X_1 = Q\\ X_{m+1} = PX_m + Q. \end{cases} \tag{20}$$

This algorithm can be implemented in parallel by considering the block matrices

$$T = \begin{pmatrix} P & Q\\ 0 & I \end{pmatrix},\ T^m = \begin{pmatrix} P^m & \sum\limits_{i=0}^{m-1} P^iQ\\ 0 & I \end{pmatrix}.$$

The upper right block of T^m defines the mth approximation X_m to $A^{\textcircled{w}_m,\bigcirc}$, i.e., $X_m = \sum_{i=0}^{m-1} P^i Q$. The matrix power T^m can be computed by the successive squaring, i.e.,

$$T_0 = T,\ T_{i+1} = T_i^2,\ i = 0, 1, \ldots, j,$$

where the integer j satisfies $2^j \geqslant m$.

The following theorem gives the sufficient condition for the convergence of the iterative process (20).

Theorem 6.1. *Let $m \in \mathbb{Z}^+$, $A \in \mathbb{C}_{n\times n}$ with $\mathrm{Ind}(A) = k$ and $\mathrm{rank}(A^k) = r$. Then the approximation*

$$X_{2^m} = \sum_{i=0}^{2^m-1} \left(I - \beta \left(A^k\right)^* A^{m+1}\right)^i \beta \left(A^k\right)^* A^{m+1} A^\dagger,$$

generated by (20), converges to $A^{\textcircled{w}_m,\bigcirc}$ if the spectral radius $\rho(I - X_1 A) \leqslant 1$. In addition, the following error estimation holds

$$\frac{\|A^{\textcircled{w}_m,\bigcirc} - X_{2^m}\|}{\|A^{\textcircled{w}_m,\bigcirc}\|} \leqslant \|\,(I - X_1 A)^{2^m}\,\|.$$

Besides,

$$\lim_{m\to\infty} sup \sqrt[2^m]{\|A^{\textcircled{w}_m,\bigcirc} - X_{2^m}\|} \leqslant \rho(I - X_1 A).$$

Proof. It is known that $A^{\textcircled{w}_m,\bigcirc} A A^{\textcircled{w}_m,\bigcirc} = A^{\textcircled{w}_m,\bigcirc}$. Next, we prove that $X_{2^m} A A^{\textcircled{w}_m,\bigcirc} = X_{2^m}$. We can see that

$$\begin{aligned}
X_{2^m} A A^{\textcircled{w}_m,\bigcirc} &= \sum_{i=0}^{2^m-1} \left(I - \beta \left(A^k\right)^* A^{m+1}\right)^i \beta \left(A^k\right)^* A^{m+1} A^\dagger A A^{\textcircled{w}_m,\bigcirc} \\
&= \sum_{i=0}^{2^m-1} \left(I - \beta \left(A^k\right)^* A^{m+1}\right)^i \beta \left(A^k\right)^* A^{m+1} A^{\textcircled{w}_m,\bigcirc} \\
&= \sum_{i=0}^{2^m-1} \left(I - \beta \left(A^k\right)^* A^{m+1}\right)^i \beta \left(A^k\right)^* A^{m+1} A^\dagger \\
&= X_{2^m}.
\end{aligned}$$

By the mathematical induction, we can get

$$I - X_{2^m} A = (I - X_1 A)^{2^m}.$$

Therefore,

$$\begin{aligned}
\|A^{\textcircled{w}_m,\bigcirc} - X_{2^m}\| = \|A^{\textcircled{w}_m,\bigcirc} - X_{2^m} A A^{\textcircled{w}_m,\bigcirc}\| &= \|(I - X_{2^m} A) A^{\textcircled{w}_m,\bigcirc}\| \\
&\leqslant \|(I - X_{2^m} A)\| \|A^{\textcircled{w}_m,\bigcirc}\| \\
&= \|(I - X_1 A)^{2^m}\| \|A^{\textcircled{w}_m,\bigcirc}\|.
\end{aligned}$$

On the basis of $\lim_{n\to\infty} \|B^n\|^{\frac{1}{n}} = \rho(B)$ for any square matrix B and any norm, it can be concluded that

$$\lim_{m\to\infty} sup \sqrt[2^m]{\|A^{\textcircled{w}_m,\bigcirc} - X_{2^m}\|} \leqslant \lim_{m\to\infty} sup \sqrt[2^m]{\|(I - X_1 A)^{2^m}\| \|A^{\textcircled{w}_m,\bigcirc}\|} = \rho(I - X_1 A).$$

If β is a real parameter such that $\max_{1\leqslant i\leqslant s} |1 - \beta\lambda_i| \leqslant 1$, where λ_i ($i = 1, 2, \ldots, s$) are the nonzero eigenvalues of $\left(A^k\right)^* A^{m+1}$, then

$$\rho(I - X_1 A) = \rho\left(I - \beta \left(A^k\right)^* A^{m+1}\right) < 1.$$

This finishes the proof. □

Example 6.2. *Consider the following matrix:*

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then, $k = \text{Ind}(A) = 3$. *Let*

$$m = 2,\ P = I - \beta \left(A^3\right)^* A^3,\ Q = \beta \left(A^3\right)^* A^3 A^\dagger,\ \beta = 0.1.$$

The eigenvalues λ_i *of* QA *are included in the set* $\{0.2, 0, 0, 0\}$. *The nonzero eigenvalues* λ_i *satisfy*

$$\max_i |1 - \beta\lambda_i| = |1 - 0.02| = 0.98 < 1.$$

Then we obtain the satisfactory approximation for $A^{\textcircled{w}_m,\bigcirc}$ *after the 6th iteration of the successive matrix squaring algorithm.*

$$T_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The upper right corner of T_6 *is an approximation of* $A^{\textcircled{w}_m,\bigcirc}$, *that is*

$$A^{\textcircled{w}_m,\bigcirc} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

7. The *m*-MP weak core inverse in certain bordered matrices

It is known that $X = A^{-1}$ is the unique solution to the following rank equality, if A is a nonsingular matrix

$$\text{rank}\begin{pmatrix} A & I \\ I & X \end{pmatrix} = \text{rank}(A). \tag{21}$$

Our intention in this section is to propose a generation of the property (21) to the m-MP weak core inverse. Results of Lemma 7.1 will be useful in verifying this result.

Lemma 7.1. *Let* $A \in \mathbb{C}_{n\times n}$, $M = \begin{pmatrix} A & AU \\ VA & B \end{pmatrix} \in \mathbb{C}_{2n\times 2n}$, *then*

$$\text{rank}(M) = \text{rank}(A) + \text{rank}(B - VAU).$$

Theorem 7.2. *Let* $m \in \mathbb{Z}^+$, $A \in \mathbb{C}_{n\times n}$ *with* $\text{Ind}(A) = k$ *and* $\text{rank}(A^k) = r$. *Then there exists a unique matrix* X *satisfies the conditions*

$$XA^k = 0,\ X^2 = X,\ \left(A^k\right)^* A^{m+1} A^\dagger X = 0, \text{rank}(X) = n - r, \tag{22}$$

as well as a certain matrix Y such that

$$YA^{\dagger}A^{k}=0,\ Y^{2}=Y,\ \left(A^{k}\right)^{*}A^{m+1}Y=0,\ \operatorname{rank}(Y)=n-r, \tag{23}$$

and a unique Z satisfying

$$\operatorname{rank}\begin{pmatrix} A & I-X \\ I-Y & Z \end{pmatrix}=\operatorname{rank}(A). \tag{24}$$

Furthermore, $Z=A^{\textcircled{w}_m,\bigcirc}$ and the matrices X, Y are defines as

$$X=I-AA^{\textcircled{w}_m,\bigcirc},\ Y=I-A^{\textcircled{w}_m,\bigcirc}A.$$

Proof. Suppose that A is expressed by Lemma 4.1. Applying Theorem 4.3, we can check that

$$X=I-AA^{\textcircled{w}_m,\bigcirc}=U\begin{pmatrix} 0 & -T^{-m}T_mNN^{\dagger} \\ 0 & I \end{pmatrix}U^{*}$$

satisfies (22). In order to show the uniqueness of X, we assume that X_0 is another matrix which fulfils (22). Let $X_1=UX_0U^{*}$ be bordered as

$$X_1=\begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix},$$

where D_1 is the $r\times r$ block. On the basis of $XA^{k}=0$ and using the invertibility of T, we gather

$$\begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix}\begin{pmatrix} T^{k} & T_k \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} D_1T^{k} & D_1T_k \\ D_3T^{k} & D_3T_k \end{pmatrix}=0,$$

which implies $D_1=0$ and $D_3=0$. Since $X^{2}=X$ and $\operatorname{rank}(X)=n-r$, we can see

$$\begin{pmatrix} 0 & D_2 \\ 0 & D_4 \end{pmatrix}\begin{pmatrix} 0 & D_2 \\ 0 & D_4 \end{pmatrix}=\begin{pmatrix} 0 & D_2D_4 \\ 0 & D_4^{2} \end{pmatrix}=\begin{pmatrix} 0 & D_2 \\ 0 & D_4 \end{pmatrix},$$

which shows that $D_2D_4=D_2$, $D_4^{2}=D_4$ and $\operatorname{rank}(D_4)=n-r$. Therefore, $D_4=I$. In addition, from $\left(A^{k}\right)^{*}A^{m+1}A^{\dagger}X=0$, i.e.,

$$\begin{aligned}
\begin{pmatrix} \left(T^{k}\right)^{*} & 0 \\ (T_k)^{*} & 0 \end{pmatrix}\begin{pmatrix} T^{m} & T_m \\ 0 & N^{m} \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & NN^{\dagger} \end{pmatrix}\begin{pmatrix} 0 & D_2 \\ 0 & I \end{pmatrix} &= \begin{pmatrix} \left(T^{k}\right)^{*} & 0 \\ (T_k)^{*} & 0 \end{pmatrix}\begin{pmatrix} T^{m} & T_mNN^{\dagger} \\ 0 & N^{m}NN^{\dagger} \end{pmatrix}\begin{pmatrix} 0 & D_2 \\ 0 & I \end{pmatrix} \\
&= \begin{pmatrix} \left(T^{k}\right)^{*} & 0 \\ (T_k)^{*} & 0 \end{pmatrix}\begin{pmatrix} 0 & T^{m}D_2+T_mNN^{\dagger} \\ 0 & N^{m}NN^{\dagger} \end{pmatrix}=\begin{pmatrix} 0 & \left(T^{k}\right)^{*}\left(T^{m}D_2+T_mNN^{\dagger}\right) \\ 0 & (T_k)^{*}\left(T^{m}D_2+T_mNN^{\dagger}\right) \end{pmatrix}=0,
\end{aligned}$$

it can be concluded that $T^{m}D_2+T_mNN^{\dagger}=0$, that is $D_2=-T^{-m}T_mNN^{\dagger}$. Hence, $X_0=X$. Similarly, we can certify that (23) is satisfied for a unique $Y=I-A^{\textcircled{w}_m,\bigcirc}A$. The matrices $X=I-AA^{\textcircled{w}_m,\bigcirc}$ and $Y=I-A^{\textcircled{w}_m,\bigcirc}A$ satisfy

$$\operatorname{rank}\begin{pmatrix} A & I-X \\ I-Y & Z \end{pmatrix}=\operatorname{rank}\begin{pmatrix} A & AA^{\textcircled{w}_m,\bigcirc} \\ A^{\textcircled{w}_m,\bigcirc}A & Z \end{pmatrix}.$$

In view of Lemma 7.1 and (24), it can be deduced that $\operatorname{rank}(Z-A^{\textcircled{w}_m,\bigcirc}AA^{\textcircled{w}_m,\bigcirc})=\operatorname{rank}(Z-A^{\textcircled{w}_m,\bigcirc})=0$, which implies $Z=A^{\textcircled{w}_m,\bigcirc}$. This finishes the proof. □

Example 7.3. *There is an example to illustrate the results of Theorem 7.2. Consider*

$$A=\begin{pmatrix}2&1&1&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&0&0&0&0\end{pmatrix}.$$

Then $\operatorname{rank}(A)=4$, $k=\operatorname{Ind}(A)=4$, $r=\operatorname{rank}(A^4)=1$. *Let* $m=2$, *then*

$$A^{\textcircled{w}_2,\bigcirc}=\begin{pmatrix}\frac{2}{5}&\frac{1}{5}&\frac{3}{10}&\frac{1}{10}&0\\\frac{1}{5}&\frac{1}{10}&\frac{3}{20}&\frac{1}{20}&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}.$$

The block matrix

$$B=\begin{pmatrix}A&I-X\\I-Y&Z\end{pmatrix}=\begin{pmatrix}A&AA^{\textcircled{w}_2,\bigcirc}\\A^{\textcircled{w}_2,\bigcirc}A&A^{\textcircled{w}_2,\bigcirc}\end{pmatrix}=\begin{pmatrix}2&1&1&0&0&1&\frac{1}{2}&\frac{3}{4}&\frac{1}{4}&0\\0&0&1&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0&0\\0&0&0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\\frac{4}{5}&\frac{2}{5}&\frac{3}{5}&\frac{3}{10}&\frac{1}{10}&\frac{2}{5}&\frac{1}{5}&\frac{3}{10}&\frac{1}{10}&0\\\frac{2}{5}&\frac{1}{5}&\frac{3}{10}&\frac{3}{20}&\frac{1}{20}&\frac{1}{5}&\frac{1}{10}&\frac{3}{20}&\frac{1}{20}&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0&0&0\end{pmatrix}$$

satisfies $\operatorname{rank}(B)=\operatorname{rank}(A)=4$. *In addition, the matrices*

$$X=I-AA^{\textcircled{w}_2,\bigcirc}=\begin{pmatrix}0&-\frac{1}{2}&-\frac{3}{4}&-\frac{1}{4}&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix},\ \operatorname{rank}(X)=4$$

and

$$Y=I-A^{\textcircled{w}_2,\bigcirc}A=\begin{pmatrix}\frac{1}{5}&-\frac{2}{5}&-\frac{3}{5}&-\frac{3}{10}&-\frac{1}{10}\\-\frac{2}{5}&\frac{4}{5}&-\frac{3}{10}&-\frac{3}{20}&-\frac{1}{20}\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix},\ \operatorname{rank}(Y)=4$$

satisfy (22) *and* (23), *respectively.*

8. Continuity of the *m*-MP weak core inverse

In this part, we give a necessary and sufficient condition for the continuity of the *m*-MP weak core inverse. In order to do this, we firstly recall the known results about the continuity of the Moore-Penrose inverse and the *m*-weak group inverse.

Lemma 8.1. *Let A, $A_p \in \mathbb{C}_{n\times n}$ and $m, p \in \mathbb{Z}^+$ satisfy $A_p \to A$ as $p \to \infty$. Then*
(1) [25] $A_p^{\dagger} \to A^{\dagger}$ as $p \to \infty$ if and only if there is $p_0 \in \mathbb{Z}^+$ such that $\operatorname{rank}(A_p) = \operatorname{rank}(A)$ for $p \geqslant p_0$;
(2) [20] $A_p^{\textcircled{W}_m} \to A^{\textcircled{W}_m}$ as $p \to \infty$ if and only if there is $p_0 \in \mathbb{Z}^+$ such that $\operatorname{rank}(A_p^l) = \operatorname{rank}(A^l)$ for $p \geqslant p_0$ and $l = \max\{\operatorname{Ind}(A), \operatorname{Ind}(A_p), \operatorname{Ind}(A_{p+1}), \ldots\}$.

Theorem 8.2. *Let A, $A_p \in \mathbb{C}_{n\times n}$ and m, $p \in \mathbb{Z}^+$ satisfy $A_p \to A$ as $p \to \infty$. Then $A_p^{\textcircled{W}_m,\bigcirc} \to A^{\textcircled{W}_m,\bigcirc}$ as $p \to \infty$ if and only if there is $p_0 \in \mathbb{Z}^+$ such that $\operatorname{rank}(A_p) = \operatorname{rank}(A)$ and $\operatorname{rank}(A_p^l) = \operatorname{rank}(A^l)$, for $p \geqslant p_0$ and $l = \max\{\operatorname{Ind}(A), \operatorname{Ind}(A_p), \operatorname{Ind}(A_{p+1}), \ldots\}$.*

Proof. "$\Leftarrow$": For $p \geqslant p_0 \in \mathbb{Z}^+$ and $l = \max\{\operatorname{Ind}(A), \operatorname{Ind}(A_p), \operatorname{Ind}(A_{p+1}), \ldots\}$, if

$$\operatorname{rank}(A_p) = \operatorname{rank}(A),\ \operatorname{rank}(A_p^l) = \operatorname{rank}(A^l),$$

if follows from Lemma 8.1 that

$$A_p^{\dagger} \to A^{\dagger},\ A_p^{\textcircled{W}_m} \to A^{\textcircled{W}_m}.$$

As a consequence,

$$A_p^{\textcircled{W}_m,\bigcirc} = A_p^{\dagger} A_p A_p^{\textcircled{W}_m} A_p A_p^{\dagger} \to A^{\dagger} A A^{\textcircled{W}_m} A A^{\dagger} = A^{\textcircled{W}_m,\bigcirc}.$$

"$\Rightarrow$": If $A_p^{\textcircled{W}_m,\bigcirc} \to A^{\textcircled{W}_m,\bigcirc}$ and $A_p \to A$, then

$$A_p A_p^{\textcircled{W}_m,\bigcirc} \to A A^{\textcircled{W}_m,\bigcirc} \text{ as } p \to \infty.$$

From [26], we realize the existence of $p_0 \in \mathbb{Z}^+$, which satisfies

$$\operatorname{rank}(A_p A_p^{\textcircled{W}_m,\bigcirc}) = \operatorname{rank}(A A^{\textcircled{W}_m,\bigcirc}) \text{ for } p \geqslant p_0.$$

For $p \geqslant p_0$ and $l = \max\{\operatorname{Ind}(A), \operatorname{Ind}(A_p), \operatorname{Ind}(A_{p+1}), \ldots\}$, we have

$$R(A_p A_p^{\textcircled{W}_m,\bigcirc}) = R(A_p^l),\ \operatorname{rank}(A A^{\textcircled{W}_m,\bigcirc}) = R(A^l).$$

Hence, for $p \geqslant p_0$, we can deduce that

$$\operatorname{rank}(A_p^{l+m+1}) = \operatorname{rank}(A_p^l) = \operatorname{rank}(A_p A_p^{\textcircled{W}_m,\bigcirc}) = \operatorname{rank}(A A^{\textcircled{W}_m,\bigcirc}) = \operatorname{rank}(A^l) = \operatorname{rank}(A^{l+m+1}),$$

which, together with Lemma 8.1, implies

$$\left(A_p^{l+m+1}\right)^{\dagger} \to \left(A^{l+m+1}\right)^{\dagger} \text{ as } p \to \infty.$$

To confirm $\operatorname{rank}(A_p) = \operatorname{rank}(A)$, we assume that $A_p^{\dagger} \nrightarrow A^{\dagger}$ firstly. Then, by

$$A_p^{l+1} \to A^{l+1},\ \left(A_p^{l+m+1}\right)^{\dagger} \to \left(A^{l+m+1}\right)^{\dagger},\ A_p^{m+1} \to A^{m+1},\ A_p^{\dagger} \nrightarrow A^{\dagger},$$

we can infer

$$A_p^{\textcircled{W}_m,\bigcirc} = A_p^{\dagger} A_p^{l+1} \left(A_p^{l+m+1}\right)^{\dagger} A_p^{m+1} A_p^{\dagger} \nrightarrow A^{\dagger} A^{l+1} \left(A^{l+m+1}\right)^{\dagger} A^{m+1} A^{\dagger} = A^{\textcircled{W}_m,\bigcirc}.$$

This is contradictory to $A_p^{\textcircled{W}_m,\bigcirc} \to A^{\textcircled{W}_m,\bigcirc}$. Therefore, we insist that $A_p^{\dagger} \to A^{\dagger}$, i.e., $\operatorname{rank}(A_p) = \operatorname{rank}(A)$. This finishes the proof. □

9. Applications

Theorem 9.1. *Let $A \in \mathbb{C}_{n\times n}$ with $\mathrm{Ind}(A) = k$ and $m \in \mathbb{Z}^+$. Then the general solution of the following consistent matrix equation*

$$\left(A^k\right)^* A^{m+1}X = \left(A^k\right)^* A^{m+1}A^\dagger B,\ B \in \mathbb{C}_{n\times n} \tag{25}$$

is expressed as

$$X = A^{\textcircled{w}_{m},\bigcirc}B + (I_n - A^{\textcircled{w}_{m},\bigcirc}A)Y,$$

where $Y \in \mathbb{C}_{n\times n}$ is arbitrary.

Proof. In Section 6, we have identified

$$\left(A^k\right)^* A^{m+1}A^{\textcircled{w}_{m},\bigcirc} = \left(A^k\right)^* A^{m+1}A^\dagger.$$

This shows that

$$\left(A^k\right)^* A^{m+1}A^{\textcircled{w}_{m},\bigcirc}A = \left(A^k\right)^* A^{m+1}.$$

Put $X = A^{\textcircled{w}_{m},\bigcirc}B + (I_n - A^{\textcircled{w}_{m},\bigcirc}A)Y$, where $Y \in \mathbb{C}_{n\times n}$. Then

$$\begin{aligned}\left(A^k\right)^* A^{m+1}X &= \left(A^k\right)^* A^{m+1}\left(A^{\textcircled{w}_{m},\bigcirc}B + (I_n - A^{\textcircled{w}_{m},\bigcirc}A)Y\right)\\ &= \left(A^k\right)^* A^{m+1}A^{\textcircled{w}_{m},\bigcirc}B + \left(A^k\right)^* A^{m+1}Y - \left(A^k\right)^* A^{m+1}A^{\textcircled{w}_{m},\bigcirc}AY\\ &= \left(A^k\right)^* A^{m+1}A^\dagger B.\end{aligned}$$

Furthermore, by Theorem 5.1 of [9], i.e., $A^{\textcircled{w}_{m}} = \left(A^D\right)^{m+1} A^k \left(A^k\right)^\dagger A^m$, we deduce

$$A^{\textcircled{w}_{m},\bigcirc} = A^\dagger A\left(A^D\right)^{m+1} A^k\left(A^k\right)^\dagger A^{m+1}A^\dagger.$$

Thus,

$$\begin{aligned}A^{\textcircled{w}_{m},\bigcirc}B &= A^\dagger A\left(A^D\right)^{m+1} A^k\left(A^k\right)^\dagger A^{m+1}A^\dagger B\\ &= A^\dagger A\left(A^D\right)^{m+1} A^k\left(A^k\right)^\dagger A^k\left(A^k\right)^\dagger A^{m+1}A^\dagger B\\ &= A^\dagger A\left(A^D\right)^{m+1} A^k\left(A^k\right)^\dagger \left(\left(A^k\right)^\dagger\right)^* \left(A^k\right)^* A^k\left(A^k\right)^\dagger A^{m+1}A^\dagger B\\ &= A^\dagger A\left(A^D\right)^{m+1} \left(\left(A^k\right)^\dagger\right)^* \left(A^k\right)^* A^{m+1}A^\dagger B\\ &= A^\dagger A\left(A^D\right)^{m+1} \left(\left(A^k\right)^\dagger\right)^* \left(A^k\right)^* A^{m+1}X\\ &= A^\dagger A\left(A^D\right)^{m+1} A^k\left(A^k\right)^\dagger A^{m+1}X\\ &= A^\dagger A\left(A^D\right)^{m+1} A^k\left(A^k\right)^\dagger A^{m+1}A^\dagger AX\\ &= A^{\textcircled{w}_{m},\bigcirc}AX,\end{aligned}$$

which implies

$$X = A^{\textcircled{w}_{m},\bigcirc}B + X - A^{\textcircled{w}_{m},\bigcirc}AX = A^{\textcircled{w}_{m},\bigcirc}B + (I_n - A^{\textcircled{w}_{m},\bigcirc}A)X.$$

Hence, X possess the pattern $X = A^{\textcircled{w}_{m},\bigcirc}B + (I_n - A^{\textcircled{w}_{m},\bigcirc}A)Y$, where $Y \in \mathbb{C}_{n\times n}$ is arbitrary. □

Theorem 9.2. *If the solution X of* (25) *satisfies $R(X) \subseteq R(A^{\dagger}A^{k})$, then X is unique and $X = A^{\textcircled{w}_m,\bigcirc}B$.*

Proof. It is obvious that $X = A^{\textcircled{w}_m,\bigcirc}B$ is a solution of (25) and

$$R(X) = R(A^{\textcircled{w}_m,\bigcirc}B) \subseteq R(A^{\textcircled{w}_m,\bigcirc}) = R(A^{\dagger}A^{k}).$$

For the uniqueness, we assume that X_1 and X_2 are two solutions of (25), and satisfy

$$R(X_1) \subseteq R(A^{\dagger}A^{k}),\ R(X_2) \subseteq R(A^{\dagger}A^{k}).$$

Using the facts

$$\left(A^{k}\right)^{*} A^{m+1}(X_1 - X_2) = 0 \text{ and } R(X_1 - X_2) \subseteq R(A^{\dagger}A^{k}),$$

we confirm that

$$R(X_1 - X_2) \subseteq N\left(\left(A^{k}\right)^{*} A^{m+1}\right) \cap R(A^{\dagger}A^{k}) = N(A^{\textcircled{w}_m,\bigcirc}A) \cap R(A^{\textcircled{w}_m,\bigcirc}A) = \{0\}.$$

This implies $X_1 = X_2$. Hence, (25) has uniquely determined solution $X = A^{\textcircled{w}_m,\bigcirc}B$ in $R(A^{\dagger}A^{k})$. □

Theorem 9.3. *Let $A \in \mathbb{C}_{n\times n}$ with* $\text{Ind}(A) = k$ *and $m \in \mathbb{Z}^{+}$. Suppose $X \in \mathbb{C}_{n\times m}$ and $D \in \mathbb{C}_{n\times m}$. If $R(D) \subseteq R(A^{k})$, then the restricted matrix equation*

$$AX = D,\ R(X) \subseteq R(A^{\dagger}A^{k}) \tag{26}$$

has the unique solution $X = A^{\textcircled{w}_m,\bigcirc}D$.

Proof. Since $R(D) \subseteq R(A^{k}) = R(AA^{\textcircled{w}_m,\bigcirc})$, it follows that $AA^{\textcircled{w}_m,\bigcirc}D = D$, which implies $A^{\textcircled{w}_m,\bigcirc}D$ is a solution of $AX = D$. And it is obvious that

$$R(X) = R(A^{\textcircled{w}_m,\bigcirc}D) \subseteq R(A^{\textcircled{w}_m,\bigcirc}) = R(A^{\dagger}A^{k}).$$

So $A^{\textcircled{w}_m,\bigcirc}D$ is a solution of (26). To prove the uniqueness, we assume Y is another solution of (26). Then we have $R(Y) \subseteq R(A^{\dagger}A^{k}) = R(A^{\textcircled{w}_m,\bigcirc}A)$. This implies

$$X = A^{\textcircled{w}_m,\bigcirc}D = A^{\textcircled{w}_m,\bigcirc}AY = Y.$$

This completes the proof. □

Theorem 9.4. *Let $A \in \mathbb{C}_{n\times n}$ with* $\text{Ind}(A) = k$ *and $m \in \mathbb{Z}^{+}$. Let $B \in \mathbb{C}_{n\times r}$ and $C^{*} \in \mathbb{C}_{n\times r}$ be of null column rank such that $R(B) = N\left(\left(A^{k}\right)^{*} A^{m+1}A^{\dagger}\right)$ and $N(C) = R(A^{\dagger}A^{k})$. Then the bordered matrix*

$$L = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$$

is nonsingular and its inverse is given by

$$L^{-1} = \begin{pmatrix} A^{\textcircled{w}_m,\bigcirc} & (I_n - A^{\textcircled{w}_m,\bigcirc}A)C^{\dagger} \\ B^{\dagger}(I_n - AA^{\textcircled{w}_m,\bigcirc}) & B^{\dagger}(AA^{\textcircled{w}_m,\bigcirc}A - A)C^{\dagger} \end{pmatrix}.$$

Proof. From $R(A^{\textcircled{w}_m,\bigcirc}) = R(A^{\dagger}A^{k}) = N(C)$, we have $CA^{\textcircled{w}_m,\bigcirc} = 0$. And since

$$R(I_n - AA^{\textcircled{w}_m,\bigcirc}) = N(AA^{\textcircled{w}_m,\bigcirc}) = N\left(\left(A^{k}\right)^{*} A^{m+1}A^{\dagger}\right) = R(B) = R(BB^{\dagger}),$$

it follows that $BB^\dagger(I_n - AA^{\textcircled{w}_m,\bigcirc}) = I_n - AA^{\textcircled{w}_m,\bigcirc}$.
Let

$$T = \begin{pmatrix} A^{\textcircled{w}_m,\bigcirc} & (I_n - A^{\textcircled{w}_m,\bigcirc}A)C^\dagger \\ B^\dagger(I_n - AA^{\textcircled{w}_m,\bigcirc}) & B^\dagger(AA^{\textcircled{w}_m,\bigcirc}A - A)C^\dagger \end{pmatrix},$$

then

$$\begin{aligned} LT &= \begin{pmatrix} AA^{\textcircled{w}_m,\bigcirc} + BB^\dagger(I_n - AA^{\textcircled{w}_m,\bigcirc}) & A(I_n - A^{\textcircled{w}_m,\bigcirc}A)C^\dagger + BB^\dagger(AA^{\textcircled{w}_m,\bigcirc}A - A)C^\dagger \\ CA^{\textcircled{w}_m,\bigcirc} & C(I_n - A^{\textcircled{w}_m,\bigcirc}A)C^\dagger \end{pmatrix} \\ &= \begin{pmatrix} AA^{\textcircled{w}_m,\bigcirc} + I_n - AA^{\textcircled{w}_m,\bigcirc} & A(I_n - A^{\textcircled{w}_m,\bigcirc}A)C^\dagger - (I_n - AA^{\textcircled{w}_m,\bigcirc})AC^\dagger \\ 0 & CC^\dagger \end{pmatrix} \\ &= \begin{pmatrix} I_n & 0 \\ 0 & I_r \end{pmatrix} \end{aligned}$$

Hence $T = L^{-1}$. □

Theorem 9.5. *Let $A \in \mathbb{C}_{n\times n}$ with $\mathrm{Ind}(A) = k$, $m \in \mathbb{Z}^+$, $B \in \mathbb{C}_{n\times r}$ and $C^* \in \mathbb{C}_{n\times r}$ be as in Theorem 9.4. Let $X \in \mathbb{C}_{n\times m}$ and $D \in \mathbb{C}_{n\times m}$ be as in Theorem 9.3. Then the unique solution of (26) is given by $X = [x_{ij}]$, where*

$$x_{ij} = \frac{det\begin{pmatrix} A(i \to d_j) & B \\ C(i \to 0) & 0 \end{pmatrix}}{det\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}}, i = 1, 2, \dots, n, j = 1, 2, \dots, m, \tag{27}$$

d_j denotes the j-th column of D and $A(i \to d_j)$ and $C(i \to 0)$ mean to substitute the i-th column of A and C by d_j and 0, respectively.

Proof. Since X is the solution of (26), we get that $R(X) \subseteq R(A^\dagger A^k) = N(C)$, which implies $CX = 0$. Then (26) can be written as

$$\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}\begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} AX & 0 \\ CX & 0 \end{pmatrix} = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}.$$

According to Theorem 9.4, we have that

$$\begin{aligned} \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}^{-1}\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} A^{\textcircled{w}_m,\bigcirc} & (I_n - A^{\textcircled{w}_m,\bigcirc}A)C^\dagger \\ B^\dagger(I_n - AA^{\textcircled{w}_m,\bigcirc}) & B^\dagger(AA^{\textcircled{w}_m,\bigcirc}A - A)C^\dagger \end{pmatrix}\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Hence $X = A^{\textcircled{w}_m,\bigcirc}D$ and (27) follows from the classical Cramer rule. □

10. Conclusions

In this paper, we present many basic properties for the m-MP weak core inverse. Representations including full-rank decomposition, limit and integral expressions of the m-MP weak core inverse are discussed. Applying the core-EP decomposition and the Hartwig-Spindelböck decomposition, we deduce two canonical forms of the m-MP weak core inverse. A variant of the successive matrix squaring iterative method, appropriate for generating the m-MP weak core inverse is developed. The relationship between the m-MP weak core inverse and certain bordered matrices is explored. And the continuity and applications of the m-MP weak core inverse are studied.

We are confident that more explorations of the m-MP weak core inverse will draw greater attention and interest, and we describe perspectives for further research.
1. Further properties, characterizations and representations of the m-MP weak core inverse.
2. We can further generalize the m-MP weak core inverse to the linear operator in Hilbert or in Banach space or to the tensor case.

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