Filomat 39:29 (2025), 10371–10380
https://doi.org/10.2298/FIL2529371L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Generalized inverses and the solutions of related equations in matrix rings

Wuxin Lu[a], Junchao Wei[a,*]

[a] *School of Mathematical Sciences, Yangzhou University, Yangzhou, Jiangsu 225002, P. R. China*

Abstract. This paper focuses on exploring several equivalent conditions for *PI* and *SEP* matrices. By constructing specific matrix equations and analyzing the existence of solutions to these equations within a given set, we determine whether a group invertible matrix belongs to the class of *PI* or *SEP* matrices.

1. Introduction

Throughout this paper, $\mathbb{C}^{n\times n}$ stands for the set of all $n \times n$ complex matrices. Let $A \in \mathbb{C}^{n\times n}$. Denotes the conjugate transpose matrix of A by A^H. A is called a group invertible matrix [6] if there exists $X \in \mathbb{C}^{n\times n}$ such that

$$AXA = A,\ XAX = X,\ XA = AX.$$

If such X exists, then it is unique, denoted by $A^{\#}$, and is called the group inverse of A.

A is said to be Moore Penrose invertible [3, 5] if there exists $X \in \mathbb{C}^{n\times n}$ such that

$$AXA = A,\ XAX = X,\ (AX)^H = AX,\ (XA)^H = XA.$$

Such X always exists uniquely by [2], denoted by A^+, and is called the Moore Penrose inverse of A.

The matrix A is called a partial isometry (abbreviated as *PI*) [1, 9] if it satisfies the equation $A = AA^HA$. Clearly, A is a *PI* if and only if $A^+ = A^H$ [9].

A is called *EP* when the group inverse $A^{\#}$ exists and $A^{\#} = A^+$ [2, 4, 10, 11]. If $A^{\#}$ exists and $A^{\#} = A^+ = A^H$, then A is called *SEP* [13]. Evidently, A is *SEP* if and only if A is both *EP* and *PI*. The studies of SEP elements in a ring with involution can refer to [7, 8, 11, 12, 14].

In this paper, we continue to study *SEP* matrix. In Section 2, we introduce some lemmas on *PI* and *SEP* matrix. In Section 3, we research the relationship between the consistency of matrix equations and *PI*, *SEP* matrix. In Section 4, with the help of group invertible matrix and *SEP* matrix, we discuss the form of the general solution to certain equation.

2020 *Mathematics Subject Classification*. Primary 15A09; Secondary 15A10; 16U90.
Keywords. PI matrix, SEP matrix, EP matrix, solution of equation, group inverse, Moore-Penrose inverse
Received: 27 February 2025; Revised: 29 August 2025; Accepted: 01 September 2025
Communicated by Dijana Mosić
* Corresponding author: Junchao Wei
Email addresses: 2230111937@qq.com (Wuxin Lu), jcweiyz@126.com (Junchao Wei)
ORCID iDs: https://orcid.org/0009-0006-7552-0934 (Wuxin Lu), https://orcid.org/0000-0002-7310-1836 (Junchao Wei)

2. Some lemmas on PI and SEP matrices

Lemma 2.1. *[7, 8] Let $A \in \mathbb{C}^{n\times n}$ be a group invertible matrix. Then*
(1) $(A^{\#})^{+} = A^{+}A^{3}A^{+}$.
(2) $(A^{+})^{\#} = (AA^{\#})^{H}A(AA^{\#})^{H}$.

Lemma 2.2. *[14] Let $A \in \mathbb{C}^{n\times n}$ be a group invertible matrix. Then*
(1) $(A^{+})^{H}AA^{\#} = (A^{+})^{H} = AA^{\#}(A^{+})^{H}$.
(2) $(A^{\#})^{H}AA^{+} = (A^{\#})^{H} = A^{+}A(A^{\#})^{H}$.
(3) $(A^{+})^{H}A^{+}A = (A^{+})^{H} = AA^{+}(A^{+})^{H}$.
(4) $(AA^{\#})^{H}A^{+} = A^{+} = A^{+}(AA^{\#})^{H}$.

Lemma 2.3. *[11] Let $A \in \mathbb{C}^{n\times n}$ be a group invertible matrix. Then*
(1) $A^{\#}A^{+}A = A^{\#} = AA^{+}A^{\#}$.
(2) $A^{\#}A^{+}A^{\#} = (A^{\#})^{3}$.
(3) $A^{\#}A^{+}A^{3} = A = A^{3}A^{+}A^{\#}$.

Lemma 2.4. *[11] Let $A \in \mathbb{C}^{n\times n}$ be a group invertible matrix. Then A is SEP if and only if one of the following equivalent conditions holds:*
(1) $AAA^{+} = AA^{H}A$.
(2) $A^{+}AA = AA^{H}A$.
(3) $A^{+}A^{H}A = A^{\#}$.
(4) $A^{\#} = AA^{H}A^{+}$.

Lemma 2.5. *[11] Let $A \in \mathbb{C}^{n\times n}$ be a group invertible matrix. Then A is a partial isometry matrix if and only if one of the following equivalent conditions holds:*
(1) $A^{+}A^{H} = A^{+}A^{+}$.
(2) $(A^{H})^{n+1} = (A^{H})^{n}A^{+}$ *for some* $n \in \mathbb{Z}^{+}$.
(3) $AA^{+} = (A^{+})^{H}A^{+}$.

3. Compatibility of matrix equation

We can construct the following equation:

$$XA^{H}A = AA^{\#}XA^{\#}A. \tag{1}$$

Theorem 3.1. *Let $A \in \mathbb{C}^{n\times n}$ be a group invertible matrix. Then A is a partial isometry matrix if and only if Eq.(3.1) has at least one solution in $\tau_A = \{A,\ A^{\#},\ (A^{+})^{H}\}$.*

Proof. " ⇒ " Assume that A is a partial isometry matrix. Then $AA^{H}A = A$, it follows

$$AA^{\#}AA^{\#}A = AA^{\#}A = A = AA^{H}A.$$

Hence, $X = A$ is a solution.

" ⇐ " From the assumption, we will show step by step as follows.

1) If $X = A$, then $AA^{H}A = AA^{\#}AA^{\#}A = A$, one gets A is *PI*.

2) If $X = A^{\#}$, then $A^{\#}A^{H}A = AA^{\#}A^{\#}A^{\#}A = A^{\#}$. It follows A is *PI* by [11, Theorem 1.5.2].

3) If $X = (A^{+})^{H}$, then $(A^{+})^{H}A^{H}A = AA^{\#}(A^{+})^{H}A^{\#}A$. By Lemma 2.2, one gets $(A^{+})^{H}A^{\#}A = (A^{+})^{H} = A^{\#}A(A^{+})^{H}$. It follows $A = (A^{+})^{H}$ and so A is *PI*. □

Theorem 3.2. *Let $A \in \mathbb{C}^{n\times n}$ be a group invertible matrix. Then A is a SEP matrix if and only if Eq.(3.1) has at least one solution in $\omega_A = \{A^{+},\ A^{H},\ (A^{\#})^{H},\ A^{+}A^{\#}A,\ (A^{\#})^{+},\ (A^{+})^{\#}\}$.*

Proof. " $\Rightarrow$ " Assume that A is a SEP matrix. Then $A^{\#} = A^{+} = A^{H}$. It follows

$$A^H A^H A = A^{\#} A^{\#} A = A^{\#} = AA^{\#} A^H A^{\#} A.$$

Hence, $X = A^H$ is a solution.

" $\Leftarrow$ " 1) If $X = A^{+}$, then $A^{+} A^H A = AA^{\#} A^{+} A^{\#} A$. By Lemma 2.3, one yields

$$A^{+} A^H A = A^{\#}.$$

Hence, A is SEP by Lemma 2.4.

2) If $X = A^H$, then $A^H A^H A = AA^{\#} A^H A^{\#} A$. Pre-multiplying the equality by AA^{+}, one gets

$$AA^{+} A^H A^H A = A^H A^H A.$$

Post-multiplying the equality by $A^{+}(A^{\#})^H (A^{\#})^H A^{+}$, one has

$$A^H A^H A(A^{+}(A^{\#})^H (A^{\#})^H A^{+}) = A^H A^H (A^{+})^H A^H (A^{\#})^H (A^{\#})^H A^{+} = (AA^{\#})^H A^{+} = A^{+},$$

then

$$AA^{+} A^{+} = A^{+}.$$

So A is EP by [11, Theorem 1.2.1], this implies $AA^{\#} A^H = A^H = A^H A^{\#} A$. Thus, one gets

$$A^H A^H A = A^H,$$

and $A = A^H A^2$. By [11, Teorem 1.5.3], A is SEP.

3) If $X = (A^{\#})^H$, then

$$(A^{\#})^H A^H A = AA^{\#} (A^{\#})^H A^{\#} A.$$

Pre-multiplying it by AA^{+}, one has $A = (A^{\#})^H A^H A$ by Lemma 2.2. Hence, A is EP by [11, Theorem 1.1.3], which induces

$$AA^{\#} (A^{\#})^H A^{\#} A = AA^{\#} (A^{+})^H A^{\#} A = (A^{+})^H$$

by Lemma 2.2. Thus, one gets $A = (A^{+})^H$ and so A is SEP.

4) If $X = A^{+} A^{\#} A$, then

$$A^{+} A^{\#} AA^H A = AA^{\#} A^{+} A^{\#} AA^{\#} A = A^{\#}.$$

Pre-multiplying the equality by $A^{+} A$, one has $A^{+} AA^{\#} = A^{\#}$. Hence, A is EP by [11, theorem 1.2.1], this leads to $A^{\#} AA^H = A^H$. Now one obtains

$$A^{\#} = A^{+} A^{\#} AA^H A = A^{+} A^H A.$$

Therefore A is SEP by Lemma 2.4.

5) If $X = (A^{\#})^{+}$, then by Lemma 2.1, one gets

$$A^{+} A^3 A^{+} A^H A = AA^{\#} A^{+} A^3 A^{+} A^{\#} A.$$

By Lemma 2.3, one obtains

$$A^{+} A^3 A^{+} A^H A = A.$$

Pre-multiplying the equality by $A^{+} A$, one has $A = A^{+} A^2$, this implies A is EP by [11, Theorem 1.2.1]. Hence, $A^{+} A^3 A^{+} = A$ and

$$AA^H A = A^{+} A^3 A^{+} A^H A = A.$$

Thus, A is SEP.

6) If $X = (A^{+})^{\#}$, then by Lemma 2.1, one yields

$$(AA^{\#})^H A(AA^{\#})^H A^H A = AA^{\#} (AA^{\#})^H A(AA^{\#})^H A^{\#} A.$$

Pre-multiplying the equality by AA^+, one gets

$$AA^+(AA^\#)^HA(AA^\#)^HA^HA = (AA^\#)^HA(AA^\#)^HA^HA.$$

Post-multiplying it by $A^+(A^+)^H$, one has

$$(AA^\#)^HA(AA^\#)^HA^HAA^+(A^+)^H = (AA^\#)^HA(AA^\#)^HA^+A = (AA^\#)^HA$$

and so

$$A = (AA^\#)^HA.$$

By [11, Theorem 1.1.3], A is *EP*, this induces $X = (A^+)^\# = (A^\#)^\# = A$. By Theorem 3.1, A is *PI*. Hence, A is *SEP*. □

Now we can change Eq.(3.1) as follows:

$$XA^HA = AA^\#XAA^+. \tag{2}$$

Theorem 3.3. *Let $A \in \mathbb{C}^{n\times n}$ be a group invertible matrix. Then A is a SEP martix if and only if Eq.(3.2) has at least one solution in $\chi_A = \{A, A^\#, A^+, A^H, (A^\#)^H, (A^+)^H\}$.*

Proof. " ⇒ " Assume that A is a *SEP* matrix, then $A^+ = A^\# = A^H$. It follows

$$AA^HA = AA^\#A = AAA^\# = AAA^+ = AA^\#AAA^+.$$

Hence, $X = A$ is a solution.

" ⇐ "1) If $X = A$, then

$$AA^HA = AA^\#AAA^+.$$

Hence, A is *SEP* by Lemma 2.4.

2) If $X = A^\#$, then

$$A^\#A^HA = AA^\#A^\#AA^+ = A^\#AA^+.$$

Pre-multiplying the equality by A^2, one has $AA^HA = AAA^+$, so A is *SEP* by Lemma 2.4.

3) If $X = A^+$, then

$$A^+A^HA = AA^\#A^+AA^+ = AA^\#A^+.$$

Post-multiplying the equality by $A^\#A$, one has $A^+A^HA = AA^\#A^\# = A^\#$ by Lemma 2.3. By Lemma 2.4, A is *SEP*.

4) If $X = A^H$, then $A^HA^HA = AA^\#A^HAA^+ = AA^\#A^H$. Post-multiplying it by $AA^\#$, one has

$$AA^\#A^H = AA^\#A^HAA^\#,$$

and then pre-multiplying the previous equality by $(A^+)^H$, one yields $AA^+ = AA^+AA^\# = AA^\#$ by Lemma 2.2, so A is *EP* by [11, Theorem 1.2.1], which induces

$$A^HA^HA = AA^\#A^H = A^+AA^H = A^H.$$

Pre-multiplying the equality by $(A^+)^H$, one gets $A^HA = A^+A$. By [11, Theorem 1.5.1], A is *PI*. Hence, A is *SEP*.

5) If $X = (A^\#)^H$, then $(A^\#)^HA^HA = AA^\#(A^\#)^HAA^+$. Pre-multiplying the equality by AA^+, one has

$$A = AA^\#(A^\#)^HAA^+ = (A^\#)^HA^HA,$$

so A is *EP* by [11, Theorem 1.2.1].This implies $AA^+ = A^\#A$ again by [11, Theorem 1.2.1] and

$$(A^\#)^HA^HA = AA^\#(A^\#)^HA^\#A.$$

Thus, A is *SEP* by Theorem 3.2.

6) If $X = (A^+)^H$, then

$$A = (A^+)^HA^HA = AA^\#(A^+)^HAA^+,$$

by Lemma 2.2. Post-multiplying the equality by AA^+, one obtains $AAA^+ = A$. By [11, Theorem 1.2.1], A is *EP*, which induces $A^\# = A^+$. Hence, A is *SEP* by 5). □

Now we can change Eq.(3.2) as follows

$$XA^HY = AA^\# XYA^+. \tag{3}$$

Theorem 3.4. *Let $A \in \mathbb{C}^{n\times n}$ be a group invertible matrix. Then A is a partial isometry matrix if and only if Eq.(3.3) has at least one solution in $\chi_A{}^2 = \{(X, Y)|X, Y \in \chi_A\}$.*

Proof. " ⇒ " Assume that A is a PI matrix, then $A^H = A^+$. It follows that

$$AA^HA^H = AA^HA^+ = AA^\# AA^HA^+.$$

Hence, $(X, Y) = (A, A^H)$ is a solution.

" ⇐ " I) If $Y = A$, A is a PI matrix by Theorem 3.3.

II) If $Y = A^\#$, then

$$XA^HA^\# = AA^\# XA^\# A^+.$$

Post-multiplying the equality by A^2, one obtains Eq.(3.1) by Lemma 2.3. Hence, when $X \in \chi_A$, A is SEP by Theorem 3.1 and Theorem 3.2.

III) If $Y = A^+$, then $XA^HA^+ = AA^\# XA^+A^+$. Post-multiplying the equality by $AA^H(A^\#)^H$, one yields

$$XA^H = AA^\# XA^+,$$

by Lemma 2.2.

1) If $X = A$, then $AA^H = AA^\# AA^+ = AA^+$, so A is PI by [11, Theorem 1.5.1].

2) If $X = A^\#$, then $A^\# A^H = AA^\# A^\# A^+ = A^\# A^+$, so A is PI by [11, Theorem 1.5.2].

3) If $X = A^+$, then

$$A^+A^H = A^+AA^+A^H = A^+A^H = A^+AA A^\# A^+A^+ = A^+A^+.$$

Hence, A is PI by Lemma 2.5.

4) If $X = A^H$, then

$$A^HA^H = AA^\# A^HA^+.$$

Pre-multiplying the equality by AA^+, one has $AAA^+ = A$, so A is EP by [11, Theorem 1.2.2]. This implies $A^+ = A^\#$, then

$$A^HA^H = A^+AA^HA^+ = A^HA^+,$$

so A is PI by Lemma 2.5.

5) If $X = (A^\#)^H$, then $(A^\#)^HA^H = AA^\#(A^\#)^HA^+$. Pre-multiplying the equality by AA^+, one has

$$(AA^+)^H = AA^+(AA^\#)^H = (AA^\#)^H,$$

so A is EP, which includes $(A^\#)^H = (A^+)^H$ by [11, Theorem 1.2.1]. Thus, one gets

$$AA^+ = (A^+)^HA^+.$$

By Lemma 2.5, A is PI.

6) If $X = (A^+)^H$, then

$$AA^+ = (A^+)^HA^+.$$

Hence, A is PI by Lemma 2.5.

IV) If $Y = A^H$, then

$$XA^HA^H = AA^\# XA^HA^+.$$

1) If $X = A$, then $AA^HA^H = AA^\# AA^HA^+ = AA^HA^+$. Pre-multiplying the equality by A^+, one has

$$A^HA^H = A^HA^+,$$

so A is PI by Lemma 2.5.

2) If $X = A^{\#}$, then

$$A^{\#}A^HA^H = AA^{\#}A^{\#}A^HA^+ = A^{\#}A^HA^+.$$

Pre-multiplying the equality by AA, one yields $AA^HA^H = AA^HA^+$. Thus, A is PI by 1).

3) If $X = A^+$, then $A^+A^HA^H = AA^{\#}A^+A^HA^+$. Pre-multiplying the equality by A^HA, one has

$$A^HA^HA^H = A^HA^HA^+,$$

by Lemma 2.2. Hence, A is PI by Lemma 2.5.

4) If $X = A^H$, then $A^HA^HA^H = AA^{\#}A^HA^HA^+$. Pre-multiplying this equality by $A^H(A^+)^H$, one has

$$A^HA^HA^H = A^HA^HA^+$$

by Lemma 2.2. Hence, A is PI by Lemma 2.5.

5) If $X = (A^{\#})^H$, then

$$A^H = (A^{\#})^HA^HA^H = AA^{\#}(A^{\#})^HA^HA^+ = AA^{\#}A^+,$$

by Lemma 2.3. Pre-multiplying the equality by A, one gets $AA^H = AA^+$, so A is PI by [11, Theorem 1.5.1].

6) If $X = (A^+)^H$, then

$$(A^+)^HA^HA^H = AA^{\#}(A^+)^HA^HA^+ = (A^+)^HA^HA^+.$$

Pre-multiplying the equality by A^H, one has $A^HA^H = A^HA^+$. Thus, A is PI by Lemma 2.5.

V) If $Y = (A^{\#})^H$, then

$$XA^H(A^{\#})^H = AA^{\#}X(A^{\#})^HA^+.$$

1) If $X = A$, then

$$AA^H(A^{\#})^H = AA^{\#}A(A^{\#})^HA^+ = A(A^{\#})^HA^+.$$

Pre-multiplying this equality by A^HA^+, one has

$$A^H = A^HA^H(A^{\#})^H = A^H(A^+A)^H(A^{\#})^HA^+ = A^H(A^{\#})^HA^+ = A^+,$$

by Lemma 2.2. Hence, A is PI.

2) If $X = A^{\#}$, then $A^{\#}A^H(A^{\#})^H = AA^{\#}A^{\#}(A^{\#})^HA^+$. Pre-multiplying this equality by AA, one has

$$AA^H(A^{\#})^H = A(A^{\#})^HA^+.$$

Hence, A is PI by 1).

3) If $X = A^+$, then

$$A^+ = A^+A^H(A^{\#})^H = AA^{\#}A^+(A^{\#})^HA^+.$$

Pre-multiplying the equality by AA^+, then $AA^+A^+ = A^+$, so A is EP by [11, Theorem 1.2.2]. This implies $A^{\#} = A^+$ and

$$A^+ = A^+(A^+)^HA^+.$$

Pre-multiplying the equality by A, one yields $AA^+ = (A^+)^HA^+$. Hence, A is PI by Lemma 2.5.

4) If $X = A^H$, then

$$A^H = A^HA^H(A^{\#})^H = AA^{\#}A^H(A^{\#})^HA^+ = AA^{\#}A^+.$$

Pre-multiplying the equality by A, one obtains $AA^H = AA^+$. Thus, A is PI by [11, Theorem 1.5.1].

5) If $X = (A^{\#})^H$, then

$$(A^{\#})^H = (A^{\#})^HA^H(A^{\#})^H = AA^{\#}(A^{\#})^H(A^{\#})^HA^+.$$

Pre-multiplying the equality by AA^+, one has $(AA^+)^H(A^{\#})^H = (A^{\#}AA^+)^H = (A^{\#})^H$, so A is EP by [11, Theorem 1.2.1]. This implies $(A^{\#})^H = (A^+)^H$ and

$$A^+ = A^+(A^+)^HA^+.$$

Hence, A is PI by 3).

6) If $X = (A^+)^H$, then

$$(A^+)^H A^H (A^\#)^H = AA^\#(A^+)^H(A^\#)^H A^+ = (A^+)^H(A^\#)^H A^+.$$

Pre-multiplying the equality by $A^H A^H$, one has

$$A^H = A^H A^H (A^\#)^H = A^H (A^\#)^H A^+ = A^+,$$

by Lemma 2.3. Hence, A is PI.

VI) If $Y = (A^+)^H$, then

$$XA^H(A^+)^H = AA^\# X(A^+)^H A^+.$$

1) If $X = A$, then

$$A = AA^+A = AA^H(A^+)^H = AA^\# A(A^+)^H A^+ = A(A^+)^H A^+.$$

Post-multiplying the equality by A, one yields $AA = A(A^+)^H A^+ A = A(A^+)^H$. By Lemma 2.5, A is PI.

2) If $X = A^\#$, then $A^\# A^H (A^+)^H = AA^\# A^\# (A^+)^H A^+ = A^\#(A^+)^H A^+$. Pre-multiplying the equality by AA, one has

$$A = A(A^+)^H A^+.$$

Hence, A is PI by 1).

3) If $X = A^+$, then $A^+A^H(A^+)^H = AA^\# A^+(A^+)^H A^+$. Pre-multiplying the equality by $A^H A$, one yields

$$A^H A^H (A^+)^H = A^H (A^+)^H A^+ = A^+.$$

Post-multiplying the equality by A^H, one has $A^H A^H = A^+ A^H$. Hence, A is PI by Lemma 2.5.

4) If $X = A^H$, then $A^H A^H (A^+)^H = AA^\# A^H (A^+)^H A^+ = AA^\# A^+$. Post-multiplying it by $A^\# A$, one yields

$$A^H A^H (A^+)^H = A^\# = AA^\# A^+,$$

by Lemma 2.3. Hence, A is EP, which induces $A^\# = A^+$ by [11, Theorem 1.2.1]. This implies

$$A^H = A^H A^H (A^+)^H = A^\# = AA^\# A^+ = A^+,$$

so A is PI.

5) If $X = (A^\#)^H$, then $(A^\#)^H A^H (A^+)^H = AA^\# (A^\#)^H (A^+)^H A^+$. Pre-multiplying the equality by AA^+, one gets

$$(A^+)^H = AA^+(A^\#)^H A^H (A^+)^H = (A^\#)^H A^H (A^+)^H = (A^+AA^\#)^H.$$

Hence, A is EP by [11, Theorem 1.2.1]. This implies

$$(A^+)^H = (A^\#)^H A^H (A^+)^H = AA^\#(A^+)^H(A^+)^H A^+ = (A^+)^H(A^+)^H A^+.$$

Pre-multiplying the equality by A^+A^H, one obtains $A^+ = A^+AA^+ = A^+(A^+)^H A^+$. By V) 3), A is PI.

6) If $X = (A^+)^H$, then

$$(A^+)^H = (A^+)^H A^H (A^+)^H = AA^\#(A^+)^H(A^+)^H A^+ = (A^+)^H(A^+)^H A^+.$$

Hence, A is PI by 5). □

4. The general solution of related equations

We now generalize Eq.(3.2) as follows:

$$XA^H A = AA^\# Y AA^+. \tag{4}$$

Theorem 4.1. *Let $A \in \mathbb{C}^{n\times n}$ be a group invertible matrix. Then the general solution to Eq.(4.1) is given by*

$$\begin{cases} X = AA^{\#}P(A^{+})^{H} + U - UA^{+}A \\ Y = PA + V - A^{+}AVAA^{+} \end{cases}, \text{ where } P,\ U,\ V \in \mathbb{C}^{n\times n} \text{ with } PA = PA^{2}A^{+}. \tag{5}$$

Proof. Frist, we have the formula (4.2) is the solution of Eq.(4.1).
In fact,

$$\begin{aligned}(AA^{\#}P(A^{+})^{H} + U - UA^{+}A)A^{H}A &= AA^{\#}PA + UA^{H}A - UA^{H}A = AA^{\#}PA \\ &= AA^{\#}PA + AA^{\#}VAA^{+} - AA^{\#}A^{+}AVAA^{+} = AA^{\#}(PA + V - A^{+}AVAA^{+})AA^{+}.\end{aligned}$$

Next, let

$$\begin{cases} X = X_0 \\ Y = Y_0, \end{cases}$$

be a solution of Eq.(4.1). Then

$$X_0A^{H}A = AA^{\#}Y_0AA^{+}.$$

Choose $P = AA^{\#}Y_0AA^{+}A^{+}, U = X_0, V = Y_0 - PA$. Then,

$$\begin{aligned}PA = AA^{\#}Y_0AA^{+}A^{+}A = X_0A^{H}AA^{+}A = X_0A^{H}A &= AA^{\#}Y_0AA^{+} \\ &= AA^{\#}Y_0AA^{+}AA^{+} = PAAA^{+}.\end{aligned}$$

Noting that

$$\begin{aligned}AA^{\#}P(A^{+})^{H} &= AA^{\#}(AA^{\#}Y_0AA^{+}A^{+})(A^{+})^{H} = AA^{\#}Y_0AA^{+}A^{+}(A^{+})^{H} \\ &= X_0A^{H}AA^{+}(A^{+})^{H} = X_0A^{H}(A^{+})^{H} = X_0A^{+}A.\end{aligned}$$

Then

$$X_0 = X_0A^{+}A + X_0 - X_0A^{+}A = AA^{\#}P(A^{+})^{H} + U - UA^{+}A.$$

Also

$$\begin{aligned}A^{+}A(Y_0 - PA)AA^{+} &= A^{+}AY_0AA^{+} - A^{+}AAA^{\#}Y_0AA^{+}A^{+}A = A^{+}AY_0AA^{+} - A^{+}AX_0A^{H}AA^{+}A \\ &= A^{+}AY_0AA^{+} - A^{+}AX_0A^{H}A = A^{+}AY_0AA^{+} - A^{+}AAA^{\#}Y_0AA^{+} = 0.\end{aligned}$$

Hence,

$$Y_0 = PA + Y_0 - PA + 0 = PA + V + A^{+}A(Y_0 - PA)AA^{+}.$$

Hence, the general solution of Eq.(4.1) is given by the formula (4.2). □

Theorem 4.2. *Let $A \in \mathbb{C}^{n\times n}$ be a group invertible matrix. Then A is a SEP matrix if and only if the general solution to Eq.(4.1) is given by*

$$\begin{cases} X = A^{H}AP(A^{+})^{H} + U - UA^{+}A \\ Y = PA + V - A^{+}AVAA^{+} \end{cases}, \text{ where } P,\ U,\ V \in \mathbb{C}^{n\times n}. \tag{6}$$

Proof. " ⇒ " Since A is SEP, which induces $A^{H}A = AA^{\#}$ and

$$A^{2}A^{+} = A$$

by [11, Theorem 1.5.3] . Hence, by Theorem 4.1, the general solution to Eq(4.1) is given by (4.3).
" ⇐ " Follows from the assumption, one gets

$$(A^{H}AP(A^{+})^{H} + U - UA^{+}A)A^{H}A = AA^{\#}(PA + V - A^{+}AVAA^{+})AA^{+},$$

i.e.,

$$A^HAPA = AA^\# PA^2A^+ \text{ for each } P \in \mathbb{C}^{n\times n}.$$

Especially, choose $P = A^+$, one has

$$A^HA = AA^+.$$

Hence, A is *SEP* by [11, Theorem 1.5.3]. □

Now we construct the equation as follows:

$$XA^HA = A^HAYAA^\#. \tag{7}$$

Theorem 4.3. *Let $A \in \mathbb{C}^{n\times n}$ be a group invertible matrix. Then the general solution to Eq.(4.4) is given by (4.3).*

Proof. Frist, we have the formula (4.3) is the solution of Eq.(4.4).
In fact,

$$\begin{aligned}&(A^HAP(A^+)^H + U - UA^+A)A^HA = A^HAPA + UA^HA - UA^HA = A^HAPA\\ &= A^HAPA + A^HAVAA^\# - A^HAA^+AVAA^+AA^\# = A^HA(PA + V - A^+AVAA^+)AA^\#.\end{aligned}$$

Next, let

$$\begin{cases} X = X_0 \\ Y = Y_0, \end{cases}$$

be a solution of Eq.(4.4). Then

$$X_0A^HA = A^HAY_0AA^\#.$$

Choose $P = A^+AY_0A^\#, U = X_0, V = Y_0 - PA$.
Noting that

$$\begin{aligned}A^HAP(A^+)^H &= A^HAA^+AY_0A^\#(A^+)^H = A^HAY_0A^\#AA^\#(A^+)^H\\ &= X_0A^HAA^\#(A^+)^H = X_0A^H(A^+)^H = X_0A^+A.\end{aligned}$$

Then

$$X_0 = X_0A^H(A^+)^H + X_0 - X_0A^H(A^+)^H = A^HAP(A^+)^H + U - UA^+A.$$

Also

$$A^+A(Y_0 - PA)AA^+ = A^+AY_0AA^+ - A^+AA^+AY_0A^\#AAA^+ = A^+AY_0AA^+ - A^+AY_0AA^+ = 0.$$

Hence,

$$Y_0 = PA + Y_0 - PA + 0 = PA + V + A^+A(Y_0 - PA)AA^+.$$

Hence, the general solution of Eq.(4.4) is given by the formula (4.3). □

From Theorem 4.2 and Theorem 4.3, we have the following conclusion.

Theorem 4.4. *Let $A \in \mathbb{C}^{n\times n}$ be a group invertible matrix. Then A is SEP if and only if Eq(4.1) and Eq(4.4) share the same solution set.*

Conflict of Interest

The authors declared that they have no conflict of interest.

References

[1] W. X. Chen, On *EP* elements, normal elements and partial isometries in rings with involution, Electron. J. Linear Algebra **23** (2012), 553–561.
[2] R. E. Hartwig, Block generalized inverses, Arch. Rational. Mech. Anal. **61** (1976), 197–251.
[3] R. E. Harte and M. Mbekhta, On generalized inverses in C^*-algebras, Studia Math. **103** (1992), 71–77.
[4] S. Karanasios, EP elements in rings and semigroup with involution and C^*-algebras, Serdica Math. J. **41** (2015), 83–116.
[5] J. J. Koliha, The Drazin and Moore-Penrose inverse in C^*-algebras, Math. Proc. R. Ir. Acad. **99**A (1999), 17–27.
[6] J. J. Koliha and P. Patrício, Elements of rings with equal spectral idempotents, J. Aust. Math. Soc. **72** (2002), 137–152.
[7] A. Q. Li and J. C. Wei, The influence of the expression form of solutions to related equations on SEP elements in a ring with involution, J. Algebra Appl. (2024) 2450113 (16 pages).
[8] A. Q. Li and J. C. Wei, The equation characterizations of generalized inverses in a ring with involution, J. Algebra Appl. (2025) 2550206 (25 pages).
[9] D. Mosic and D. S. Djordjević, Partial isometries and EP elements in rings with involution, Electron. J. Linear Algebra **18** (2009), 761–772.
[10] D. Mosic and D. S. Djordjević, New characterizations of EP, generalized normal and generalized Hermitian elements in rings, Appl. Math. Comput. **218** (2012), 6702–6710.
[11] D. Mosic, Generalized inverses, Faculty of Sciences and Mathematics, University of Niš, Niš, 2018.
[12] Z. C. Xu, R. J. Chen and J. C. Wei, Strongly EP elements in a ring with involution, Filomat **34** (2020), 2101–2107.
[13] R. J. Zhao, H. Yao and J. C. Wei, Characterizations of partial isometries and two special kinds of EP elements, Czechoslovak Math. J. **70** (2020), 539–551.
[14] D. D. Zhao and J. C. Wei, Strongly EP elements in rings with involution, J. Algebra Appl. **21** (2022), 2250088 (10 pages).