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Strongly EP elements which are characterized by projections, a–commutative and w–core inverses in a ring with involution

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Abstract. In this paper, we characterize strongly EP elements in a ring with involution by various methods. Especially, we construct projections with parameter variables in a specific set to characterize strongly EP elements. Then we introduce the concept of a–commutativity and w–core inverse to study the charactirization of strongly EP elements.

1. Introduction

Let R be an associative ring with an identity. A mapping $*$: $R \longrightarrow R$ is called an involution if $*$ is an anti-automorphism of order 2; that is, for any $a, b \in R$,

$$(a^*)^* = a,\ (a+b)^* = a^* + b^*,\ (ab)^* = a^*b^*.$$

In this case, R is said to be a $*$–ring or an involution ring.

An element a is said to be Moore–Penrose invertible element if there exists $a^\dagger \in R$ such that

$$aa^\dagger a = a,\ a^\dagger aa^\dagger = a^\dagger,\ (aa^\dagger)^* = aa^\dagger,\ (a^\dagger a)^* = a^\dagger a.$$

Such an element $a^\dagger$ is uniquely determined if it exists, and is called the Moore–Penrose inverse (or MP–inverse) of a (see [6, 10, 14]). The set of all Moore–Penrose invertible elements of R will be denoted by $R^\dagger$.

An element $a \in R$ is called group invertible element, if there is $a^\# \in R$ such taht

$$aa^\# a = a,\ a^\# aa^\# = a^\#,\ aa^\# = a^\# a.$$

$a^\#$ is called the group inverse of a and it is uniquely determined by the above equations [1]. We write $R^\#$ to denote the set of all group invertible elements of R.

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$a \in R$ is called a partial isometry, if $aa^*a = a$. We write R^{PI} to denote the set of all partial isometries of R. It is known that $a \in R^\dagger$ is partial isometry if and only if $a^\dagger = a^*$ (see [7]).

$a \in R^\# \cap R^\dagger$ is called EP if $a^\# = a^\dagger$ (see [9]). We denote the set of all EP elements of R by R^{EP}.

$a \in R^\# \cap R^\dagger$ is called strongly EP if $a^\# = a^\dagger = a^*$. We denote the set of all strongly EP elements of R by R^{SEP}. Evidently, $a \in R^{SEP}$ if and only if $a \in R^{EP}$ and $a \in R^{PI}$.

The study of generalized inverses in a ring with involution is an important ingredient in the ring theory. Many researchers have done a lots of results in this area. For instances, Mosić et al. presented a number of meaningful characterizations of EP elements and partial isometries in [5–7, 10]. In recent years, Wei et al. provided many new characterizetions of strongly EP elements. For example, in [3], Zhao and Wei gave some sufficient and necessary conditions for an element to be a strongly EP element through some transformations of equations. In addition, Guan and Wei characterized strongly EP elements by using the solution of the generalized inverse equation in a specific set in [13]. More interesting results on generalized inverses in rings with involution can also be found in [2, 4, 8, 11, 16, 17].

Inspired by these results, this paper mainly study some new ways to characterize strongly EP elements. The paper is organized as follows: in Section 2, based on [15, Corollary 2.3], we construct projections with parameter variables and give some equivalent conditions for an element $a \in R^\# \cap R^\dagger$ to be a strongly EP element. In Section 3, we introduce the concept of a–commutativity and characterize strongly EP elements by constructing a–commutative elements. In Section 4, we study the w–core inverses and use it to investigate strongly EP elements.

2. Construct Projections with parameter variables to characterize Strongly *EP* elements

Let R be a $*-$ring. An element $a \in R$ is called projection, if $a^2 = a = a^*$. We write $PE(R)$ to denote the set of all projections of R. In [15], it is shown that $a \in PE(R)$ if and only if $a = aa^*$ or $a = a^*a$. We begin with the following lemma which follows from [15, Corollary 2.3].

Lemma 2.1. *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{SEP}$ if and only if $aa^*a^\#aa^\dagger = a^\dagger aa^\#$.*

Inspired by Lemma 2.1, we have the following corollary.

Theorem 2.2. *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{SEP}$ if and only if $aa^*a^\#a \in PE(R)$.*

Proof. "$\Longrightarrow$" Assume that $a \in R^{SEP}$. Then $aa^*a^\#aa^\dagger = a^\dagger aa^\#$ by Lemma 2.1, it follows that $aa^*a^\#a = (aa^*a^\#aa^\dagger)a = (a^\dagger aa^\#)a = a^\dagger a \in PE(R)$.

"$\Longleftarrow$" From the assumption, we have

$$aa^*a^\#a = (aa^*a^\#a)^*,$$

this gives

$$aa^*a^\#a = aa^\dagger(aa^*a^\#a) = aa^\dagger(aa^*a^\#a)^* = (aa^*a^\#aaa^\dagger)^* = (aa^*)^* = aa^*,$$

so

$$a^* = a^\dagger aa^* = a^\dagger aa^*a^\#a = a^*a^\#a.$$

Hence, $a \in R^{EP}$ by [7, Theorem 1.2.1], which implies

$$aa^* = aa^*a^\#a \in PE(R).$$

Thus, $aa^* = (aa^*)^2$, which induces

$$a = aa^*(a^\dagger)^* = (aa^*)^2(a^\dagger)^* = aa^*a.$$

Thus, $a \in R^{SEP}$. $\square$

Obversing Theorem 2.2, we can give the following result.

Theorem 2.3. *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{SEP}$ if and only if $xa^*ax^{\#} \in PE(R)$ for some $x \in \chi_a = \{a,\ a^{\#},\ a^{\dagger},\ a^*,\ (a^{\dagger})^*,\ (a^{\#})^*\}$.*

Proof. " $\Longrightarrow$ " It follows from Theorem 2.2 that $x = a \in \chi_a$ is a solution.

" $\Longleftarrow$ " If there exists $x_0 \in \chi_a$ such that $x_0a^*ax_0^{\#} \in PE(R)$, then

$$x_0a^*ax_0^{\#} = x_0a^*ax_0^{\#}(x_0a^*ax_0^{\#})^* = x_0a^*ax_0^{\#}(x_0^{\#})^*a^*ax_0^*.$$

Then we can discuss the following cases.

(1) If $x_0 \in \tau_a = \{a,\ a^{\#},\ (a^{\dagger})^*\}$, then $x_0^{\dagger}x_0 = a^{\dagger}a,\ x_0^*aa^{\dagger} = x_0^*$ and $x_0a^{\#}ax_0^{\#} = aa^{\#}$ by [13, Lemma 4.1]. It follows that

$$\begin{aligned} a^*ax_0^{\#} &= a^{\dagger}aa^*ax_0^{\#} = x_0^{\dagger}x_0a^*ax_0^{\#} = x_0^{\dagger}x_0a^*ax_0^{\#}(x_0^{\#})^*a^*ax_0^* \\ &= a^{\dagger}aa^*ax_0^{\#}(x_0^{\#})^*a^*ax_0^* = a^*ax_0^{\#}(x_0^{\#})^*a^*ax_0^* \end{aligned}$$

and

$$ax_0^{\#} = (a^{\dagger})^*a^*ax_0^{\#} = (a^{\dagger})^*a^*ax_0^{\#}(x_0^{\#})^*a^*ax_0^* = ax_0^{\#}(x_0^{\#})^*a^*ax_0^*.$$

So

$$\begin{aligned} aa^{\#} &= x_0a^{\#}ax_0^{\#} = x_0a^{\#}ax_0^{\#}(x_0^{\#})^*a^*ax_0^* = aa^{\#}(x_0^{\#})^*a^*ax_0^* \\ &= (aa^{\#}(x_0^{\#})^*a^*ax_0^*)aa^{\dagger} = aa^{\#}aa^{\dagger} = aa^{\dagger}. \end{aligned}$$

Hence, $a \in R^{EP}$.

(a) If $x_0 = a$, then

$$aa^{\#} = aa^{\#}(a^{\#})^*a^*aa^* = aa^*,$$

so $a \in R^{SEP}$ by [7, Theorem 1.5.3].

(b) If $x_0 = a^{\#}$, then

$$aa^{\#} = aa^{\#}a^*a^*a(a^{\#})^* = a^*a^*a(a^{\#})^*$$

and

$$a^* = aa^{\#}a^* = a^*a^*a(a^{\#})^*a^* = a^*a^*a.$$

Aplying the involution on the last equality, one has $a = a^2a^*$. Thus, $a \in R^{SEP}$ by [7, Theorem 1.5.3].

(c) If $x_0 = (a^{\dagger})^*$, then

$$aa^{\#} = aa^{\#}(a^{\dagger})^{\#}a^*aa^{\dagger} = aa^{\#}aa^* = aa^*.$$

So $a \in R^{SEP}$.

(2) If $x_0 \in \gamma_a = \{a^{\dagger},\ a^*,\ (a^{\#})^*\}$, then $x_0^{\#}x_0 = (aa^{\#})^*,\ x_0^*a^{\dagger}a = x_0^*$ and $x_0a^{\dagger}ax_0^{\#} = (aa^{\#})^*$ by [13, Lemma 4.1], this leads to

$$\begin{aligned} a^*ax_0^{\#} &= (aa^{\#})^*a^*ax_0^{\#} = x_0^{\#}x_0a^*ax_0^{\#} \\ &= x_0^{\#}x_0a^*ax_0^{\#}(x_0^{\#})^*a^*ax_0^* = a^*ax_0^{\#}(x_0^{\#})^*a^*ax_0^* \end{aligned}$$

and

$$ax_0^{\#} = (a^{\dagger})^*a^*ax_0^{\#} = (a^{\dagger})^*a^*ax_0^{\#}(x_0^{\#})^*a^*ax_0^* = ax_0^{\#}(x_0^{\#})^*a^*ax_0^*.$$

This induces

$$\begin{aligned} (aa^{\#})^* &= x_0a^{\dagger}ax_0^{\#} = x_0a^{\dagger}ax_0^{\#}(x_0^{\#})^*a^*ax_0^* = (aa^{\#})^*(x_0^{\#})^*a^*ax_0^* \\ &= ((aa^{\#})^*(x_0^{\#})^*a^*ax_0^*)a^{\dagger}a = (aa^{\#})^*a^{\dagger}a = a^{\dagger}a. \end{aligned}$$

Hence, $a \in R^{EP}$ by [7, Theorem 1.1.3], it follows that if $x_0 = a^{\dagger} = a^{\#}$ or $x_0 = (a^{\#})^* = (a^{\dagger})^*$, then $a \in R^{SEP}$ by (1).

If $x_0 = a^*$, then

$$aa^{\#} = (aa^{\#})^* = (aa^{\#})^*((a^*)^{\#})^*a^*a(a^*)^* = aa^{\#}a^{\#}a^*a^2 = a^{\#}a^*a^2$$

and

$$a = a^2a^{\#} = aa^{\#}a^*a^2 = a^*a^2.$$

Thus, $a \in R^{SEP}$ by [7, Theorem 1.5.3].

Therefore, in any case, we have $a \in R^{SEP}$. □

Corollary 2.4. *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{SEP}$ if and only if $xa^*x^{\#}a \in PE(R)$ for some $x \in \tau_a$.*

Proof. " $\Longrightarrow$ " Assume that $a \in R^{SEP}$, then $aa^*a^{\#}a \in PE(R)$ by Theorem 2.2. Hence, $x = a \in \tau_a$ is a solution.

" $\Longleftarrow$ " Noting that $x^{\#}a = ax^{\#}$ for $x = a$ or $x = a^{\#}$. In this case, $a \in R^{SEP}$ by Theorem 2.3.

Now take $x = (a^{\dagger})^*$, then $(a^{\dagger})^*a^*((a^{\dagger})^*)^{\#}a \in PE(R)$. Since $(a^{\dagger})^{\#} = (aa^{\#})^*a(aa^{\#})^*$ by [13, Lemma 2.2], it follows that

$$(a^{\dagger})^*a^*((a^{\dagger})^*)^{\#}a = (aa^{\dagger})^*((aa^{\#})^*a(aa^{\#})^*)^*a = aa^{\dagger}aa^{\#}a^*aa^{\#}a = aa^{\#}a^*a.$$

So $aa^{\#}a^*a \in PE(R)$, this induces

$$aa^{\#}a^*a = (aa^{\#}a^*a)^*aa^{\#}a^*a$$

and

$$aa^{\#} = aa^{\#}a^{\dagger}a = aa^{\#}a^*aa^{\dagger}(a^{\dagger})^* = (aa^{\#}a^*a)^*aa^{\#}a^*aa^{\dagger}(a^{\dagger})^*$$

$$= (aa^{\#}a^*a)^*aa^{\#} = a^*a(aa^{\#})^*aa^{\#}.$$

It follows that

$$aa^{\dagger} = aa^{\#}aa^{\dagger} = a^*a(aa^{\#})^*aa^{\#}aa^{\dagger} = a^*a(aa^{\#})^* = a^{\dagger}a(a^*a(aa^{\#})^*) = a^{\dagger}a^2a^{\dagger}.$$

Hence, $a \in R^{EP}$, which leads to $aa^{\dagger} = a^*a(aa^{\#})^*aa^{\#} = a^*a$. Thus, $a \in R^{SEP}$. □

Theorem 2.5. *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{SEP}$ if and only if $xx^*aa^{\#} \in PE(R)$ for some $x \in \tau_a$.*

Proof. " $\Longrightarrow$ " Since $a \in R^{SEP}$, we have $aa^*aa^{\#} \in PE(R)$ by Theorem 2.2. Taking $x = a \in \tau_a$, we are done.

" $\Longleftarrow$ " (1) If $x = a$, then $aa^*aa^{\#} \in PE(R)$. This induces $a \in R^{SEP}$ by Theorem 2.2.

(2) If $x = a^{\#}$, then $a^{\#}(a^{\#})^*aa^{\#} \in PE(R)$, this gives

$$a^{\#}(a^{\#})^*aa^{\#} = (a^{\#}(a^{\#})^*aa^{\#})^*a^{\#}(a^{\#})^*aa^{\#}.$$

Multiplying the equality on the right by $aa^{\dagger}a^*a^{\dagger}a$, one gets

$$a^{\#} = (aa^{\#})^*a^{\#}(a^{\#})^*a^{\#} = a^{\dagger}a((aa^{\#})^*a^{\#}(a^{\#})^*a^{\#}) = a^{\dagger}aa^{\#}.$$

Hence, $a \in R^{EP}$ by [6, Theorem 2.1], which induces

$$a^{\#} = (aa^{\#})^*a^{\#}(a^{\#})^*a^{\#} = aa^{\#}a^{\#}(a^{\#})^*a^{\#} = a^{\#}(a^{\#})^*a^{\#} = a^{\#}(a^{\dagger})^*a^{\#}$$

and

$$a = aa^{\#}a = aa^{\#}(a^{\dagger})^*a^{\#}a = (a^{\dagger})^*.$$

Thus, $a \in R^{SEP}$.

(3) If $x = (a^{\dagger})^*$, then $(a^{\dagger})^*((a^{\dagger})^*)^*aa^{\#} \in PE(R)$, i.e., $(a^{\dagger})^*a^{\#} \in PE(R)$. This gives

$$(a^{\dagger})^*a^{\#} = (a^{\dagger})^*a^{\#}((a^{\dagger})^*a^{\#})^* = (a^{\dagger})^*a^{\#}(a^{\#})^*a^{\dagger},$$

$$a = a^2a^{\#} = a^2a^*(a^{\dagger})^*a^{\#} = a^2a^*(a^{\dagger})^*a^{\#}(a^{\#})^*a^{\dagger} = a(a^{\#})^*a^{\dagger}$$

and

$$a^{\dagger}a = a^{\dagger}a(a^{\#})^*a^{\dagger} = (a^{\#})^*a^{\dagger} = ((a^{\#})^*a^{\dagger})aa^{\dagger} = a^{\dagger}a^2a^{\dagger}.$$

Hence, $a \in R^{EP}$ and $a^* = a^*a^{\dagger}a = a^*(a^{\#})^*a^{\dagger} = a^{\dagger}$. Thus, $a \in R^{SEP}$. □

Theorem 2.6. *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{SEP}$ if and only if $ax^*xa^{\#} \in PE(R)$ for some $x \in \chi_a$.*

Proof. " $\Longrightarrow$ " Assume that $a \in R^{SEP}$, then $aa^*aa^\# \in PE(R)$ by Theorem 2.2, that is choosing $x = a \in \chi_a$, as desired.

" $\Longleftarrow$ " From the assumption, there exists $x_0 \in \chi_a$ such that

$$ax_0^*x_0a^\# = (ax_0^*x_0a^\#)^* = (a^\#)^*x_0^*x_0a^* = ((a^\#)^*x_0^*x_0a^*)aa^\dagger = ax_0^*x_0a^\#aa^\dagger.$$

Then we can discuss the following cases.

(1) If $x_0 \in \tau_a$, then we have

$$a^\dagger ax_0^* = x_0^*,\ (x_0^\dagger)^*x_0^* = aa^\dagger,\ x_0^\#x_0 = aa^\#,\ aa^\dagger x_0 = x_0.$$

It follows that

$$x_0^*x_0a^\# = a^\dagger ax_0^*x_0a^\# = a^\dagger ax_0^*x_0a^\#aa^\dagger = x_0^*x_0a^\#aa^\dagger,$$

$$x_0a^\# = aa^\dagger x_0a^\# = (x_0^\dagger)^*x_0^*x_0a^\# = (x_0^\dagger)^*x_0^*x_0a^\#aa^\dagger = aa^\dagger x_0a^\#aa^\dagger = x_0a^\#aa^\dagger,$$

and

$$a^\# = aa^\#a^\# = x_0^\#x_0a^\# = x_0^\#x_0a^\#aa^\dagger = aa^\#a^\#aa^\dagger = a^\#aa^\dagger.$$

Hence, $a \in R^{EP}$ by [7, Theorem 1.2].

(a) If $x_0 = a$, then $aa^*aa^\# \in PE(R)$. By Theorem 2.2, $a \in R^{SEP}$.

(b) If $x_0 = a^\#$, then $a(a^\#)^*a^\#a^\# \in PE(R)$, one gets

$$a(a^\#)^*a^\#a^\# = a(a^\#)^*a^\#a^\#(a(a^\#)^*a^\#a^\#)^*.$$

Multiplying the equality on the left by $a^3a^*a^\dagger$, one has

$$a = a(a(a^\#)^*a^\#a^\#)^* = a^2(a^\#)^*a^\#a^\#$$

and

$$a^\# = a^\#a^\#a = a^\#a^\#a^2(a^\#)^*a^\#a^\# = (a^\#)^*a^\#a^\#.$$

This induces

$$a = a^\#a^2 = (a^\#)^*a^\#a^\#a^2 = (a^\#)^*.$$

Hence, $a \in R^{SEP}$.

(c) If $x_0 = (a^\dagger)^*$, then $a((a^\dagger)^*)^*(a^\dagger)^*a^\# \in PE(R)$, that is $(a^\dagger)^*a^\# \in PE(R)$. Hence, $a \in R^{SEP}$ by Theorem 2.5 (3).

(2) If $x_0 \in \gamma_a$, then, similar to (1), we obtain $a \in R^{EP}$. It follows that $a \in R^{SEP}$ for $x_0 = a^\dagger = a^\#$ or $x_0 = (a^\#)^* = (a^\dagger)^*$ by (b) and (c).

If $x_0 = a^*$, then $a^2a^*a^\# \in PE(R)$, this infers

$$a^2a^*a^\# = a^2a^*a^\#(a^2a^*a^\#)^*.$$

Multiplying the equality on the left by $a^2(a^\dagger)^*a^\#a^\#$, one has

$$a = a(a^2a^*a^\#)^* = a(a^\#)^*aa^*a^*.$$

It follows that

$$(a^\dagger)^* = a^\dagger a(a^\dagger)^* = a^\dagger a(a^\#)^*aa^*a^*(a^\dagger)^* = (a^\dagger)^*aa^*.$$

So $a^\dagger = aa^*a^\dagger$. By [7, Theorem 1.5.3], $a \in R^{SEP}$.

□

3. Using a–commutative elements to characterize Strongly EP elements

Let $a,\ x,\ y \in R$. Then $x,\ y$ are called a–commutative if $ax = ya$.

Evidently, $a \in E(R)$ (the set of all idempotents of R) if and only if $a,\ 1$ are a–commutative if and only if $a,\ 2a-1$ are a–commutative.

Inspired by Lemma 2.1, we have the following theorem.

Theorem 3.1. *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{SEP}$ if and only if $aa^{\#},\ (aa^{\#})^*aa^*$ are $a^{\dagger}$–commutative.*

Proof. " $\Longrightarrow$ " Since $a \in R^{SEP}$, one gets $aa^*a^{\#}a \in PE(R)$ by Theorem 2.2 and $aa^*a^{\#}aa^{\dagger} = a^{\dagger}aa^{\#}$ by Lemma 2.1. It follows that

$$a^{\dagger}aa^{\#} = aa^*a^{\#}aa^{\dagger} = (aa^*a^{\#}a)^*a^{\dagger} = (aa^{\#})^*aa^*a^{\dagger}.$$

Hence, $aa^{\#},\ (aa^{\#})^*aa^*$ are $a^{\dagger}$–commutative.

" $\Longleftarrow$ " From the assumption, we have

$$a^{\dagger}aa^{\#} = (aa^{\#})^*aa^*a^{\dagger} = ((aa^{\#})^*aa^*a^{\dagger})aa^{\dagger} = a^{\dagger}aa^{\#}aa^{\dagger} = a^{\dagger}.$$

Hence, $a \in R^{EP}$ by [7, Theorem 1.2.1], which implies

$$(aa^{\#})^*aa^* = aa^{\#}aa^* = aa^* = (aa^*)^* = ((aa^{\#})^*aa^*)^* = aa^*aa^{\#},$$

and so

$$aa^*aa^{\#}a^{\dagger} = (aa^{\#})^*aa^*a^{\dagger} = a^{\dagger}aa^{\#}.$$

Thus, $a \in R^{SEP}$ by Lemma 2.1. □

Lemma 3.2. *Let $a,\ x,\ y \in R$. If $x,\ y$ are a–commutative, then $x^k,\ y^k$ are a–commutative for all $k \in \mathbb{Z}^+$.*

Proof. It can be varified by induction on k. □

Since $(aa^{\#})^k = aa^{\#}$, Theorem 3.1 and Lemma 3.2 imply the following theorem.

Theorem 3.3. *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{SEP}$ if and only if $aa^{\#},\ ((aa^{\#})^*aa^*)^k$ are $a^{\dagger}$–commutative for $k = 2,\ 3$.*

Proof. " $\Longrightarrow$ " It is an immediate result of Theorem 3.1 and Lemma 3.2.

" $\Longleftarrow$ " Assume that $aa^{\#},\ ((aa^{\#})^*aa^*)^k$ are $a^{\dagger}$–commutative for $k = 2,\ 3$, we have

$$a^{\dagger}(aa^{\#})^2 = ((aa^{\#})^*aa^*)^2a^{\dagger}$$

and

$$a^{\dagger}(aa^{\#})^3 = ((aa^{\#})^*aa^*)^3a^{\dagger},$$

that is

$$a^{\dagger}aa^{\#} = (aa^{\#})^*(aa^*)^2a^{\dagger}$$

and

$$a^{\dagger}aa^{\#} = (aa^{\#})^*(aa^*)^3a^{\dagger},$$

it follows that

$$(aa^{\#})^*(aa^*)^2a^{\dagger} = (aa^{\#})^*(aa^*)^3a^{\dagger}.$$

Multiplying the last equality on the left by $a^{\dagger}(a^{\dagger})^*a^{\dagger}$, one gets

$$a^*a^{\dagger} = a^*aa^*a^{\dagger}.$$

Multiplying the equality on the right by $a(aa^{\#})^*$, one obtains

$$a^* = a^*aa^*.$$

Hence, $a \in R^{PI}$, it follows that $a^{\dagger} = a^*$. This yields

$$a^{\dagger}aa^{\#} = (aa^{\#})^*(aa^*)^2a^{\dagger} = (aa^{\#})^*aa^{\dagger}a^{\dagger} = a^{\dagger}.$$

So $a \in R^{SEP}$. □

Lemma 3.4. *Let $a,\ x,\ y \in R$. If $x,\ y$ are a–commutative, then $x + ya,\ y + ay$ are a–commutative.*

Proof. Since $x,\ y$ are a–commutative, one has

$$a(x + ya) = ax + aya = ya + aya = (y + ay)a$$

and hence, $x + ya,\ y + ay$ are a–commutative. □

Inspired by Theorem 3.1 and Lemma 3.4, we have the following theorem.

Theorem 3.5. *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{SEP}$ if and only if $aa^{\#} + (aa^{\#})^* aa^* a^{\dagger}$, $(aa^{\#})^* aa^* + a^*$ are $a^{\dagger}$–commutative.*

Proof. " $\Longrightarrow$ " Since $a \in R^{SEP}$, $aa^{\#}$, $(aa^{\#})^* aa^*$ are $a^{\dagger}$–commutative by Theorem 3.1. Then we have $aa^{\#} + (aa^{\#})^* aa^* a^{\dagger}$, $(aa^{\#})^* aa^* + a^{\dagger}(aa^{\#})^* aa^*$ are $a^{\dagger}$–commutative by Lemma 3.4, that is $aa^{\#} + (aa^{\#})^* aa^* a^{\dagger}$, $(aa^{\#})^* aa^* + a^*$ are $a^{\dagger}$–commutative.

" $\Longleftarrow$ " From the hypothesis, one gets $a^{\dagger}(aa^{\#} + (aa^{\#})^* aa^* a^{\dagger}) = ((aa^{\#})^* aa^* + a^*)a^{\dagger}$, that is

$$a^{\dagger} aa^{\#} = (aa^{\#})^* aa^* a^{\dagger}.$$

Hence, $aa^{\#}$, $(aa^{\#})^* aa^*$ are $a^{\dagger}$–commutative, which induces $a \in R^{SEP}$ by Theorem 3.1. □

Lemma 3.6. *Let $a,\ x,\ y \in R$ and $x,\ y$ are a–commutative. If $x,\ y \in R^{\#}$, then $x^{\#},\ y^{\#}$ are a–commutative.*

Proof. From the assumption, we have $ax = ya$. Then

$$ax^{\#} = axx^{\#}x^{\#} = yax^{\#}x^{\#} = y^{\#}y^2ax^{\#}x^{\#}.$$

By Lemma 3.2, one gets $y^2a = ax^2$, it implies

$$ax^{\#} = y^{\#}ax^2x^{\#}x^{\#} = y^{\#}axx^{\#}.$$

$$y^{\#}a = y^{\#}y^{\#}ya = y^{\#}y^{\#}ax = y^{\#}y^{\#}ax^2x^{\#} = y^{\#}y^{\#}yaxx^{\#} = y^{\#}axx^{\#}.$$

Hence, $ax^{\#} = y^{\#}a$, that is $x^{\#},\ y^{\#}$ are a–commutative. □

Noting that $(aa^{\#})^{\#} = aa^{\#}$, $((aa^{\#})^* aa^*)^{\#} = (a^{\#})^* a^{\dagger}$. Then Theorem 3.1 and Lemma 3.6 imply the following theorem.

Theorem 3.7. *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{SEP}$ if and only if $aa^{\#}$, $(a^{\#})^* a^{\dagger}$ are $a^{\dagger}$–commutative.*

Theorem 3.8. *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{SEP}$ if and only if $aa^{\dagger}$, $aa^*a^{\#}a$ are x–commutative for some $x \in \chi_a$.*

Proof. " $\Longrightarrow$ " Since $a \in R^{SEP}$, one gets $aaa^{\dagger} = a$, $aa^*a^{\#}aa = aa^*a = a$. Hence, $aaa^{\dagger} = aa^*a^{\#}aa$, that is $aa^{\dagger}$, $aa^*a^{\#}a$ are a–commutative. Thus, $x = a \in \chi_a$ is a solution.

" $\Longleftarrow$ " If there exists $x_0 \in \chi_a$ such that $aa^{\dagger}$, $aa^*a^{\#}a$ are x_0–commutative, then

$$x_0aa^{\dagger} = aa^*a^{\#}ax_0.$$

(1) If $x_0 \in \tau_a$, then $x_0a^{\dagger}a = x_0$ and $x_0^{\#}x_0 = aa^{\#}$, it follows that

$$x_0aa^{\dagger}a^{\dagger}a = aa^*a^{\#}ax_0a^{\dagger}a = aa^*a^{\#}ax_0 = x_0aa^{\dagger}$$

and

$$aa^{\dagger}a^{\dagger}a = aa^{\#}aa^{\dagger}a^{\dagger}a = x_0^{\#}x_0aa^{\dagger}a^{\dagger}a = x_0^{\#}x_0aa^{\dagger} = aa^{\#}aa^{\dagger} = aa^{\dagger}.$$

Hence, $a \in R^{EP}$. If follows that

$$aa^{\#} = aa^{\#}aa^{\dagger} = x_0^{\#}x_0aa^{\dagger} = x_0^{\#}aa^*a^{\#}ax_0 = x_0^{\#}aa^*x_0.$$

(a) If $x_0 = a$, then $aa^\# = a^\# aa^* a = a^* a$. So $a \in R^{SEP}$ by [7, Theorem 1.5.3].
(b) If $x_0 = a^\#$, then $aa^\# = a^2 a^* a^\#$, this infers

$$a = aa^\# a = a^2 a^* a^\# a = a^2 a^*.$$

Hence, $a \in R^{SEP}$ by [7, Theorem 1.5.3].
(c) If $x_0 = (a^\dagger)^* = (a^\#)^*$, then $aa^\# = a^* aa^* (a^\#)^* = a^* a$. So $a \in R^{SEP}$.
(2) If $x_0 \in \gamma_a$, then $x_0 aa^\dagger = x_0$ and $x_0 x_0^\dagger = a^\dagger a$, one obtains

$$a^\dagger a = x_0 x_0^\dagger = x_0 aa^\dagger x_0^\dagger = aa^* a^\# ax_0 x_0^\dagger = aa^* a^\# aa^\dagger a = aa^* a^\# a.$$

Hence, $aa^* a^\# a \in PE(R)$. By Theorme 2.2, $a \in R^{SEP}$. □

4. Using w–core inverse to characterize strongly *EP* elements

Let R be a $*$–ring and $a, w \in R$. If there exists $x \in R$ such that

$$x = awx^2, \ a = xawa, \ (awx)^* = awx,$$

then a is called w–core invertible and x is called the w–core inverse of a [12]. Denote by $a_w^{\textcircled{\#}} = x$.

Particularly, if a is a 1–core invertible element, then a is called core invertible and x is called the core inverse of a and denote it by $a^{\textcircled{\#}}$. Using the w–core invertibility, the following theorem gives a new characterization of SEP elements.

Theorem 4.1. *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{SEP}$ if and only if $a_{a^*}^{\textcircled{\#}} = aa^\#$.*

Proof. " $\Longrightarrow$ " Since $a \in R^{SEP}$, one has $aa^* aa^\# a^\dagger = a^\dagger aa^\#$ by Lemma 2.1, which implies

$$aa^* aa^\# = aa^* aa^\# a^\dagger a = a^\dagger aa^\# a = a^\dagger a = aa^\dagger.$$

Hence,

$$(aa^* aa^\#)^* = aa^* aa^\#,$$
$$aa^* (aa^\#)^2 = aa^\dagger aa^\# = aa^\#,$$

and

$$(aa^\#) aa^* a = aa^* a = a.$$

Thus, $a_{a^*}^{\textcircled{\#}} = aa^\#$.
" $\Longleftarrow$ " From the assumption, one gets

$$aa^* aa^\# = (aa^* aa^\#)^*,$$

$$aa^\# = aa^* (aa^\#)^2 = aa^* aa^\#.$$

Hence, $aa^* aa^\# \in PE(R)$ and, by Theorem 2.2, $a \in R^{SEP}$. □

Lemma 4.2. *Let $a \in R^\dagger$. Then $a_{a^*}^{\textcircled{\#}} = (a^\dagger)^* a^\dagger$.*

Proof. By the definition of w–core inverse, we can easily check

$$aa^* (a^\dagger)^* a^\dagger = aa^\dagger aa^\dagger = aa^\dagger = (aa^* (a^\dagger)^* a^\dagger)^*,$$

$$aa^* ((a^\dagger)^* a^\dagger)^2 = aa^\dagger (a^\dagger)^* a^\dagger = (a^\dagger)^* a^\dagger,$$

and

$$(a^\dagger)^* a^\dagger aa^* a = (a^\dagger)^* a^* a = a.$$

So $a_{a^*}^{\textcircled{\#}} = (a^\dagger)^* a^\dagger$. □

By Theorem 4.1 and Lemma 4.2, we have the following corollary.

Corollary 4.3. *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{SEP}$ if and only if $aa^{\#} = (a^{\dagger})^* a^{\dagger}$.*

Lemma 4.4. *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a^{\textcircled{\#}}_{a^*a^{\#}} = a(a^{\dagger})^* a^{\dagger}$.*

Proof. According to the definition of w–core inverse, we can verify directly

$$aa^*a^{\#}a(a^{\dagger})^*a^{\dagger} = aa^*(a^{\dagger})^*a^{\dagger} = aa^{\dagger} = (aa^*a^{\#}a(a^{\dagger})^*a^{\dagger})^*,$$

$$aa^*a^{\#}(a(a^{\dagger})^*a^{\dagger})^2 = aa^{\dagger}a(a^{\dagger})^*a^{\dagger} = a(a^{\dagger})^*a^{\dagger},$$

and

$$(a(a^{\dagger})^*a^{\dagger})aa^*a^{\#}a = a(a^{\dagger})^*a^*a^{\#}a = aa^{\#}a = a.$$

Hence, $a^{\textcircled{\#}}_{a^*a^{\#}} = a(a^{\dagger})^*a^{\dagger}$. □

Combining Corollary 4.3 and Lemma 4.4, we can easily get the following corollary.

Corollary 4.5. *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{SEP}$ if and only if $a^{\textcircled{\#}}_{a^*a^{\#}} = a$.*

Proof. " $\Longrightarrow$ " Since $a \in R^{SEP}$, we have $aa^{\#} = (a^{\dagger})^*a^{\dagger}$ by Corollary 4.3. Hence, $a^{\textcircled{\#}}_{a^*a^{\#}} = a((a^{\dagger})^*a^{\dagger}) = a(aa^{\#}) = a$.

" $\Longleftarrow$ " Assume that $a^{\textcircled{\#}}_{a^*a^{\#}} = a$, then we have $a(a^{\dagger})^*a^{\dagger} = a$ by Lemma 4.4. This infers

$$a = a^{\#}a^2 = a^{\#}(a(a^{\dagger})^*a^{\dagger})a = (a^{\dagger})^*.$$

Hence, $a \in R^{PI}$, this implies $a^{\dagger} = a^*$. Now we have $a = a(a^{\dagger})^*a^{\dagger} = a^2a^{\dagger}$. Thus, $a \in R^{SEP}$. □

Lemma 4.6. *Let $a \in R^{\#} \cap R^{\dagger}$. Then $(aa^*)^{\textcircled{\#}}_{a^{\#}} = a(a^{\dagger})^*a^{\dagger}$.*

Proof. Similar to the proof of Lemma 4.4, we can easily verify the result. □

Corollary 4.5 and Lemma 4.6 lead to the following theorem.

Theorem 4.7. *Let $a \in R^{\#} \cap R^{\dagger}$. Then the following statments are equivalent:*
(1) $a \in R^{SEP}$;
(2) $(aa^*)^{\textcircled{\#}}_{a^{\#}} = a$;
(3) $(a^{\dagger}a)^{\textcircled{\#}}_{a^{\#}} = a(a^{\dagger})^*a^{\dagger}$;
(4) $a^{\dagger}a^2 = a(a^{\dagger})^*a^{\dagger}$.

Proof. (1)$\Longrightarrow$(2) Since $a \in R^{SEP}$, by Lemma 4.6 we have

$$(aa^*)^{\textcircled{\#}}_{a^{\#}} = a(a^{\dagger})^*a^{\dagger} = aaa^{\#} = a.$$

(2)$\Longrightarrow$(3) Assume that $(aa^*)^{\textcircled{\#}}_{a^{\#}} = a$, then $a(a^{\dagger})^*a^{\dagger} = a$ by Lemma 4.6. Hence, $a \in R^{SEP}$ by Corollary 4.5, which infers $a^{\dagger}a = aa^*$. By Lemma 4.6, we have

$$(a^{\dagger}a)^{\textcircled{\#}}_{a^{\#}} = (aa^*)^{\textcircled{\#}}_{a^{\#}} = a(a^{\dagger})^*a^{\dagger}.$$

(3)$\Longrightarrow$(4) Noting that $(a^{\dagger}a)^{\textcircled{\#}}_{a^{\#}} = a^{\dagger}a^2$, then we have $a^{\dagger}a^2 = a(a^{\dagger})^*a^{\dagger}$ by (3).

(4)$\Longrightarrow$(1) Assume that $a^{\dagger}a^2 = a(a^{\dagger})^*a^{\dagger}$, this leads to

$$a = a^{\#}a^2 = a^{\#}(a^{\dagger}a^2)a = a^{\#}a(a^{\dagger})^*a^{\dagger}a = (a^{\dagger})^*.$$

Thus, $a \in R^{PI}$, which infers $a^{\dagger}a^2 = a(a^{\dagger})^*a^{\dagger} = a^2a^{\dagger}$. Then

$$aa^{\dagger} = a^{\#}a^2a^{\dagger} = a^{\#}a^{\dagger}a^2 = a^{\#}a.$$

Hence, $a \in R^{SEP}$. □

Lemma 4.4 and Lemma 4.6 induces us to give the following result, which proof is trivial.

Proposition 4.8. *Let R be a $*$−ring and $a,\ w,\ b,\ x \in R$. Then $a^{\textcircled{\#}}_{wb} = x$ if and only if $(aw)^{\textcircled{\#}}_{b} = x$ and $aR = awR$.*

References

[1] A. Ben-Israel and T. N. E. Greville, Generalized Inverses, Theory and Applications, 2nd ed., CMS Books Math./Ouvrages Math. SMC15, Springer, NewYork, 2003.
[2] A. Q. Li, M. G. Guan and J. C. Wei, *SEP* elements and solutions of related equations in a ring with involution, Filomat. 37(24)(2023), 8213-8228.
[3] D. D. Zhao and J. C. Wei, Strongly *EP* elements in rings with involution, J. Algebra Appl. 21(5)(2022), Article ID 2250088.
[4] D. Mosić and D. S. Djordjević, Moore-Penrose-invertible normal and Hermitian elements in rings, Linear Algebra Appl. 431(5-7)(2009), 732-745.
[5] D. Mosić and D. S. Djordjević, Further results on partial isometries and *EP* elements in rings with involution, Math. Comput. Modelling 54(1-2)(2011), 460-465.
[6] D. Mosić, D. S. Djordjević and J. J. Koliha, *EP* elements in rings, Linear Algebra Appl. 431(5-7)(2009), 527-535.
[7] D. Mosić, Generalized inverses, Faculty of Sciences and Mathematics, University of Niš, Niš, 2018.
[8] D. Mosić, S. Z. Xu and J. Benítez, Further results of special elements in rings with involution, Filomat. 34(10)(2020), 3381-3393.
[9] G. Szép, Simultaneous triangularization of projector matrices, Acta Math. Hungar. 48(3-4)(1986), 285-288.
[10] J. J. Koliha, D. Cvetković and D. S. Djordjević, Moore-Penrose inverse in rings with involution, Linear Algebra Appl. 426(2-3)(2007), 371-381.
[11] J. J. Koliha, The Drazin and Moore-Penrose inverse in C^*-algebras, Math. Proc. R. Ir. Acad. 99A(1)(1999), 17-27.
[12] L. Y. Wu, H. H. Zhu, Weighted w-core inverses in rings, Czecho. Math. J. 73 (2023) 581-602.
[13] M. G. Guan and J. C. Wei, Generalized inverse equations and *SEP* elements in a ring with involution, Georgian Math. J. 31(1)(2024) 25-33.
[14] R. Harte and M. Mbekhta, On generalized inverses in C^*-algebras, Studia Math. 103(1)(1992), 71-77.
[15] Y. C. Qu, S. T. Fan and J. C. Wei, Projections, one-sided idempotents and *SEP* elements in a ring with involution, Georgian Math. J. 2025.
[16] Y. K. Zhou and J. L. Chen, Generalized inverses and units in a unitary ring, Acta Math. Sin.(Engl. Ser.)40(4)(2024), 1000-1014.
[17] Y. Wang, X. R. Wang and J. C. Wei, $W-$core inverses in a ring with involution, Filomat. 38(27)(2024), 9507-9517.