Filomat 39:29 (2025), 10391–10400
https://doi.org/10.2298/FIL2529391H

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Quasi covering dimension of finite distributive lattices

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Abstract. In this paper, we study the quasi covering dimension dim_q of finite distributive lattices. By the join-irreducible elements, we characterize the dense elements of a finite lattice. Based on the notion of the widths of posets, we prove that for every finite distributive lattice L, $dim_q(L) = \max\{width(\uparrow a \cap \mathcal{J}(L)) \mid a \in \mathrm{Min}(\mathcal{J}(L))\} - 1$, where $\mathrm{Min}(\mathcal{J}(L))$ is the set of all minimal elements of join-irreducible elements of L. Finally, we study the quasi covering dimension of the linear sum and rectangular product of two finite distributive lattices.

1. Introduction

Dimension theory of posets is a branch of order theory that focuses on defining and studying the concept of dimension in certain classes of posets. It plays an important role in the study of posets. There are dimensions of posets, lattices and frames: order dimension, Krull dimension, small inductive dimension, covering dimension, and quasi-covering dimension have been studied in [2, 5–8, 10–18, 22–24]. In [8], Charalambous introduced the notion of σ-frames and established a meaningful dimension theory for normal σ-frames. In [10], Dube et al. developed the theory of covering dimension dim for finite lattices. Especially, they used the notion of minimal covers to characterize the covering dimension for finite lattices. In [24], Zhang et al. proved that $dim(L_1 \times L_2) = \max\{dim(L_1), dim(L_2)\}$ for all finite lattices L_1 and L_2. They gave a negative answer to the question being that whether the relation $dim(L_1 \circ L_2) \leq dim(L_1) + dim(L_2) + 1$ holds for all finite lattices L_1 and L_2.

In [15], Georgiou et al. introduced and studied a new kind of dimension in the realm of frames, called quasi covering dimension. In [6], Boyadzhiev et al. developed the theory for quasi covering dimension dim_q of finite lattices. In particular, they proved that for every finite lattice L, $dim_q(L) = \max\{dim(\downarrow x) \mid x \text{ is a dense element of } L\}$. So we can use the dense elements and the covering dimension to study the quasi covering dimension for finite lattices. Boyadzhiev et al. also showed in [6] that $dim_q(X \times Y) = \max\{dim_q(X), dim_q(Y)\}$ for all finite lattices X and Y. So far, we do not obtain the linear sum theorem and the rectangular product theorem of the quasi covering dimension of finite lattices.

2020 *Mathematics Subject Classification*. Primary 06A07; Secondary 06B05, 06D50
Keywords. Quasi covering dimension, finite distributive lattice, join-irreducible element, linear sum, rectangular product
Received: 30 January 2025; Revised: 02 August 2025; Accepted: 01 September 2025
Communicated by Ljubiša D. R. Kočinac
This work is supported by the National Natural Science Foundation of China (Grant no. 12471438) and the Fundamental Research Funds for the Central Universities (Grant no. GK202501014).
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The study of finite distributive lattices holds particular importance in the dimension theory of posets due to their well-behaved structure and rich connections to algebra, topology, combinatorics, and logic (see [4, 9, 19–21]). By Birkhoff's representation theorem of finite distributive lattices (see [3]), up to isomorphism, finite distributive lattices are in a one-to-one correspondence with finite posets via their sets of join-irreducible elements, which allows for a combinatorial interpretation of their properties. This makes them an ideal setting for investigating dimension-theoretic concepts. It was proved in [22] that for any finite distributive lattice L, $dim(L) = ord_L(\mathrm{Max}(\mathcal{J}(L)))$, where $\mathrm{Max}(\mathcal{J}(L))$ is the set of all maximal elements of join-irreducible elements of L. The sum and product theorems of the covering dimension of finite distributive lattices were also obtained in [22]. Additionally, for every finite topological space, its covering dimension equals the covering dimension of its open set lattice (see [8]). These findings motivate us to study the quasi covering dimension of finite distributive lattices.

In this paper, we focus on the quasi covering dimension dim_q of finite distributive lattices. By the notion of the widths of posets, we prove that for every finite distributive lattice L,

$$dim_q(L) = \max\{width(\uparrow a \cap \mathcal{J}(L)) | a \in \mathrm{Min}(\mathcal{J}(L))\} - 1,$$

where $\mathrm{Min}(\mathcal{J}(L))$ is the set of all minimal elements of join-irreducible elements of L. Moreover, we give the linear sum theorem and the rectangular product theorem of the quasi covering dimension of finite distributive lattices.

2. Preliminaries

Our references for elementary properties of lattices are [9, 20]. Let P be a poset and $Q \subseteq P$. Then $a \in Q$ is a *maximal element* of Q if $a \leq x$ and $x \in Q$ imply $a = x$. We denote the set of maximal elements of Q by $\mathrm{Max}(Q)$. A *minimal element* of $Q \subseteq P$ and $\mathrm{Min}(Q)$ are defined dually. The poset P is an *antichain* if $x \leq y$ in P only if $x = y$. The *width* of a poset P is n if there is an antichain in P of n elements and all antichains in P have $\leq n$ elements; in formula, $width(P) = n$. It is obvious that $\mathrm{Max}(P)$ and $\mathrm{Min}(P)$ are antichains. Let P be a poset and $Q \subseteq P$. If for every $a \in P$, there exists a subset A of Q such that $a = \bigvee A$, then Q is called join-dense in P.

Let P be a poset and $A \subseteq P$. A is called a *down-set* if, whenever $x \in A$, $y \in P$ and $y \leq x$, we have $y \in A$. For $A \subseteq P$, we write $A^l = \{y \in P | y \leq x \text{ for every } x \in A\}$ and $\downarrow A = \{y \in P | \text{there exists } x \in A \text{ such that } y \leq x\}$. Dually, we can define up-sets, A^u and $\uparrow A$. The set of down-sets of P will be denoted by $Down(P)$. Obviously, $(Down(P), \subseteq)$ is a frame. For every $\mathcal{A} \subseteq Down(P)$,

$$\bigvee \mathcal{A} = \bigcup \mathcal{A}, \ \bigwedge \mathcal{A} = \bigcap \mathcal{A}.$$

$\emptyset$ and P are the bottom element and the top element of $(Down(P), \subseteq)$, respectively. We use $\mathrm{Down}(P)$ to denote the down-set lattice $(Down(P), \subseteq)$.

Let P be a poset and $x, y \in P$. We say that x is *covered* by y (or x is a *lower cover* of y), and write $x \prec y$, if $x < y$ and $x \leq z < y$ implies $z = x$. Let L be a lattice with the bottom element 0_L. $a \in L$ is called an *atom* if $0_L \prec a$. The set of all atoms of L is denoted by $Atom(L)$.

Let L be a bounded lattice. The top element and the bottom element of L will be denoted by 1_L and 0_L, respectively. L is called *prime* if $x \wedge y = 0_L$ implies that $x = 0_L$ or $y = 0_L$ for all $x, y \in L$.

Definition 2.1. ([8]) Let L be a finite lattice. A subset C of L is called a *cover* of L if $\bigvee C = 1_L$ and $0_L \notin C$. A cover D of L is called a *refinement* of a cover C of L if for each $d \in D$, there exists $c \in C$ such that $d \leq c$.

Definition 2.2. ([8]) Let L be a finite lattice. The order of a subset C of L, $ord_L(C)$, is defined to be k, where $k \in \{0, 1, \dots\}$, if and only if the infimum of any $k + 2$ distinct elements of L is 0_L and there exist $k + 1$ distinct elements of L whose infimum is not 0_L.

Definition 2.3. ([8]) The function dim, called *covering dimension*, with domain the class of all finite lattices and range the set $\{0, 1, \dots\}$, is defined as follows:

(1) $dim(L) \leq k$, where $k \in \{0, 1, \dots\}$, if and only if for every cover C of L, there exists a cover R of L such that R is a refinement of C and $ord_L(R) \leq k$.

(2) $dim(L) = k$, where $k \in \{1, 2, \dots\}$, if $dim(L) \leq k$ and $dim(L) \nleq k - 1$.

Definition 2.4. ([6]) Let L be a finite lattice. A subset C of L is called a *quasi cover* of L if $\bigvee C = x$, where x is a dense element of L (that is, $x \wedge y \neq 0_L$, for every $y \in L \setminus \{0_L\}$). Two quasi covers C_1 and C_2 of L are called *similar* (in short, we write $C_1 \sim C_2$) if $\bigvee C_1 = \bigvee C_2$. A quasi cover D of L is called a *refinement* of a quasi cover C of L if for each $d \in D$, there exists $c \in C$ such that $d \leq c$.

Definition 2.5. ([6]) The function dim_q, called quasi covering dimension, with domain the class of all finite lattices and range the set $\{0, 1, \dots\}$, is defined as follows:

(1) $dim_q(L) \leq k$, where $k \in \{0, 1, \dots\}$, if and only if for every quasi cover C of L, there exists a quasi cover R of L such that $C \sim R$, R is a refinement of C and $ord_L(R) \leq k$.

(2) $dim_q(L) = k$, where $k \in \{0, 1, \dots\}$, if $dim_q(L) \leq k$ and $dim_q(L) \nleq k - 1$.

Proposition 2.6. ([6]) *For every finite lattice L, we have*

$$dim_q(L) = \max\{dim(\downarrow x) \mid x \text{ is a dense element of } L\}.$$

3. Quasi covering dimension of finite distributive lattices

In this section, we present an alternative approach of the quasi covering dimension paying attention to finite distributive lattices. This goal will be achieved by join-irreducible elements and the meaning of width.

Definition 3.1. ([9]) Let L be a lattice. An element $x \in L$ is *join-irreducible* if

(1) $x \neq 0_L$ (in case L has the bottom element 0_L);

(2) $x = a \vee b$ implies that $x = a$ or $x = b$ for all $a, b \in L$.

We denote the set of all join-irreducible elements of a lattice L by $\mathcal{J}(L)$.

Proposition 3.2. ([9]) *Let L be a finite lattice. Then $\mathcal{J}(L)$ is join-dense in L.*

Proposition 3.3. ([22]) *Let L be a finite lattice. Then for every $A \subseteq L$, $\bigwedge A = 0_L$ if and only if $A^l \cap \mathcal{J}(L) = \emptyset$.*

Corollary 3.4. *Let L be a finite lattice. Then for every $a, b \in L$, $a \wedge b = 0_L$ if and only if $(\downarrow a \cap \mathcal{J}(L)) \cap (\downarrow b \cap \mathcal{J}(L)) = \emptyset$.*

Proposition 3.5. *Let L be a finite lattice. Then x is a dense element of L if and only if* $\mathrm{Min}(\mathcal{J}(L)) \subseteq \downarrow x \cap \mathcal{J}(L)$.

Proof. Necessity. Assume that $\mathrm{Min}(\mathcal{J}(L)) \nsubseteq \downarrow x \cap \mathcal{J}(L)$. Then there exists $m \in \mathrm{Min}(\mathcal{J}(L))$ and $m \notin \downarrow x$. By the definition of minimal elements, we have that

$$(\downarrow m \cap \mathcal{J}(L)) \cap (\downarrow x \cap \mathcal{J}(L)) = \{m\} \cap (\downarrow x \cap \mathcal{J}(L)) = \emptyset.$$

By Corollary 3.4, $x \wedge m = 0_L$. This contradicts with the fact that x is a dense element of L. Thus $\mathrm{Min}(\mathcal{J}(L)) \subseteq \downarrow x \cap \mathcal{J}(L)$.

Sufficiency. Suppose that $\mathrm{Min}(\mathcal{J}(L)) \subseteq \downarrow x \cap \mathcal{J}(L)$. For every $y \in L \setminus \{0_L\}$, by Proposition 3.2, there exists $A \subseteq \mathcal{J}(L)$ such that $y = \bigvee A$. Then for every $a \in A$, there exists $a' \in \mathrm{Min}(\mathcal{J}(L))$ such that $a' \leq a$. This means that $\downarrow y \cap \mathrm{Min}(\mathcal{J}(L)) \neq \emptyset$. Therefore,

$$(\downarrow x \cap \mathcal{J}(L)) \cap (\downarrow y \cap \mathcal{J}(L)) \neq \emptyset.$$

By Corollary 3.4, $x \wedge y \neq 0_L$. So x is a dense element of L. □

Proposition 3.6. ([3]) *(Birkhoff's representation theorem for finite distributive lattices) Let L be a finite distributive lattice. Then the map*

$$\varphi : a \longmapsto spec(a)$$

is an isomorphism between L and $\mathrm{Down}(\mathcal{J}(L))$, *where $spec(a) = \{x \in \mathcal{J}(L) \mid x \leq a\} = \downarrow a \cap \mathcal{J}(L)$ is the spectrum of a.*

Proposition 3.7. ([22]) *Let L be a finite distributive lattice. Then $dim(L) = ord_L(\mathrm{Max}(\mathcal{J}(L)))$.*

Lemma 3.8. *Let L be a distributive lattice. Then for every sublattice M of L, M is a distributive lattice. In particular, for each $x \in L$, $\downarrow x$ is a distributive lattice.*

Proof. Since the join and meet in M coincide with those in L, the proof is straightforward. □

Proposition 3.9. *Let L be a finite lattice. Then for every $x \in L$, $\mathcal{J}(\downarrow x) = \downarrow x \cap \mathcal{J}(L)$.*

Proof. Let $a \in \mathcal{J}(\downarrow x)$ and $a = b \vee c$, where $b, c \in L$. It is obvious that $b, c \in \downarrow x$. Since $a \in \mathcal{J}(\downarrow x)$, $a = b$ or $a = c$. Then $a \in \downarrow x \cap \mathcal{J}(L)$. Thus $\mathcal{J}(\downarrow x) \subseteq \downarrow x \cap \mathcal{J}(L)$. Let $a \in \downarrow x \cap \mathcal{J}(L)$ and $a = b \vee c$, where $b, c \in \downarrow x$. Since $a \in \mathcal{J}(L)$, $a = b$ or $a = c$. Then $a \in \mathcal{J}(\downarrow x)$. So $\downarrow x \cap \mathcal{J}(L) \subseteq \mathcal{J}(\downarrow x)$. □

Let L be a finite lattice. We put

$$\mathrm{A}(\mathcal{J}(L)) = \max\{ord_L(A) \mid A \text{ is an antichain of } \mathcal{J}(L)\}.$$

Then we have the following proposition.

Proposition 3.10. *Let L be a finite lattice. Then*

$$\mathrm{A}(\mathcal{J}(L)) = \max\{width(\uparrow a \cap \mathcal{J}(L)) \mid a \in \mathrm{Min}(\mathcal{J}(L))\} - 1.$$

Proof. Let $\mathrm{A}(\mathcal{J}(L)) = k$, where $k \in \{0, 1, \ldots\}$. Assume that there exists $a \in \mathrm{Min}(\mathcal{J}(L))$ such that $width(\uparrow a \cap \mathcal{J}(L)) - 1 > k$. Then there exists an antichain $C \subseteq \uparrow a \cap \mathcal{J}(L)$ such that $|C| > k+1$, where $|C|$ denote the cardinality of C. By Proposition 3.3, we have that $\bigwedge C \neq 0_L$. Therefore, we find an antichain C of $\mathcal{J}(L)$ such that $ord_L(C) > k$, which contradicts with the fact that $\mathrm{A}(\mathcal{J}(L)) = k$. So $\max\{width(\uparrow a \cap \mathcal{J}(L)) \mid a \in \mathrm{Min}(\mathcal{J}(L))\} - 1 \leq k$.

Conversely, let $\max\{width(\uparrow a \cap \mathcal{J}(L)) \mid a \in \mathrm{Min}(\mathcal{J}(L))\} - 1 = k$, where $k \in \{0, 1, \ldots\}$. Assume that $\mathrm{A}(\mathcal{J}(L)) > k$. Then there exists an antichain A of $\mathcal{J}(L)$ such that $ord_L(A) > k$. Thus there exists an antichain $C \subseteq A \subseteq \mathcal{J}(L)$ such that $\bigwedge C \neq 0_L$ and $|C| > k+1$. By Proposition 3.3, we have that $C^l \cap \mathcal{J}(L) \neq \emptyset$. Therefore, there exists $a \in \mathrm{Min}(\mathcal{J}(L))$ such that $a \in C^l$. Then $C \subseteq \uparrow a \cap \mathcal{J}(L)$, and thus $width(\uparrow a \cap \mathcal{J}(L)) > k+1$. This contradicts with the fact that

$$\max\{width(\uparrow a \cap \mathcal{J}(L)) \mid a \in \mathrm{Min}(\mathcal{J}(L))\} - 1 = k.$$

So $\mathrm{A}(\mathcal{J}(L)) \leq k$. □

Lemma 3.11. *Let L be a finite lattice, and let M be a sublattice of L with $0_L \in M$. Then for every subset $A \subseteq M$, $ord_M(A) = ord_L(A)$.*

Proof. Since the join and meet in M coincide with those in L, the proof is straightforward. □

Theorem 3.12. *Let L be a finite distributive lattice. Then*

$$dim_q(L) = \max\{width(\uparrow a \cap \mathcal{J}(L)) \mid a \in \mathrm{Min}(\mathcal{J}(L))\} - 1.$$

Proof. On one hand,

$$\begin{aligned}
&dim_q(L)\\
=&\max\{dim(\downarrow x) \mid x \text{ is a dense element of } L\} \text{ (By Proposition 2.6)}\\
=&\max\{ord_{\downarrow x}(\mathrm{Max}(\mathcal{J}(\downarrow x))) \mid x \text{ is a dense element of } L\} \text{ (By Proposition 3.7)}\\
=&\max\{ord_{\downarrow x}(\mathrm{Max}(\downarrow x \cap \mathcal{J}(L))) \mid x \text{ is a dense element of } L\} \text{ (By Proposition 3.9)}\\
=&\max\{ord_L(\mathrm{Max}(\downarrow x \cap \mathcal{J}(L))) \mid x \text{ is a dense element of } L\} \text{ (By Lemma 3.11)}\\
\leq&\mathrm{A}(\mathcal{J}(L))\\
=&\max\{width(\uparrow a \cap \mathcal{J}(L)) \mid a \in \mathrm{Min}(\mathcal{J}(L))\} - 1 \text{ (By Proposition 3.10).}
\end{aligned}$$

On the other hand, suppose that

$$\max\{width(\uparrow a\cap\mathcal{J}(L))\,|\,a\in \mathrm{Min}(\mathcal{J}(L))\}-1=k,$$

where $k\in\{0,1,\dots\}$. It suffices to prove that $dim_q(L)\geq k$. Let $a\in\mathrm{Min}(\mathcal{J}(L))$ such that

$$width(\uparrow a\cap\mathcal{J}(L))=k+1.$$

Then there exists an antichain $C\subseteq\uparrow a\cap\mathcal{J}(L)$ with $|C|=k+1$. Let

$$P=(\downarrow C\cap\mathcal{J}(L))\cup\mathrm{Min}(\mathcal{J}(L)).$$

It is obvious that P is a down-set of $\mathcal{J}(L)$. By Proposition 3.6, there exists $x\in L$ such that $spec(x)=\downarrow x\cap\mathcal{J}(L)=P$ and $x=\bigvee P$. By Proposition 3.5, x is a dense element of L. As

$$\mathcal{J}(\downarrow x)=\downarrow x\cap\mathcal{J}(L)=(\downarrow C\cap\mathcal{J}(L))\cup\mathrm{Min}(\mathcal{J}(L)),$$

we can conclude that

$$\mathrm{Max}(\mathcal{J}(\downarrow x))=C\cup\{m\in\mathrm{Min}(\mathcal{J}(L))\,|\,m\notin\downarrow C\}.$$

Since $a\in\mathrm{Min}(\mathcal{J}(L))$ and $C\subseteq\uparrow a\cap\mathcal{J}(L)$, $a\in C^l\cap\mathcal{J}(L)$. By Proposition 3.3, $\bigwedge C\neq 0_L$. Then

$$ord_L(\mathrm{Max}(\mathcal{J}(\downarrow x)))\geq|C|-1=k.$$

For every subset A of $\mathrm{Max}(\mathcal{J}(\downarrow x))$ with $k+2$ elements, since $|C|=k+1$, there exists $b\in A\cap\{m\in\mathrm{Min}(\mathcal{J}(L))\,|\, m\notin\downarrow C\}$. By Proposition 3.3, we can obtain that $\bigwedge A=0_L$. Then $ord_L(\mathrm{Max}(\mathcal{J}(\downarrow x)))=k$. By Proposition 3.7, we have that

$$dim(\downarrow x)=ord_{\downarrow x}(\mathrm{Max}(\mathcal{J}(\downarrow x)))=ord_L(\mathrm{Max}(\mathcal{J}(\downarrow x)))=k.$$

So

$$k=dim(\downarrow x)\leq\max\{dim(\downarrow x)\,|\,x \text{ is a dense element of } L\}=dim_q(L).$$

□

Example 3.13. (1) Let L be a finite distributive lattice. The Hasse diagram of L is Fig. 1.

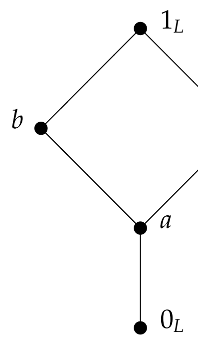

Fig. 1 Hasse diagram of L

Then the Hasse diagram of $\mathcal{J}(L)$ is Fig. 2.

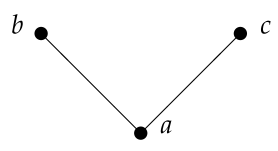

Fig. 2 Hasse diagram of $\mathcal{J}(L)$

We can see that $\mathrm{Max}(\mathcal{J}(L)) = \{b, c\}$ and $\mathrm{Min}(\mathcal{J}(L)) = \{a\}$. Then

$$dim(L) = ord_L(\mathrm{Max}(\mathcal{J}(L))) = 1.$$

Since $width(\uparrow a \cap \mathcal{J}(L)) = 2$, we have that

$$dim_q(L) = width(\uparrow a \cap \mathcal{J}(L)) - 1 = 2 - 1 = 1.$$

(2) Let L be a finite distributive lattice. The Hasse diagram of L is Fig. 3.

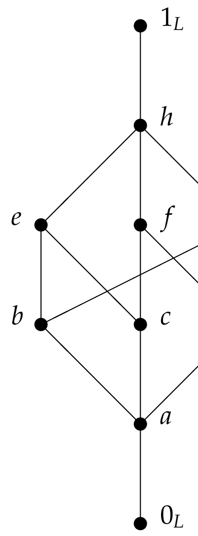

Fig. 3 Hasse diagram of L

Then the Hasse diagram of $\mathcal{J}(L)$ is Fig. 4.

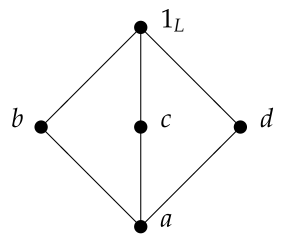

Fig. 4 Hasse diagram of $\mathcal{J}(L)$

We can obtain that $\mathrm{Max}(\mathcal{J}(L)) = \{1_L\}$ and $\mathrm{Min}(\mathcal{J}(L)) = \{a\}$. Then

$$dim(L) = ord_L(\mathrm{Max}(\mathcal{J}(L))) = 0.$$

Since $width(\uparrow a \cap \mathcal{J}(L)) = 3$, we have that

$$dim_q(L) = width(\uparrow a \cap \mathcal{J}(L)) - 1 = 3 - 1 = 2.$$

(3) Let L be a finite distributive lattice. The Hasse diagram of L is Fig. 5.

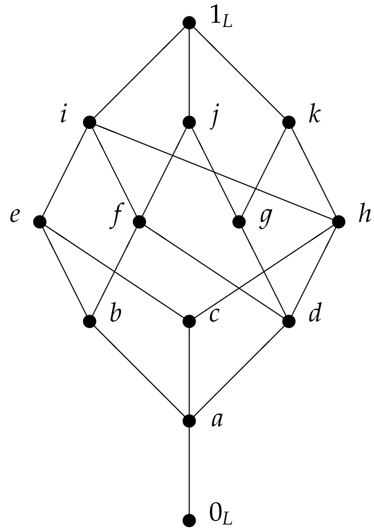

Fig. 5 Hasse diagram of L

Then the Hasse diagram of $\mathcal{J}(L)$ is Fig. 6.

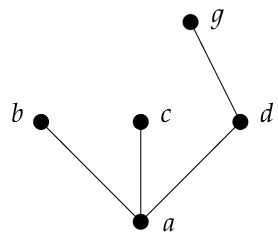

Fig. 6 Hasse diagram of $\mathcal{J}(L)$

We can conclude that $\mathrm{Max}(\mathcal{J}(L)) = \{b, c, g\}$ and $\mathrm{Min}(\mathcal{J}(L)) = \{a\}$. Then

$$dim(L) = ord_L(\mathrm{Max}(\mathcal{J}(L))) = 2.$$

Since $width(\uparrow a \cap \mathcal{J}(L)) = 3$,

$$dim_q(L) = width(\uparrow a \cap \mathcal{J}(L)) - 1 = 3 - 1 = 2.$$

4. Sum and product theorems

Definition 4.1. ([9]) Let $(L_1, \leq_1)$ and $(L_2, \leq_2)$ be disjoint posets. The *linear sum* $L_1 \oplus L_2$ is the poset $(L_1 \cup L_2, \leq)$, where the relation $\leq$ is defined as follows:

$$x \leq y \Longleftrightarrow \begin{cases} x, y \in L_1 \textit{ and } x \leq_1 y \textit{ or} \\ x, y \in L_2 \textit{ and } x \leq_2 y \textit{ or} \\ x \in L_1, y \in L_2. \end{cases}$$

It is easy to see that if $(L_1, \leq_1)$ and $(L_2, \leq_2)$ are lattices, then $L_1 \oplus L_2$ is a lattice.

Proposition 4.2. ([22]) *Let L_1 and L_2 be distributive lattices. Then $L_1 \oplus L_2$ is a distributive lattice.*

Lemma 4.3. *Let $P_1, P_2, \ldots, P_n$ be posets. Then*

$$width(\bigoplus_{i=1}^{n} P_i) = \max\{width(P_i) \mid i \in \{1, 2, \ldots, n\}\}.$$

Proof. By Definition 4.1, the proof is straightforward. □

Proposition 4.4. ([22]) *Let L_1 and L_2 be bounded lattices. Then $\mathcal{J}(L_1 \oplus L_2) = \{0_{L_2}\} \cup \mathcal{J}(L_1) \cup \mathcal{J}(L_2)$.*

Theorem 4.5. *Let L_1 and L_2 be finite distributive lattices. Then*

$$dim_q(L_1 \oplus L_2) = \max\{dim_q(L_1), width(\mathcal{J}(L_2)) - 1\}.$$

Proof. By Theorem 3.12, we have that

$$dim_q(L_1 \oplus L_2) = \max\{width(\uparrow a \cap \mathcal{J}(L_1 \oplus L_2)) \,|\, a \in \mathrm{Min}(\mathcal{J}(L_1 \oplus L_2))\} - 1.$$

By Definition 4.1 and Proposition 4.4, we conclude that

$$\mathrm{Min}(\mathcal{J}(L_1 \oplus L_2)) = \mathrm{Min}(\mathcal{J}(L_1)).$$

Then

$$dim_q(L_1 \oplus L_2) = \max\{width(\uparrow a \cap \mathcal{J}(L_1 \oplus L_2)) \,|\, a \in \mathrm{Min}(\mathcal{J}(L_1))\} - 1.$$

For every $a \in \mathrm{Min}(\mathcal{J}(L_1))$, by Proposition 4.4,

$$\uparrow a \cap \mathcal{J}(L_1 \oplus L_2) = (\uparrow a \cap \mathcal{J}(L_1)) \cup \{0_{L_2}\} \cup \mathcal{J}(L_2).$$

For every $a \in \mathrm{Min}(\mathcal{J}(L_1))$, by Lemma 4.3, we can obtain that

$$width(\uparrow a \cap \mathcal{J}(L_1 \oplus L_2)) = \max\{width(\uparrow a \cap \mathcal{J}(L_1)), width(\mathcal{J}(L_2))\}.$$

Therefore,

$$\begin{aligned}
&dim_q(L_1 \oplus L_2)\\
&= \max\{\max\{width(\uparrow a \cap \mathcal{J}(L_1)), width(\mathcal{J}(L_2))\} \,|\, a \in \mathrm{Min}(\mathcal{J}(L_1))\} - 1\\
&= \max\{\max\{width(\uparrow a \cap \mathcal{J}(L_1)) \,|\, a \in \mathrm{Min}(\mathcal{J}(L_1))\} - 1, width(\mathcal{J}(L_2)) - 1\}\\
&= \max\{dim_q(L_1), width(\mathcal{J}(L_2)) - 1\}.
\end{aligned}$$

□

Definition 4.6. ([1]) Let $(L_1, \leq_1)$ and $(L_2, \leq_2)$ be bounded lattices. The *rectangular product* $L_1 \Box L_2$ is the poset $(L_1 \Box L_2, \leq)$, where

$$L_1 \Box L_2 = \{(x, y) \in L_1 \times L_2 \,|\, x \neq 0_{L_1} \text{and } y \neq 0_{L_2}\} \cup \{(0_{L_1}, 0_{L_2})\}$$

and the relation $\leq$ is defined as follows:

$$(x_1, x_2) \leq (y_1, y_2) \Longleftrightarrow x_1 \leq_1 y_1 \text{ and } x_2 \leq_2 y_2,$$

for every $(x_1, x_2), (y_1, y_2) \in (L_1 \Box L_2, \leq)$.

It is easy to see that

$$(x_1, x_2) \wedge (y_1, y_2) = \begin{cases} (x_1 \wedge y_1, x_2 \wedge y_2), & \text{if } x_1 \wedge y_1 \neq 0_{L_1} \text{and } x_2 \wedge y_2 \neq 0_{L_2} \\ (0_{L_1}, 0_{L_2}), & \text{otherwise} \end{cases}$$

and $(x_1, x_2) \vee (y_1, y_2) = (x_1 \vee y_1, x_2 \vee y_2)$.

Remark 4.7. It follows from Definition 4.6 that when L_1 (*resp.*, L_2) is isomorphic to **2** (**2** is a chain with two elements), $L_1 \Box L_2$ is isomorphic to L_2 (*resp.*, L_1). We have that

$$dim_q(L_1 \Box L_2) = \begin{cases} dim_q(L_2), & \text{if } L_1 \text{ is isomorphic } \mathbf{2} \\ dim_q(L_1), & \text{if } L_2 \text{ is isomorphic } \mathbf{2}. \end{cases}$$

Proposition 4.8. ([22]) *Let L_1 and L_2 be finite lattices. Then*

$$\mathcal{J}(L_1 \square L_2) = \{(x,b) \mid x \in \mathcal{J}(L_1), b \in Atom(L_2)\} \cup \{(a,y) \mid a \in Atom(L_1), y \in \mathcal{J}(L_2)\}.$$

Proposition 4.9. ([1]) *Let L_1 and L_2 be bounded lattices. Then the following statements are equivalent:*
(1) $L_1 \square L_2$ is distributive.
(2) L_1 and L_2 are distributive and
*(a) L_1 or L_2 is isomorphic to **2** or*
(b) L_1 and L_2 are prime.

Proposition 4.10. ([22]) *Let L be a finite lattice. Then L is prime if and only if $|Atom(L)| = 1$.*

Theorem 4.11. *Let L_1, L_2 be finite prime distributive lattices. If neither L_1 nor L_2 are isomorphic to **2**, then*

$$dim_q(L_1 \square L_2) = width(\mathcal{J}(L_1)) + width(\mathcal{J}(L_2)) - 1.$$

Proof. If neither L_1 nor L_2 are isomorphic to **2**, by Proposition 3.2, we have that $|\mathcal{J}(L_1)| > 1$ and $|\mathcal{J}(L_2)| > 1$. By Proposition 4.10, $|Atom(L_1)| = |Atom(L_2)| = 1$. Let a_1 and a_2 be the atoms of L_1 and L_2, respectively. By Proposition 4.8,

$$\mathcal{J}(L_1 \square L_2) = \{(x,a_2) \mid x \in \mathcal{J}(L_1)\} \cup \{(a_1,y) \mid y \in \mathcal{J}(L_2)\}.$$

Since an atom is a join-irreducible element, (a_1, a_2) is the bottom element of $\mathcal{J}(L_1 \square L_2)$. Then

$$\mathrm{Min}(\mathcal{J}(L_1 \square L_2)) = (a_1, a_2).$$

Let $width(\mathcal{J}(L_1)) = k_1$ and $width(\mathcal{J}(L_2)) = k_2$. By Theorem 3.12, it suffices to prove that

$$width(\uparrow(a_1,a_2) \cap \mathcal{J}(L_1 \square L_2)) = k_1 + k_2.$$

Since a_1 and a_2 are the atoms of L_1 and L_2, respectively, a_1 and a_2 are the bottom elements of $\mathcal{J}(L_1)$ and $\mathcal{J}(L_2)$, respectively. As $|\mathcal{J}(L_1)| > 1$ and $|\mathcal{J}(L_2)| > 1$, there exists an antichain $A \subseteq \mathcal{J}(L_1) \setminus \{a_1\}$ and an antichain $B \subseteq \mathcal{J}(L_2) \setminus \{a_2\}$ such that $|A| = k_1$ and $|B| = k_2$. By Definition 4.6, we can conclude that for every $a \in \{(x,a_2) \mid x \in \mathcal{J}(L_1)\}$ and every $b \in \{(a_1,y) \mid y \in \mathcal{J}(L_2)\}$, if $a, b \neq (a_1, a_2)$, a and b are incomparable. Then $\{(a_1,b) \mid b \in B\} \cup \{(a,a_2) \mid a \in A\}$ is an antichain of $\uparrow(a_1,a_2) \cap \mathcal{J}(L_1 \square L_2)$ such that

$$|\{(a_1,b) \mid b \in B\} \cup \{(a,a_2) \mid a \in A\}| = k_1 + k_2.$$

Assume that there exists an antichain $C \subseteq \uparrow(a_1,a_2) \cap \mathcal{J}(L_1 \square L_2)$ such that $|C| > k_1 + k_2$. It is obvious that $|C| > 2$. We can conclude that $(a_1,a_2) \notin C$. Take $C_1 = \{(x,y) \in C \mid x = a_1\}$ and $C_2 = \{(x,y) \in C \mid y = a_2\}$. Since $C_1 \cap C_2 = \emptyset$ and $C_1 \cup C_2 = C$, $|C_1| > k_2$ or $|C_2| > k_1$. Without loss of generality, suppose that $|C_1| > k_2$. Then $\{y \in L_2 \mid (a_1, y) \in C_1\}$ is an antichain of $\mathcal{J}(L_2)$ such that

$$|\{y \in L_2 \mid (a_1,y) \in C_1\}| > k_2.$$

This contradicts with the fact that $width(\mathcal{J}(L_2)) = k_2$. So

$$width(\uparrow(a_1,a_2) \cap \mathcal{J}(L_1 \square L_2)) = k_1 + k_2.$$

Therefore, $dim_q(L_1 \square L_2) = width(\mathcal{J}(L_1)) + width(\mathcal{J}(L_2)) - 1$. $\square$

Acknowledgements

The authors would like to thank the Section Editor, Professor Ljubiša Kočinac, and the anonymous reviewer for their valuable comments and suggestions.

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