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Remarks on relativization of star-K-Menger spaces

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Abstract. A subspace Y of a space X is said to have the *relatively star-K-Menger* (resp., *relatively star-C-Menger*) property in X if for every sequence $(\mathcal{U}_n : n \in \omega)$ of open covers of X, there exists a sequence $(K_n : n \in \omega)$ of compact (resp., countably compact) subsets of X such that $(\mathrm{St}(K_n, \mathcal{U}_n) : n \in \omega)$ forms an open cover of Y. In this paper, we investigate the topological properties of relatively star-K-Menger and relatively star-C-Menger subspaces. We establish a variety of preservation results under natural topological operations, and derive cardinal restrictions by bounding the extent of such subspaces in terms of the classical invariants $\mathfrak{b}$ and $\mathfrak{d}$. Examples demonstrate non-preservation in products and distinctions from related properties, establishing a unified framework for relative star-selection principles.

1. introduction and preliminaries

The star-covering properties were first systematically studied by van Douwen et al. [9], introducing key concepts that have shaped further research. Matveev's survey [17] provides a detailed overview, highlighting their role in the development of generalized covering properties.

A systematic study of relative topological properties was initiated by Arhangel'skii, in a series of his papers (see [1, 2]). Given a topological property P, one can associate a relative version of it formulated in terms of location of Y in X in such a natural way that when $Y = X$, the relative property coincides with P.

The main investigation of relative selection principles was commenced by Kocinac (see [11, 13, 15]), and then a number of papers were published in this area ([3, 6, 7, 16, 19–24, 26]). The relative versions of some star selection principles were introduced and studied by Bonanzinga and Pansera in [4]. Later Sen [18] studied the relative versions of star-Δ-Menger spaces.

The purpose of this paper is to investigate the relationship between relatively star-K-Menger spaces and related spaces, and study topological properties of relatively star-K-Menger spaces.

Throughout the paper by "a space" we mean "a topological space", a family $\mathcal{U}$ of subsets X is an open cover of $A \subseteq X$ if the elements of $\mathcal{U}$ are open in X and $A \subset \bigcup \mathcal{U} = \bigcup\{U : U \in \mathcal{U}\}$.

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Given a space X, a collection $\mathcal{F}$ of its subsets and $A \subseteq X$, *the star of A with respect to $\mathcal{F}$*, denoted by $\mathrm{St}(A, \mathcal{F})$, is the set $\mathrm{St}(A, \mathcal{F}) = \bigcup\{F \in \mathcal{F} : F \cap A \neq \emptyset\}$. We write $\mathrm{St}(x, \mathcal{F})$ instead of $\mathrm{St}(\{x\}, \mathcal{F})$ for all $x \in X$.

A nonempty collection $\mathcal{I}$ of subsets of space X is called an ideal in X if it has the following properties: (i) If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$ (*hereditary*) (ii) If $A \in \mathcal{I}$ and $B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$ (finite additivity).

We first recall the classical notions of spaces that are used in this paper.

Definition 1.1. A space X is said to be

1. $\mathcal{K}$-starcompact [27] if for every open cover $\mathcal{U}$ of X there exists a compact subset F of X such that $\mathrm{St}(F, \mathcal{U}) = X$.
2. C-starcompact [28] if for every open cover $\mathcal{U}$ of X there exists a countably compact subset C of X such that $\mathrm{St}(C, \mathcal{U}) = X$.

Throughout the paper, the cardinality of a set is denoted by $|A|$. Let ω denote the first infinite cardinal, ω_1 the first uncountable cardinal, $\mathfrak{c}$ the cardinality of the set of real numbers. For a cardinal κ, let κ^+ be the smallest cardinal greater than κ. For each pair of ordinals α, β with $\alpha < \beta$, we write $[\alpha, \beta) = \{\gamma : \alpha \leq \gamma < \beta\}$, $(\alpha, \beta] = \{\gamma : \alpha < \gamma \leq \beta\}$, $(\alpha, \beta) = \{\gamma : \alpha < \gamma < \beta\}$, $[\alpha, \beta] = \{\gamma : \alpha \leq \gamma \leq \beta\}$. As usual, a cardinal is an initial ordinal and an ordinal is the set of smaller ordinals. A cardinal is often viewed as a space with the usual order topology. Other terms and symbols follow [10].

Definition 1.2. A space X is said to be star-Menger [12] if for each sequence $(\mathcal{U}_n : n \in \omega)$ of open covers of X, there is a sequence $(\mathcal{V}_n : n \in \omega)$ such that for each $n \in \omega$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $X = \bigcup_{n\in\omega} \mathrm{St}(\cup\mathcal{V}_n, \mathcal{U}_n)$.

Definition 1.3. A subspace Y of a space X is said to be relatively star-Menger (in short, RSM) [4] in X if for each sequence $(\mathcal{U}_n : n \in \omega)$ of open covers of X, there is a sequence $(\mathcal{V}_n : n \in \omega)$ such that for each $n \in \omega$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and $Y \subset \bigcup_{n\in\omega} \mathrm{St}(\cup\mathcal{V}_n, \mathcal{U}_n)$.

Consider a subcollection Δ of the power set $\mathcal{P}(X)$ of X, which is closed under finite unions and contains singletons.

Definition 1.4. ([18]) A space X is said to have the *star-Δ-Menger property* in X, if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X, there exists a sequence $(A_n : n \in \mathbb{N})$ of members of Δ such that

$$X = \bigcup_{n\in\mathbb{N}} \mathrm{St}(A_n, \mathcal{U}_n).$$

Definition 1.5. ([18]) A subset A of a space X is said to have the *relative star-Δ-Menger property* in X, or A is a *relatively star-Δ-Menger subspace* of X, if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X, there exists a sequence $(A_n : n \in \mathbb{N})$ of members of Δ such that

$$A \subseteq \bigcup_{n\in\mathbb{N}} \mathrm{St}(A_n, \mathcal{U}_n).$$

If we put Δ=the collection of finite subsets of X (resp., the collection of compact subsets of X, the collection of countably compact subsets of X) in Definition 1.4, we get the notions of strongly star Menger [12] (resp., star-K-Menger [12], star-C-Menger [30]).

If we put Δ=the collection of finite subsets of X (resp., the collection of compact subsets of X, the collection of countably compact subsets of X) in Definition 1.5, we get the notions of relatively strongly star Menger [4] (resp., relatively star-K-Menger [18], relatively star-C-Menger [18]).

It is clear by the definition that every relatively strongly star-Menger subspace is relatively star-K-Menger, every relatively star-K-Menger subspace is relatively star-C-Menger subspace and relatively star-K-Menger subspace is relatively star-Menger.

The set of all relatively star-K-Menger subspaces of X is denoted by RSKM(X).

The set of all relatively star-C-Menger subspaces of X is denoted by RSCM(X).

Definition 1.6. A subspace Y of a space X is said to be relatively (∗) [4, 5] if for every open cover $\mathcal{U}$ of X there exists a compact subset $K \subset X$ such that $Y \subseteq \mathrm{St}(K, \mathcal{U})$.

Similarly we may define the following.

Definition 1.7. A subspace Y of a space X is said to be relatively (∗∗) if for every open cover $\mathcal{U}$ of X there exists a countably compact subset $C \subset X$ such that $Y \subseteq \mathrm{St}(C, \mathcal{U})$.

Note that if a subspace Y of a space X is relatively (∗) in X, then it is relatively (∗∗) in X.

An ideal I of subsets of a space X is said to be admissible or free if $\{x\} \in I$ for each $x \in X$. An ideal I is $\sigma-$ideal if it is closed under countable union. An ideal I of subsets of a space X is proper if $X \notin I$.

For a space X, the extent $e(X)$ of X is the supremum of cardinalities of closed discrete subsets of X.

2. Relatively star-K-Menger and relatively star-C-Menger

In this section, we examine the relationships between relatively star- K-Menger and relatively star-C-Menger spaces, along with other related classes.

Theorem 2.1. ([18]) *The set of all relatively star-Δ-Menger subsets of a space X forms an admissible σ-ideal in X.*

Corollary 2.2. *The set of all relatively star-K-Menger (resp., relatively star-C-Menger) forms an admissible σ-ideal in X.*

It can be observed that when X is star-K-Menger(resp., star-C-Menger), then RSKM(X) (resp., RSCM(X)) is not a proper ideal.

Corollary 2.3. *A space X is star-K-Menger (resp., star-C-Menger) if and only if RSKM(X) = $\mathcal{P}$(X) (resp., RSCM(X) = $\mathcal{P}$(X)).*

Lemma 2.4. *A space X is relatively* (∗) *(resp., relatively* (∗∗)*) if and only if X is K-starcompact (resp., C-starcompact).*

The following theorem follows directly from the definitions.

Theorem 2.5. *If a subspace Y is relatively* (∗) *in X, then Y is relatively star-K-Menger subspace of X.*

However, the converse of above theroem is not true.

Example 2.6. There exists a relatively star-K-Menger subspace of X which is not relatively (∗) in X.

Proof. Let $X = \omega$ be the countably infinite discrete space. Then X is a Tychonoff star-K-Menger space, since every countable space is Menger. Hence by Theorem 2.1, X is a relatively star-K-Menger subspace. Since X is not a K-starcompact space, by Lemma 2.4, it is not a relatively (∗) subspace in itself. □

Theorem 2.7. *If a subspace Y is relatively* (∗∗) *in X, then Y is relatively star-C-Menger subspace of X.*

The following example shows that the converse of above theorem is not true.

Example 2.8. Let

$$X = [0, \omega] \times [0, \omega] \setminus \{\omega, \omega\}$$

be the subspace of the product space $[0, \omega] \times [0, \omega]$. Then X is a Tychonoff space.

Since X is a countable space, hence it is a Menger space. Hence X is a star-C-Menger space. Hence by Theorem 2.1, X is a relatively star-C-Menger subspace of X.

Song [29] shows that X is not a C-starcompact space. Hence by Lemma 2.8, X is not a relatively (∗∗) in X.

Definition 2.9. ([5]) A subspace Y of a space X is said to be relatively countable (*) if for every open cover $\mathcal{U}$ of X there exists a countable subset W of X such that $Y \subseteq \mathrm{St}(W, \mathcal{U})$.

Example 2.10. There exists a Tychonoff space containing a relatively star-K-Menger subspace which is not relatively strongly star-Menger.

Proof. Let $D = \{d_\alpha : \alpha < \mathfrak{c}\}$ be a discrete space of cardinality $\mathfrak{c}$ and let $aD = \mathrm{D} \cup \{d^*\}$ be one-point compactification of D. Let

$$Y = (aD \times [0, \mathfrak{c}^+)) \cup (D \times \{c^+\})$$

be the subspace of the product space $X = aD \times [0, \mathfrak{c}^+]$. Clearly Y is a Tychonoff space. In [26, Example 2.2], Song shows that Y is star-K-Menger space. Hence by Theorem 2.1, Y is a relatively star-K-Menger susbpace. Then by corollary 2.3, $\mathrm{M} = D \times \{c^+\}$ is a relatively star-K-Menger subspace of Y.

Next we show that $\mathrm{M} = D \times \{c^+\}$ is not a relatively strongly star-Menger subspace of Y. We only show that M is not a relatively countable (*), since every relatively strongly star-Menger subspace is relatively countable (*). Let us consider the open cover

$$\mathcal{U} = \{\{d_\alpha\} \times [0, \mathfrak{c}^+] : \alpha < \mathfrak{c}\} \cup \{aD \times [0, \mathfrak{c}^+)\}$$

of X. We need to show that $\mathrm{St}(F, \mathcal{U}) \neq X$ for any countable subset F of X. To show this, let F be any countable subset of X. Then there exists $\alpha_0 < \mathfrak{c}$ such that $F \cap (\{d_{\alpha_0}\} \times [0, \mathfrak{c}^+]) = \emptyset$. Hence $\left\langle d_{\alpha_0}, \mathfrak{c}^+ \right\rangle \notin \mathrm{St}(F, \mathcal{U})$, since $\{d_{\alpha_0}\} \times [0, \mathfrak{c}^+]$ is the only element of $\mathcal{U}$ containing the point $\left\langle d_{\alpha_0}, \mathfrak{c}^+ \right\rangle$, which shows that M is not relatively countable (*) subspace. □

The following theorem follows directly from the definitions.

Theorem 2.11. *For a topological space X, RSKM(X) ⊆ RSCM(X).*

The following example shows that the converse of above theorem is not true.

Example 2.12. There exists a Tychonoff space containing a relatively star-C-Menger subspace which is not relatively star-K-Menger.

Proof. Let $X_1 = [0, \mathfrak{c}]$ be with the usual topology. Then X_1 is compact. Let $X_2 = [0, \mathfrak{c}]$ as follows: for each $\alpha < \mathfrak{c}, \alpha$ is isolated and a set U containing $\mathfrak{c}$ is open if and only if $X \setminus U$ is finite. Then X_2 is compact.

Let

$$X = (X_1 \times X_2) \setminus \{\langle \mathfrak{c}, \mathfrak{c} \rangle\}$$

be the subspace of the product space $X_1 \times X_2$. Song [27, Example 2.2] shows that X is a star-C-Menger space. Thus by Corollary 2.6, $\mathrm{M} = \{\langle \mathfrak{c}, \alpha \rangle : \alpha < \mathfrak{c}\}$ is a relatively star-C-Menger subspace of X.

Now we show that M is not a relatively star-K-Menger. For each $\alpha < \mathfrak{c}$, let

$$U_\alpha = (\alpha, \mathfrak{c}] \times \{\alpha\}.$$

Then $U_\alpha \cap U_{\alpha'} = \emptyset$ for $\alpha \neq \alpha'$. For each $n \in \omega$, let

$$\mathcal{U}_n = \{U_\alpha : \alpha < \mathfrak{c}\} \cup \{[0, \mathfrak{c}) \times X_2\}.$$

Consider the sequence $(\mathcal{U}_n : n \in \omega)$ of open covers of X. Let $(K_n : n \in \omega)$ be any sequence of compact subsets of X. For each $n \in \omega$, since K_n is compact and $\{\langle \mathfrak{c}, \alpha \rangle : \alpha < \mathfrak{c}\}$ is a discrete closed subset of X, the set $K_n \cap \{\langle \mathfrak{c}, \alpha \rangle : \alpha < \mathfrak{c}\}$ is finite. Then there exists $\alpha_n < \mathfrak{c}$ such that

$$K_n \cap \{\langle \mathfrak{c}, \alpha \rangle : \alpha < \mathfrak{c}\} = \emptyset.$$

Let $\alpha' = sup\{\alpha_n : n \in \omega\}$. Then $\alpha' < \mathfrak{c}$ and

$$(\bigcup_{n \in \omega} K_n) \cap \{\langle \mathfrak{c}, \alpha \rangle : \alpha > \alpha'\} = \emptyset.$$

Let L = $\bigcup_{n\in\omega}$ K$_n$. Then L is σ-compact, hence L is Lindelöf. Let

$$L' = L \setminus \bigcup\{U_\alpha : \alpha > \alpha'\}.$$

If L′ = ∅, we can pick $\alpha_0 > \alpha'$, then L ∩ $U_{\alpha_0} = \emptyset$, hence $\langle \mathfrak{c}, \alpha_0\rangle \notin \mathrm{St}(K_n, \mathcal{U}_n)$ for each $n \in \omega$, since U_{α_0} is the only element of $\mathcal{U}_n$ containing the point $\langle \mathfrak{c}, \alpha_0\rangle$ for each $n \in \omega$. On the other hand, if L′ ≠ ∅, since L′ is closed in L, L′ is Lindelöf and L′ ⊆ $[0, \mathfrak{c}) \times X_2$. Therefore, $\pi(L')$ is a Lindelöf subset of a countably compact space $[0, \mathfrak{c})$, where $\pi : [0, \mathfrak{c}) \times X_2 \to [0, \mathfrak{c})$ is the projection. Hence there exists an $\alpha_1 < \mathfrak{c}$ such that $\pi(L') \cap (\alpha_1, \mathfrak{c}) = \emptyset$. Choose $\beta < \mathfrak{c}$ such that $\beta > max\{\alpha_1, \alpha'\}$. Then $\langle \mathfrak{c}, \beta\rangle \notin \mathrm{St}(K_n, \mathcal{U}_n)$ for each $n \in \omega$. Since U_β is the only element of $\mathcal{U}_n$ containing the point $\langle c, \beta\rangle$ and K$_n$ ∩ $U_\beta = \emptyset$ for each $n \in \omega$. This shows that M is not a relatively star-K-Menger subspace. □

Every relatively star-K-Menger subspace is relatively star-Menger. However, the following example shows that the converse is not true.

Example 2.13. There exists a Hausdorff space containing a relatively star-Menger subspace which is not relatively star-K-Menger.

Proof. Let

$$A = \{a_\alpha : \alpha < \mathfrak{c}\}, B = \{b_n : n \in \omega\},$$

$$\text{and } Y = \{\langle a_{\alpha,}, b_n\rangle : \alpha < \mathfrak{c}, n \in \omega\},$$

and let

$$X = Y \cup A \cup \{a\} \text{ where } a \notin Y \cup A.$$

We topologize X as follows: every point of Y is isolated; a basic neighbourhood of a point $a_\alpha \in A$ for each $\alpha < \mathfrak{c}$ takes the form

$$U_{a_\alpha}(n) = \{a_\alpha\} \cup \{\langle a_\alpha, b_m\rangle : m > n\} \text{ for } n \in \omega$$

and a basic neighbourhood of a point a takes the form

$$U_a(F) = \{a\} \cup \bigcup\{\langle a_\alpha, b_n\rangle : a_\alpha \in A \setminus F, n \in \omega\} \text{ for a countable subset F of A.}$$

Clearly X is a Hausdorff space. Hence A = $\{a_\alpha : \alpha < \mathfrak{c}\}$ being a subspace of X is a Hausdorff space. In [28, Example 2.3], Song shows that X is a star-Menger space. Hence, A is a relatively star-Menger subspace of X.

Next we show that A = $\{a_\alpha : \alpha < \mathfrak{c}\}$ is not a relatively star-K-Menger subspace of X. For each $\alpha < \mathfrak{c}$, let

$$U_\alpha = \{a_\alpha\} \cup \{\langle a_\alpha, b_n\rangle : n \in \omega\} \text{ and } U = U_a(\emptyset).$$

Then U_α is open and closed in X by the construction of the topology of X and

$$U_\alpha \cap U_{\alpha'} = \emptyset \text{ for } \alpha \neq \alpha'.$$

For each n ∈ ω, let

$$\mathcal{U}_n = \{U_\alpha : \alpha < \mathfrak{c}\} \cup \{U\}.$$

Let us consider the sequence $(\mathcal{U}_n : n \in \omega)$ of open covers of X. We only show that $\bigcup_{n\in\omega} \mathrm{St}(K_n, \mathcal{U}_n) \neq$ X for any sequence $(K_n : n \in \omega)$ of compact subsets of X. Let $(K_n : n \in \omega)$ be any sequence of compact subsets of X. For each $n \in \omega$, since K_n is compact, then there exists $\alpha_n < \mathfrak{c}$ such that $K_n \cap U_\alpha = \emptyset$ for each $\alpha > \alpha_n$. Let $\alpha' = \sup\{\alpha_n : n \in \omega\}$. If we pick $\beta > \alpha'$. Then $U_\beta \cap K_n = \emptyset$ for each $n \in \omega$. Hence $a_\beta \notin \mathrm{St}(K_n, \mathcal{U}_n)$ for each $n \in \omega$, since U_β is the only element of $\mathcal{U}_n$ containing the point a_β, which shows that A is not a relatively star-K-Menger. □

3. Topological properties of relatively star-K-Menger and relatively star-C-Menger

In this section, we study some topological properties of relatively star-K-Menger and relatively star-C-Menger spaces.

Theorem 3.1. *Let $Y \subseteq X$ be dense, and let $A \subseteq Y$. Then*

$$A \text{ is relatively star-K-Menger in } X \iff A \text{ is relatively star-K-Menger in } Y.$$

The same equivalence holds with "K" replaced by "C".

Proof. ($\Rightarrow$) Let $(\mathcal{V}_n : n \in \mathbb{N})$ be a sequence of open covers of Y. For each n, define

$$\mathcal{U}_n := \{U \subseteq X : U \text{ open in } X,\ U \cap Y \in \mathcal{V}_n\}.$$

Since Y is dense, $\mathcal{U}_n$ covers X, and A is relatively star-K-Menger in X, then there exist a sequence $(K_n : n \in \mathbb{N})$ of compact sets of X such that $A \subseteq \bigcup_n \mathrm{St}_X(K_n, \mathcal{U}_n)$. Set $K'_n := K_n \cap Y$, compact in Y. Then

$$\mathrm{St}_Y(K'_n, \mathcal{V}_n) = \mathrm{St}_X(K_n, \mathcal{U}_n) \cap Y.$$

Hence $A \subseteq \bigcup_n \mathrm{St}_Y(K'_n, \mathcal{V}_n)$.

($\Leftarrow$) Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X, and put $\mathcal{V}_n := \{U \cap Y : U \in \mathcal{U}_n\}$, open covers of Y. Since A is relatively star-K-Menger in Y, thus there exist a sequence $(K_n : n \in \mathbb{N})$ of compact sets of Y such that $A \subseteq \bigcup_{n\in\mathbb{N}} \mathrm{St}_Y(K_n, \mathcal{V}_n)$. Since K_n is also compact in X and

$$\mathrm{St}_Y(K_n, \mathcal{V}_n) \subseteq \mathrm{St}_X(K_n, \mathcal{U}_n).$$

Thus $A \subseteq \bigcup_{n\in\mathbb{N}} \mathrm{St}_X(K_n, \mathcal{U}_n)$. □

Theorem 3.2. *Let $F \subseteq X$ be closed and $A \subseteq F$. Then A is relatively star-K-Menger in X if and only if A is relatively star-K-Menger in F.*

Similarly for the "C"-version.

Proof. ($\Rightarrow$) Given $(\mathcal{V}_n : n \in \mathbb{N})$ is a sequence of open covers of F, extend to $\mathcal{U}_n := \{U \subseteq X : U \cap F \in \mathcal{V}_n\} \cup \{X \setminus F\}$. SinceA is relatively star-K-Menger in X, then there exist a sequence $(K_n : n \in \mathbb{N})$ of compact sets of X such that $K_n \cap F$ compact in F, and

$$\mathrm{St}_F(K_n \cap F, \mathcal{V}_n) = \mathrm{St}_X(K_n, \mathcal{U}_n) \cap F.$$

($\Leftarrow$) Given $(\mathcal{U}_n : n \in \mathbb{N})$ is a sequence of open covers of X. Then $\mathcal{V}_n = \{U \cap F : U \in \mathcal{U}_n\}$ is an open cover of F. Since A is relatively star-K-Menger in F, thus there exist a sequence $(K_n : n \in \mathbb{N})$ of compact sets of F such that $A \subseteq \bigcup_n \mathrm{St}_F(K_n, \mathcal{V}_n) \subseteq \bigcup_n \mathrm{St}_X(K_n, \mathcal{U}_n)$. □

We now present some results involving cardinal invariants.

Theorem 3.3. *Let X be a topological space and $A \subseteq X$. If $|A| < \mathfrak{d}$, then A has the relative star-K-Menger property in X.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X. For each $a \in A$, choose a sequence $\langle U_n^a : n \in \mathbb{N}\rangle$ such that $a \in U_n^a \in \mathcal{U}_n$. This defines a function $f_a \colon \mathbb{N} \to \bigcup_n \mathcal{U}_n$, by $f_a(n) = U_n^a$. The family $\{f_a : a \in A\}$ has cardinality $|A| < \mathfrak{d}$. Hence, there exists $g \in \omega^\omega$ dominating all f_a.

Define $A_n = \{a \in A : a \in U^a_{g(n)}\}$. Since A_n is finite (otherwise, we would contradict the domination), it is compact. Thus,

$$A \subseteq \bigcup_{n\in\mathbb{N}} \mathrm{St}(A_n, \mathcal{U}_n),$$

proving that A has the relative star-K-Menger property in X. □

Theorem 3.4. *Let X be a topological space and $A \subseteq X$. If $|A| < \mathfrak{b}$, then A has the relative star-C-Menger property in X.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X. For each $a \in A$, choose a sequence $\langle U_n^a : n \in \mathbb{N}\rangle$ with $a \in U_n^a \in \mathcal{U}_n$. This defines $f_a \in \omega^\omega$ as before. Since $|A| < \mathfrak{b}$, the family $\{f_a : a \in A\}$ is bounded by some $g \in \omega^\omega$.

Define $A_n = \{a \in A : a \in U^a_{g(n)}\}$, then A_n is countably compact. Hence,

$$A \subseteq \bigcup_{n\in\mathbb{N}} \mathrm{St}(A_n, \mathcal{U}_n).$$

Thus A has the relative star-C-Menger property in X. □

Corollary 3.5. *Every subset A of X with $|A| < \min\{\mathfrak{b}, \mathfrak{d}\}$ is both relatively star-K-Menger and relatively star-C-Menger in X.*

Theorem 3.6. *Assume $\mathfrak{b} = \mathfrak{d}$. Then a subset $A \subseteq X$ is relatively star-K-Menger if and only if it is relatively star-C-Menger.*

Proof. The forward direction follows from the fact that every compact set is countably compact. For the converse, if A is relatively star-C-Menger, then by assumption $|A| < \mathfrak{d} = \mathfrak{b}$. By Theorem 3.3, A is relatively star-K-Menger. □

Let X be any topological space. The Alexandroff duplicate $A(X)$ of a space X is $X \times \{0, 1\}$; each point of $X\times\{1\}$ is isolated and a basic neighborhood of a point $\langle x, 0\rangle \in X\times\{0\}$ is of the form $(U\times\{0\})\cup((U\times\{1\})\setminus\{\langle x, 1\rangle\})$, where U is a neighbourhood of x in X.

Theorem 3.7. *Let M be a subspace of space X such that the Alexandroff duplicate A(M) of M is relatively star-K-Menger subspace. Then M is a relatively star-K-Menger subspace of X.*

Proof. Since M is homeomorphic to $M \times\{0\}$ and $M \times\{0\} \subseteq M \times\{0, 1\} \in$ RSKM(A(X)), by Corollary 2.3, we have that M is a relatively star-K-Menger subspace of X. □

In a similar way we obtain the following.

Theorem 3.8. *Let M be a subspace of a space X such that $A(M) \in RSCM(A(X))$, then $M \in RSCM(X)$.*

In [15, Theorem 4.1] Sen showed that there exists a Tychonoff relatively star-K-Menger (resp., relatively star-C-Menger) subspace M of X such that A(M) is not relatively star-K-Menger (resp., relatively star-C-Menger) subspace.

Lemma 3.9. ([8]) *For a subspace M of a T_1 space X, e(M) = e(A(M)).*

Theorem 3.10. ([18]) *If X is a T_1 space and A(M) is a relatively star-K-Menger subspace of A(X), for some closed subset M of X, then $e(M) < \omega_1$.*

Theorem 3.11. *If M is a relatively star-K-Menger subspace of a space X with $e(M) < \omega_1$, then $A(M) \in RSKM(A(X))$.*

Proof. Let $(\mathcal{U}_n : n \in \omega)$ be a sequence of open covers of A(X). For each $n \in \omega$ and each $x \in M$, choose an open neighbourhood

$$W_{n_x} = (V_{n_x} \times \{0, 1\}) \setminus \{\left\langle x, 1\right\rangle\}$$

of $\langle x, 0\rangle$ satisfying that there exists some $U \in \mathcal{U}_n$ such that $W_{n_x} \subseteq U$, where V_{n_x} is an open subset of X containing x. For each $n \in \omega$, let $\mathcal{V}_n = \{V_{n_x} : x \in X\}$. Then $(\mathcal{V}_n : n \in \omega)$ is a sequence of open covers of X. Since M $\in$ RSKM(X), there exists a sequence $(K_n : n \in \omega)$ of compact subsets of X such that M $\subseteq \bigcup_{n\in\omega} \mathrm{St}(K_n, \mathcal{V}_n)$. Let for each $n \in \omega$, $K_n^{'} = K_n \times \{0, 1\}$. Then $K_n^{'}$ is a compact subset of $A(X)$ and $M \times \{0\} \subseteq \bigcup_{n\in\omega} \mathrm{St}(K_n^{'}, \mathcal{U}_n)$. Let $A = A(X) \setminus \bigcup_{n\in\omega} \mathrm{St}(K_n', \mathcal{U}_n)$. Then A is a discrete closed subset of A(M). Since for a T_1 space, $e(M) = e(A(M))$, the set A is countable and we can enumerate A as $(a_n : n \in \omega)$. Let for each $n \in \omega$, $K_n^{''} = (K_n^{'} \times \{0, 1\}) \cup \{a_n\}$. Then $K_n^{''}$ is a compact subset of $A(X)$ and $A(M) = \bigcup_{n\in\omega} \mathrm{St}(K_n^{''}, \mathcal{U}_n)$, which shows that $A(M)$ is a relatively star-K-Menger subspace. □

On the same lines of proof in above theorem, we obtain.

Corollary 3.12. *If M is a relatively star-C-Menger subspace of a space X with $e(M) < \omega_1$, then $A(M) \in$ RSCM(A(X)).*

We have the following corollary from Theorem 3.4 and 3.5.

Corollary 3.13. *If M is a relatively star-K-Menger subspace of a T_1-space X, then A(M) is a relatively star-K-Menger if and only if $e(M) < \omega_1$.*

We can get the same result for relatively star-C-Menger subspaces of the space X after necessary modifications.

Corollary 3.14. *If M is a relatively star-C-Menger subspace of a T_1-space X, then A(M) is a relatively star-C-Menger if and only if $e(M) < \omega_1$.*

In [18], the author showed that the preimage of a star-Δ-Menger space under an open perfect map is star-Δ-Menger, similarly we can prove the following result.

Theorem 3.15. *Let f be a perfect open mapping from a space X to Y. If $A \in RSKM(Y)$, then $f^{-1}(A) \in RSKM(X)$.*

By Theorem 3.9, we have the following corollary.

Corollary 3.16. *Let M be a relatively star-K-Menger subspace of X and Y a compact space. Then $M \times Y$ is a relatively star-K-Menger subspace of $X \times Y$.*

Corollary 3.17. *Let M be a relatively star-K-Menger subspace of X and Y a σ-compact space, then $M \times Y$ is a relatively star-K-Menger subspace of $X \times Y$.*

Proof. The proof follows from the fact that RSKM(X) is an σ-ideal and hence closed under countable unions. □

The following well-known example from [31] shows that the product of two countably compact (and hence relatively star-K-Menger) subspaces need not be relatively star-K-Menger. Here we give the proof roughly for the sake of completeness.

Example 3.18. There exists two countably compact subspaces M and N of space X such that $M \times N$ is not relatively star-K-Menger.

Proof. Let D($\mathfrak{c}$) be a discrete space of cardinality $\mathfrak{c}$. We can define $M = \bigcup_{\alpha<\omega_1} E_\alpha$ and $N = \bigcup_{\alpha<\omega_1} F_\alpha$, where E_α and F_α are the subsets of $\beta(D)$ which are defined inductively so as to satisfy the following conditions (1), (2) and (3):

1. $E_\alpha \cap F_\beta = D$ if $\alpha \neq \beta$;
2. $|E_\alpha| \leq \mathfrak{c}$ and $|F_\beta| \leq \mathfrak{c}$;
3. every infinite subset of E_α (resp., F_α) has an accumulation point in $E_{\alpha+1}$ (resp., $F_{\alpha+1}$).

These sets E_α and F_α are well-defined, since every infinite closed set in $\beta(D)$ has cardinality $2^{\mathfrak{c}}$. In fact, the diagonal $\{\langle d, d\rangle : d \in D\}$ is a discrete clopen subset of M × N of cardinality $\mathfrak{c}$ so that it is not a relatively star-K-Menger. Hence $M \times N$ is not relatively star-K-Menger, since subsets of relatively star-K-Menger are relatively star-K-Menger. □

Example 3.19. There exists a countably compact (and hence relatively star-K-Menger) subspace M of a space X and a Lindelöf space Y such that M × Y is not a relatively star-K-Menger subspace of X × Y.

Proof. Let $X = [0, \omega_1]$ with the usual order topology and $Y = \omega_1 + 1$ with the following one: each point $\alpha < \omega_1$ is isolated and a set U containing ω_1 is open if and only if $Y \setminus U$ is countable. Then Y is Lindelöf, and since X is star-K-Menger, the subspace $M = [0, \omega]$ of X is relatively star-K-Menger in X.

For each $\alpha < \omega_1$, let

$$U_\alpha = [0, \alpha] \times [\alpha, \omega_1] \text{ and } V_\alpha = (\alpha, \omega_1) \times \{\alpha\}.$$

Then

$$U_\alpha \cap V_{\alpha'} = \emptyset \text{ for any } \alpha < \omega_1 \text{ and } \alpha' < \omega_1$$

and

$$V_\alpha \cap V_{\alpha'} = \emptyset \text{ if } \alpha \neq \alpha'.$$

For each $n \in \omega$, let

$$\mathcal{U}_n = \{U_\alpha : \alpha < \omega_1\} \cup \{V_\alpha : \alpha < \omega_1\}.$$

Then $\mathcal{U}_n$ is an open cover of $X \times Y$. Let us consider the sequence $(\mathcal{U}_n : n \in \omega)$ of the open covers of $X \times Y$. It suffices to show that $\bigcup_{n\in\omega} \mathrm{St}(K_n, \mathcal{U}_n) \neq M \times Y$ for any sequence $(K_n : n \in \omega)$ of finite subsets of X. Since K_n is compact for each $n \in \omega$. Thus $\pi(K_n)$ is a compact subset of X, where $\pi : X \times Y \to X$ is the projection. Thus there exists $\alpha_n < \omega_1$ such that

$$K_n \cap ((\alpha_n, \omega_1) \times Y) = \emptyset.$$

Let $\beta = \sup \{\alpha_n : n \in \omega\}$. By cofinality, $\beta < \omega$ and

$$(\bigcup K_n) \cap ((\beta, \omega_1) \times Y) = \emptyset.$$

If we pick $\alpha > \beta$, then $\langle \alpha, \alpha\rangle \notin \mathrm{St}(K_n, \mathcal{U}_n)$ for each $n \in \omega$, since V_α is the only element of $\mathcal{U}_n$ containing the point $\langle \alpha, \alpha\rangle$ and $U_\alpha \cap K_n = \emptyset$ for each $n \in \omega$. This shows that $M \times Y$ is not relatively star-K-Menger. □

On the similar lines, we can prove the following result.

Example 3.20. There exists a countably compact (and hence relatively star-C-Menger) subspace of space X and a Lindelöf space Y such that M × Y is not a relatively star-C-Menger subspace of X × Y.

4. Conclusion

This paper has advanced the study of relative selection principles by thoroughly investigating the properties of relatively star-K-Menger and relatively star-C-Menger spaces. We have established their fundamental characteristics, demonstrated that they form σ-ideals, and clarified their distinctness through a series of counterexamples. Furthermore, we explored their behavior under key topological operations such as taking dense or closed subspaces, products, and the Alexandroff duplicate, often establishing characterizations linked to cardinal invariants. These results provide a foundation for further research in this area of selection principles and covering properties.

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