Filomat 39:29 (2025), 10411–10417
https://doi.org/10.2298/FIL2529411S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Involutions on Wall manifolds

Ashim Kumar Sarmah[a,*], Konthoujam Somorjit Singh[a]

[a]*Department of Mathematics, National Institute of Technology Manipur, Langol, Imphal 795004, India*

Abstract. We determine the fixed point sets of involutions on a Wall manifold and also construct examples that illustrate the realization of our main results.

1. Introduction

Let F be the fixed point set of a given topological transformation group (X, G), where X is a compact Hausdorff space and G is a compact Lie group. The study of fixed point sets of transformation groups has been a fascinating problem among researchers in algebraic topology. Many researchers have made significant contributions to this field, establishing various results and theorems. Smith ([9]) proved that fixed point set F of $G = \mathbb{Z}_p$, (p is a prime) acting on X, exhibits the same mod p cohomological properties of a product of two spheres, that is, the cohomology characteristic of the space inherited by the fixed point set is the same. Extending Smith's prior research ([9]), Swan ([11]) gave sufficient conditions to determine the fixed point set of $G = \mathbb{Z}_p$, p is a prime, actions on *mod p* cohomology product of two even dimensional spheres. Su ([10]) described the many possibilities for the cohomology of the component of a fixed point set and established a conjecture that if $(X, G = \mathbb{Z}_p)$ satisfies Poincaré duality then each component of F also satisfies Poincaré duality. Bredon ([3, 4]) proved Su's result for the special case when X is totally nonhomologous to zero (TNHZ) in X_G (Borel Space), and he also determined the fixed point set F when $G = \mathbb{Z}_p$ acts on *mod p* cohomology projective space and the product of two spheres. Puppe ([8]) proved that if the cohomology ring $H^*(X)$ with $\mathbb{Z}_p$-coefficients is generated as an algebra by k elements, then each component of the fixed point set F has a cohomology that can also be described using at most k generators, when X is considered totally non-homologous to zero in X_G with respect to the Čech cohomology. David ([5]) deals with some methods to study arbitrary topological torus actions and applies those methods to discuss the fixed point sets of torus actions on product of two odd-dimensional spheres. Moreover, Allday ([1]) studied the action of a torus group G on a space X with a rational cohomology algebra of a product of three odd-dimensional spheres, when X is not totally non-homologous in X_G such a way that F is non-empty. Peltier and Beem ([7]) obtained the possible cases of the fixed point set of involutions on Dold manifold $P(1, n)$. Dimpi et al. ([6]) determined the fixed point set of involutions of the product of three

2020 *Mathematics Subject Classification*. Primary 58C30; Secondary 55T10, 55S10, 57S17.
Keywords. Fixed point, Leray-Serre spectral sequence, Steenrod square, totally nonhomologous to zero.
Received: 04 March 2025; Revised: 15 August 2025; Accepted: 04 September 2025
Communicated by Ljubiša D. R. Kočinac
* Corresponding author: Ashim Kumar Sarmah
Email addresses: ashimkumarsarmah6@gmail.com (Ashim Kumar Sarmah), ksomorjitmaths@gmail.com (Konthoujam Somorjit Singh)
ORCID iDs: https://orcid.org/0009-0004-5147-9725 (Ashim Kumar Sarmah), https://orcid.org/0009-0004-6945-9458 (Konthoujam Somorjit Singh)

spheres, which is a generalizations of the Su's ([10]) result. In this piece of work, we find the possible cases of fixed point sets of involutions on a Wall manifold of type $Q(m,n)$, for $m = 1$.

2. Preliminaries

It is well known that the Dold manifold is a quotient space of $\mathbb{R}P^n \times \mathbb{C}P^m$ under the free involution $T([x,[z]]) = [(-x,[\bar{z}])]$, where m and n are non negative integers. The $(n+2m+1)$-dimensional quotient space $P(m,n) \times [0,1]/\sim$ is the Wall manifold $Q(m,n)$, where $([(x,y),z],0) \sim ([(x,-y),z],1)$ denotes the equivalence relation.

A compact Lie group G acts freely on contractible space E_G and acts diagonally on the product $(X \times E_G)$ gives the Borel construction as a orbit space $X_G = (X \times E_G)/G$ on X. Let B_G be the classifying space of G associated with the universal G-bundle $G \to E_G \to B_G$ then the projection $X \times E_G \to E_G$ gives a fibration $X_G \to B_G$ with fiber X. Note that for $G = \mathbb{Z}_2$, $H^*(B_G;\mathbb{Z}_2) \cong \mathbb{Z}_2[t]$, where $\deg t = 1$.

A space X is considered as totally non-homologous to zero (TNHZ) in X_G if $i_1^* : H^*(X_G;R) \to H^*(X;R)$ is surjective, where $i_1 : X \hookrightarrow X_G$ is an inclusion map and R is a commutative ring. Throughout this paper, cohomology is Čech cohomology with coefficients in $\mathbb{Z}_2$.

The following propositions are employed to establish our main results concerning fixed point sets.

Proposition 2.1. ([3]) *Suppose that X is totally non-homologous to zero(TNHZ) in X_G and let $\{a_i\}$ be a set of homogeneous elements of $H^*(X_G)$ such that $i^*(a_i)$ forms a $\mathbb{Z}_2$-basis of $H^*(X)$ under the inclusion map $i : X \hookrightarrow X_G$. Then, $H^*(X_G)$ is a free $H^*(B_G)$-module generated by $\{a_i\}$.*

Proposition 2.2. ([2]) *Let $G = \mathbb{Z}_2$ act on a finitistic space X and suppose that $\sum rkH^i(X) \leq \infty$. Then the following statements are equivalent:*

(1) *X is TNHZ in X_G.*

(2) *$\sum rkH^i(F) = \sum rkH^i(X)$.*

(3) *G acts trivially on $H^*(X)$ and spectral sequence $E^{r,q}$ of $X_G \to B_G$ degenerates.*

Proposition 2.3. ([8]) *Let $G = \mathbb{Z}_2$ and X be a paracompact G-space and $H^*(X)$ is generated by n elements with an ideal of relations generated by r relations. If X is TNHZ in X_G, then the cohomology ring of each connected component of the fixed point set is generated by at most n elements with an ideal of relations generated by at most r relations.*

3. Main theorems

In this section, our main aim is to determine the fixed point set of an involution on a finitistic space X with the same cohomology type as a Wall manifold $Q(1,n)$. We also construct some examples and show the realizability of our main result.

Theorem 3.1. *Let F_i be any fixed point set component of an involution on a finitistic space $X \sim_2 Q(1,n)$. If X is TNHZ in X_G, then the cohomology ring of each component F_i is generated by one of the following:*

(1) *a single element of degree 1 or 2.*

(2) *two elements of degree 1 or two elements of degrees 1 and 2.*

(3) *three elements of degrees 1,1 and 2 or three elements of degree 1.*

In this first case, if $a_1 \in H^1(F_1)$, $b_1 \in H^1(F_1)$ and $c_1 \in H^2(F_1)$ are the generators, then $sq^1(c_1) = c_1 b_1$.

Proof. As $X \sim_2 Q(1,n)$ is TNHZ in X_G, by Proposition 2.2, the cohomology ring $H^*(X_G)$ is generated by the set $\{1, \alpha\delta^i, \beta\delta^i, \delta^i, \alpha\beta\delta^i\}$, where $0 \leq i \leq n$ and $i^*(\alpha) = a, i^*(\beta) = b$ and $i^*(\delta) = c$. Consider $j^*(\alpha) = ut + x$, $j^*(\beta) = vt + y$ and $j^*(\delta) = \lambda t^2 + zt + w$, where $u, v, \lambda \in \mathbb{Z}_2$ and $j : F_G \hookrightarrow X_G$ is the inclusion map. Let F_i represent one of the component of the fixed point set F and consider inclusion map $j_i : F_{iG} \hookrightarrow X_G$. The cohomology classes of the component F_i can be expressed as: $j_i^*(\alpha) = u_i t + x_i$, $j_i^*(\beta) = v_i t + y_i$ and $j_i^*(\delta) = \lambda_i t^2 + z_i t + w_i$, where $u_i, v_i, \lambda_i \in \mathbb{Z}_2$. The fact that $j_i^* : H^k(X_G) \to H^k(F_{iG})$ is onto for sufficiently large k and by Proposition 2.3, allows us to conclude that any three of x_i, y_i, z_i and w_i suffice to generator $H^*(F_i)$. Furthermore, let a_1, $b_1 \in H^1(F_1)$ and $c_1 \in H^2(F_1)$ be the generators. If $x_1 = 0$, then y_1,z_1 and w_1 are the generators of $H^*(F)$. For sufficiently large value of r, we have

$$z_1 t^r = j_i^* \left(\sum_{\nu=1}^{n} A_\nu \alpha \delta^\nu t^{r-2\nu} + \sum_{\nu=1}^{n} B_\nu \beta \delta^\nu t^{r-2} + \sum_{\nu=1}^{n} C_\nu \alpha\beta \delta^\nu t^{r-2\nu+1} + \sum_{\nu=1}^{n} D_\nu \delta^\nu t^{r+1-2\nu} \right)$$

where $A_\nu, B_\nu, C_\nu, D_\nu \in \mathbb{Z}_2$. If we expand the right-hand side of the above equation using the values of $j_1^*(\alpha)$, $j_1^*(\beta)$, $j_1^*(\delta)$, and comparing the coefficient of t^r we find that $D_1 = 1$, so that

$$\begin{aligned}(\lambda_1 t^2 + w_1) t^{r-1} \quad = \quad & \sum_{\nu=1}^{n} A_\nu (u_1 (\lambda_1 t^2 + z_1 t + w_1)^\nu t^{r-2\nu+1} + \sum_{\nu=1}^{n} B_\nu (v_1 t + y_1)(\lambda_1 t^2 + z_1 t + w_1)^\nu t^{r-2} \\ & + \sum_{\nu=1}^{n} C_\nu u_1 (v_1 t + y_1)(\lambda_1 t^2 + z_1 t + w_1)^\nu t^{r-2\nu+2} + \sum_{\nu=2}^{n} D_\nu (\lambda_1 t^2 + z_1 t + w_1)^\nu t^{r+1-2\nu}.\end{aligned}$$

Now comparing the coefficient of t^{r-1}, we obtain $w_1 = A_2 u z_1^2 + (B_1 + C_1) z_1 y_1$, which contradicts our hypothesis. Similarly, we can show that $y_1 \neq 0$. So $z_1 = 0$, therefore $sq^1(w_1) = w_1 y_1$. □

The following lemma gives the possible number of components of a fixed point set.

Lemma 3.2. *Let $X \sim_2 Q(1,n)$ be a finitistic space. Then, the fixed point set F of an involution on X has at most five components.*

Proof. We can consider the following two relations:

$$\beta^2 = A_1 \alpha t + A_2 \beta t + \alpha\beta \qquad (1)$$

and

$$sq^1(\delta) = \Lambda_1 \alpha t^2 + \Lambda_2 \beta t^2 + \Lambda_3 \alpha\beta t + \Lambda_4 \delta t + \beta\delta, \qquad (2)$$

where $A_i, \Lambda_i \in \mathbb{Z}_2$. By comparing the coefficients of t^2 in (1) and the coefficients of t^3 in (2), we obtain the relations $uv = A_1 u + (A_2 - 1)v$ and $v\lambda = \Lambda_1 u + \Lambda_2 v + \Lambda_3 uv + \Lambda_4 \lambda$. From these relations, we deduced that $H^0(F)$ is generated by the set $\{1, u, v, \lambda, u\lambda\}$. This implies that the fixed point set F has at most five components. □

Now, we prove our main theorems.

Theorem 3.3. *Let $X \sim_2 Q(1,n)$ be a finitistic space and F be the fixed point set of an involution on X. Then one of the following occurs:*

(1) *$H^*(F)$ is generated by $x, y, z \in H^1(F)$ with relations $x^2 = 0, y^2 = xy$ and $z^{n+1} = \epsilon_1 x z^n + \epsilon_2 y z^n + \epsilon_3 xyz^{n-1}$, where $\epsilon_i \in \mathbb{Z}_2$.*

(2) *$F \sim_2 Q(1,r) \bigsqcup Q(1, n-r-1)$, $P(1,n) \bigsqcup P(1,n)$, $(\mathbb{R}P^1 \times \mathbb{R}P^n) \bigsqcup (\mathbb{R}P^1 \times \mathbb{R}P^n)$, $(\mathbb{C}P^n \times \mathbb{R}P^1) \bigsqcup (\mathbb{C}P^n \times \mathbb{R}P^1)$ or $\mathbb{R}P(n\eta \oplus \sigma) \bigsqcup \mathbb{R}P(n\eta \oplus \sigma))$, where η and σ are trivial and non trivial line bundles over $\mathbb{R}P^1$, respectively.*

(3) *$F \sim_2 (\mathbb{R}P^1 \times \mathbb{R}P^n) \bigsqcup \mathbb{C}P^n \bigsqcup \mathbb{C}P^n$, $(\mathbb{R}P^1 \times \mathbb{R}P^n) \bigsqcup (\mathbb{R}P^1 \times \mathbb{R}P^r) \bigsqcup (\mathbb{R}P^1 \times \mathbb{R}P^{n-r-1})$.*

(4) *$F \sim_2 (P(1,r) \bigsqcup P(1,n-r-1)) \bigsqcup (P(1,r) \bigsqcup P(1,n-r-1))$.*

(5) $F \sim_2 (\mathbb{R}P^1 \times \mathbb{R}P^n) \bigsqcup (\mathbb{C}P^r \bigsqcup \mathbb{C}P^{n-r-1}) \bigsqcup (\mathbb{C}P^r \bigsqcup \mathbb{C}P^{n-r-1})$.

Proof. The proof of the theorem proceeds by considering various cases for the structure of the fixed point set F under the involution. In each case, we analyze the cohomology ring and determine the relations between its generators.

Case 1. Suppose the cohomology ring of one of the component of fixed point set is generated by three independent classes of degree 1. Let $\alpha^2 = B_1\alpha t + B_2\beta t$, where $B_1, B_2 \in \mathbb{Z}_2$. By comparing the coefficient of t, we have $B_1 = B_2 = 0$. So $\alpha^2 = 0$, which implies that $u = 0$ and $x^2 = 0$. By comparing the coefficients of t^2 and t in (1), we can show that $v = 0$ and $y^2 = xy$. From (2), we get $\lambda = 0$, thus F has only one component. Since j^* is injective, and $\alpha\beta\delta^i, \delta^i, \alpha\delta^i$ and $\beta\delta^i$ are, respectively, generators of $H^{2i+2}(X_G)$, $H^{2i}(X_G)$ and $H^{2i+1}(X_G)$, for $1 \leq i \leq n$ and xyz^i, z^i, xz^i, yz^i are generators of $H^{2i+2}(F), H^{2i}(F)$ and $H^{2i+1}(F)$, respectively, for $0 \leq i \leq n$. Since F is a poincaré duality space, $H^i(F) = 0$ for all $i > 4n+4$. Moreover, since $\dim_{\mathbb{Z}_2}(H^*(F)) = 4n+4$, it follows that $z^{n+1} = \epsilon_1 xz^n + \epsilon_2 yz^n + \epsilon_3 xyz^{n-1}$, where $\epsilon_i \in \mathbb{Z}_2$.

Case 2. Let $H^*(F) = H^*(F_1) \oplus H^*(F_2)$, where F_1 and F_2 are components of F. Then the following subcases are possible:

(a) Suppose $\alpha^2 = 0$, then $j^*(\alpha) = (x_1, x_2)$ with $x_i^2 = 0$. If $\beta^2 = \alpha\beta$ then $j^*(\beta) = (y_1, y_2)$ with $y_i^2 = x_i y_i$. From (2), $j^*(\delta) = (t^2 + w_1, w_2)$, where $sq^1(w_i) = y_i w_i$. If instead $\beta^2 = \alpha\beta + \beta t + \alpha t$, then $j^*(\beta) = (t, t)$, and from (2), we obtain $j^*(\delta) = (w_1, w_2)$.

(b) If $\alpha^2 = \alpha t$, then $j^*(\alpha) = (t, 0)$ and $\beta^2 = \alpha\beta + \beta t$. Thus, $j^*(\beta) = (y_1, y_2)$ with $y_i^2 = 0$. Now, the following possibilities arise from (2):

(1) $j^*(\delta) = (w_1, w_2), sq^1(w_i) = y_i w_i$.
(2) $j^*(\delta) = (z_1 t + y_1 z_1 + z_1^2, z_2 t + y_2 z_2 + z_2^2)$.

Case 3. Suppose $H^*(F) = H^*(F_1) \oplus H^*(F_2) \oplus H^*(F_3)$, where F_1, F_2 and F_3 are components of F. The following subcases arise:

(c) In this subcase, each component F_i satisfies the conditions of case 2(b). We have $j^*(\alpha) = (t, t, t)$ and $j^*(\beta) = (y_1, y_2, y_3)$ with $y_i^2 = 0$. Hence, by (2), we obtain two possibilities for $j^*(\delta) = (t^2 + z_1 t + z_1 y_1 + z_1^2, t^2 + y_2 t + w_2, t^2 + y_3 t + w_3)$ or $j^*(\delta) = (t^2 + z_1 t + z_1 y_1 + z_1^2, t^2 + z_2 t + z_2 y_2 + z_2^2, t^2 + z_3 t + z_3 y_3 + z_3^2)$.

(d) By Case 2 (b), we have $j^*(\alpha) = (0, t, t)$ and $j^*(\beta) = (y_1, t, t)$ with $y_1^2 = 0$.Then by (1), $j^*(\delta) = (z_1 t + y_1 z_1 + z_1^2, t^2 + w_2, t^2 + w_3)$.

Case 4. Let $H^*(F) = H^*(F_1) \oplus H^*(F_2) \oplus H^*(F_3) \oplus H^*(F_4)$, where each F_i is a component of F. In this case, each F_i satisfies the conditions of Case 2 (b). Hence, we take $j^*(\alpha) = (t, t, t, t)$ and $j^*(\beta) = (y_1, y_2, y_3, y_4)$. From (2), we have $j^*(\delta) = (t^2 + y_1 t + w_1, t^2 + y_2 t + w_2, t^2 + y_3 t + w_3, t^2 + y_4 t + w_4)$ with $sq^1(w_i) = y_i w_i$.

Case 5. Let $H^*(F) = H^*(F_1) \oplus H^*(F_2) \oplus H^*(F_3) \oplus H^*(F_4) \oplus H^*(F_5)$, where each F_i is a component of F. By Case 2 (b), $\alpha^2 = \alpha t$ and $\beta^2 = \alpha\beta + \beta t$. Then $j^*(\alpha) = (t, t, t, t, t)$ and $j^*(\beta) = (y_1, t, t, t, t)$, from (1) it follows that $j^*(\delta) = (t^2 + z_1 t + y_1 t + z_1^2, w_2, w_3, w_4, w_5)$. $\square$

Theorem 3.4. *Let F be the fixed point set of an involution on a finitistic space X where $X \sim_2 Q(1, n)$. If F consists of five components and if $F_1 \sim_2 \mathbb{RP}^1 \times \mathbb{RP}^n$, then $F_2 \bigsqcup F_3 \sim_2 \mathbb{CP}^s \bigsqcup \mathbb{C}P^{n-s-1}$ and $F_4 \bigsqcup F_5 \sim_2 \mathbb{CP}^s \bigsqcup \mathbb{C}P^{n-s-1}$.*

Proof. Let $F_2 \bigsqcup F_3 \sim_2 \mathbb{CP}^{s_2} \bigsqcup \mathbb{C}P^{s_3}$ and $F_4 \bigsqcup F_5 \sim_2 \mathbb{CP}^{s_4} \bigsqcup \mathbb{C}P^{s_5}$. By Theorem 3.3, Case (5), we have $j_i^*(\alpha) = t$, $j_i^*(\beta) = t$ and $j_i^*(\delta) = w_i$ for $i = 2, 3, 4, 5$. First consider the case $F_2 \bigsqcup F_3 \sim_2 \mathbb{CP}^{s_2} \bigsqcup \mathbb{C}P^{s_3}$. We consider the following relation:

$$\delta^{n+1} = \sum_{\nu=0}^{n} (B_\nu \alpha t + C_\nu \beta t + D_\nu \alpha\beta)\delta^{n-\nu} t^{2\nu} + \sum_{\nu=1}^{n+1} \Lambda_\nu \delta^{n+1-\nu} t^{2\nu}. \qquad (3)$$

For $i = 2$ in relation (3), we obtain

$$w_2^{n+1} = \sum_{i=1}^{n} (B_i + C_i + D_i) w_2^{n.-i} t^{2i+2} + \sum_{i=1}^{n+1} \Lambda_i w_2^{n+1-i} t^{2i}.$$

This implies that $B_i + C_i + D_i + \Lambda_{i+1} = \Lambda_1 = 0$ for all $n - s_2 \leq i \leq n+1$. For $i = 3$ gives $s_3 + 1 \leq n - s_2$. Applying same argument for $i = 4, 5$, we obtain $s_5 + 1 \leq n - s_4$. Since $s_2 + s_3 + s_4 + s_5 = 2n - 2$, it follows that $s_3 + 1 = n - s_2$ and $s_5 + 1 = n - s_4$. $\square$

Next, we prove a sequence of lemmas.

Lemma 3.5. *Let $X \sim_2 Q(1,n)$ be a finitistic space. If one of the components of the fixed point set F of an involution on X is $P(1,s)$, where $1 \le s \le n$, then the remaining components of F are of the type $P(1,r)$, where $1 \le r \le n$.*

Proof. Assume that F_1 is one of the components of F such that $F_1 \sim_2 P(1,s)$.
According to Case 2 (a) and (2), we obtain $A_1 = A_2 + A_3 = A_4 = 0$ and $sq^1(w_1) = y_1 w_1$. Now, consider the remaining possible components of F, other than those of type $P(1,r)$, are $F_i \sim_2 \mathbb{RP}^1 \times \mathbb{RP}^r$, $\mathbb{RP}^1 \times \mathbb{CP}^r$, $\mathbb{RP}^r$, or $\mathbb{CP}^r$, where $i \neq 1$. If one of these remaining components is $F_i \sim_2 \mathbb{RP}^1 \times \mathbb{RP}^r$, then according to Case 2 (b), we obtain $j_1^*(\alpha) = t$, $j_1^*(\beta) = y_1$, and $j_1^*(\delta) = z_2 t + y_2 z_2 + z_1^2$ with $y_1^2 = 0$. From (2), it follows that $\Lambda_1 = \Lambda_2 = \Lambda_3 = 0$ and $\Lambda_4 = 1$, which leads to a contradiction. Similar contradictions arise for the other possible cases listed above. □

Lemma 3.6. *Let F be the fixed point set of an involution on a finitistic space X such that $X \sim_2 Q(1,n)$. If F consists of two or three components, then the following cases are not possible:*

(1) $F \sim_2 (\mathbb{RP}^1 \times \mathbb{CP}^n) \bigsqcup (\mathbb{RP}^1 \times \mathbb{CP}^n)$,

(2) $F \sim_2 (\mathbb{RP}^1 \times \mathbb{RP}^n) \bigsqcup (\mathbb{RP}^1 \times \mathbb{CP}^n)$,

(3) $F \sim_2 P(1,n) \bigsqcup (\mathbb{RP}^1 \times \mathbb{RP}^n)$ *or* $F \sim_2 P(1,n) \bigsqcup (\mathbb{RP}^1 \times \mathbb{CP}^n)$.

(4) $RP(n\eta \oplus \sigma) \bigsqcup \mathbb{C}P^n \bigsqcup \mathbb{C}P^n$, *where η and σ are trivial and non trivial line bundles over $\mathbb{R}P^1$, respectively.*

Proof. Let $H^*(F) = H^*(F_1) \oplus H^*(F_2)$, where F_1 and F_2 are the components of F.
Case 1. Assume $F_1 \sim_2 \mathbb{RP}^1 \times \mathbb{CP}^n$. By Case 2 (b), we have $j_1^*(\alpha) = t$ and $j_1^*(\beta) = y_1$ with $y_1^2 = 0$. Using condition (2), it follows that $j_1^*(\delta) = w_1$ and $sq^1(w_1) = y_1 w_1$. This leads to a contradiction since $sq^1(w_1) = 0$.
Case 2. Suppose $F_1 \sim_2 \mathbb{RP}^1 \times \mathbb{RP}^n$. According to previous lemma , we have $\Lambda_1 = \Lambda_2 = \Lambda_3 = 0$ and $\Lambda_4 = 1$. Now, if $F_2 \sim_2 \mathbb{RP}^1 \times \mathbb{CP}^n$, then by Case 2 (b), we obtain $j_2^*(\alpha) = t$ and $j_1^*(\beta) = y_2$ with $y_2^2 = 0$. Therefore, by condition (2), we can conclude $j^*(\delta) = w_2$ and $\Lambda_1 = \Lambda_2 + \Lambda_3 = \Lambda_4 = 0$, which is a contradiction.
Case 3 is ruled out by the previous lemma.
Case 4. Suppose $RP(n\eta \oplus \sigma) \bigsqcup \mathbb{C}P^n \bigsqcup \mathbb{C}P^n$, where η is trivial and σ is the non trivial line bundles over $\mathbb{R}P^1$, respectively. From Theorem 3.3, Case 3(d), it follows that $j_1^*(\alpha) = 0$, $j^*(\beta) = y_1$ with $y_1^2 = 0$ and $j_1^*(\delta) = z_1(t + y_1 + z_1)$ and $j_i^*(\alpha) = j_i^*(\beta) = t$, while for $i = 2, 3$, we have $j_i^*(\delta) = t^2 + w_i$ for $i = 2, 3$. Substituting the values $j_2^*(\alpha)$, $j_2^*(\beta)$ and $j_2^*(\delta)$ into relation (3) gives $\Lambda_1 = 0$. Similarly, using the values $j_1^*(\alpha)$, $j_1^*(\beta)$ and $j_1^*(\delta)$ along with the condition $z_1^{n+1} = y_1 z_1^n$, we find that $\Lambda_1 = 1$ when n is odd, which leads to a contradiction. Note that $z^{n+1} = 0$ when n is even. □

Lemma 3.7. *Let F be the fixed point set of an involution on a finitistic space X where $X \sim_2 Q(1,n)$. If F consists of either three or five components and if one of these components is $F_1 \sim_2 \mathbb{RP}^1 \times \mathbb{RP}^n$ or $\mathbb{RP}(n\eta \oplus \sigma)$, then the remaining components of F must be of cohomology type $\mathbb{CP}^s$ or $\mathbb{RP}^1 \times \mathbb{RP}^s$, where $1 \le s \le n$.*

Proof. Let F_1 be a component of F such that $F_1 \sim_2 \mathbb{RP}^1 \times \mathbb{RP}^n$. According to Lemma 3.5, the other components F_i $(i \neq 1)$ cannot have a cohomology type $\mathbb{RP}^1 \times \mathbb{CP}^s$ or $P(1,s)$. If one of the remaining components, say F_i, is such that $F_i \sim_2 \mathbb{RP}^n$ for $i \neq 1$, then by Theorem 3.3 (Case 5) and equation (2), it follows that $j^*(\alpha) = t$, $j^*(\beta) = 0$, and $j^*(\delta) = t^2 + zt + z^2$. This implies to $\Lambda_1 = \Lambda_4 = 1$. However, Lemma 3.5 asserts that $\Lambda_1 = \Lambda_2 = \Lambda_3 = 0$ and $\Lambda_4 = 1$, which results in a contradiction. Thus, the remaining components must be of cohomology type $\mathbb{CP}^s$ or $\mathbb{RP}^1 \times \mathbb{RP}^s$. □

Now, we provide some examples to realize the main theorem 3.3.

Example 3.8. Define an involution T_1 on $P(1,n) \times \mathbb{S}^1$ by

$$T_1([(x,y),[z_0,z_1,\ldots,z_n]],z) = ([(x,y),[\bar{z}_0,\bar{z}_1,\ldots,\bar{z}_n]],z)$$

which commutes with the involution S. Thus, T_1 induces an involution on the quotient space $Q(1,n)$ with fixed point set $F = Q(\mathbb{R}P^1 \times \mathbb{R}P^n)$, where $((y, y'), [x], z) \sim ((y, -y'), [x], -z)$. This represents the realization of case (1) of Theorem 3.3. when $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0$.

Let $E = CP(n\eta \oplus \sigma)$, where η and σ are trivial and non trivial complex bundles over $\mathbb{Q}(1,1)$, respectively. Then $H^*(E)$ is generated by $x, y \in H^1(E)$ and $z \in H^2(E)$ with relations $x^2 = 0, y^2 = xy$ and $z^{n+1} = xyz^n$. Now, replace $z - xy$ by w, when n is odd, then $H^*(E)$ is generated by x, y, w with relation $x^2 = y^2 + xy = w^{n+1} = 0$. Consider diagonal actions of involutions defined by conjugation on each fibers of $n\eta$ and σ, which gives a fixed point set $F = RP(n\eta \oplus \sigma)$, where η and σ are trivial and non trivial real line bundles over $\mathbb{Q}(1,1)$. This represents the realization of case (1) of Theorem 3.3 when $\epsilon_1 = \epsilon_2 = 0$ and $\epsilon_3 = 1$.

An involution T_2 defined by

$$T_2([x, [z_0, z_1, \ldots, z_n], z]) = [x, [z_0, z_1, \ldots, z_r, -z_{r+1}, \ldots, -z_n], z],$$

which commutes with S. This induces an involution on $Q(1,n)$ with fixed point set $F = Q(1,r) \bigsqcup Q(1, r-1)$. This represents the realization of second case of Theorem 3.3.

An involution T_3 defined by

$$T_3([x, [z_0, z_1, \ldots, z_n], z]) = [x, [z_0, z_1, \ldots, z_n], \bar{z}],$$

which commutes with S and so it induced an involution on $Q(1,n)$ with fixed point set $F = P(1,n) \bigsqcup P(1,n)$.This represents the realization of second case of case (2) of Theorem 3.3.

An involution T_4 defined by

$$T_4([x, [z_0, z_1, \ldots, z_n], z]) = [x, [\bar{z}_0, \bar{z}_1, \ldots, \bar{z}_n], \bar{z}],$$

which commutes with S and so it induced an involution on $Q(1,n)$ with fixed point set $F = (\mathbb{R}P^1 \times \mathbb{R}P^n) \bigsqcup (\mathbb{R}P^1 \times \mathbb{R}P^n)$. This represents the realization of third case of case (2) of Theorem 3.3.

Finally, an involution T_5 defined by

$$T_5([x, [z_0, z_1, \ldots, z_n], z]) = [-x, [z_0, z_1, \ldots, z_r, -z_{r+1}, \ldots, -z_n], z],$$

which commutes with S. So it induced an involution on $Q(1,n)$ with fixed point set $F = (\mathbb{C}P^n \times \mathbb{R}P^1) \bigsqcup (\mathbb{C}P^n \times \mathbb{R}P^1)$. This represents the realization of fourth case of case (2) of Theorem 3.3.

An involution T_6 defined by

$$T_6([x, [z_0, z_1, \ldots, z_n], z]) = [-x, [z_0, z_1, \ldots, z_n], \bar{z}],$$

commutes with S and so induces an involution on $Q(1,n)$ with fixed point set $F = (\mathbb{R}P^1 \times \mathbb{R}P^n) \bigsqcup \mathbb{C}P^n \bigsqcup \mathbb{C}P^n$. This represents the realization of first case of case (3) of Theorem 3.3.

An involution T_7 defined by

$$T_7([x, [z_0, z_1, \ldots, z_n], z]) = [x, [\bar{z}_0, \bar{z}_1, \ldots, \bar{z}_r, -\bar{z}_{r+1}, \ldots, -\bar{z}_n], \bar{z}],$$

commute with S and induced involution on $Q(1,n)$ with fixed point set $F = (\mathbb{R}P^1 \times \mathbb{R}P^n) \bigsqcup (\mathbb{R}P^1 \times \mathbb{R}P^r) \bigsqcup (\mathbb{R}P^1 \times \mathbb{R}P^{n-r-1})$. This represents the realization of second case of case (2) of Theorem 3.3.

An involution T_8 defined by

$$T_8([x, [z_0, z_1, \ldots, z_n], z]) = [x, [z_0, z_1, \ldots, z_r, -z_{r+1}, \ldots, -z_n], \bar{z}],$$

commutes with S and induced involution on $Q(1,n)$ with fixed point set $F = (P(1,r) \bigsqcup P(1, n-r-1)) \bigsqcup (P(1,r) \bigsqcup P(1, n-r-1))$. This represents the realization of case (4) of Theorem 3.3.

An involution T_9 defined by

$$T_9([x, [z_0, z_1, \ldots, z_n], z]) = [-x, [z_0, z_1, \ldots, z_r, -z_{r+1}, \ldots, -z_n], \bar{z}],$$

commutes with S and so induces an involution on $Q(1,n)$ with fixed point set $F = (\mathbb{R}P^1 \times \mathbb{R}P^n) \bigsqcup (\mathbb{C}P^r \bigsqcup \mathbb{C}P^{n-r-1}) \bigsqcup (\mathbb{C}P^r \bigsqcup \mathbb{C}P^{n-r-1})$. This represents the realization of case (5) of Theorem 3.3.

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