Filomat 39:29 (2025), 10419–10428
https://doi.org/10.2298/FIL2529419A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On the automorphisms of crossed modules

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Abstract. The notion of a *crossed module*, originally formulated by J. H. C. Whitehead within the framework of combinatorial homotopy theory, has emerged as a fundamental structure in modern algebra and topology. Over the decades, crossed modules have been recognized as powerful tools with broad applicability across diverse mathematical disciplines, including category theory, group homology and cohomology, homotopy theory, abstract algebra, and K-theory. They serve as a unifying concept that bridges algebraic and topological methods, offering a rich framework for both theoretical exploration and practical computation. Subsequent research has significantly expanded upon Whitehead's foundational work. For instance, Alp provided a detailed exposition of the *actor crossed module* in the context of algebroids, shedding light on its structural properties and categorical significance. Furthermore, Dehghanizadeh and Davvaz have systematically investigated several specialized classes of crossed modules, such as nilpotent, solvable, and n-complete crossed modules, as well as their representation theory, thereby extending the scope of applications and deepening our understanding of these algebraic objects. In the present paper, we continue this line of inquiry by introducing and studying novel concepts related to the *center* of a crossed module, the group of *central automorphisms*, and the family of *n-central automorphism groups*. Our analysis not only formalizes these notions within a rigorous categorical framework but also establishes several new results that clarify the interplay between centrality conditions and the intrinsic algebraic structure of crossed modules. These contributions aim to enrich the current theory and pave the way for future developments in the intersection of algebra, topology, and category theory.

1. Introduction

The category of crossed module was introduced by Whitehead [14]. Crossed modules have numerous applications in homology and cohomology of groups, homotopy theory, algebra, K-theory, and Lie algebras Also the equivalent category of crossed module which was the category of cat^1- object and introduced by Loday [12]. Computer application of these two categories is also presented by Alp and Wensley [1, 2]. Recent half-century studies and all of these applications showed that studying of crossed module category and all kind of automorphisms are very important. Importance of these works and applications gave birth to think about the subject of this paper. A special braided monoidal category is Yetter-Drinfeld category

2020 *Mathematics Subject Classification*. Primary 18D35; Secondary 20L05, 55U35.
Keywords. Automorphism, center, $n-$central automorphism, crossed module.
Received: 14 May 2024; Revised: 16 August 2025; Accepted: 02 September 2025
Communicated by Ljubiša D. R. Kočinac
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can be defined as follows: Let H be a Hopf algebra over a field k. Let Δ denote the coproduct and S the antipode of H. Let V be a vector space over k. Then V is called a (left left) Yetter-Drinfeld module over H if

- $(V, .)$ is a left $H-$module, where $. : H \otimes V \to V$ denotes the left action of H on V,
- $(V, \delta\ )$ is a left $H-$comodule, where $\delta : V \to H \otimes V$ denotes the left coaction of H on V,
- the maps $.$ and δ satisfy the compatibility condition $\delta(h.v) = h_{(1)}v_{(-1)}S(h_{(3)}) \otimes h_{(2)}.v_{(0)}$ for all $h \in H, v \in V$, where, using Sweedler notation, $(\Delta \otimes \mathrm{id})\Delta(h) = h_{(1)} \otimes h_{(2)} \otimes h_{(3)} \in H \otimes H \otimes$ H denotes the twofold coproduct of $h \in H$, and $\delta(v) = v_{(-1)} \otimes v_{(0)}$.

Let $(A, *)$ be a binary algebraic structure. Left translations are mappings $L_a : A \to A, x \mapsto a * x$. $(A, *)$ is left self-distributive if all L_a's are endomorphisms. That means we have: $a * (x * y) = (a * x) * (a * y)$ [10]. We recall some definitions and properties of the crossed module category. A crossed module (T, G, ∂) consist of a group homomorphism $\partial : T \longrightarrow G$ together with an action $(g, t) \longrightarrow {}^{g}t$ of G on T satisfying $\partial({}^{g}t) = g\partial(t)g^{-1}$ and ${}^{\partial(s)}t = sts^{-1}$, for all $g \in G$ and $s, t \in T$, for example see [13]. Simplicial crossed modules and mapping cones have been studied by Conduché [6]. q-perfect crossed modules have been studied in [10]. Also, Gilbert studied automorphisms and crossed modules in [11]. In the study of crossed modules, semi-direct products are of great importance. For more information, see [7].

In addition, the inner automorphism map $\tau : N \to Aut(N)$, other standard examples of crossed modules are:

- The inclusion of a normal subgroup $N \to G$,
- A G-module M with the zero homomorphism $M \to G$,
- And any epimorphism $E \to G$ with central kernel.

While the center of crossed modules is presented in section 2, Automorphisms of crossed modules and the obtained results are presented in section 3.

2. Center of crossed modules

Many results have been presented about crossed modules in [3–5, 8]. The Center, $n-$Center, Central automorphisms, and $n-$Central subjects were observed in this paper. This section includes automorphisms, central and $n-$Central automorphisms of a crossed module concept.

Definition 2.1. A crossed module morphism

$$< \alpha, \phi >: (T, G, \partial) \to (T', G', \partial')$$

is a commutative diagram of homomorphisms of groups

$$\begin{array}{ccc} T & \xrightarrow{\alpha} & T' \\ \partial \downarrow & & \downarrow \partial' \\ G & \xrightarrow[\phi]{} & G' \end{array}$$

such that for all $x \in G$ and $t \in T$; we have $\alpha({}^{x}t) = {}^{\phi(x)}\, \alpha(t)$.

Definition 2.2. Suppose that (T, G, ∂) is a crossed module. Center of (T, G, ∂) is the crossed module kernel $Z(T, G, \partial)$ of $< \eta, \gamma >$. Thus $Z(T, G, \partial)$ is the crossed module $(T^{G}, St_G(T) \cap Z(G), \partial)$ where T^{G} denotes the fixed point subgroup of T; that is,

$$T^{G} = \{\, t \in T \mid \ {}^{x}t = t \text{ for all } x \in G\}.$$

$St_G(T)$ is the stabilizer in G of T, that is:

$$St_G(T) = \{\, x \in G \mid \ {}^{x}t = t \text{ for all } t \in T\}$$

and $Z(T)$ is the center of G. Note that T^{G} is central in T.

Definition 2.3. Suppose that (T, G, ∂) is a crossed module. *n-center of* (T, G, ∂), $Z^n(T, G, \partial)$, *for n a nonnegative integer g is the crossed module* $\left((T^G)^n, Z^n(G) \cap St_G(T), \partial\right)$ *where*

$$(T^G)^n = \{\, t \in T \mid\ t^n = 1 \quad and \quad {}^{g}t = t\,;\ \ \forall g \in G\}$$

$$Z^n(G) = \{\, g \in Z(G) \mid \quad g^n = 1\}$$

$$St_G(T) = \{g \in G \mid\ {}^{g}t = t,\ \forall\, t \in T\}$$

The n-central of (T, G, ∂) *is a normal crossed submodule called n-central crossed submodule of* (T, G, ∂).

3. Automorphisms of crossed modules

In this section, we introduce the concept of Adeny-Yen crossed module map and central and n-central automorphisms of a crossed module and obtain results for them.

Definition 3.1. Suppose that (T, G, ∂) is a crossed module and $Z(T, G, \partial)$; center of it and $< \alpha, \phi > \in \mathrm{Aut}(T, G, \partial)$. If $< \bar{\alpha}, \bar{\phi} >$ induced of $< \alpha, \phi >$ in $\mathrm{Aut}\left(\frac{T}{T^G}, \frac{G}{St_G(T) \cap Z(G)}, \bar{\partial}\right)$; is identity, then$< \alpha, \phi >$ is called *central automorphism of crossed module* (T, G, ∂).

Definition 3.2. Let (T, G, ∂) is a crossed module and $Z^n(T, G, \partial)$; n-central of it;

$$Z^n(T, G, \partial) = \left((T^G)^n, Z^n(G) \cap St_G(T), \partial\right)$$

; and $< \alpha, \phi > \in \mathrm{Aut}(T, G, \partial)$. If $< \alpha, \phi >$ induces $< \bar{\alpha}, \bar{\phi} >$ in $\mathrm{Aut}\left(\frac{T}{(T^G)^n}, \frac{G}{St_G(T) \cap Z^n(G)}, \bar{\partial}\right)$; is identity; then $< \alpha, \phi >$ is called *n-central automorphism of crossed module* (T, G, ∂).
On the other hand:

$$\begin{aligned} < \alpha, \phi >: \quad & (T, G, \partial) \longrightarrow (T, G, \partial) \\ & \alpha : T \longrightarrow T \quad \text{isomorphism} \\ & \phi : G \longrightarrow G \quad \text{isomorphism} \end{aligned}$$

such that

- $\theta\partial = \partial\alpha$
- $\alpha({}^{g}t) = {}^{\theta(g)}\alpha(t); \qquad \forall t \in T$

$$\begin{array}{ccc} T & \xrightarrow{\ \alpha\ } & T' \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle\partial} \\ G & \xrightarrow[\ \theta\ ]{} & G' \end{array}$$

and in quotient crossed module; $\mathrm{Aut}\left(\frac{T}{(T^G)^n}, \frac{G}{Z^n(G) \cap St_G(T)}, \bar{\partial}\right)$ such that

$$\bar{\partial} : \frac{T}{(T^G)^n} \longrightarrow \frac{G}{Z^n(G) \cap St_G(T)}$$

$$\bar{\partial}\left(t(T^G)^n\right) = \partial(t)\,(Z^n(G) \cap St_G(T))$$

and

$$\begin{array}{ccc} \frac{T}{(T^G)^n} & \xrightarrow{\bar{\alpha}} & \frac{T}{(T^G)^n} \\ \bar{\partial}\downarrow & & \downarrow\bar{\partial} \\ \frac{G}{Z^n(G)\cap St_G(T)} & \xrightarrow[\bar{\phi}]{} & \frac{G}{Z^n(G)\cap St_G(T)} \end{array}$$

such that

- $\bar{\phi}\bar{\partial} = \bar{\partial}\bar{\alpha}$
- $\bar{\alpha}\left({}^{gZ^n(G)\cap St_G(T)}t(T^G)^n \right) = {}^{\bar{\phi}(gZ^n(G)\cap St_G(T))}\bar{\alpha}(t(T^G)^n)$

and

$$\bar{\alpha}(t(T^G)^n) = \alpha(t)(T^G)^n$$

$$\bar{\phi}\,(gZ^n(G) \cap St_G(T)) = \phi(g)Z^n(G) \cap St_G(T)$$

if $\bar{\alpha}$ and $\bar{\phi}$; induced identity; then $< \alpha, \phi >$ is called the n-central automorphism of crossed module (T, G, ∂).

Theorem 3.3. *If (T, G, ∂) is a crossed module, then*

$$Z(T \rtimes G) \leq C_T(G)Z(G).$$

Proof. The proof is straightforward and easily obtained using Definition of 2.2 and membership from the left. □

Theorem 3.4. *If (T, G, ∂) is a crossed module and $(|T|, |G|) = 1$, then $f|_T \in Aut_C(T)$ and $f|_G \in Aut_C(G)$, for all $f \in Aut_C(T \rtimes G)$).*

Proof. Suppose $f \in Aut_C(T \rtimes G)$ and $t \in T$, then $f|_T$ is in $Aut(T)$ and therefore $t^{-1}f(t) \in Z(T \rtimes G) \cap G$. In addition we have $Z(T \rtimes G) \cap G \leq Z(T)$. Hence we have $f|_T \in Aut_C(T)$. If $f \in Aut_C(T \rtimes G)$, $g \in G$ and $f(g) = g't$ such that $g' \in G, t \in T$, then $g^{-1}g' \in Z(G)$ and $t \in C_T(G)$. If $|T| = r$, then we have $f(g^r) = g'r$, for some $g' \in G$. But for all $g \in G$, we have an element$g_1 \in G$ such that $g = g_1^r$ and $f(g) \in G$. This means that G under the central automorphism f of $T \rtimes G$, is an invariant subgroup. Hence $f|_G \in Aut(G)$ and $g^{-1}f(g) \in Z(T \rtimes G) \cap G \leq Z(G)$ and therefore $f|_G \in Aut_C(G)$ □

Theorem 3.5. *If (T, G, ∂) is a crossed module and $C_T(G) = 1$, then $f|_G \in Aut_C(G)$ and $\pi|_G \circ f|_T \in Hom(T, Z(T \rtimes G) \cap G)$, for all $f \in Aut_C(T \rtimes G)$, where $\pi|_G$ is the projection map from $T \rtimes G$ into G.*

Proof. If $f \in Aut_C(T \rtimes G)$ and $g \in G$, then $f(g) = g_1t_1$ and $g^{-1}f(g) = g^{-1}g_1t_1 \in Z(T \rtimes G)$. Hence by theorem 3.3, we have $g^{-1}g_1 \in Z(G)$ and $t_1 \in C_T(G) = 1$. Therefore $f|_G \in Aut_C(G)$. If $t \in T$, then from $C_T(G) = 1$, we get it $f(t) = gt$ and $\pi|_G \circ f|_T \in Hom(T, Z(T \rtimes G) \cap G)$, wehere is $g \in G$. □

Definition 3.6. A non-group G, that has non-trivial abelian direct factor is said to be purely non-abelian.

Definition 3.7. A Adeny-Yen crossed module map is an onto map $< \phi_1, \phi_2 >$ from $Aut_C(T, G, \partial)$ to $Hom((T, G, \partial), Z(T, G, \partial))$ such that,

$$< \phi_1, \phi_2 >< \alpha, \theta >=< \phi_1, \phi_2 >_{<\alpha,\theta>}$$

and $< \phi_1, \phi_2 >_{<\alpha,\theta>}$ is crossed module homomorphism of (T, G, ∂) into $Z(T, G, \partial) = (T^G, St_G(T) \cap Z(G), \partial)$. Furthermore,

$$< \phi_1, \phi_2 >_{<\alpha,\theta>}=< \phi_{1<\alpha,\theta>}, \phi_{2<\alpha,\theta>} >,$$

where

$$\begin{aligned} \phi_{1<\alpha,\theta>} \quad &: \quad T \longrightarrow T^G \\ & \quad\ \phi_{1<\alpha,\theta>}(t) = t^{-1}\alpha(t) \end{aligned}$$

and

$$\begin{aligned} \phi_{2<\alpha,\theta>} \quad &: \quad G \longrightarrow St_G(T) \cap Z(G) \\ & \quad\ \phi_{2<\alpha,\theta>}(g) = g^{-1}\theta(g) \end{aligned}$$

Let C^* be the set of all central automorphisms of (T, G, ∂) fixing $Z(T, G, \partial)$ element wise.

Theorem 3.8. *For purely non-abelian groups T and G, Adeny-Yen crossed module map is one-to-one correspondence of* $Aut_C(T, G, \partial)$ *onto*

$$Hom((T, G, \partial), Z(T, G, \partial)).$$

Proof. [9]. □

Theorem 3.9. *For any non-abelian groups T and G the restriction of the Adeny-Yen crossed module map*

$$< \phi_1, \phi_2 >: C^* \longrightarrow Hom\,((T, G, \partial), (Z(T, G, \partial))\,,$$

is a homomorphism crossed module.

Proof. Suppose $< \alpha_1, \theta_1 >$ and $< \alpha_2, \theta_2 > \in C^*$. On the other hand,

$$\begin{array}{ccc} T & \xrightarrow{\alpha_1} & T^G \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle\partial} \\ G & \xrightarrow[\theta_1]{} & St_G(T) \cap Z(G) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} T & \xrightarrow{\alpha_2} & T^G \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle\partial} \\ G & \xrightarrow[\theta_2]{} & St_G(T) \cap Z(G). \end{array}$$

Then for any $t \in T,\ g \in G$ we have

$$\begin{aligned} \phi_{1<\alpha_1,\theta_1><\alpha_2,\theta_2>}(t) &= t^{-1}(\alpha_1 \circ \alpha_2)(t) \\ &= t^{-1}(\alpha_1(\alpha_2(t)) \end{aligned}$$

and

$$\begin{aligned} \phi_{1<\alpha_1,\theta_1>} \circ \phi_{1<\alpha_2,\theta_2>}(t) &= \phi_{1<\alpha_1,\theta_1>}(t^{-1}\alpha_2(t)) \\ &= (t^{-1}\alpha_2(t))^{-1}\alpha_1(t^{-1}\alpha_2(t)) \\ &= \alpha_2(t^{-1})t\alpha_1(t^{-1})\alpha_1(\alpha_2(t)) \\ &= \alpha_2(t^{-1})t^{-1}\alpha_1(\alpha_2(t)) \\ &= t^{-1}\alpha_1(\alpha_2(t)) \end{aligned}$$

Moreover,

$$\begin{aligned} \phi_{1<\alpha_1,\theta_1><\alpha_2,\theta_2>}(g) &= g^{-1}(\alpha_1 \circ \alpha_2)(g) \\ &= g^{-1}(\alpha_1(\alpha_2(g)) \end{aligned}$$

and

$$\begin{aligned} \phi_{1<\alpha_1,\theta_1>} \circ \phi_{1<\alpha_2,\theta_2>}(g) &= \phi_{1<\alpha_1,\theta_1>}(g^{-1}\alpha_2(g)) \\ &= (g^{-1}\alpha_2(g))^{-1}\alpha_1(g^{-1}\alpha_2(g)) \\ &= \alpha_2(g^{-1})g\alpha_1(g^{-1})\alpha_1(\alpha_2(g)) \\ &= g\alpha_2(g^{-1})\alpha_1(g^{-1})\alpha_1(\alpha_2(g)) \\ &= g^{-1}\alpha_1(g^{-1})\alpha_1(\alpha_2(g)) \\ &= g^{-1}\alpha_1(\alpha_2(g)). \end{aligned}$$

So $< \phi_1, \phi_2 >_{<\alpha_1,\theta_1><\alpha_2,\theta_2>} = < \phi_1, \phi_2 >_{<\alpha_1,\theta_1>} \circ < \phi_1, \phi_2 >_{<\alpha_2,\theta_2>}$. □

Definition 3.10. Given a crossed module $\mathcal{X} = (\partial : T \to G)$. We denote by Der($\mathcal{X}$) the set of all derivations from G to T, i.e. all maps $\chi : G \to T$ such that for all $q, r \in G$,

$$\chi(qr) = (\chi q)^r \chi(r).$$

Definition 3.11. The Whitehead group $\mathcal{W}(\mathcal{X})$ is defined to be group of units of Der($\mathcal{X}$). The elements of $\mathcal{W}(\mathcal{X})$ will be called regular derivations.

Example 3.12. If T is a G-module, then the trivial homomorphism $T \to G$ is a crossed module and Der($\mathcal{X}$) is the usual abelian group of derivations.

Example 3.13. Together with the conjugation action of a group G of itself, the identity map $\mathcal{X} = (id : G \to G)$ is a crossed mudule. An automorphism α of G determines its displacement derivation $\delta_\alpha \in \mathcal{W}(\mathcal{X})$ given by $\delta_\alpha(r) = \alpha(r)r^{-1}$ and the correspodence $\alpha \to \delta_\alpha$ is an isomorphism $\delta : \mathrm{Aut}(G) \to \mathcal{W}(\mathcal{X})$.

Definition 3.14. The actor crossed module $\mathcal{A}(\mathcal{X})$ is defined to be the crossed module

$$\mathcal{A}(\mathcal{X}) = (\Delta : \mathcal{W}(\mathcal{X}) \to \mathrm{Aut}(\mathcal{X})).$$

Theorem 3.15. *Let (T, G, ∂) has trivial n-central, then its actor $\mathcal{A}(T, G, \partial)$ also has trivial n-central.*

Proof. Let us assume that $Z^n(T, G, \partial) = 1$, so that $(T^G)^n = 1$ and $Z^n(G) \cap St_G(T) = 1$. Now the n-centre of $\mathcal{A}(T, G, \partial)$ is the crossed module

$$\mathcal{A}(T, G, \partial) = (D(G, T), \mathrm{Aut}(T, G, \partial), \Delta)$$

$$Z^n(\mathcal{A}(T, G, \partial)) =$$

$$\left(\left(D(G, T)^{\mathrm{Aut}(T,G,\partial)}\right)^n, Z^n(\mathrm{Aut}(T, G, \partial)) \cap St_{\mathrm{Aut}(T,G,\partial)}(D(G, T), \Delta\right).$$

So assume that $\chi \in \left(D(G, T)^{\mathrm{Aut}(T,G,\partial)}\right)^n$. Then for all

$$< \alpha, \phi > \in \mathrm{Aut}(T, G, \partial) \quad , \ ^{<\alpha,\phi>}\chi = \chi.$$

and $\chi^n = 1$. In particular, this true for all $< \alpha_y, \phi_y >$ where $y \in G$. But

$$< \alpha_y, \phi_y >_\chi = \eta_{\chi(y)^{-1}} \circ \chi$$

So that $^{<\alpha_y,\phi_y>}\chi = \chi$ implies that $\eta_{\chi(y)^{-1}} = 1$ for all $y \in G$. that is $\chi(y)^{-1}\, {}^x\chi(y) = 1$ for all $x, y \in G$. Now since $(T^G)^n = 1$, χ is the trivial derivation and so

$$\left(D(G, T)^{<\alpha_y,\phi_y>}\right)^n = 1.$$

Now suppose that

$$< \alpha, \partial > \in Z^n(\mathrm{Aut}(T, G, \partial)) \cap St_{\mathrm{Aut}(T,G,\partial)}(D(G, T)).$$

Then $^{<\alpha,\phi>}\chi = \chi$ for all $\chi D(G, T)$. In particular, $^{<\alpha,\phi>}\eta_t = \eta_t$ for all $t \in T$, that is $\eta_{\alpha(t)} = \eta_t$ which implies that $t^{-1}\alpha(t) \in (T^G)^n = 1$ for all $t \in T$. Thus $\alpha = 1_T$, the identity automorphism of T. Now $< \alpha, \phi > \in Z(\mathrm{Aut}(T, G, \partial))$ and hence for all $y \in G$,

$$< \alpha, \phi >< \alpha_y, \phi_y >=< \alpha_y, \phi_y >< \alpha, \phi >$$

and

$$< \alpha, \phi >^n = 1$$

implying that $\phi\phi_y = \phi_y\phi$ for all $y \in G$. So $\phi(xyx^{-1}) = y\phi(x)y^{-1}$ for all $x, y \in G$. Since ϕ is an automorphism of G, we have $y^{-1}\phi(y) \in Z(G)$ for all $y \in G$. Now since $< \alpha, \phi >$ is a crossed module morphism, $\alpha({}^yt) = {}^{\phi(y)}\alpha(t)$. But $\alpha^n = 1_T$ so that ${}^yt = \phi(y)t$ for all $y \in G$ and $t \in T$. Therefore $y^{-1}\phi(y) \in Z^n(G) \cap St_G(T) = 1$ so that $\phi^n = 1_G$. Therefore $< \alpha^n, \phi^n >=< 1_T, 1_G >$ completing the proof. □

Theorem 3.16. *There is a homomorphism of groups*

$$\Delta : \mathcal{W}(\mathcal{X}) \to Aut(\mathcal{X})$$
$$\chi \mapsto < \sigma, \rho >$$

and with the action $\chi^{<\alpha,\phi>} = \alpha^{-1}\chi\phi$, $\mathcal{A} < \mathcal{X} >= (\Delta : \mathcal{W}(\mathcal{X}) \to Aut(\mathcal{X}))$, *is a crossed module.*

Proof. [1]. □

In the holomorph Hol $X(= X \rtimes \mathrm{Aut}(X))$ of a group X, the action of $\mathrm{Aut}(X)$ on X becomes conjugation. Thus there is a one to one correspondence between characteristic subgroups of Hol X contained in X. The holomorph Hol (T, G, ∂) of a crossed module (T, G, ∂) is the semi-direct product

$$(T, G, \partial) \rtimes \mathcal{A}(T, G, \partial) = (T \rtimes D(G, T), G \rtimes \mathrm{Aut}(T, G, \partial), \partial \times \Delta),$$

with action given by

$$^{(g,<\alpha,\phi>)}(t, \chi) = (^{g_\alpha}(t\chi\phi^{-1}(g^{-1})), \alpha\chi\phi^{-1})$$

for all $g \in G$, $t \in T$, $< \alpha, \phi >\in \mathrm{Aut}(T, G, \partial)$ and $\chi \in D(G, T)$. A subcrossed module (T', G', ∂) of the crossed module (T, G, ∂) is characteristic in (T, G, ∂) if restriction defines a morphism $\mathcal{A}(T, G, \partial) \longrightarrow \mathcal{A}(T', G', \partial)$. On the other hand: a subcrossed module of (T, G, ∂) is characteristic if and only if its image in $\mathrm{Hol}(T, G, \partial)$ is a normal subcrossed module.

Proposition 3.17. *The whitehead group of* χ *is isomorphic to the group of units* $\mathbb{Z}_n^\star$ *of* $\mathbb{Z}_n$*, which is isomorphic to*

$$Aut(\mathbb{Z}_n, +) = \{k \longrightarrow (\alpha_k : 1 \longrightarrow k) | \quad 1 \leq k \leq n\}$$

with identity α_1.

Proof. [1]. □

Proposition 3.18. *If* G *is a cyclic group, then* $Aut\, C_n(G) = Aut(G)$.

Proof. In cyclic group $G =< a >$; $Z_n(G) = G$, for all $n \in \mathbb{N}$. Now for all $\alpha \in \mathrm{Aut}(G)$; $\bar{\alpha} : \frac{G}{Z_n(G)} \longrightarrow \frac{C}{Z_n(G)}$ such that $\bar{\alpha}(gZ_n(G)) = \alpha(g)Z_n(G)$ we have $g^{-1}\alpha(g) \in Z_n(G) = G$, so $\alpha \in \mathrm{Aut}_{C_n}(G)$. Therefore $\mathrm{Aut}_{C_0}(G) \subseteq \mathrm{Aut}_{C_n}(G)$ completing the proof. □

Theorem 3.19. *Let* χ *be a crossed module and* $\mathcal{W}(\chi)$*, whitehead group of* χ*. Then* $Aut_{C_n}(\mathcal{W}) = Aut(\mathcal{W})$.

Proof. By propositions 3.17 and 3.18, proof is straightforward. □

A groupoid is a small category in which every arrow is invertible.
In the notation used [2], a finite groupoid $C = (C_1, C_0)$ consists of the following:

- a set $\mathrm{Ob}(C) = C_0$ of objects,
- a set $\mathrm{Arr}(C) = C_1$ of objects,
- source and target maps $\partial_1^-, \partial_1^+ : C_1 \longrightarrow C_0$, so that we write $(\alpha : u \longrightarrow v)$ whenever $\partial_1^-\alpha = u$ and $\partial_1^+\alpha = v$, and denote by $C(u, v)$ the hom-set of arrows with source u and target v;
- a function $\partial_0 : C_0 \longrightarrow C_1$, $u \longrightarrow (1_u : u \longrightarrow u)$, the identity arrow at u,
- an associative partial composition: $C_1 \times_\bullet C_1 \longrightarrow C_1$; with $\alpha\beta$ defined whenever $\partial_1^+\alpha = \partial_1^-\beta$, such that $\partial_1^-(\alpha\beta) = \partial_1^-\alpha$ and $\partial_1^+(\alpha\beta) = \partial_1^+\beta$, so that $C(u) := C(u, u)$ is a group with identity 1_u, called the object group at u,
- for each arrow $(\alpha : u \longrightarrow v)$ an inverse arrow $(\alpha^{-1} : v \longrightarrow u)$ such that $\alpha\alpha^{-1} = 1_u$ and $\alpha^{-1}\alpha = 1_v$.

A morphism of groupoids, as for general categories, is called a functor. Thus a functor $g = (g_1, g_0) : C \longrightarrow D$ is a pair of maps $(g_1 : C_1 \longrightarrow D_1, g_\bullet : C_0 \longrightarrow D_0)$ such that $g_1 1_u = 1_{g_0 u}$ and $g_1(\alpha\beta) = (g_1\alpha)(g_1\beta)$ whenever the composite arrow is defined. For example:

1. The categories of groups and groupoids, and their morphisms, are written **Gp**, **Gpd** respectively. There is a functor Gpd : **Gp** $\longrightarrow$ **Gpd**, $C \longrightarrow C_0$, $C \longrightarrow (C : \bullet \longrightarrow \bullet)$, where C_0 is a groupoid with a single object $\bullet$.
2. For X a set, the trivial groupoid $O(X) = (O_1, O_0)$ on X has $O_0 = X$ and $O_1 = \{1_X \mid \ x \in X\}$. We denote $O(\{1, \dots, n\})$ by O_n.
3. The unit groupoid I has objects $\{0, 1\}$ and four arrows. The two non-identity arrows are $(l : 0 \longrightarrow 1)$ and its inverse $(l^{-1} : 1 \longrightarrow 0)$.
4. The product $C \times D$ of groupoids C, D has objects $C_0 \times D_0$, arrows $C_1 \times D_1$, and composition $(\alpha_1, \beta_1)(\alpha_2, \beta_2) = (\alpha_1\alpha_2, \beta_1\beta_2)$, so that $(\alpha, \beta)^{-1} = (\alpha^{-1}, \beta^{-1})$. In particular, $C = C_0 \times I_n$ may be thought of as the groupoid with n objects $\{1, 2, \dots, n\}$; $n^2|C|$ arrows $\{(p, c, q) \mid \ c \in C, \ 1 \le p, q \le n\}$; source $\partial_1^-(p, c, q) = p$, target $\partial_1^+(p, c, q) = q$, composition $(p, c, q)(q, c^1, r) = (p, cc^1, r)$, and inverses $(p, c, q)^{-1} = (q, c^{-1}, p)$.
5. The crossed module of groupoids corresponding to $\chi = (\delta : B \longrightarrow C)$ is $\chi_\bullet = (\delta_\bullet : B_\bullet \longrightarrow C_\bullet)$ where $\delta_\bullet(\bullet, b, \bullet) = (\bullet, \delta b, \bullet)$ and $(\bullet, b, \bullet)^{(\bullet, c, \bullet)} = (\bullet, b^c, \bullet)$ since $C = C_\bullet \times I_n$ acts on itself, we obtain the identity crossed module $(\text{id} : C \longrightarrow C)$.
6. If $\chi = (\delta : B \longrightarrow C)$ is a crossed module of groups, let $B = B_\bullet \times 0_n$ and $C = C_\bullet \times I_n$. Then B is a C-groupoid-system, and $\mathcal{C} = (\partial : B \longrightarrow C) \cong \chi_\bullet \times \mathcal{I}_n$ is a crossed module with $\partial(p, b, p) = (p, \delta b, p)$. If $B = B_\bullet \times I_m$ then $(\partial : B \longrightarrow C_\bullet)$ is a crossed module with $\partial(u, , b, v) = \delta b$ and $(u, b, v)^c = (u, b^c, v)$.
7. The direct product $\mathcal{C} \times \mathcal{C}'$ is $(\partial \times \partial' : C_2 \times C_2' \longrightarrow C_1 \times C_1')$, with $(\beta, \beta')^{(\alpha, \alpha')} := (\beta^\alpha, \beta'^{\alpha'})$.
8. The connected tree groupoid I_n has objects $\{1, 2, \dots, n\}$ and arrows $\{(p, q) \mid \ 1 \le p, q \le n\}$ where $\partial_1^-(p.q) = p$, $\partial_1^+(p, q) = q$, $(p, q)(q, r) = (p, r)$, and $(p, q)^{-1} = (q, p)$. Note that $I_2 \cong I$.

Proposition 3.20. *The automorphism group of $C = C_\bullet \times I_n$ is given by*

$$Aut\, C \cong ((S_n \times Aut\, C) \ltimes C^n)/k_1(C)$$

where $k_1(C) = \{\big(((\ ), \wedge c), (c^{-1}, \dots, c^{-1})\big) \mid \ c \in C\} \cong C$ and $()$ is the identity permutation.

Proof. [2] □

The following result generalises before proposition:

Proposition 3.21. *The automorphism group of*

$$\mathcal{C} = (\partial : (B_\bullet \times 0_n) \longrightarrow (C_\bullet \times I_n))$$

is given by

$$Aut\, \mathcal{C} \cong ((S_n \times Aut\,) \ltimes C^n)\,/K_2(\mathcal{C})$$

where

$$\chi = (\partial : B \longrightarrow C)$$

and

$$K_2(\mathcal{C}) = \{\big(((\ ), (\wedge c, \wedge c)), (c^{-1}, \dots, c^{-1})\big) \mid \ c \in C\} \cong C.$$

Proof. [2] □

Proposition 3.22. *Let $C = C_\bullet \times I_m$ be abelian. Then,*

$$Aut_{C_n}(C) \cong ((S_m \times Aut\, C) \ltimes C^m)\,/K_1(C)$$

where $K_1(C)$ defined in proposition 3.20.

Proof. By Proposition 3.21 proof is straightforward. □

Proposition 3.23. *If $B_\bullet \times 0_m$ and $C_\bullet \times I_m$ are abelian, and*

$$C = (\partial : (B_\bullet \times 0_m) \longrightarrow (C_\bullet \times I_m)),$$

then

$$Aut_{C_n}(C) \cong ((S_m \times Aut(\chi) \ltimes C^m)/K_2(G)),$$

where

$$\chi = (\partial : B \longrightarrow C)$$

and

$$K_2(C) = \{\Big(((\ ), (\wedge c, \wedge c)), (c^{-1}, \ldots, c^{-1})\Big) \mid \quad c \in C\} \cong C.$$

Proof. By Proposition 3.22, it is obvious. □

4. Conclusion

This work has undertaken a systematic investigation of the algebraic and categorical structures associated with the center, nnn-center, central automorphism group, and nnn-central automorphism group of crossed modules. Building upon the foundational definitions and structural properties of crossed modules, we have derived several original results that refine the understanding of these subgroups and their mutual relationships. The methods presented herein not only provide a unified framework for analyzing these invariants but also yield constructive criteria for their identification in specific algebraic settings. The significance of these findings extends beyond the immediate scope of crossed modules. In particular, the algebraic phenomena identified in the present study suggest natural pathways for generalization to more intricate categorical structures, such as crossed polymodules and higher-dimensional crossed objects. Such extensions are anticipated to interact meaningfully with ongoing developments in higher-dimensional algebra, homotopical group theory, and non-abelian cohomology. Furthermore, the techniques and perspectives advanced here may find applications in the study of symmetry, obstruction theory, and the classification of algebraic structures in related homological contexts. Future research will focus on formulating and proving analogous results in the broader framework of crossed polymodules, with particular attention to the preservation or transformation of the central and nnn-central properties under natural functors between such categories. This line of inquiry promises to deepen the connections between the theory of crossed modules and the broader landscape of categorical algebra, thereby enriching both the theoretical foundations and potential applications of these structures.

Acknowledgements

The authors wish to express their deepest gratitude to the anonymous referees for their careful reading of the manuscript and for providing a wealth of insightful comments, constructive criticisms, and precise recommendations. Their rigorous evaluation and attention to detail have substantially improved the mathematical accuracy, conceptual depth, and overall clarity of the paper. The authors also gratefully acknowledge the handling editor for their professional guidance and for facilitating a fair and efficient review process. The generous investment of time, expertise, and scholarly commitment by both the reviewers and the editor has been invaluable in shaping the final version of this work.

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