Filomat 39:29 (2025), 10429–10459
https://doi.org/10.2298/FIL2529429D
Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On a solvable two-dimensional system of difference equations

Melike Dilbeyen[a], **Merve Kara**[b,*]

[a]*Department of Mathematics, Institute of Science, Karamanoğlu Mehmet" Bey University 70200, Karaman, Türkiye*
[b]*Department of Mathematics, Kamil Özdağ Science Faculty, Karamanoğlu Mehmet Bey University, 70200, Karaman, Türkiye*

Abstract. In this paper, we investigate two-dimensional system of difference equations with variable coefficients. Firstly, we obtain the general solutions of mentioned system of difference equations. Later, the solutions of this system of difference equations are acquired when the coefficients are constant. Additionally, the solutions are expressed when the parameters α and γ are equal to 1 or not equal to 1. Further, we research the asymptotic behavior of the well- defined solutions of aforementioned system of difference equations. Lastly, the forbidden set of the initial conditions is acquired by using obtained formulas.

1. Introduction and preliminaries

Difference equations emerge as descriptions of formations in nature. Because; most of the time-varying variables have discrete measurements. While it sometimes emerges as a mathematical model of a physical event or nature event, it sometimes can arise from numerical approaches. Difference equations and system of difference equations attract the attention of many branches of science because they include real-life modeling. For example; biology, physics, engineering, economics. So, theory of difference equations have call attention to a lot of authors in previous years [1–10, 12, 15–22, 25]. Such as, in an earlier paper, Yazlik have solved the following difference equation

$$z_{n+1} = \frac{z_{n-2}z_{n-3}z_{n-4}}{z_n z_{n-1}(\pm 1 \pm z_{n-2}z_{n-3}z_{n-4})}, n \in \mathbb{N}_0,$$

where the initial values $z_{-4}, z_{-3} \cdots , z_0$ are real numbers, in [23].
In [14], Stević et al. studied the solution of the following difference equation and asymptotic behavior of the solution

$$x_n = \frac{x_{n-3}x_{n-4}x_{n-5}}{x_{n-1}x_{n-2}(a_n + b_n x_{n-3}x_{n-4}x_{n-5})}, n \in \mathbb{N}_0, \tag{1}$$

2020 *Mathematics Subject Classification*. Primary 39A10; Secondary 39A20, 39A30.
Keywords. Asymptotic behavior, solution, system of difference equations.
Received: 13 August 2025; Revised: 14 August 2025; Accepted: 26 August 2025
Communicated by Maria Alessandra Ragusa
This study is a part of the first author's Ph.D. Thesis.
* Corresponding author: Merve Kara
Email addresses: melikedilbeyen@gmail.com (Melike Dilbeyen), mervekara@kmu.edu.tr (Merve Kara)
ORCID iDs: https://orcid.org/0009-0003-8304-8287 (Melike Dilbeyen), https://orcid.org/0000-0001-8081-0254 (Merve Kara)

where a_n and b_n are real sequences and the initial conditions x_{-i}, $i=\overline{1,5}$ are real numbers.
The authors of [24] solved the following non-linear difference equation

$$u_n = \frac{u_{n-4}u_{n-5}u_{n-6}}{u_{n-1}u_{n-2}(a+bu_{n-3}u_{n-4}u_{n-5}u_{n-6})},\ n\in\mathbb{N}_0, \tag{2}$$

where the initial values and the parameters are real numbers, in closed form. Also, they obtained asymptotic behavior of well-defined solution of difference equation (2).
In [11], Karakaya, et al. showed that the following system of difference equations

$$u_n = \frac{u_{n-4}v_{n-5}u_{n-6}}{v_{n-1}u_{n-2}(a+bv_{n-3}u_{n-4}v_{n-5}u_{n-6})},\ v_n = \frac{v_{n-4}u_{n-5}v_{n-6}}{u_{n-1}v_{n-2}(c+du_{n-3}v_{n-4}u_{n-5}v_{n-6})},\ n\in\mathbb{N}_0,$$

where the parameters a,b,c,d and the initial conditions $u_{-k}, v_{-k}, i=\overline{1,6}$ are reel numbers, some sub-classes of non-linear system of difference equations are solvable in closed form.
Here, we will study the following system of difference equations

$$x_n = \frac{x_{n-3}y_{n-4}x_{n-5}}{y_{n-1}x_{n-2}(\alpha_n+\beta_n x_{n-3}y_{n-4}x_{n-5})},\quad y_n = \frac{y_{n-3}x_{n-4}y_{n-5}}{x_{n-1}y_{n-2}(\gamma_n+\delta_n y_{n-3}x_{n-4}y_{n-5})},\ n\in\mathbb{N}_0, \tag{3}$$

where $(\alpha_n)_{n\in\mathbb{N}_0}$, $(\beta_n)_{n\in\mathbb{N}_0}$, $(\gamma_n)_{n\in\mathbb{N}_0}$ and $(\delta_n)_{n\in\mathbb{N}_0}$ are real sequences and initial values x_k, y_k, $k=\overline{-5,-1}$ are real numbers. Note that system (3) is two-dimensional system of difference equations form of equation (1). In this study, we obtain the general solutions of system (3). The solutions of system (3) are obtained when the parameters are constant. In addition, the solutions are acquired when the parameters α and γ are equal to 1 or not equal to 1. Moreover, we study the asymptotic behavior of the well- defined solutions of system (3). Finally, the forbidden set of the initial conditions are defined by using obtained formulas.
Throughout this study, we use the following lemma.

Lemma 1.1. *[13] Let $(\gamma_n)_{n\in\mathbb{N}_0}$ and $(\delta_n)_{n\in\mathbb{N}_0}$ be two sequences of real numbers and y_{km+l}, $l=\overline{0,k-1}$ be solutions of the equations*

$$y_{km+l} = \gamma_{km+l}y_{k(m-1)+l} + \delta_{km+l},\ m\in\mathbb{N}_0. \tag{4}$$

Then for each fixed $l=\overline{0,k-1}$ and $m\geq -1$, equation (4) *has the general solution*

$$y_{km+l} = y_{l-k}\prod_{j=0}^{m}\gamma_{kj+l} + \sum_{s=0}^{m}\delta_{ks+l}\prod_{j=s+1}^{m}\gamma_{kj+l}. \tag{5}$$

Further, if $(\gamma_n)_{n\in\mathbb{N}_0}$ and $(\delta_n)_{n\in\mathbb{N}_0}$ are constant and $l=\overline{0,k-1}$, then

$$y_{km+l} = \begin{cases} \gamma^{m+1}y_{l-k} + \delta\frac{1-\gamma^{m+1}}{1-\gamma}, & \text{if } \gamma\neq 1,\\ y_{l-k}+\delta(m+1), & \text{if } \gamma = 1.\end{cases} \tag{6}$$

2. Formulas for well-defined solutions of system (3)

Let $\{(x_n,y_n)\}_{n\geq -5}$ be a solutions of system (3). If at least one of the initial values x_{-i}, y_{-i}, $i=\overline{1,5}$, is equal to zero, then the solutions of system (3) is not defined. For instance, if $x_{-5}=0$, then $x_0=0$ and so y_1 is not defined. Similarly, if $y_{-5}=0$, then $y_0=0$ and so x_1 is not defined. For $i=\overline{1,4}$, the other cases are similar. On the other hand, if $x_{n_0}=0$ ($n_0\in\mathbb{N}_0$), $x_n\neq 0$, for $-5\leq n\leq n_0-1$, and x_k and y_k are defined for $-5\leq k\leq n_0-1$, then according to the first equation in (3) we get that $y_{n_0-4}=0$. If $n_0-4\leq -1$, then $y_{-j_0}=0$, for $j_0=\overline{1,4}$. If $n_0>3$, then according to the second equation in (3) we have that $y_{n_0-7}=0$ or $y_{n_0-9}=0$. If $n_0-7\leq -1$, then $y_{-j_1}=0$, for $j_1=\overline{1,5}$. If $n_0-9\leq -1$, then $y_{-j_1}=0$, for $j_1=\overline{1,5}$. Repeating this process, we have that

$y_{-j_1} = 0$, for $j_1 = \overline{1,5}$. Similarly, if $y_{n_1} = 0$ ($n_1 \in \mathbb{N}_0$), $y_n \neq 0$, for $-5 \leq n \leq n_1 - 1$, and x_k and y_k are defined for $-5 \leq k \leq n_1 - 1$, one can easily show that $x_{-j_1} = 0$, for $j_1 = \overline{1,5}$. Therefore, for every well-defined solutions of system (3), we have that

$$x_n y_n \neq 0,\ n \geq -5, \tag{7}$$

if and only if $x_{-i}y_{-i} \neq 0$, for $i = \overline{1,5}$.

Now, if we assume that $(x_n, y_n)_{n\geq -5}$ is a well-defined solutions of system (3). From system (3) we obtain

$$x_n y_{n-1} x_{n-2} = \frac{x_{n-3}y_{n-4}x_{n-5}}{\alpha_n + \beta_n x_{n-3}y_{n-4}x_{n-5}}, \quad y_n x_{n-1} y_{n-2} = \frac{y_{n-3}x_{n-4}y_{n-5}}{\gamma_n + \delta_n y_{n-3}x_{n-4}y_{n-5}},\ n \in \mathbb{N}_0,$$

and

$$\frac{1}{x_n y_{n-1} x_{n-2}} = \frac{\alpha_n}{x_{n-3}y_{n-4}x_{n-5}} + \beta_n, \quad \frac{1}{y_n x_{n-1} y_{n-2}} = \frac{\gamma_n}{y_{n-3}x_{n-4}y_{n-5}} + \delta_n,\ n \in \mathbb{N}_0.$$

If we use the following change of variables

$$u_n = \frac{1}{x_n y_{n-1} x_{n-2}},\ v_n = \frac{1}{y_n x_{n-1} y_{n-2}},\ n \geq -3, \tag{8}$$

which transforms system (3) into the following linear third-order difference equations

$$u_n = \alpha_n u_{n-3} + \beta_n,\ v_n = \gamma_n v_{n-3} + \delta_n,\ n \in \mathbb{N}_0. \tag{9}$$

If we carry out the decomposition of indices $n \to 3m + k$, $m \in \mathbb{N}_0$ and $k = \overline{0,2}$, to equations in (9), then it can be written as follows

$$u_m^{(k)} = \alpha_{3m+k} u_{m-1}^{(k)} + \beta_{3m+k}, \quad v_m^{(k)} = \gamma_{3m+k} v_{m-1}^{(k)} + \delta_{3m+k}, \tag{10}$$

where $u_m^{(k)} = u_{3m+k}$, $v_m^{(k)} = v_{3m+k}$, $m \geq -1$ and $k = \overline{0,2}$. By using equation (5) in Lemma 1.1, the solutions of equations in (10) can be written as follows

$$u_{3m+k} = u_{k-3}\prod_{h=0}^{m}\alpha_{3h+k} + \sum_{s=0}^{m}\beta_{3s+k}\prod_{h=s+1}^{m}\alpha_{3h+k},\ v_{3m+k} = v_{k-3}\prod_{h=0}^{m}\gamma_{3h+k} + \sum_{s=0}^{m}\delta_{3s+k}\prod_{h=s+1}^{m}\gamma_{3h+k},$$

for $m \in \mathbb{N}_0$, $k = \overline{0,2}$. By using the transformations in (8), we have

$$x_n = \frac{1}{u_n y_{n-1} x_{n-2}},\ y_n = \frac{1}{v_n x_{n-1} y_{n-2}},\ n \geq -3,$$

$$x_n = \frac{v_{n-1}y_{n-3}}{u_n} = \frac{v_{n-1}}{u_n v_{n-3} x_{n-4} y_{n-5}} = \frac{v_{n-1}u_{n-4}}{u_n v_{n-3}} x_{n-6},\ n \geq 1, \tag{11}$$

$$y_n = \frac{u_{n-1}x_{n-3}}{v_n} = \frac{u_{n-1}}{v_n u_{n-3} y_{n-4} x_{n-5}} = \frac{u_{n-1}v_{n-4}}{v_n u_{n-3}} y_{n-6},\ n \geq 1. \tag{12}$$

Now, we can apply the decomposition of indices $n \to 6m + t$, $m \in \mathbb{N}_0$, $t = \overline{1,6}$, to equations in (11) and (12), then we obtain

$$x_{6m+t} = \frac{v_{6m+t-1}u_{6m+t-4}}{u_{6m+t}v_{6m+t-3}} x_{6(m-1)+t}, \quad y_{6m+t} = \frac{u_{6m+t-1}v_{6m+t-4}}{v_{6m+t}u_{6m+t-3}} y_{6(m-1)+t}. \tag{13}$$

From system (13), we get the following equations

$$x_{6m+t} = x_{t-6}\prod_{r=0}^{m}\frac{v_{6r+t-1}u_{6r+t-4}}{u_{6r+t}v_{6r+t-3}}, \quad y_{6m+t} = y_{t-6}\prod_{r=0}^{m}\frac{u_{6r+t-1}v_{6r+t-4}}{v_{6r+t}u_{6r+t-3}}, \ m\in\mathbb{N}_0,\ t=\overline{1,6}. \tag{14}$$

If we take $t = 3i + j$, where $i \in \{0, 1\}$ and $j = \overline{1,3}$, then equations in (14) can be written in the following form

$$\begin{cases} x_{6m+3i+j} = x_{3i+j-6}\prod_{r=0}^{m}\frac{v_{3(2r+i)+j-1}u_{3(2r+i-1)+j-1}}{u_{3\left(2r+i+\left\lfloor\frac{j}{3}\right\rfloor\right)+j-3\left\lfloor\frac{j}{3}\right\rfloor}v_{3\left(2r+i+\left\lfloor\frac{j-3}{3}\right\rfloor\right)+j-3\left\lfloor\frac{j}{3}\right\rfloor}}, \\ y_{6m+3i+j} = y_{3i+j-6}\prod_{r=0}^{m}\frac{u_{3(2r+i)+j-1}v_{3(2r+i-1)+j-1}}{v_{3\left(2r+i+\left\lfloor\frac{j}{3}\right\rfloor\right)+j-3\left\lfloor\frac{j}{3}\right\rfloor}u_{3\left(2r+i+\left\lfloor\frac{j-3}{3}\right\rfloor\right)+j-3\left\lfloor\frac{j}{3}\right\rfloor}}, \end{cases} \quad m\in\mathbb{N}_0. \tag{15}$$

Let

$$\begin{aligned} &X_{1j} = x_{j-4}y_{j-5}x_{j-6},\ X_{2j} = x_{j-3-3\lfloor\frac{j}{3}\rfloor}y_{j-4-3\lfloor\frac{j}{3}\rfloor}x_{j-5-3\lfloor\frac{j}{3}\rfloor}, \\ &Y_{1j} = y_{j-4}x_{j-5}y_{j-6},\ Y_{2j} = y_{j-3-3\lfloor\frac{j}{3}\rfloor}x_{j-4-3\lfloor\frac{j}{3}\rfloor}y_{j-5-3\lfloor\frac{j}{3}\rfloor}. \end{aligned} \tag{16}$$

for $j = \overline{1,3}$.
Using formulas in (15) we obtain the following formulas for general solution of the system (3)

$$\begin{aligned} x_{6m+3i+j} = x_{3i+j-6}\prod_{r=0}^{m} & \frac{(Y_{1j})^{-1}\prod_{h=0}^{2r+i}\gamma_{3h+j-1}+\sum_{s=0}^{2r+i}\delta_{3s+j-1}\prod_{h=s+1}^{2r+i}\gamma_{3h+j-1}}{(Y_{2j})^{-1}\prod_{h=0}^{2r+i+\lfloor\frac{j-3}{3}\rfloor}\gamma_{3h+j-3\lfloor\frac{j}{3}\rfloor}+\sum_{s=0}^{2r+i+\lfloor\frac{j-3}{3}\rfloor}\delta_{3s+j-3\lfloor\frac{j}{3}\rfloor}\prod_{h=s+1}^{2r+i+\lfloor\frac{j-3}{3}\rfloor}\gamma_{3h+j-3\lfloor\frac{j}{3}\rfloor}} \\ & \times\frac{(X_{1j})^{-1}\prod_{h=0}^{2r+i-1}\alpha_{3h+j-1}+\sum_{s=0}^{2r+i-1}\beta_{3s+j-1}\prod_{h=s+1}^{2r+i-1}\alpha_{3h+j-1}}{(X_{2j})^{-1}\prod_{h=0}^{2r+i+\lfloor\frac{j}{3}\rfloor}\alpha_{3h+j-3\lfloor\frac{j}{3}\rfloor}+\sum_{s=0}^{2r+i+\lfloor\frac{j}{3}\rfloor}\beta_{3s+j-3\lfloor\frac{j}{3}\rfloor}\prod_{h=s+1}^{2r+i+\lfloor\frac{j}{3}\rfloor}\alpha_{3h+j-3\lfloor\frac{j}{3}\rfloor}}, \end{aligned} \tag{17}$$

$$\begin{aligned} y_{6m+3i+j} = y_{3i+j-6}\prod_{r=0}^{m} & \frac{(X_{1j})^{-1}\prod_{h=0}^{2r+i}\alpha_{3h+j-1}+\sum_{s=0}^{2r+i}\beta_{3s+j-1}\prod_{h=s+1}^{2r+i}\alpha_{3h+j-1}}{(X_{2j})^{-1}\prod_{h=0}^{2r+i+\lfloor\frac{j-3}{3}\rfloor}\alpha_{3h+j-3\lfloor\frac{j}{3}\rfloor}+\sum_{s=0}^{2r+i+\lfloor\frac{j-3}{3}\rfloor}\beta_{3s+j-3\lfloor\frac{j}{3}\rfloor}\prod_{h=s+1}^{2r+i+\lfloor\frac{j-3}{3}\rfloor}\alpha_{3h+j-3\lfloor\frac{j}{3}\rfloor}} \\ & \times\frac{(Y_{1j})^{-1}\prod_{h=0}^{2r+i-1}\gamma_{3h+j-1}+\sum_{s=0}^{2r+i-1}\delta_{3s+j-1}\prod_{h=s+1}^{2r+i-1}\gamma_{3h+j-1}}{(Y_{2j})^{-1}\prod_{h=0}^{2r+i+\lfloor\frac{j}{3}\rfloor}\gamma_{3h+j-3\lfloor\frac{j}{3}\rfloor}+\sum_{s=0}^{2r+i+\lfloor\frac{j}{3}\rfloor}\delta_{3s+j-3\lfloor\frac{j}{3}\rfloor}\prod_{h=s+1}^{2r+i+\lfloor\frac{j}{3}\rfloor}\gamma_{3h+j-3\lfloor\frac{j}{3}\rfloor}}, \end{aligned} \tag{18}$$

for $m \in \mathbb{N}_0$, $i \in \{0, 1\}$.

2.1. *The solutions of system* (3) *with constant coefficients*

In this section we take account of the case when the sequences $\alpha_n, \beta_n, \gamma_n$ and δ_n are constant. In this case, system (3) becomes

$$x_n = \frac{x_{n-3}y_{n-4}x_{n-5}}{y_{n-1}x_{n-2}(\alpha+\beta x_{n-3}y_{n-4}x_{n-5})}, \quad y_n = \frac{y_{n-3}x_{n-4}y_{n-5}}{x_{n-1}y_{n-2}(\gamma+\delta y_{n-3}x_{n-4}y_{n-5})}, \ n\in\mathbb{N}_0. \tag{19}$$

Now, using formulas (17) and (18) the solutions of the system (19) become

$$\begin{aligned} x_{6m+3i+j} = x_{3i+j-6}\prod_{r=0}^{m} & \frac{(Y_{1j})^{-1}\prod_{h=0}^{2r+i}\gamma+\sum_{s=0}^{2r+i}\delta\prod_{h=s+1}^{2r+i}\gamma}{(Y_{2j})^{-1}\prod_{h=0}^{2r+i+\lfloor\frac{j-3}{3}\rfloor}\gamma+\sum_{s=0}^{2r+i+\lfloor\frac{j-3}{3}\rfloor}\delta\prod_{h=s+1}^{2r+i+\lfloor\frac{j-3}{3}\rfloor}\gamma} \\ & \times\frac{(X_{1j})^{-1}\prod_{h=0}^{2r+i-1}\alpha+\sum_{s=0}^{2r+i-1}\beta\prod_{h=s+1}^{2r+i-1}\alpha}{(X_{2j})^{-1}\prod_{h=0}^{2r+i+\lfloor\frac{j}{3}\rfloor}\alpha+\sum_{s=0}^{2r+i+\lfloor\frac{j}{3}\rfloor}\beta\prod_{h=s+1}^{2r+i+\lfloor\frac{j}{3}\rfloor}\alpha}, \end{aligned} \tag{20}$$

$$y_{6m+3i+j} = y_{3i+j-6} \prod_{r=0}^{m} \frac{(X_{1j})^{-1} \prod_{h=0}^{2r+i} \alpha + \sum_{s=0}^{2r+i} \beta \prod_{h=s+1}^{2r+i} \alpha}{(X_{2j})^{-1} \prod_{h=0}^{2r+i+\lfloor \frac{j-3}{3} \rfloor} \alpha + \sum_{s=0}^{2r+i+\lfloor \frac{j-3}{3} \rfloor} \beta \prod_{h=s+1}^{2r+i+\lfloor \frac{j-3}{3} \rfloor} \alpha} \times \frac{(Y_{1j})^{-1} \prod_{h=0}^{2r+i-1} \gamma + \sum_{s=0}^{2r+i-1} \delta \prod_{h=s+1}^{2r+i-1} \gamma}{(Y_{2j})^{-1} \prod_{h=0}^{2r+i+\lfloor \frac{j}{3} \rfloor} \gamma + \sum_{s=0}^{2r+i+\lfloor \frac{j}{3} \rfloor} \delta \prod_{h=s+1}^{2r+i+\lfloor \frac{j}{3} \rfloor} \gamma}, \tag{21}$$

for $m \in \mathbb{N}_0$, $i \in \{0,1\}$ and $j = \overline{1,3}$.

Now, we obtain the solutions of system (19) for four cases. We obtain the following forms from formulas in (20) and (21) for $m \in \mathbb{N}_0$, $i \in \{0,1\}$ and $j = \overline{1,3}$,

$$\begin{cases} x_{6m+3i+j} = x_{3i+j-6} \prod_{r=0}^{m} \frac{X_{2j}Y_{2j}}{X_{1j}Y_{1j}} \frac{\gamma^{2r+i+1}((1-\gamma)-\delta Y_{1j})+\delta Y_{1j}}{\gamma^{2r+i+1+\lfloor \frac{j-3}{3} \rfloor}\big((1-\gamma)-\delta Y_{2j}\big)+\delta Y_{2j}} \frac{\alpha^{2r+i}((1-\alpha)-\beta X_{1j})+\beta X_{1j}}{\alpha^{2r+i+1+\lfloor \frac{j}{3} \rfloor}((1-\alpha)-\beta X_{2j})+\beta X_{2j}}, \\ y_{6m+3i+j} = y_{3i+j-6} \prod_{r=0}^{m} \frac{X_{2j}Y_{2j}}{X_{1j}Y_{1j}} \frac{\alpha^{2r+i+1}((1-\alpha)-\beta X_{1j})+\beta X_{1j}}{\alpha^{2r+i+1+\lfloor \frac{j-3}{3} \rfloor}((1-\alpha)-\beta X_{2j})+\beta X_{2j}} \frac{\gamma^{2r+i}((1-\gamma)-\delta Y_{1j})+\delta Y_{1j}}{\gamma^{2r+i+1+\lfloor \frac{j}{3} \rfloor}((1-\gamma)-\delta Y_{2j})+\delta Y_{2j}}, \end{cases} \tag{22}$$

if $\alpha \neq 1$ and $\gamma \neq 1$,

$$\begin{cases} x_{6m+3i+j} = x_{3i+j-6} \prod_{r=0}^{m} \frac{X_{2j}Y_{2j}}{X_{1j}Y_{1j}} \frac{\gamma^{2r+i+1}((1-\gamma)-\delta Y_{1j})+\delta Y_{1j}}{\gamma^{2r+i+1+\lfloor \frac{j-3}{3} \rfloor}((1-\gamma)-\delta Y_{2j})+\delta Y_{2j}} \frac{1+\beta(2r+i)X_{1j}}{1+\beta(2r+i+1+\lfloor \frac{j}{3} \rfloor)X_{2j}}, \\ y_{6m+3i+j} = y_{3i+j-6} \prod_{r=0}^{m} \frac{X_{2j}Y_{2j}}{X_{1j}Y_{1j}} \frac{1+\beta(2r+i+1)X_{1j}}{1+\beta(2r+i+1+\lfloor \frac{j-3}{3} \rfloor)X_{2j}} \frac{\gamma^{2r+i}((1-\gamma)-\delta Y_{1j})+\delta Y_{1j}}{\gamma^{2r+i+1+\lfloor \frac{j}{3} \rfloor}\big((1-\gamma)-\delta Y_{2j}\big)+\delta Y_{2j}}, \end{cases} \tag{23}$$

if $\alpha = 1$ and $\gamma \neq 1$,

$$\begin{cases} x_{6m+3i+j} = x_{3i+j-6} \prod_{r=0}^{m} \frac{X_{2j}Y_{2j}}{X_{1j}Y_{1j}} \frac{1+\delta(2r+i+1)Y_{1j}}{1+\delta(2r+i+1+\lfloor \frac{j-3}{3} \rfloor)Y_{2j}} \frac{\alpha^{2r+i}((1-\alpha)-\beta X_{1j})+\beta X_{1j}}{\alpha^{2r+i+1+\lfloor \frac{j}{3} \rfloor}((1-\alpha)-\beta X_{2j})+\beta X_{2j}}, \\ y_{6m+3i+j} = y_{3i+j-6} \prod_{r=0}^{m} \frac{X_{2j}Y_{2j}}{X_{1j}Y_{1j}} \frac{\alpha^{2r+i+1}((1-\alpha)-\beta X_{1j})+\beta X_{1j}}{\alpha^{2r+i+1+\lfloor \frac{j-3}{3} \rfloor}((1-\alpha)-\beta X_{2j})+\beta X_{2j}} \frac{1+\delta(2r+i)Y_{1j}}{1+\delta(2r+i+1+\lfloor \frac{j}{3} \rfloor)Y_{2j}}, \end{cases} \tag{24}$$

if $\alpha \neq 1$ and $\gamma = 1$,

$$\begin{cases} x_{6m+3i+j} = x_{3i+j-6} \prod_{r=0}^{m} \frac{X_{2j}Y_{2j}}{X_{1j}Y_{1j}} \frac{1+\delta(2r+i+1)Y_{1j}}{1+\delta(2r+i+1+\lfloor \frac{j-3}{3} \rfloor)Y_{2j}} \frac{1+\beta(2r+i)X_{1j}}{1+\beta(2r+i+1+\lfloor \frac{j}{3} \rfloor)X_{2j}}, \\ y_{6m+3i+j} = y_{3i+j-6} \prod_{r=0}^{m} \frac{X_{2j}Y_{2j}}{X_{1j}Y_{1j}} \frac{1+\beta(2r+i+1)X_{1j}}{1+\beta(2r+i+1+\lfloor \frac{j-3}{3} \rfloor)X_{2j}} \frac{1+\delta(2r+i)Y_{1j}}{1+\delta(2r+i+1+\lfloor \frac{j}{3} \rfloor)Y_{2j}}, \end{cases} \tag{25}$$

if $\alpha = 1$ and $\gamma = 1$.

2.1.1. *Asymptotic behavior of solutions of system* (19)

In this section, we will examine the asymptotic behavior of well-defined solutions of system (19). Before giving the results that was obtained by us, we show some notations that we will use in the results. Let

$$M_{1j} = \frac{(1-\alpha) - \beta X_{1j}}{\alpha^{\lfloor \frac{j-3}{3} \rfloor}\big((1-\alpha) - \beta X_{2j}\big)}, \; M_{2j} = \frac{(1-\alpha) - \beta X_{1j}}{\alpha^{1+\lfloor \frac{j}{3} \rfloor}\big((1-\alpha) - \beta X_{2j}\big)},$$

$$N_{1j} = \frac{(1-\gamma) - \delta Y_{1j}}{\gamma^{\lfloor \frac{j}{3} \rfloor+1}\big((1-\gamma) - \delta Y_{2j}\big)}, \; N_{2j} = \frac{(1-\gamma) - \delta Y_{1j}}{\gamma^{\lfloor \frac{j-3}{3} \rfloor}\big((1-\gamma) - \delta Y_{2j}\big)},$$

where $j = \overline{1,3}$. In the first result, we consider the case $\alpha \neq -1 \neq \gamma$, $\beta \neq 0 \neq \delta$.

Theorem 2.1. *Assume that $\alpha \neq -1 \neq \gamma$, $\beta \neq 0 \neq \delta$ and $(x_{6m+3i+j}, y_{6m+3i+j})_{m\geq -1}$ for $i \in \{0,1\}$, $j = \overline{1,3}$ is a well-defined solutions of system* (19)*. Then the following statements are true.*

(a) If $|\alpha| > 1$, $X_{1j} \neq \frac{1-\alpha}{\beta} = X_{2j}$, $Y_{1j} = \frac{1-\gamma}{\delta} = Y_{2j}$, then $|x_{6m+3i+j}| \to \infty$, $|y_{6m+3i+j}| \to \infty$ as $m \to \infty$.

(b) If $|\alpha| > 1$, $X_{1j} = \frac{1-\alpha}{\beta} \neq X_{2j}$, $Y_{1j} = \frac{1-\gamma}{\delta} = Y_{2j}$, then $x_{6m+3i+j} \to 0$, $y_{6m+3i+j} \to 0$ as $m \to \infty$.

(c) If $|\gamma| > 1$, $X_{1j} = \frac{1-\alpha}{\beta} = X_{2j}$, $Y_{1j} \neq \frac{1-\gamma}{\delta} = Y_{2j}$, then $|x_{6m+3i+j}| \to \infty$, $|y_{6m+3i+j}| \to \infty$ as $m \to \infty$.

(d) If $|\gamma| > 1$, $X_{1j} = \frac{1-\alpha}{\beta} = X_{2j}$, $Y_{1j} = \frac{1-\gamma}{\delta} \neq Y_{2j}$, then $x_{6m+3i+j} \to 0$, $y_{6m+3i+j} \to 0$ as $m \to \infty$.

(e) Assume that $|\alpha| > 1$, $X_{1j} \neq \frac{1-\alpha}{\beta} \neq X_{2j}$, $Y_{1j} = \frac{1-\gamma}{\delta} = Y_{2j}$.

- *(i) If $|M_{1j}| < 1$, then $y_{6m+3i+j} \to 0$ as $m \to \infty$.*
- *(ii) If $|M_{1j}| > 1$, then $|y_{6m+3i+j}| \to \infty$ as $m \to \infty$.*
- *(iii) If $|M_{2j}| < 1$, then $x_{6m+3i+j} \to 0$ as $m \to \infty$.*
- *(iv) If $|M_{2j}| > 1$, then $|x_{6m+3i+j}| \to \infty$ as $m \to \infty$.*
- *(v) If $M_{1j} = 1$, then the sequence $y_{6m+3i+j}$ is constant.*
- *(vi) If $M_{2j} = 1$, then the sequence $x_{6m+3i+j}$ is constant.*
- *(vii) If $M_{1j} = -1$, then the sequences $y_{12m+3i+j}$ and $y_{12m+3i+j+6}$ are convergent.*
- *(viii) If $M_{2j} = -1$, then the sequences $x_{12m+3i+j}$ and $x_{12m+3i+j+6}$ are convergent.*

(f) Assume that $|\gamma| > 1$, $X_{1j} = \frac{1-\alpha}{\beta} = X_{2j}$, $Y_{1j} \neq \frac{1-\gamma}{\delta} \neq Y_{2j}$.

- *(i) If $|N_{1j}| < 1$, then $y_{6m+3i+j} \to 0$ as $m \to \infty$.*
- *(ii) If $|N_{1j}| > 1$, then $|y_{6m+3i+j}| \to \infty$ as $m \to \infty$.*
- *(iii) If $|N_{2j}| < 1$, then $x_{6m+3i+j} \to 0$ as $m \to \infty$.*
- *(iv) If $|N_{2j}| > 1$, then $|x_{6m+3i+j}| \to \infty$ as $m \to \infty$.*
- *(v) If $N_{1j} = 1$, then the sequence $y_{6m+3i+j}$ is constant.*
- *(vi) If $N_{2j} = 1$, then the sequence $x_{6m+3i+j}$ is constant.*
- *(vii) If $N_{1j} = -1$, then the sequences $y_{12m+3i+j}$ and $y_{12m+3i+j+6}$ are convergent.*
- *(viii) If $N_{2j} = -1$, then the sequences $x_{12m+3i+j}$ and $x_{12m+3i+j+6}$ are convergent.*

(g) If $|\alpha\gamma| > 1$, $X_{1j} \neq \frac{1-\alpha}{\beta} = X_{2j}$ and $Y_{1j} \neq \frac{1-\gamma}{\delta} = Y_{2j}$, then $|x_{6m+3i+j}| \to \infty$ and $|y_{6m+3i+j}| \to \infty$ as $m \to \infty$.

(h) Assume that $|\alpha| > 1$, $|\gamma| > 1$, $X_{1j} \neq \frac{1-\alpha}{\beta} = X_{2j}$, $Y_{1j} = \frac{1-\gamma}{\delta} \neq Y_{2j}$.

- *(i) If $|\frac{\alpha}{\gamma}| < 1$, then $x_{6m+3i+j} \to 0$ and $y_{6m+3i+j} \to 0$ as $m \to \infty$.*
- *(ii) If $|\frac{\alpha}{\gamma}| > 1$, then $|x_{6m+3i+j}| \to \infty$ and $|y_{6m+3i+j}| \to \infty$ as $m \to \infty$.*

(j) If $|\alpha| > 1$, $|\gamma| > 1$, $X_{1j} \neq \frac{1-\alpha}{\beta} = X_{2j}$ and $Y_{1j} \neq \frac{1-\gamma}{\delta} \neq Y_{2j}$, then $|x_{6m+3i+j}| \to \infty$ and $|y_{6m+3i+j}| \to \infty$ as $m \to \infty$.

(k) Assume that $|\alpha| > 1$, $|\gamma| > 1$, $X_{1j} = \frac{1-\alpha}{\beta} \neq X_{2j}$, $Y_{1j} \neq \frac{1-\gamma}{\delta} = Y_{2j}$.

- *(i) If $|\frac{\gamma}{\alpha}| < 1$, then $x_{6m+3i+j} \to 0$ and $y_{6m+3i+j} \to 0$ as $m \to \infty$.*
- *(ii) If $|\frac{\gamma}{\alpha}| > 1$, then $|x_{6m+3i+j}| \to \infty$ and $|y_{6m+3i+j}| \to \infty$ as $m \to \infty$.*

(l) *If* $|\alpha| > 1$, $|\gamma| > 1$, $X_{1j} = \frac{1-\alpha}{\beta} \neq X_{2j}$ *and* $Y_{1j} = \frac{1-\gamma}{\delta} \neq Y_{2j}$, *then* $x_{6m+3i+j} \to 0$ *and* $y_{6m+3i+j} \to 0$ *as* $m \to \infty$.

(m) *If* $|\alpha| > 1$, $|\gamma| > 1$, $X_{1j} = \frac{1-\alpha}{\beta} \neq X_{2j}$ *and* $Y_{1j} \neq \frac{1-\gamma}{\delta} \neq Y_{2j}$, *then* $x_{6m+3i+j} \to 0$ *and* $y_{6m+3i+j} \to 0$ *as* $m \to \infty$.

(n) *If* $|\alpha| > 1$, $|\gamma| > 1$, $X_{1j} \neq \frac{1-\alpha}{\beta} \neq X_{2j}$ *and* $Y_{1j} = \frac{1-\gamma}{\delta} \neq Y_{2j}$, *then* $x_{6m+3i+j} \to 0$ *and* $y_{6m+3i+j} \to 0$ *as* $m \to \infty$.

(o) *If* $|\alpha| > 1$, $|\gamma| > 1$, $X_{1j} \neq \frac{1-\alpha}{\beta} \neq X_{2j}$, $Y_{1j} \neq \frac{1-\gamma}{\delta} = Y_{2j}$, *then* $|x_{6m+3i+j}| \to \infty$ *and* $|y_{6m+3i+j}| \to \infty$ *as* $m \to \infty$.

(p) *If* $|\alpha| > 1$, $|\gamma| > 1$, $X_{1j} = \frac{1-\alpha}{\beta} = X_{2j}$ *and* $Y_{1j} = \frac{1-\gamma}{\delta} = Y_{2j}$, *then the sequences* $x_{6m+3i+j}$ *and* $y_{6m+3i+j}$ *are constant.*

(r) *Assume that* $|\alpha| > 1$, $|\gamma| > 1$, $X_{1j} \neq \frac{1-\alpha}{\beta} \neq X_{2j}$, $Y_{1j} \neq \frac{1-\gamma}{\delta} \neq Y_{2j}$.

- *(i)* *If* $|N_{1j}M_{1j}| < 1$, *then* $y_{6m+3i+j} \to 0$ *as* $m \to \infty$.
- *(ii)* *If* $|N_{1j}M_{1j}| > 1$, *then* $|y_{6m+3i+j}| \to \infty$ *as* $m \to \infty$.
- *(iii)* *If* $|N_{2j}M_{2j}| < 1$, *then* $x_{6m+3i+j} \to 0$ *as* $m \to \infty$.
- *(iv)* *If* $|N_{2j}M_{2j}| > 1$, *then* $|x_{6m+3i+j}| \to \infty$ *as* $m \to \infty$.
- *(v)* *If* $N_{1j}M_{1j} = 1$, *then the sequence* $y_{6m+3i+j}$ *is constant.*
- *(vi)* *If* $N_{2j}M_{2j} = 1$, *then the sequence* $x_{6m+3i+j}$ *is constant.*
- *(vii)* *If* $N_{1j}M_{1j} = -1$, *then the sequences* $y_{12m+3i+j}$ *and* $y_{12m+3i+j+6}$ *are convergent.*
- *(viii)* *If* $N_{2j}M_{2j} = -1$, *then the sequences* $x_{12m+3i+j}$ *and* $x_{12m+3i+j+6}$ *are convergent.*

Proof. (a) If $X_{1j} \neq \frac{1-\alpha}{\beta} = X_{2j}$, $Y_{1j} = \frac{1-\gamma}{\delta} = Y_{2j}$, by using (22), then we have

$$\begin{cases} x_{6m+3i+j} &= x_{3i+j-6} \prod_{r=0}^{m} \frac{\alpha^{2r+i}((1-\alpha)-\beta X_{1j})+\beta X_{1j}}{\beta X_{1j}}, \\ y_{6m+3i+j} &= y_{3i+j-6} \prod_{r=0}^{m} \frac{\alpha^{2r+i+1}((1-\alpha)-\beta X_{1j})+\beta X_{1j}}{\beta X_{1j}}, \end{cases} \tag{26}$$

for $m \in \mathbb{N}_0$, $i \in \{0,1\}$ and $j = \overline{1,3}$. From (26) and $|\alpha| > 1$, we obtain

$$\lim_{m\to\infty} |x_{6m+3i+j}| = \infty, \quad \lim_{m\to\infty} |y_{6m+3i+j}| = \infty. \tag{27}$$

(b) If $|\alpha| > 1$, $X_{1j} = \frac{1-\alpha}{\beta} \neq X_{2j}$, $Y_{1j} = \frac{1-\gamma}{\delta} = Y_{2j}$, by using (22),then we get

$$\begin{cases} x_{6m+3i+j} &= x_{3i+j-6} \prod_{r=0}^{m} \frac{\beta X_{2j}}{\alpha^{2r+i+1+\lfloor \frac{j}{3} \rfloor}((1-\alpha)-\beta X_{2j})+\beta X_{2j}}, \\ y_{6m+3i+j} &= y_{3i+j-6} \prod_{r=0}^{m} \frac{\beta X_{2j}}{\alpha^{2r+i+1+\lfloor \frac{j-3}{3} \rfloor}((1-\alpha)-\beta X_{2j})+\beta X_{2j}}, \end{cases} \tag{28}$$

for $m \in \mathbb{N}_0$, $i \in \{0,1\}$ and $j = \overline{1,3}$. From (28) and $|\alpha| > 1$, we have

$$\lim_{m\to\infty} x_{6m+3i+j} = 0, \quad \lim_{m\to\infty} y_{6m+3i+j} = 0.$$

The proofs of $(c), (d)$ are similar to proofs of $(a), (b)$. So, they are omitted.

(e) $(i), (ii), (iii), (iv)$ Let

$$l_r^i := \frac{\alpha^{2r+i}((1-\alpha)-\beta X_{1j})+\beta X_{1j}}{\alpha^{2r+i+1+\lfloor \frac{j}{3} \rfloor}((1-\alpha)-\beta X_{2j})+\beta X_{2j}}, \quad k_r^i := \frac{\alpha^{2r+i+1}((1-\alpha)-\beta X_{1j})+\beta X_{1j}}{\alpha^{2r+i+1+\lfloor \frac{j-3}{3} \rfloor}((1-\alpha)-\beta X_{2j})+\beta X_{2j}},$$

for $i \in \{0,1\}$ and $j=\overline{1,3}$. Then, we obtain

$$\lim_{r\to\infty} l_r^i = \lim_{r\to\infty} \frac{\frac{(1-\alpha)-\beta X_{1j}}{\alpha^{1+\lfloor \frac{j}{3} \rfloor}} + \frac{\beta X_{1j}}{\alpha^{2r+i+1+\lfloor \frac{j}{3} \rfloor}}}{(1-\alpha)-\beta X_{2j} + \frac{\beta X_{2j}}{\alpha^{2r+i+1+\lfloor \frac{j}{3} \rfloor}}} = M_{2j}, \tag{29}$$

$$\lim_{r\to\infty} k_r^i = \lim_{r\to\infty} \frac{\frac{(1-\alpha)-\beta X_{1j}}{\alpha^{\lfloor \frac{j-3}{3} \rfloor}} + \frac{\beta X_{1j}}{\alpha^{2r+i+1+\lfloor \frac{j-3}{3} \rfloor}}}{(1-\alpha)-\beta X_{2j} + \frac{\beta X_{2j}}{\alpha^{2r+i+1+\lfloor \frac{j-3}{3} \rfloor}}} = M_{1j}, \tag{30}$$

for $i \in \{0,1\}$ and $j=\overline{1,3}$. From (22), (29) and (30); four statements easily follow.
(v), (vi) In this case,

$$l_r^i = \frac{\frac{(1-\alpha)-\beta X_{1j}}{\alpha^{1+\lfloor \frac{j}{3} \rfloor}} + \frac{\beta X_{1j}}{\alpha^{2r+i+1+\lfloor \frac{j}{3} \rfloor}}}{(1-\alpha)-\beta X_{2j} + \frac{\beta X_{2j}}{\alpha^{2r+i+1+\lfloor \frac{j}{3} \rfloor}}} = 1, \quad k_r^i = \frac{\frac{(1-\alpha)-\beta X_{1j}}{\alpha^{\lfloor \frac{j-3}{3} \rfloor}} + \frac{\beta X_{1j}}{\alpha^{2r+i+1+\lfloor \frac{j-3}{3} \rfloor}}}{(1-\alpha)-\beta X_{2j} + \frac{\beta X_{2j}}{\alpha^{2r+i+1+\lfloor \frac{j-3}{3} \rfloor}}} = 1, \tag{31}$$

for $i \in \{0,1\}$ and $j=\overline{1,3}$. From (22) and (31), two results easily follow.
(vii) Since $M_{1j}=-1$ and by using the following asymptotic relations

$$\frac{1}{1+x} = 1 - x + O\left(x^2\right), \quad \ln(1+x) = x - \frac{x^2}{2} + O\left(x^3\right), \tag{32}$$

we get

$$\begin{aligned} k_r^i &= -\frac{\alpha^{2r+i+1+\lfloor \frac{j-3}{3} \rfloor}((1-\alpha)-\beta X_{2j}) + \beta X_{2j}}{\alpha^{2r+i+1+\lfloor \frac{j-3}{3} \rfloor}((1-\alpha)-\beta X_{2j}) + \beta X_{2j}} \\ &= -\left(1 + \frac{\beta X_{2j}}{\alpha^{2r+i+1+\lfloor \frac{j-3}{3} \rfloor}((1-\alpha)-\beta X_{2j})}\right) \frac{1}{\left(1 + \frac{\beta X_{2j}}{\alpha^{2r+i+1+\lfloor \frac{j-3}{3} \rfloor}((1-\alpha)-\beta X_{2j})}\right)} \\ &= -\left(1 + \frac{\beta X_{2j}}{\alpha^{2r+i+1+\lfloor \frac{j-3}{3} \rfloor}((1-\alpha)-\beta X_{2j})}\right)\left(1 - \frac{\beta X_{2j}}{\alpha^{2r+i+1+\lfloor \frac{j-3}{3} \rfloor}((1-\alpha)-\beta X_{2j})} + O\left(\frac{1}{\alpha^{4r}}\right)\right) \\ &= -\left(1 + O\left(\frac{1}{\alpha^{2r}}\right)\right) \end{aligned} \tag{33}$$

for large enough r. From (33), the presumption $|\alpha| > 1$ and by a known criterion for the convergence of products the result follows.
$(viii)$ Since $M_{2j}=-1$ and by using the asymptotic relation (32),
we have

$$\begin{aligned} l_r^i &= -\left(\frac{\alpha^{2r+i+1+\lfloor \frac{j}{3} \rfloor}((1-\alpha)-\beta X_{2j}) + \beta X_{2j}}{\alpha^{2r+i+1+\lfloor \frac{j}{3} \rfloor}((1-\alpha)-\beta X_{2j}) + \beta X_{2j}}\right) \\ &= -\left(1 + \frac{\beta X_{2j}}{\alpha^{2r+i+1+\lfloor \frac{j}{3} \rfloor}((1-\alpha)-\beta X_{2j})}\right)\left(\frac{1}{1 + \frac{\beta X_{2j}}{\alpha^{2r+i+1+\lfloor \frac{j}{3} \rfloor}((1-\alpha)-\beta X_{2j})}}\right) \\ &= -\left(1 + \frac{\beta X_{2j}}{\alpha^{2r+i+1+\lfloor \frac{j}{3} \rfloor}((1-\alpha)-\beta X_{2j})}\right)\left(1 - \frac{\beta X_{2j}}{\alpha^{2r+i+1+\lfloor \frac{j}{3} \rfloor}((1-\alpha)-\beta X_{2j})} + O\left(\frac{1}{\alpha^{4r}}\right)\right) \end{aligned}$$

$$= -\left(1 + O\left(\frac{1}{\alpha^{2r}}\right)\right) \tag{34}$$

for large enough r. From (34), the presumption $|\alpha| > 1$ and by a known criterion for the convergence of products the result follows.

(f) (i),(ii),(iii),(iv) Let

$$a_r^i := \frac{\gamma^{2r+i+1}((1-\gamma) - \delta Y_{1j}) + \delta Y_{1j}}{\gamma^{2r+i+1+\lfloor \frac{j-3}{3} \rfloor}((1-\gamma) - \delta Y_{2j}) + \delta Y_{2j}}, \quad p_r^i := \frac{\gamma^{2r+i}((1-\gamma) - \delta Y_{1j}) + \delta Y_{1j}}{\gamma^{2r+i+1+\lfloor \frac{j}{3} \rfloor}((1-\gamma) - \delta Y_{2j}) + \delta Y_{2j}}, \tag{35}$$

for $i \in \{0, 1\}$ and $j = \overline{1, 3}$. Then we have

$$\lim_{r\to\infty} a_r^i = \lim_{r\to\infty} \frac{\frac{((1-\gamma)-\delta Y_{1j})}{\gamma^{\lfloor \frac{j-3}{3} \rfloor}} + \frac{\delta Y_{1j}}{\gamma^{2r+i+1+\lfloor \frac{j-3}{3} \rfloor}}}{(1-\gamma) - \delta Y_{2j} + \frac{\delta Y_{2j}}{\gamma^{2r+i+1+\lfloor \frac{j-3}{3} \rfloor}}} = N_{2j}, \ \lim_{r\to\infty} p_r^i = \lim_{r\to\infty} \frac{\frac{((1-\gamma)-\delta Y_{1j})}{\gamma^{1+\lfloor \frac{j}{3} \rfloor}} + \frac{\delta Y_{1j}}{\gamma^{2r+i+1+\lfloor \frac{j}{3} \rfloor}}}{(1-\gamma) - \delta Y_{2j} + \frac{\delta Y_{2j}}{\gamma^{2r+i+1+\lfloor \frac{j}{3} \rfloor}}} = N_{1j}, \tag{36}$$

for $i \in \{0, 1\}$ and $j = \overline{1, 3}$. From (22) and (36); four statements easily follow.

(v),(vi) In this case,

$$a_r^i = \frac{\frac{((1-\gamma)-\delta Y_{1j})}{\gamma^{\lfloor \frac{j-3}{3} \rfloor}} + \frac{\delta Y_{1j}}{\gamma^{2r+i+1+\lfloor \frac{j-3}{3} \rfloor}}}{(1-\gamma) - \delta Y_{2j} + \frac{\delta Y_{2j}}{\gamma^{2r+i+1+\lfloor \frac{j-3}{3} \rfloor}}} = 1, \quad p_r^i = \frac{\frac{((1-\gamma)-\delta Y_{1j})}{\gamma^{1+\lfloor \frac{j}{3} \rfloor}} + \frac{\delta Y_{1j}}{\gamma^{2r+i+1+\lfloor \frac{j}{3} \rfloor}}}{(1-\gamma) - \delta Y_{2j} + \frac{\delta Y_{2j}}{\gamma^{2r+i+1+\lfloor \frac{j}{3} \rfloor}}} = 1, \tag{37}$$

for $i \in \{0, 1\}$ and $j = \overline{1, 3}$. From (22), (37); two results easily follow.

(vii) Since $N_{1j} = -1$ and by using the asymptotic relation (32), we get

$$\begin{aligned}
p_r^i &= -\left(\frac{\gamma^{2r+i+1+\lfloor \frac{j}{3} \rfloor}((1-\gamma) - \delta Y_{2j}) + \delta Y_{2j}}{\gamma^{2r+i+1+\lfloor \frac{j}{3} \rfloor}((1-\gamma) - \delta Y_{2j}) + \delta Y_{2j}}\right) \\
&= -\left(1 + \frac{\delta Y_{2j}}{\gamma^{2r+i+1+\lfloor \frac{j}{3} \rfloor}((1-\gamma) - \delta Y_{2j})}\right) \frac{1}{\left(1 + \frac{\delta Y_{2j}}{\gamma^{2r+i+1+\lfloor \frac{j}{3} \rfloor}((1-\gamma)-\delta Y_{2j})}\right)} \\
&= -\left(1 + \frac{\delta Y_{2j}}{\gamma^{2r+i+1+\lfloor \frac{j}{3} \rfloor}((1-\gamma) - \delta Y_{2j})}\right)\left(1 - \frac{\delta Y_{2j}}{\gamma^{2r+i+1+\lfloor \frac{j}{3} \rfloor}((1-\gamma) - \delta Y_{2j})} + O\left(\frac{1}{\gamma^{4r}}\right)\right) \\
&= -\left(1 + O\left(\frac{1}{\gamma^{2r}}\right)\right)
\end{aligned} \tag{38}$$

for large enough r. From (38), the presumption $|\gamma| > 1$ and by a known criterion for the convergence of products the result follows.

(viii) Since $N_{2j} = -1$ and by using the asymptotic relation (32), we obtain

$$\begin{aligned}
a_r^i &= -\left(\frac{\gamma^{2r+i+1+\lfloor \frac{j-3}{3} \rfloor}((1-\gamma) - \delta Y_{2j}) + \delta Y_{2j}}{\gamma^{2r+i+1+\lfloor \frac{j-3}{3} \rfloor}((1-\gamma) - \delta Y_{2j}) + \delta Y_{2j}}\right) \\
&= -\left(1 + \frac{\delta Y_{2j}}{\gamma^{2r+i+1+\lfloor \frac{j-3}{3} \rfloor}((1-\gamma) - \delta Y_{2j})}\right) \frac{1}{\left(1 + \frac{\delta Y_{2j}}{\gamma^{2r+i+1+\lfloor \frac{j-3}{3} \rfloor}((1-\gamma)-\delta Y_{2j})}\right)}
\end{aligned}$$

$$= -\left(1 + \frac{\delta Y_{2j}}{\gamma^{2r+i+1+\lfloor \frac{j-3}{3} \rfloor}((1-\gamma) - \delta Y_{2j})}\right)\left(1 - \frac{\delta Y_{2j}}{\gamma^{2r+i+1+\lfloor \frac{j-3}{3} \rfloor}((1-\gamma) - \delta Y_{2j})} + O\left(\frac{1}{\gamma^{4r}}\right)\right)$$
$$= -\left(1 + O\left(\frac{1}{\gamma^{2r}}\right)\right) \tag{39}$$

for large enough r. From (39), the presumption $|\gamma| > 1$ and by a known criterion for the convergence of products the result follows.

(g) In this condition, from (22), we have

$$\begin{cases} x_{6m+3i+j} = x_{3i+j-6} \prod_{r=0}^{m} \frac{1}{X_{1j}Y_{1j}} \frac{(\alpha\gamma)^{2r+i}\left(\gamma((1-\gamma)-\delta Y_{1j}) + \frac{\delta Y_{1j}}{\gamma^{2r+i}}\right)}{\delta} \frac{\left(((1-\alpha)-\beta X_{1j}) + \frac{\beta X_{1j}}{\alpha^{2r+i}}\right)}{\beta}, \\ y_{6m+3i+j} = y_{3i+j-6} \prod_{r=0}^{m} \frac{1}{X_{1j}Y_{1j}} \frac{(\alpha\gamma)^{2r+i}\left(\alpha((1-\alpha)-\beta X_{1j}) + \frac{\beta X_{1j}}{\alpha^{2r+i}}\right)}{\beta} \frac{\left(((1-\gamma)-\delta Y_{1j}) + \frac{\delta Y_{1j}}{\gamma^{2r+i}}\right)}{\delta}. \end{cases}$$

Note that, if $|\alpha\gamma| > 1$, we have that $\lim_{m\to\infty}|x_{6m+3i+j}| = \infty$ and $\lim_{m\to\infty}|y_{6m+3i+j}| = \infty$.

(h) (i),(ii) In this case, by using (22), we acquire

$$\begin{cases} x_{6m+3i+j} = x_{3i+j-6} \prod_{r=0}^{m} \frac{Y_{2j}\delta}{X_{1j}\beta} \left(\frac{\alpha}{\gamma}\right)^{2r+i} \frac{((1-\alpha)-\beta X_{1j}) + \frac{\beta X_{1j}}{\alpha^{2r+i}}}{\gamma^{1+\lfloor \frac{j-3}{3} \rfloor}((1-\gamma)-\delta Y_{2j}) + \frac{\delta Y_{2j}}{\gamma^{2r+i}}}, \\ y_{6m+3i+j} = y_{3i+j-6} \prod_{r=0}^{m} \frac{Y_{2j}\delta}{X_{1j}\beta} \left(\frac{\alpha}{\gamma}\right)^{2r+i+1} \frac{((1-\alpha)-\beta X_{1j}) + \frac{\beta X_{1j}}{\alpha^{2r+i+1}}}{\gamma^{\lfloor \frac{j}{3} \rfloor}((1-\gamma)-\delta Y_{2j}) + \frac{\delta Y_{2j}}{\gamma^{2r+i+1}}}. \end{cases} \tag{40}$$

From (40), two statements easily follow.

(j) In this case, from (22), we obtain

$$\begin{cases} x_{6m+3i+j} = x_{3i+j-6} \prod_{r=0}^{m} \frac{Y_{2j}}{X_{1j}Y_{1j}} \frac{((1-\gamma)-\delta Y_{1j}) + \frac{\delta Y_{1j}}{\gamma^{2r+i+1}}}{\gamma^{\lfloor \frac{j-3}{3} \rfloor}((1-\gamma)-\delta Y_{2j}) + \frac{\delta Y_{2j}}{\gamma^{2r+i+1}}} \frac{\alpha^{2r+i}((1-\alpha)-\beta X_{1j}) + \beta X_{1j}}{\beta}, \\ y_{6m+3i+j} = y_{3i+j-6} \prod_{r=0}^{m} \frac{Y_{2j}}{X_{1j}Y_{1j}} \frac{\alpha^{2r+i+1}((1-\alpha)-\beta X_{1j}) + \beta X_{1j}}{\beta} \frac{((1-\gamma)-\delta Y_{1j}) + \frac{\delta Y_{1j}}{\gamma^{2r+i}}}{\gamma^{1+\lfloor \frac{j}{3} \rfloor}((1-\gamma)-\delta Y_{2j}) + \frac{\delta Y_{2j}}{\gamma^{2r+i}}}. \end{cases} \tag{41}$$

Note that, in this case, $|\alpha| > 1$, $|\gamma| > 1$ and we get $\lim_{m\to\infty}|x_{6m+3i+j}| = \infty$, $\lim_{m\to\infty}|y_{6m+3i+j}| = \infty$.

(k) (i),(ii) In this case, from (22), we acquire

$$\begin{cases} x_{6m+3i+j} = x_{3i+j-6} \prod_{r=0}^{m} \frac{X_{2j}\beta}{Y_{1j}\delta} \left(\frac{\gamma}{\alpha}\right)^{2r+i+1} \frac{((1-\gamma)-\delta Y_{1j}) + \frac{\delta Y_{1j}}{\gamma^{2r+i+1}}}{\alpha^{\lfloor \frac{j}{3} \rfloor}((1-\alpha)-\beta X_{2j}) + \frac{\beta X_{2j}}{\alpha^{2r+i+1}}}, \\ y_{6m+3i+j} = y_{3i+j-6} \prod_{r=0}^{m} \frac{X_{2j}\beta}{Y_{1j}\delta} \left(\frac{\gamma}{\alpha}\right)^{2r+i} \frac{((1-\gamma)-\delta Y_{1j}) + \frac{\delta Y_{1j}}{\gamma^{2r+i}}}{\alpha^{1+\lfloor \frac{j-3}{3} \rfloor}((1-\alpha)-\beta X_{2j}) + \frac{\beta X_{2j}}{\alpha^{2r+i}}}. \end{cases} \tag{42}$$

From (42), two statements easily follow.

(l) In this condition, from (22), we have

$$x_{6m+3i+j} = x_{3i+j-6} \prod_{r=0}^{m} X_{2j}Y_{2j} \frac{\delta\beta}{(\alpha\gamma)^{2r+i+1}\left(\gamma^{\lfloor \frac{j-3}{3} \rfloor}\left((1-\gamma) - \delta Y_{2j}\right) + \frac{\delta Y_{2j}}{\gamma^{2r+i+1}}\right)\left(\alpha^{\lfloor \frac{j}{3} \rfloor}\left((1-\alpha) - \beta X_{2j}\right) + \frac{\beta X_{2j}}{\alpha^{2r+i+1}}\right)},$$

$$y_{6m+3i+j} = y_{3i+j-6} \prod_{r=0}^{m} X_{2j}Y_{2j} \frac{\delta\beta}{(\alpha\gamma)^{2r+i+1}\left(\alpha^{\lfloor \frac{j-3}{3} \rfloor}\left((1-\alpha) - \beta X_{2j}\right) + \frac{\beta X_{2j}}{\alpha^{2r+i+1}}\right)\left(\gamma^{\lfloor \frac{j}{3} \rfloor}\left((1-\gamma) - \delta Y_{2j}\right) + \frac{\delta Y_{2j}}{\gamma^{2r+i+1}}\right)}.$$

Note that, if $|\alpha| > 1$ and $|\gamma| > 1$, then $|\alpha\gamma| > 1$, we have that $\lim_{m\to\infty} x_{6m+3i+j} = 0$, $\lim_{m\to\infty} y_{6m+3i+j} = 0$.

(m) In this case, from (22), we get

$$x_{6m+3i+j} = x_{3i+j-6}\prod_{r=0}^{m}\frac{X_{2j}Y_{2j}}{Y_{1j}}\frac{((1-\gamma)-\delta Y_{1j})+\frac{\delta Y_{1j}}{\gamma^{2r+i+1}}}{\gamma^{\lfloor\frac{j-3}{3}\rfloor}\left((1-\gamma)-\delta Y_{2j}\right)+\frac{\delta Y_{2j}}{\gamma^{2r+i+1}}}\frac{\beta}{\alpha^{2r+i+1+\lfloor\frac{j}{3}\rfloor}((1-\alpha)-\beta X_{2j})+\beta X_{2j}},$$
$$y_{6m+3i+j} = y_{3i+j-6}\prod_{r=0}^{m}\frac{X_{2j}Y_{2j}}{Y_{1j}}\frac{\beta}{\alpha^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}((1-\alpha)-\beta X_{2j})+\beta X_{2j}}\frac{((1-\gamma)-\delta Y_{1j})+\frac{\delta Y_{1j}}{\gamma^{2r+i}}}{\gamma^{1+\lfloor\frac{j}{3}\rfloor}((1-\gamma)-\delta Y_{2j})+\frac{\delta Y_{2j}}{\gamma^{2r+i}}}.$$

Note that, $|\alpha| > 1$, $|\gamma| > 1$ and we get $\lim_{m\to\infty} x_{6m+3i+j} = 0, \lim_{m\to\infty} y_{6m+3i+j} = 0$.

(n) In this case, from (22), we obtain

$$x_{6m+3i+j} = x_{3i+j-6}\prod_{r=0}^{m}\frac{X_{2j}Y_{2j}}{X_{1j}}\frac{\delta}{\gamma^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}((1-\gamma)-\delta Y_{2j})+\delta Y_{2j}}\frac{((1-\alpha)-\beta X_{1j})+\frac{\beta X_{1j}}{\alpha^{2r+i}}}{\alpha^{1+\lfloor\frac{j}{3}\rfloor}((1-\alpha)-\beta X_{2j})+\frac{\beta X_{2j}}{\alpha^{2r+i}}},$$
$$y_{6m+3i+j} = y_{3i+j-6}\prod_{r=0}^{m}\frac{X_{2j}Y_{2j}}{X_{1j}}\frac{((1-\alpha)-\beta X_{1j})+\frac{\beta X_{1j}}{\alpha^{2r+i+1}}}{\alpha^{\lfloor\frac{j-3}{3}\rfloor}((1-\alpha)-\beta X_{2j})+\frac{\beta X_{2j}}{\alpha^{2r+i+1}}}\frac{\delta}{\gamma^{2r+i+1+\lfloor\frac{j}{3}\rfloor}((1-\gamma)-\delta Y_{2j})+\delta Y_{2j}}.$$

Note that, in this case, $|\alpha| > 1$, $|\gamma| > 1$ and we get $\lim_{m\to\infty} x_{6m+3i+j} = 0, \lim_{m\to\infty} y_{6m+3i+j} = 0$.

(o) In this case, by using (22), we obtain

$$x_{6m+3i+j} = x_{3i+j-6}\prod_{r=0}^{m}\frac{X_{2j}}{X_{1j}Y_{1j}}\frac{\gamma^{2r+i+1}((1-\gamma)-\delta Y_{1j})+\delta Y_{1j}}{\delta}\frac{((1-\alpha)-\beta X_{1j})+\frac{\beta X_{1j}}{\alpha^{2r+i}}}{\alpha^{1+\lfloor\frac{j}{3}\rfloor}((1-\alpha)-\beta X_{2j})+\frac{\beta X_{2j}}{\alpha^{2r+i}}},$$
$$y_{6m+3i+j} = y_{3i+j-6}\prod_{r=0}^{m}\frac{X_{2j}}{X_{1j}Y_{1j}}\frac{((1-\alpha)-\beta X_{1j})+\frac{\beta X_{1j}}{\alpha^{2r+i+1}}}{\alpha^{\lfloor\frac{j-3}{3}\rfloor}((1-\alpha)-\beta X_{2j})+\frac{\beta X_{2j}}{\alpha^{2r+i+1}}}\frac{\gamma^{2r+i}((1-\gamma)-\delta Y_{1j})+\delta Y_{1j}}{\delta}.$$

Note that, in this case, $|\alpha| > 1$, $|\gamma| > 1$ and we get $\lim_{m\to\infty}|x_{6m+3i+j}| = \infty$, $\lim_{m\to\infty}|y_{6m+3i+j}| = \infty$.

(p) In this case, equations in (22) can be written to the following form

$$x_{6m+3i+j} = x_{3i+j-6}, \quad y_{6m+3i+j} = y_{3i+j-6}, \tag{43}$$

for $m \in \mathbb{N}_0$, $i \in \{0, 1\}$ and $j = \overline{1,3}$. From (43) the result easily follows.

(r) (i), (ii),(iii), (iv) In this case, from (22), we obtain

$$\lim_{r\to\infty} l_r^i a_r^i = \lim_{r\to\infty}\frac{\frac{((1-\alpha)-\beta X_{1j})}{\alpha^{1+\lfloor\frac{j}{3}\rfloor}}+\frac{\beta X_{1j}}{\alpha^{2r+i+1+\lfloor\frac{j}{3}\rfloor}}}{((1-\alpha)-\beta X_{2j})+\frac{\beta X_{2j}}{\alpha^{2r+i+1+\lfloor\frac{j}{3}\rfloor}}}\frac{\frac{((1-\gamma)-\delta Y_{1j})}{\gamma^{\lfloor\frac{j-3}{3}\rfloor}}+\frac{\delta Y_{1j}}{\gamma^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}}}{((1-\gamma)-\delta Y_{2j})+\frac{\delta Y_{2j}}{\gamma^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}}} = M_{2j}N_{2j},$$
$$\lim_{r\to\infty} k_r^i p_r^i = \lim_{r\to\infty}\frac{\frac{((1-\alpha)-\beta X_{1j})}{\alpha^{\lfloor\frac{j-3}{3}\rfloor}}+\frac{\beta X_{1j}}{\alpha^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}}}{((1-\alpha)-\beta X_{2j})+\frac{\beta X_{2j}}{\alpha^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}}}\frac{\frac{((1-\gamma)-\delta Y_{1j})}{\gamma^{1+\lfloor\frac{j}{3}\rfloor}}+\frac{\delta Y_{1j}}{\gamma^{2r+i+1+\lfloor\frac{j}{3}\rfloor}}}{((1-\gamma)-\delta Y_{2j})+\frac{\delta Y_{2j}}{\gamma^{2r+i+1+\lfloor\frac{j}{3}\rfloor}}} = M_{1j}N_{1j},$$

and

$$\lim_{m\to\infty} x_{6m+3i+j} = \lim_{m\to\infty} x_{3i+j-6}\left(\frac{X_{2j}Y_{2j}}{X_{1j}Y_{1j}}\right)^{m+1}\prod_{r=0}^{m} M_{2j}N_{2j}$$
$$= \lim_{m\to\infty} x_{3i+j-6}\left(\frac{X_{2j}Y_{2j}}{X_{1j}Y_{1j}}\right)^{m+1}(M_{2j}N_{2j})^{m+1}, \tag{44}$$

$$\lim_{m\to\infty} y_{6m+3i+j} = \lim_{m\to\infty} y_{3i+j-6}\left(\frac{X_{2j}Y_{2j}}{X_{1j}Y_{1j}}\right)^{m+1}\prod_{r=0}^{m} M_{1j}N_{1j}$$
$$= y_{3i+j-6}\left(\frac{X_{2j}Y_{2j}}{X_{1j}Y_{1j}}\right)^{m+1}(M_{1j}N_{1j})^{m+1}. \tag{45}$$

From (22), (44) and (45), four statements easily follow.

(v),(vi) If $M_{1j}N_{1j} = 1$ and $M_{2j}N_{2j} = 1$ from (22) we obtain

$$\lim_{m\to\infty} x_{6m+3i+j} = \lim_{m\to\infty} x_{3i+j-6}\left(\frac{X_{2j}Y_{2j}}{X_{1j}Y_{1j}}\right)^{m+1}, \lim_{m\to\infty} y_{6m+3i+j} = \lim_{m\to\infty} y_{3i+j-6}\left(\frac{X_{2j}Y_{2j}}{X_{1j}Y_{1j}}\right)^{m+1}. \tag{46}$$

From (46), this statement easily follows.

(vii),(viii) If $M_{1j}N_{1j} = -1$ and $M_{2j}N_{2j} = -1$ from (22) we get

$$\lim_{m\to\infty} x_{6m+3i+j} = \lim_{m\to\infty} x_{3i+j-6}\left(\frac{X_{2j}Y_{2j}}{X_{1j}Y_{1j}}\right)^{m+1}(-1)^{m+1},$$
$$\lim_{m\to\infty} y_{6m+3i+j} = \lim_{m\to\infty} y_{3i+j-6}\left(\frac{X_{2j}Y_{2j}}{X_{1j}Y_{1j}}\right)^{m+1}(-1)^{m+1}. \tag{47}$$

From (47) the sequences $x_{6m+3i+j}$ and $y_{6m+3i+j}$ converge to sub-sequences of m depending on whether they are odd or even number.

□

Now we introduce the asymtotic behavior of well-defined solutions of system (19) for the case $|\alpha| > 1$, $\alpha = \gamma$, $\beta \neq 0 \neq \delta$ and $j = \overline{1,3}$, by using the next notation:

$$L_j = \frac{\Big((1-\alpha) - \delta Y_{1j}\Big)\Big((1-\alpha) - \beta X_{1j}\Big)}{\alpha^{1+\lfloor \frac{j-3}{3}\rfloor + \lfloor \frac{j}{3}\rfloor}\Big((1-\alpha) - \delta Y_{2j}\Big)\Big((1-\alpha) - \beta X_{2j}\Big)}.$$

Theorem 2.2. *Assume that $|\alpha| > 1$, $\alpha = \gamma$, $\beta \neq 0 \neq \delta$ and $(x_{6m+3i+j}, y_{6m+3i+j})_{m\geq -1}$ for $i \in \{0,1\}$, $j = \overline{1,3}$ is a well-defined solutions of system* (19)*. Then the following statements are true.*

(a) If $|L_j| < 1$, then $x_{6m+3i+j} \to 0$ and $y_{6m+3i+j} \to 0$ as $m \to \infty$.

(b) If $|L_j| > 1$, then $|x_{6m+3i+j}| \to \infty$ and $|y_{6m+3i+j}| \to \infty$ as $m \to \infty$.

(c) If $L_j = 1$, then the sequences $x_{6m+3i+j}$ and $y_{6m+3i+j}$ are convergent.

(d) If $L_j = -1$, then the sequences $x_{12m+3i+j}$, $x_{12m+3i+j+6}$, $y_{12m+3i+j}$, $y_{12m+3i+j+6}$ are convergent.

Proof. $(a), (b)$
According to conditions, the solutions in (22) become

$$\begin{cases} x_{6m+3i+j} = x_{3i+j-6}\prod_{r=0}^{m}\frac{X_{2j}Y_{2j}}{X_{1j}Y_{1j}}\frac{\alpha^{2r+i+1}((1-\alpha)-\delta Y_{1j})+\delta Y_{1j}}{\alpha^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}((1-\alpha)-\delta Y_{2j})+\delta Y_{2j}}\frac{\alpha^{2r+i}((1-\alpha)-\beta X_{1j})+\beta X_{1j}}{\alpha^{2r+i+1+\lfloor\frac{j}{3}\rfloor}((1-\alpha)-\beta X_{2j})+\beta X_{2j}}, \\ y_{6m+3i+j} = y_{3i+j-6}\prod_{r=0}^{m}\frac{X_{2j}Y_{2j}}{X_{1j}Y_{1j}}\frac{\alpha^{2r+i+1}((1-\alpha)-\beta X_{1j})+\beta X_{1j}}{\alpha^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}((1-\alpha)-\beta X_{2j})+\beta X_{2j}}\frac{\alpha^{2r+i}((1-\alpha)-\delta Y_{1j})+\delta Y_{1j}}{\alpha^{2r+i+1+\lfloor\frac{j}{3}\rfloor}((1-\alpha)-\delta Y_{2j})+\delta Y_{2j}}. \end{cases}$$

Let

$$f_r^i := \frac{\alpha^{2r+i+1}((1-\alpha)-\delta Y_{1j})+\delta Y_{1j}}{\alpha^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}((1-\alpha)-\delta Y_{2j})+\delta Y_{2j}}\frac{\alpha^{2r+i}((1-\alpha)-\beta X_{1j})+\beta X_{1j}}{\alpha^{2r+i+1+\lfloor\frac{j}{3}\rfloor}((1-\alpha)-\beta X_{2j})+\beta X_{2j}}, \tag{48}$$

$$s_r^i := \frac{\alpha^{2r+i+1}((1-\alpha)-\beta X_{1j})+\beta X_{1j}}{\alpha^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}((1-\alpha)-\beta X_{2j})+\beta X_{2j}}\frac{\alpha^{2r+i}((1-\alpha)-\delta Y_{1j})+\delta Y_{1j}}{\alpha^{2r+i+1+\lfloor\frac{j}{3}\rfloor}((1-\alpha)-\delta Y_{2j})+\delta Y_{2j}}. \tag{49}$$

We get

$$\lim_{r\to\infty} f_r^i = \lim_{r\to\infty}\frac{((1-\alpha)-\delta Y_{1j})+\frac{\delta Y_{1j}}{\alpha^{2r+i+1}}}{\frac{((1-\alpha)-\delta Y_{2j})}{\alpha^{-\lfloor\frac{j-3}{3}\rfloor}}+\frac{\delta Y_{2j}}{\alpha^{2r+i+1}}}\frac{\frac{((1-\alpha)-\beta X_{1j})}{\alpha^{1+\lfloor\frac{j}{3}\rfloor}}+\frac{\beta X_{1j}}{\alpha^{2r+i+1+\lfloor\frac{j}{3}\rfloor}}}{((1-\alpha)-\beta X_{2j})+\frac{\beta X_{2j}}{\alpha^{2r+i+1+\lfloor\frac{j}{3}\rfloor}}} = L_j, \tag{50}$$

$$\lim_{r\to\infty} s_r^i = \lim_{r\to\infty}\frac{((1-\alpha)-\beta X_{1j})+\frac{\beta X_{1j}}{\alpha^{2r+i+1}}}{\frac{((1-\alpha)-\beta X_{2j})}{\alpha^{-\lfloor\frac{j-3}{3}\rfloor}}+\frac{\beta X_{2j}}{\alpha^{2r+i+1}}}\frac{\frac{((1-\alpha)-\delta Y_{1j})}{\alpha^{1+\lfloor\frac{j}{3}\rfloor}}+\frac{\delta Y_{1j}}{\alpha^{2r+i+1+\lfloor\frac{j}{3}\rfloor}}}{((1-\alpha)-\delta Y_{2j})+\frac{\delta Y_{2j}}{\alpha^{2r+i+1+\lfloor\frac{j}{3}\rfloor}}} = L_j, \tag{51}$$

for $j = \overline{1,3}$, $i \in \{0,1\}$. From (22), (50) and (51) we have that

$$\lim_{m\to\infty} x_{6m+3i+j} = \lim_{m\to\infty} x_{3i+j-6}\left(\frac{X_{2j}Y_{2j}}{X_{1j}Y_{1j}}\right)^{m+1}\prod_{r=0}^{m} L_j, \quad \lim_{m\to\infty} y_{6m+3i+j} = \lim_{m\to\infty} y_{3i+j-6}\left(\frac{X_{2j}Y_{2j}}{X_{1j}Y_{1j}}\right)^{m+1}\prod_{r=0}^{m} L_j,$$

for $j = \overline{1,3}$, $i \in \{0,1\}$. From the last two equations, two statements easily follow.
(c) From (32), (48) and (49), we have

$$\begin{aligned} f_r^i =& \frac{\alpha^{2r+i+1}((1-\alpha)-\delta Y_{1j})+\delta Y_{1j}}{\alpha^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}((1-\alpha)-\delta Y_{2j})+\delta Y_{2j}}\frac{\alpha^{2r+i}((1-\alpha)-\beta X_{1j})+\beta X_{1j}}{\alpha^{2r+i+1+\lfloor\frac{j}{3}\rfloor}((1-\alpha)-\beta X_{2j})+\beta X_{2j}} \\ =& \frac{\alpha^{2r+i+1}((1-\alpha)-\delta Y_{1j})\left(1+\frac{\delta Y_{1j}}{\alpha^{2r+i+1}((1-\alpha)-\delta Y_{1j})}\right)}{\alpha^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}((1-\alpha)-\delta Y_{2j})\left(1+\frac{\delta Y_{2j}}{\alpha^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}((1-\alpha)-\delta Y_{2j})}\right)} \\ &\times\frac{\alpha^{2r+i}((1-\alpha)-\beta X_{1j})\left(1+\frac{\beta X_{1j}}{\alpha^{2r+i}((1-\alpha)-\beta X_{1j})}\right)}{\alpha^{2r+i+1+\lfloor\frac{j}{3}\rfloor}((1-\alpha)-\beta X_{2j})\left(1+\frac{\beta X_{2j}}{\alpha^{2r+i+1+\lfloor\frac{j}{3}\rfloor}((1-\alpha)-\beta X_{2j})}\right)} \\ =& \left(1+\frac{\delta Y_{1j}}{\alpha^{2r+i+1}((1-\alpha)-\delta Y_{1j})}\right)\left(1-\frac{\delta Y_{2j}}{\alpha^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}((1-\alpha)-\delta Y_{2j})}+O\left(\frac{1}{\alpha^{4r}}\right)\right) \\ &\times\left(1+\frac{\beta X_{1j}}{\alpha^{2r+i}((1-\alpha)-\beta X_{1j})}\right)\left(1-\frac{\beta X_{2j}}{\alpha^{2r+i+1+\lfloor\frac{j}{3}\rfloor}((1-\alpha)-\beta X_{2j})}+O\left(\frac{1}{\alpha^{4r}}\right)\right) \end{aligned}$$

$$=1+\frac{\delta Y_{1j}}{\alpha^{2r+i+1}((1-\alpha)-\delta Y_{1j})}-\frac{\delta Y_{2j}}{\alpha^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}((1-\alpha)-\delta Y_{2j})}+\frac{\beta X_{1j}}{\alpha^{2r+i}((1-\alpha)-\beta X_{1j})}$$
$$-\frac{\beta X_{2j}}{\alpha^{2r+i+1+\lfloor\frac{j}{3}\rfloor}((1-\alpha)-\beta X_{2j})}+O\left(\frac{1}{\alpha^{4r}}\right),$$

$$s_r^i=\frac{\alpha^{2r+i+1}((1-\alpha)-\beta X_{1j})+\beta X_{1j}}{\alpha^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}((1-\alpha)-\beta X_{2j})+\beta X_{2j}}\frac{\alpha^{2r+i}((1-\alpha)-\delta Y_{1j})+\delta Y_{1j}}{\alpha^{2r+i+1+\lfloor\frac{j}{3}\rfloor}((1-\alpha)-\delta Y_{2j})+\delta Y_{2j}}$$
$$=\frac{\alpha^{2r+i+1}((1-\alpha)-\beta X_{1j})\left(1+\frac{\beta X_{1j}}{\alpha^{2r+i+1}((1-\alpha)-\beta X_{1j})}\right)}{\alpha^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}((1-\alpha)-\beta X_{2j})\left(1+\frac{\beta X_{2j}}{\alpha^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}((1-\alpha)-\beta X_{2j})}\right)}$$
$$\times\frac{\alpha^{2r+i}((1-\alpha)-\delta Y_{1j})\left(1+\frac{\delta Y_{1j}}{\alpha^{2r+i}((1-\alpha)-\delta Y_{1j})}\right)}{\alpha^{2r+i+1+\lfloor\frac{j}{3}\rfloor}((1-\alpha)-\delta Y_{2j})\left(1+\frac{\delta Y_{2j}}{\alpha^{2r+i+1+\lfloor\frac{j}{3}\rfloor}((1-\alpha)-\delta Y_{2j})}\right)}$$
$$=\left(1+\frac{\beta X_{1j}}{\alpha^{2r+i+1}((1-\alpha)-\beta X_{1j})}\right)\left(1-\frac{\beta X_{2j}}{\alpha^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}((1-\alpha)-\beta X_{2j})}+O\left(\frac{1}{\alpha^{4r}}\right)\right)$$
$$\times\left(1+\frac{\delta Y_{1j}}{\alpha^{2r+i}((1-\alpha)-\delta Y_{1j})}\right)\left(1-\frac{\delta Y_{2j}}{\alpha^{2r+i+1+\lfloor\frac{j}{3}\rfloor}((1-\alpha)-\delta Y_{2j})}+O\left(\frac{1}{\alpha^{4r}}\right)\right)$$
$$=1+\frac{\beta X_{1j}}{\alpha^{2r+i+1}((1-\alpha)-\beta X_{1j})}-\frac{\beta X_{2j}}{\alpha^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}((1-\alpha)-\beta X_{2j})}+\frac{\delta Y_{1j}}{\alpha^{2r+i}((1-\alpha)-\delta Y_{1j})}$$
$$-\frac{\delta Y_{2j}}{\alpha^{2r+i+1+\lfloor\frac{j}{3}\rfloor}((1-\alpha)-\delta Y_{2j})}+O\left(\frac{1}{\alpha^{4r}}\right),$$

for sufficiently large r, $j=\overline{1,3}$ and $i\in\{0,1\}$. From which the convergence of the sequences $x_{6m+3i+j}$ and $y_{6m+3i+j}$ can be seen easily.
(d) From (32), (48) and (49), we have

$$f_r^i=\frac{\alpha^{2r+i+1}((1-\alpha)-\delta Y_{1j})+\delta Y_{1j}}{\alpha^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}((1-\alpha)-\delta Y_{2j})+\delta Y_{2j}}\frac{\alpha^{2r+i}((1-\alpha)-\beta X_{1j})+\beta X_{1j}}{\alpha^{2r+i+1+\lfloor\frac{j}{3}\rfloor}((1-\alpha)-\beta X_{2j})+\beta X_{2j}}$$
$$=\frac{\alpha^{2r+i+1}((1-\alpha)-\delta Y_{1j})\left(1+\frac{\delta Y_{1j}}{\alpha^{2r+i+1}((1-\alpha)-\delta Y_{1j})}\right)}{\alpha^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}((1-\alpha)-\delta Y_{2j})\left(1+\frac{\delta Y_{2j}}{\alpha^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}((1-\alpha)-\delta Y_{2j})}\right)}$$
$$\times\frac{\alpha^{2r+i}((1-\alpha)-\beta X_{1j})\left(1+\frac{\beta X_{1j}}{\alpha^{2r+i}((1-\alpha)-\beta X_{1j})}\right)}{\alpha^{2r+i+1+\lfloor\frac{j}{3}\rfloor}((1-\alpha)-\beta X_{2j})\left(1+\frac{\beta X_{2j}}{\alpha^{2r+i+1+\lfloor\frac{j}{3}\rfloor}((1-\alpha)-\beta X_{2j})}\right)}$$
$$=-\left(1+\frac{\delta Y_{1j}}{\alpha^{2r+i+1}((1-\alpha)-\delta Y_{1j})}\right)\left(1-\frac{\delta Y_{2j}}{\alpha^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}((1-\alpha)-\delta Y_{2j})}+O\left(\frac{1}{\alpha^{4r}}\right)\right)$$
$$\times\left(1+\frac{\beta X_{1j}}{\alpha^{2r+i}((1-\alpha)-\beta X_{1j})}\right)\left(1-\frac{\beta X_{2j}}{\alpha^{2r+i+1+\lfloor\frac{j}{3}\rfloor}((1-\alpha)-\beta X_{2j})}+O\left(\frac{1}{\alpha^{4r}}\right)\right)$$

$$= -1 - \frac{\delta Y_{1j}}{\alpha^{2r+i+1}((1-\alpha)-\delta Y_{1j})} + \frac{\delta Y_{2j}}{\alpha^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}((1-\alpha)-\delta Y_{2j})} - \frac{\beta X_{1j}}{\alpha^{2r+i}((1-\alpha)-\beta X_{1j})}$$
$$+ \frac{\beta X_{2j}}{\alpha^{2r+i+1+\lfloor\frac{j}{3}\rfloor}((1-\alpha)-\beta X_{2j})} - O\left(\frac{1}{\alpha^{4r}}\right),$$

$$s_r^i = \frac{\alpha^{2r+i+1}((1-\alpha)-\beta X_{1j})+\beta X_{1j}}{\alpha^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}((1-\alpha)-\beta X_{2j})+\beta X_{2j}} \frac{\alpha^{2r+i}((1-\alpha)-\delta Y_{1j})+\delta Y_{1j}}{\alpha^{2r+i+1+\lfloor\frac{j}{3}\rfloor}((1-\alpha)-\delta Y_{2j})+\delta Y_{2j}}$$
$$= \frac{\alpha^{2r+i+1}((1-\alpha)-\beta X_{1j})\left(1+\frac{\beta X_{1j}}{\alpha^{2r+i+1}((1-\alpha)-\beta X_{1j})}\right)}{\alpha^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}((1-\alpha)-\beta X_{2j})\left(1+\frac{\beta X_{2j}}{\alpha^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}((1-\alpha)-\beta X_{2j})}\right)}$$
$$\times \frac{\alpha^{2r+i}((1-\alpha)-\delta Y_{1j})\left(1+\frac{\delta Y_{1j}}{\alpha^{2r+i}((1-\alpha)-\delta Y_{1j})}\right)}{\alpha^{2r+i+1+\lfloor\frac{j}{3}\rfloor}((1-\alpha)-\delta Y_{2j})\left(1+\frac{\delta Y_{2j}}{\alpha^{2r+i+1+\lfloor\frac{j}{3}\rfloor}((1-\alpha)-\delta Y_{2j})}\right)}$$
$$= -\left(1+\frac{\beta X_{1j}}{\alpha^{2r+i+1}((1-\alpha)-\beta X_{1j})}\right)\left(1-\frac{\beta X_{2j}}{\alpha^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}((1-\alpha)-\beta X_{2j})}+O\left(\frac{1}{\alpha^{4r}}\right)\right)$$
$$\times\left(1+\frac{\delta Y_{1j}}{\alpha^{2r+i}((1-\alpha)-\delta Y_{1j})}\right)\left(1-\frac{\delta Y_{2j}}{\alpha^{2r+i+1+\lfloor\frac{j}{3}\rfloor}((1-\alpha)-\delta Y_{2j})}+O\left(\frac{1}{\alpha^{4r}}\right)\right)$$
$$= -1 - \frac{\beta X_{1j}}{\alpha^{2r+i+1}((1-\alpha)-\beta X_{1j})} + \frac{\beta X_{2j}}{\alpha^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}((1-\alpha)-\beta X_{2j})} - \frac{\delta Y_{1j}}{\alpha^{2r+i}((1-\alpha)-\delta Y_{1j})}$$
$$+ \frac{\delta Y_{2j}}{\alpha^{2r+i+1+\lfloor\frac{j}{3}\rfloor}((1-\alpha)-\delta Y_{2j})} - O\left(\frac{1}{\alpha^{4r}}\right),$$

for sufficiently large r, $j = \overline{1,3}$ and $i \in \{0,1\}$, from which the convergence of the sequences $x_{12m+3i+j}$, $x_{12m+3i+j+6}$, $y_{12m+3i+j}$ and $y_{12m+3i+j+6}$ can be obtained easily.

Here we introduce the asymptotic behavior of well-defined solutions of system (19) for the case $|\alpha| > 1$, $\gamma = -\alpha$, $\beta \neq 0 \neq \delta$, by using the notations:

$$K_j = \frac{(-\alpha)^{-\lfloor\frac{j-3}{3}\rfloor}\Big((1+\alpha)-\delta Y_{1j}\Big)\Big((1-\alpha)-\beta X_{1j}\Big)}{\alpha^{1+\lfloor\frac{j}{3}\rfloor}\Big((1+\alpha)-\delta Y_{2j}\Big)\Big((1-\alpha)-\beta X_{2j}\Big)},\ F_j = \frac{(\alpha)^{-\lfloor\frac{j-3}{3}\rfloor}\Big((1+\alpha)-\delta Y_{1j}\Big)\Big((1-\alpha)-\beta X_{1j}\Big)}{(-\alpha)^{1+\lfloor\frac{j}{3}\rfloor}\Big((1+\alpha)-\delta Y_{2j}\Big)\Big((1-\alpha)-\beta X_{2j}\Big)}.$$

□

Theorem 2.3. *Assume that* $|\alpha| > 1$, $\gamma = -\alpha$, $\beta \neq 0 \neq \delta$ *and* $(x_{6m+3i+j}, y_{6m+3i+j})_{m\geq -1}$ *for* $i \in \{0,1\}$, $j = \overline{1,3}$ *is a well-defined solutions of system* (19). *Then the following statements are true.*

(a) If $|K_j| < 1$, *then* $x_{6m+3i+j} \to 0$ *as* $m \to \infty$.

(b) If $|F_j| < 1$, *then* $y_{6m+3i+j} \to 0$ *as* $m \to \infty$.

(c) If $|K_j| > 1$, *then* $|x_{6m+3i+j}| \to \infty$ *as* $m \to \infty$.

(d) If $|F_j| > 1$, *then* $|y_{6m+3i+j}| \to \infty$ *as* $m \to \infty$.

(e) If $K_j = 1$, *then the sequence* $x_{6m+3i+j}$ *is convergent.*

(f) If $F_j = 1$, then the sequence $y_{6m+3i+j}$ is convergent.

(g) If $K_j = -1$, then the sequences $x_{12m+3i+j}$ and $x_{12m+3i+j+6}$ are convergent.

(h) If $F_j = -1$, then the sequences $y_{12m+3i+j}$ and $y_{12m+3i+j+6}$ are convergent.

Proof. $(a), (b), (c), (d)$: According to conditions, the solutions in (22) become

$$\begin{cases} x_{6m+3i+j} = x_{3i+j-6} \prod_{r=0}^{m} \frac{X_{2j}Y_{2j}}{X_{1j}Y_{1j}} \frac{(-\alpha)^{2r+i+1}((1+\alpha)-\delta Y_{1j})+\delta Y_{1j}}{(-\alpha)^{2r+i+1+\lfloor \frac{j-3}{3} \rfloor}((1+\alpha)-\delta Y_{2j})+\delta Y_{2j}} \frac{\alpha^{2r+i}((1-\alpha)-\beta X_{1j})+\beta X_{1j}}{\alpha^{2r+i+1+\lfloor \frac{j}{3} \rfloor}((1-\alpha)-\beta X_{2j})+\beta X_{2j}}, \\ y_{6m+3i+j} = y_{3i+j-6} \prod_{r=0}^{m} \frac{X_{2j}Y_{2j}}{X_{1j}Y_{1j}} \frac{\alpha^{2r+i+1}((1-\alpha)-\beta X_{1j})+\beta X_{1j}}{\alpha^{2r+i+1+\lfloor \frac{j-3}{3} \rfloor}((1-\alpha)-\beta X_{2j})+\beta X_{2j}} \frac{(-\alpha)^{2r+i}((1+\alpha)-\delta Y_{1j})+\delta Y_{1j}}{(-\alpha)^{2r+i+1+\lfloor \frac{j}{3} \rfloor}((1+\alpha)-\delta Y_{2j})+\delta Y_{2j}}. \end{cases} \tag{52}$$

Let

$$c_r^i := \frac{(-\alpha)^{2r+i+1}((1+\alpha)-\delta Y_{1j})+\delta Y_{1j}}{(-\alpha)^{2r+i+1+\lfloor \frac{j-3}{3} \rfloor}((1+\alpha)-\delta Y_{2j})+\delta Y_{2j}} \frac{\alpha^{2r+i}((1-\alpha)-\beta X_{1j})+\beta X_{1j}}{\alpha^{2r+i+1+\lfloor \frac{j}{3} \rfloor}((1-\alpha)-\beta X_{2j})+\beta X_{2j}}, \tag{53}$$

$$d_r^i := \frac{\alpha^{2r+i+1}((1-\alpha)-\beta X_{1j})+\beta X_{1j}}{\alpha^{2r+i+1+\lfloor \frac{j-3}{3} \rfloor}((1-\alpha)-\beta X_{2j})+\beta X_{2j}} \frac{(-\alpha)^{2r+i}((1+\alpha)-\delta Y_{1j})+\delta Y_{1j}}{(-\alpha)^{2r+i+1+\lfloor \frac{j}{3} \rfloor}((1+\alpha)-\delta Y_{2j})+\delta Y_{2j}}. \tag{54}$$

Then we have

$$\lim_{r\to\infty} c_r^i = \lim_{r\to\infty} \frac{((1+\alpha)-\delta Y_{1j}) + \frac{\delta Y_{1j}}{(-\alpha)^{2r+i+1}}}{\frac{((1+\alpha)-\delta Y_{2j})}{(-\alpha)^{-\lfloor \frac{j-3}{3} \rfloor}} + \frac{\delta Y_{2j}}{(-\alpha)^{2r+i+1}}} \frac{\frac{((1-\alpha)-\beta X_{1j})}{\alpha^{1+\lfloor \frac{j}{3} \rfloor}} + \frac{\beta X_{1j}}{\alpha^{2r+i+1+\lfloor \frac{j}{3} \rfloor}}}{((1-\alpha)-\beta X_{2j}) + \frac{\beta X_{2j}}{\alpha^{2r+i+1+\lfloor \frac{j}{3} \rfloor}}} = K_j, \tag{55}$$

$$\lim_{r\to\infty} d_r^i = \lim_{r\to\infty} \frac{((1-\alpha)-\beta X_{1j}) + \frac{\beta X_{1j}}{\alpha^{2r+i+1}}}{\frac{((1-\alpha)-\beta X_{2j})}{\alpha^{-\lfloor \frac{j-3}{3} \rfloor}} + \frac{\beta X_{2j}}{\alpha^{2r+i+1}}} \frac{\frac{((1+\alpha)-\delta Y_{1j})}{(-\alpha)^{1+\lfloor \frac{j}{3} \rfloor}} + \frac{\delta Y_{1j}}{(-\alpha)^{2r+i+1+\lfloor \frac{j}{3} \rfloor}}}{((1+\alpha)-\delta Y_{2j}) + \frac{\delta Y_{2j}}{(-\alpha)^{2r+i+1+\lfloor \frac{j}{3} \rfloor}}} = F_j, \tag{56}$$

for $j = \overline{1,3}$, $i \in \{0,1\}$. From (52), (55) and (56) we have that

$$\lim_{m\to\infty} x_{6m+3i+j} = \lim_{m\to\infty} x_{3i+j-6} \left(\frac{X_{2j}Y_{2j}}{X_{1j}Y_{1j}}\right)^{m+1} \prod_{r=0}^{m} K_j, \ \lim_{m\to\infty} y_{6m+3i+j} = \lim_{m\to\infty} y_{3i+j-6} \left(\frac{X_{2j}Y_{2j}}{X_{1j}Y_{1j}}\right)^{m+1} \prod_{r=0}^{m} F_j.$$

Four results can be easily seen using limits of the general terms (55) and (56).
$(e), (f)$ From (32), (53) and (54), we have

$$\begin{aligned} c_r^i =& \frac{(-\alpha)^{2r+i+1}((1+\alpha)-\delta Y_{1j})+\delta Y_{1j}}{(-\alpha)^{2r+i+1+\lfloor \frac{j-3}{3} \rfloor}((1+\alpha)-\delta Y_{2j})+\delta Y_{2j}} \frac{\alpha^{2r+i}((1-\alpha)-\beta X_{1j})+\beta X_{1j}}{\alpha^{2r+i+1+\lfloor \frac{j}{3} \rfloor}((1-\alpha)-\beta X_{2j})+\beta X_{2j}} \\ =& \frac{(-\alpha)^{2r+i+1}((1+\alpha)-\delta Y_{1j})\left(1+\frac{\delta Y_{1j}}{(-\alpha)^{2r+i+1}((1+\alpha)-\delta Y_{1j})}\right)}{(-\alpha)^{2r+i+1+\lfloor \frac{j-3}{3} \rfloor}((1+\alpha)-\delta Y_{2j})\left(1+\frac{\delta Y_{2j}}{(-\alpha)^{2r+i+1+\lfloor \frac{j-3}{3} \rfloor}((1+\alpha)-\delta Y_{2j})}\right)} \\ &\times \frac{\alpha^{2r+i}((1-\alpha)-\beta X_{1j})\left(1+\frac{\beta X_{1j}}{\alpha^{2r+i}((1-\alpha)-\beta X_{1j})}\right)}{\alpha^{2r+i+1+\lfloor \frac{j}{3} \rfloor}((1-\alpha)-\beta X_{2j})\left(1+\frac{\beta X_{2j}}{\alpha^{2r+i+1+\lfloor \frac{j}{3} \rfloor}((1-\alpha)-\beta X_{2j})}\right)} \\ =& \left(1+\frac{\delta Y_{1j}}{(-\alpha)^{2r+i+1}((1+\alpha)-\delta Y_{1j})}\right)\left(1-\frac{\delta Y_{2j}}{(-\alpha)^{2r+i+1+\lfloor \frac{j-3}{3} \rfloor}((1+\alpha)-\delta Y_{2j})}+O\left(\frac{1}{(-\alpha)^{4r}}\right)\right) \end{aligned}$$

$$\times\left(1+\frac{\beta X_{1j}}{\alpha^{2r+i}((1-\alpha)-\beta X_{1j})}\right)\left(1-\frac{\beta X_{2j}}{\alpha^{2r+i+1+\lfloor\frac{j}{3}\rfloor}((1-\alpha)-\beta X_{2j})}+O\left(\frac{1}{\alpha^{4r}}\right)\right)$$

$$=1+\frac{\delta Y_{1j}}{(-\alpha)^{2r+i+1}((1+\alpha)-\delta Y_{1j})}-\frac{\delta Y_{2j}}{(-\alpha)^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}((1+\alpha)-\delta Y_{2j})}+\frac{\beta X_{1j}}{\alpha^{2r+i}((1-\alpha)-\beta X_{1j})}$$

$$-\frac{\beta X_{2j}}{\alpha^{2r+i+1+\lfloor\frac{j}{3}\rfloor}((1-\alpha)-\beta X_{2j})}+O\left(\frac{1}{\alpha^{4r}}\right),$$

$$d_r^i=\frac{\alpha^{2r+i+1}((1-\alpha)-\beta X_{1j})+\beta X_{1j}}{\alpha^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}((1-\alpha)-\beta X_{2j})+\beta X_{2j}}\frac{(-\alpha)^{2r+i}((1+\alpha)-\delta Y_{1j})+\delta Y_{1j}}{(-\alpha)^{2r+i+1+\lfloor\frac{j}{3}\rfloor}((1+\alpha)-\delta Y_{2j})+\delta Y_{2j}}$$

$$=\frac{\alpha^{2r+i+1}((1-\alpha)-\beta X_{1j})\left(1+\frac{\beta X_{1j}}{\alpha^{2r+i+1}((1-\alpha)-\beta X_{1j})}\right)}{\alpha^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}((1-\alpha)-\beta X_{2j})\left(1+\frac{\beta X_{2j}}{\alpha^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}((1-\alpha)-\beta X_{2j})}\right)}$$

$$\times\frac{(-\alpha)^{2r+i}((1+\alpha)-\delta Y_{1j})\left(1+\frac{\delta Y_{1j}}{(-\alpha)^{2r+i}((1+\alpha)-\delta Y_{1j})}\right)}{(-\alpha)^{2r+i+1+\lfloor\frac{j}{3}\rfloor}((1+\alpha)-\delta Y_{2j})\left(1+\frac{\delta Y_{2j}}{(-\alpha)^{2r+i+1+\lfloor\frac{j}{3}\rfloor}((1+\alpha)-\delta Y_{2j})}\right)}$$

$$=\left(1+\frac{\beta X_{1j}}{\alpha^{2r+i+1}((1-\alpha)-\beta X_{1j})}\right)\left(1-\frac{\beta X_{2j}}{\alpha^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}((1-\alpha)-\beta X_{2j})}+O\left(\frac{1}{\alpha^{4r}}\right)\right)$$

$$\times\left(1+\frac{\delta Y_{1j}}{(-\alpha)^{2r+i}((1+\alpha)-\delta Y_{1j})}\right)\left(1-\frac{\delta Y_{2j}}{(-\alpha)^{2r+i+1+\lfloor\frac{j}{3}\rfloor}((1+\alpha)-\delta Y_{2j})}+O\left(\frac{1}{(-\alpha)^{4r}}\right)\right)$$

$$=1+\frac{\beta X_{1j}}{\alpha^{2r+i+1}((1-\alpha)-\beta X_{1j})}-\frac{\beta X_{2j}}{\alpha^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}((1-\alpha)-\beta X_{2j})}+\frac{\delta Y_{1j}}{(-\alpha)^{2r+i}((1+\alpha)-\delta Y_{1j})}$$

$$-\frac{\delta Y_{2j}}{(-\alpha)^{2r+i+1+\lfloor\frac{j}{3}\rfloor}((1+\alpha)-\delta Y_{2j})}+O\left(\frac{1}{\alpha^{4r}}\right),$$

for sufficiently large r, from which the convergence of the sequences $x_{6m+3i+j}$ and $y_{6m+3i+j}$ for $j=\overline{1,3}$, $i\in\{0,1\}$, can be seen easily.

(g), (h) From (32), (53) and (54), we have

$$c_r^i=\frac{(-\alpha)^{2r+i+1}((1+\alpha)-\delta Y_{1j})+\delta Y_{1j}}{(-\alpha)^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}((1+\alpha)-\delta Y_{2j})+\delta Y_{2j}}\frac{\alpha^{2r+i}((1-\alpha)-\beta X_{1j})+\beta X_{1j}}{\alpha^{2r+i+1+\lfloor\frac{j}{3}\rfloor}((1-\alpha)-\beta X_{2j})+\beta X_{2j}}$$

$$=\frac{(-\alpha)^{2r+i+1}((1+\alpha)-\delta Y_{1j})\left(1+\frac{\delta Y_{1j}}{(-\alpha)^{2r+i+1}((1+\alpha)-\delta Y_{1j})}\right)}{(-\alpha)^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}((1+\alpha)-\delta Y_{2j})\left(1+\frac{\delta Y_{2j}}{(-\alpha)^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}((1+\alpha)-\delta Y_{2j})}\right)}$$

$$\times\frac{\alpha^{2r+i}((1-\alpha)-\beta X_{1j})\left(1+\frac{\beta X_{1j}}{\alpha^{2r+i}((1-\alpha)-\beta X_{1j})}\right)}{\alpha^{2r+i+1+\lfloor\frac{j}{3}\rfloor}((1-\alpha)-\beta X_{2j})\left(1+\frac{\beta X_{2j}}{\alpha^{2r+i+1+\lfloor\frac{j}{3}\rfloor}((1-\alpha)-\beta X_{2j})}\right)}$$

$$=-\left(1+\frac{\delta Y_{1j}}{(-\alpha)^{2r+i+1}((1+\alpha)-\delta Y_{1j})}\right)\left(1-\frac{\delta Y_{2j}}{(-\alpha)^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}((1+\alpha)-\delta Y_{2j})}+O\left(\frac{1}{(-\alpha)^{4r}}\right)\right)$$

$$\times\left(1+\frac{\beta X_{1j}}{\alpha^{2r+i}((1-\alpha)-\beta X_{1j})}\right)\left(1-\frac{\beta X_{2j}}{\alpha^{2r+i+1+\lfloor\frac{j}{3}\rfloor}((1-\alpha)-\beta X_{2j})}+O\left(\frac{1}{\alpha^{4r}}\right)\right)$$

$$=-1-\frac{\delta Y_{1j}}{(-\alpha)^{2r+i+1}((1+\alpha)-\delta Y_{1j})}+\frac{\delta Y_{2j}}{(-\alpha)^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}((1+\alpha)-\delta Y_{2j})}-\frac{\beta X_{1j}}{\alpha^{2r+i}((1-\alpha)-\beta X_{1j})}$$

$$+\frac{\beta X_{2j}}{\alpha^{2r+i+1+\lfloor\frac{j}{3}\rfloor}((1-\alpha)-\beta X_{2j})}-O\left(\frac{1}{\alpha^{4r}}\right),$$

$$d_r^i=\frac{\alpha^{2r+i+1}((1-\alpha)-\beta X_{1j})+\beta X_{1j}}{\alpha^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}((1-\alpha)-\beta X_{2j})+\beta X_{2j}}\frac{(-\alpha)^{2r+i}((1+\alpha)-\delta Y_{1j})+\delta Y_{1j}}{(-\alpha)^{2r+i+1+\lfloor\frac{j}{3}\rfloor}((1+\alpha)-\delta Y_{2j})+\delta Y_{2j}}$$

$$=\frac{\alpha^{2r+i+1}((1-\alpha)-\beta X_{1j})\left(1+\frac{\beta X_{1j}}{\alpha^{2r+i+1}((1-\alpha)-\beta X_{1j})}\right)}{\alpha^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}((1-\alpha)-\beta X_{2j})\left(1+\frac{\beta X_{2j}}{\alpha^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}((1-\alpha)-\beta X_{2j})}\right)}$$

$$\times\frac{(-\alpha)^{2r+i}((1+\alpha)-\delta Y_{1j})\left(1+\frac{\delta Y_{1j}}{(-\alpha)^{2r+i}((1+\alpha)-\delta Y_{1j})}\right)}{(-\alpha)^{2r+i+1+\lfloor\frac{j}{3}\rfloor}((1+\alpha)-\delta Y_{2j})\left(1+\frac{\delta Y_{2j}}{(-\alpha)^{2r+i+1+\lfloor\frac{j}{3}\rfloor}((1+\alpha)-\delta Y_{2j})}\right)}$$

$$=-\left(1+\frac{\beta X_{1j}}{\alpha^{2r+i+1}((1-\alpha)-\beta X_{1j})}\right)\left(1-\frac{\beta X_{2j}}{\alpha^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}((1-\alpha)-\beta X_{2j})}+O\left(\frac{1}{\alpha^{4r}}\right)\right)$$

$$\times\left(1+\frac{\delta Y_{1j}}{(-\alpha)^{2r+i}((1+\alpha)-\delta Y_{1j})}\right)\left(1-\frac{\delta Y_{2j}}{(-\alpha)^{2r+i+1+\lfloor\frac{j}{3}\rfloor}((1+\alpha)-\delta Y_{2j})}+O\left(\frac{1}{(-\alpha)^{4r}}\right)\right)$$

$$=-1-\frac{\beta X_{1j}}{\alpha^{2r+i+1}((1-\alpha)-\beta X_{1j})}+\frac{\beta X_{2j}}{\alpha^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}((1-\alpha)-\beta X_{2j})}-\frac{\delta Y_{1j}}{(-\alpha)^{2r+i}((1+\alpha)-\delta Y_{1j})}$$

$$+\frac{\delta Y_{2j}}{(-\alpha)^{2r+i+1+\lfloor\frac{j}{3}\rfloor}((1+\alpha)-\delta Y_{2j})}-\left(\frac{1}{\alpha^{4r}}\right),$$

for sufficiently large r, from which the convergence of the sequences $x_{12m+3i+j}$, $x_{12m+3i+j+6}$, $y_{12m+3i+j}$ and $y_{12m+3i+j+6}$ for $j=\overline{1,3}$, $i\in\{0,1\}$, can be obtained easily. □

The following theorem associated to the case $|\alpha|<1$, $|\gamma|<1$, $\beta X_{1j}\neq 0\neq\beta X_{2j}$, $\delta Y_{1j}\neq 0\neq\delta Y_{2j}$.

Theorem 2.4. *Assume that $|\alpha|<1$, $|\gamma|<1$, $\beta X_{1j}\neq 0\neq\beta X_{2j}$, $\delta Y_{1j}\neq 0\neq\delta Y_{2j}$ and $(x_{6m+3i+j},y_{6m+3i+j})_{m\geq -1}$ for $i\in\{0,1\}$, $j=\overline{1,3}$ is a well-defined solutions of system* (19). *Then the sequences $(x_{6m+3i+j})_{m\geq-1}$ and $(y_{6m+3i+j})_{m\geq-1}$ are convergent.*

Proof. In this case, by using (22) and (32), we have

$$x_{6m+3i+j}=x_{3i+j-6}\prod_{r=0}^{m}\frac{X_{2j}Y_{2j}}{X_{1j}Y_{1j}}\frac{\gamma^{2r+i+1}((1-\gamma)-\delta Y_{1j})+\delta Y_{1j}}{\gamma^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}((1-\gamma)-\delta Y_{2j})+\delta Y_{2j}}\frac{\alpha^{2r+i}((1-\alpha)-\beta X_{1j})+\beta X_{1j}}{\alpha^{2r+i+1+\lfloor\frac{j}{3}\rfloor}((1-\alpha)-\beta X_{2j})+\beta X_{2j}}$$

$$=x_{3i+j-6}\prod_{r=0}^{m}\frac{\left(1+\frac{\gamma^{2r+i+1}\left((1-\gamma)-\delta Y_{1j}\right)}{\delta Y_{1j}}\right)}{\left(1+\frac{\gamma^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}\left((1-\gamma)-\delta Y_{2j}\right)}{\delta Y_{2j}}\right)}\frac{\left(1+\frac{\alpha^{2r+i}\left((1-\alpha)-\beta X_{1j}\right)}{\beta X_{1j}}\right)}{\left(1+\frac{\alpha^{2r+i+1+\lfloor\frac{j}{3}\rfloor}\left((1-\alpha)-\beta X_{2j}\right)}{\beta X_{2j}}\right)}$$

$$
\begin{aligned}
&= x_{3i+j-6}\prod_{r=0}^{m}\left(1+\frac{\gamma^{2r+i+1}\big((1-\gamma)-\delta Y_{1j}\big)}{\delta Y_{1j}}\right)\left(1-\frac{\gamma^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}\big((1-\gamma)-\delta Y_{2j}\big)}{\delta Y_{2j}}+O(\gamma^{4r})\right) \qquad (57)\\
&\times\left(1+\frac{\alpha^{2r+i}\big((1-\alpha)-\beta X_{1j}\big)}{\beta X_{1j}}\right)\left(1-\frac{\alpha^{2r+i+1+\lfloor\frac{j}{3}\rfloor}\big((1-\alpha)-\beta X_{2j}\big)}{\beta X_{2j}}+O(\alpha^{4r})\right)\\
&= x_{3i+j-6}\prod_{r=0}^{m}\left(1+O(\gamma^{2r})\right)\left(1+O(\alpha^{2r})\right)\\
&= x_{3i+j-6}\prod_{r=0}^{m}\left(1+O\left((\alpha\gamma)^{2r}\right)\right),
\end{aligned}
$$

$$
\begin{aligned}
y_{6m+3i+j} &= y_{3i+j-6}\prod_{r=0}^{m}\frac{X_{2j}Y_{2j}}{X_{1j}Y_{1j}}\frac{\alpha^{2r+i+1}((1-\alpha)-\beta X_{1j})+\beta X_{1j}}{\alpha^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}((1-\alpha)-\beta X_{2j})+\beta X_{2j}}\frac{\gamma^{2r+i}((1-\gamma)-\delta Y_{1j})+\delta Y_{1j}}{\gamma^{2r+i+1+\lfloor\frac{j}{3}\rfloor}((1-\gamma)-\delta Y_{2j})+\delta Y_{2j}}\\
&= y_{3i+j-6}\prod_{r=0}^{m}\frac{\left(1+\frac{\alpha^{2r+i+1}\left((1-\alpha)-\beta X_{1j}\right)}{\beta X_{1j}}\right)}{\left(1+\frac{\alpha^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}\left((1-\alpha)-\beta X_{2j}\right)}{\beta X_{2j}}\right)}\frac{\left(1+\frac{\gamma^{2r+i}\left((1-\gamma)-\delta Y_{1j}\right)}{\delta Y_{1j}}\right)}{\left(1+\frac{\gamma^{2r+i+1+\lfloor\frac{j}{3}\rfloor}\left((1-\gamma)-\beta Y_{2j}\right)}{\delta Y_{2j}}\right)}
\end{aligned}
$$

$$
\begin{aligned}
&= y_{3i+j-6}\prod_{r=0}^{m}\left(1+\frac{\alpha^{2r+i+1}\big((1-\alpha)-\beta X_{1j}\big)}{\beta X_{1j}}\right)\left(1-\frac{\alpha^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}\big((1-\alpha)-\beta X_{2j}\big)}{\beta X_{2j}}+O(\alpha^{4r})\right) \qquad (58)\\
&\times\left(1+\frac{\gamma^{2r+i}\big((1-\gamma)-\delta Y_{1j}\big)}{\delta Y_{1j}}\right)\left(1-\frac{\gamma^{2r+i+1+\lfloor\frac{j}{3}\rfloor}\big((1-\gamma)-\delta Y_{2j}\big)}{\delta Y_{2j}}+O(\gamma^{4r})\right)\\
&= y_{3i+j-6}\prod_{r=0}^{m}\left(1+O(\alpha^{2r})\right)\left(1+O(\gamma^{2r})\right)\\
&= y_{3i+j-6}\prod_{r=0}^{m}\left(1+O\left((\gamma\alpha)^{2r}\right)\right).
\end{aligned}
$$

From (57) and (58) it is clearly seen that $(x_{6m+3i+j})_{m\geq-1}$ and $(y_{6m+3i+j})_{m\geq-1}$, for $i\in\{0,1\}$, $j=\overline{1,3}$, are convergent. □

Let $\hat{N}_j := \frac{1}{\beta X_{1j}}-\frac{1}{\beta X_{2j}}-1-\lfloor\frac{j}{3}\rfloor$, $\tilde{N}_j := \frac{1}{\beta X_{1j}}-\frac{1}{\beta X_{2j}}-\lfloor\frac{j-3}{3}\rfloor$.
The cases $|\gamma|>1$, $\alpha=1$ are considered in the following theorem.

Theorem 2.5. *Assume that $|\gamma|>1$, $\alpha=1$ and $(x_{6m+3i+j}, y_{6m+3i+j})_{m\geq-1}$ for $i\in\{0,1\}$, $j=\overline{1,3}$ is a well-defined solutions of system* (19)*, then the following statements are true.*

(a) Suppose that $Y_{1j}\neq\frac{1-\gamma}{\delta}\neq Y_{2j}$ and $N_{2j}=1$,

- *(i) If $\hat{N}_j<0$, $|Y_{2j}|<|Y_{1j}|$, then $x_{6m+3i+j}\to 0$ as $m\to\infty$.*
- *(ii) If $\hat{N}_j>0$, $|Y_{2j}|>|Y_{1j}|$, then $|x_{6m+3i+j}|\to\infty$ as $m\to\infty$.*
- *(iii) If $\hat{N}_j=0$, $Y_{2j}=Y_{1j}$, then $x_{6m+3i+j}$ is convergent.*

(b) *Suppose that* $Y_{1j} \neq \frac{1-\gamma}{\delta} \neq Y_{2j}$ *and* $N_{1j} = 1$,

(i) *If* $\tilde{N}_j < 0$, $|Y_{2j}| < |Y_{1j}|$, *then* $y_{6m+3i+j} \to 0$ *as* $m \to \infty$.

(ii) *If* $\tilde{N}_j > 0$, $|Y_{2j}| > |Y_{1j}|$, *then* $|y_{6m+3i+j}| \to \infty$ *as* $m \to \infty$.

(iii) *If* $\tilde{N}_j = 0$, $Y_{2j} = Y_{1j}$, *then* $y_{6m+3i+j}$ *is convergent.*

(c) *If* $Y_{1j} = \frac{1-\gamma}{\delta} \neq Y_{2j}$, *then* $x_{6m+3i+j} \to 0$ *and* $y_{6m+3i+j} \to 0$ *as* $m \to \infty$.

(d) *If* $Y_{1j} \neq \frac{1-\gamma}{\delta} = Y_{2j}$, *then* $|x_{6m+3i+j}| \to \infty$ *and* $|y_{6m+3i+j}| \to \infty$ *as* $m \to \infty$.

(e) *If* $Y_{1j} = \frac{1-\gamma}{\delta} = Y_{2j}$ *and* $N_{2j} = 1$, *then* $x_{6m+3i+j}$ *is convergent.*

(f) *If* $Y_{1j} = \frac{1-\gamma}{\delta} = Y_{2j}$ *and* $N_{1j} = 1$, *then* $y_{6m+3i+j}$ *is convergent.*

Proof. (*a*)
(*i*), (*ii*), (*iii*) According to condition, the first equation of (23) become

$$x_{6m+3i+j} = x_{3i+j-6} \prod_{r=0}^{m} \frac{X_{2j}Y_{2j}}{X_{1j}Y_{1j}} \frac{\frac{((1-\gamma)-\delta Y_{1j})}{\gamma^{\lfloor \frac{j-3}{3} \rfloor}} + \frac{\delta Y_{1j}}{\gamma^{2r+i+1+\lfloor \frac{j-3}{3} \rfloor}}}{((1-\gamma)-\delta Y_{2j}) + \frac{\delta Y_{2j}}{\gamma^{2r+i+1+\lfloor \frac{j-3}{3} \rfloor}}} \frac{1 + \frac{i+\frac{1}{\beta X_{1j}}}{2r}}{1 + \frac{i+1+\lfloor \frac{j}{3} \rfloor + \frac{1}{\beta X_{2j}}}{2r}}.$$

If $Y_{1j} \neq \frac{1-\gamma}{\delta} \neq Y_{2j}$ and $N_{2j} = 1$, by using (32), we have

$$\begin{aligned}
\lim_{m\to\infty} x_{6m+3i+j} &= \lim_{m\to\infty} x_{3i+j-6} \prod_{r=0}^{m} \frac{X_{2j}Y_{2j}}{X_{1j}Y_{1j}} \frac{\frac{((1-\gamma)-\delta Y_{1j})}{\gamma^{\lfloor \frac{j-3}{3} \rfloor}} + \frac{\delta Y_{1j}}{\gamma^{2r+i+1+\lfloor \frac{j-3}{3} \rfloor}}}{((1-\gamma)-\delta Y_{2j}) + \frac{\delta Y_{2j}}{\gamma^{2r+i+1+\lfloor \frac{j-3}{3} \rfloor}}} \frac{1 + \frac{i+\frac{1}{\beta X_{1j}}}{2r}}{1 + \frac{i+1+\lfloor \frac{j}{3} \rfloor + \frac{1}{\beta X_{2j}}}{2r}} \\
&= \lim_{m\to\infty} x_{3i+j-6} \prod_{r=0}^{m} \frac{Y_{2j}}{Y_{1j}} \left(1 + \frac{i + \frac{1}{\beta X_{1j}}}{2r}\right)\left(1 - \frac{i+1+\lfloor \frac{j}{3} \rfloor + \frac{1}{\beta X_{2j}}}{2r} + O\left(\frac{1}{r^2}\right)\right) \\
&= \lim_{m\to\infty} x_{3i+j-6} \prod_{r=0}^{m} \frac{Y_{2j}}{Y_{1j}} \left(1 + \frac{\hat{N}_j}{2r} + O\left(\frac{1}{r^2}\right)\right), \qquad (59)
\end{aligned}$$

for a large r. From (59), the results easily follow.
(*b*)
(*i*), (*ii*), (*iii*)According to condition, the second equation in (23) becomes

$$y_{6m+3i+j} = y_{3i+j-6} \prod_{r=0}^{m} \frac{X_{2j}Y_{2j}}{X_{1j}Y_{1j}} \frac{1+\beta(2r+i+1)X_{1j}}{1+\beta(2r+i+1+\lfloor \frac{j-3}{3} \rfloor)X_{2j}} \frac{\gamma^{2r+i}((1-\gamma)-\delta Y_{1j})+\delta Y_{1j}}{\gamma^{2r+i+1+\lfloor \frac{j}{3} \rfloor}\left((1-\gamma)-\delta Y_{2j}\right)+\delta Y_{2j}}.$$

If $Y_{1j} \neq \frac{1-\gamma}{\delta} \neq Y_{2j}$ and $N_{1j} = 1$, by using (32), we get

$$\begin{aligned}
\lim_{m\to\infty} y_{6m+3i+j} &= \lim_{m\to\infty} y_{3i+j-6} \prod_{r=0}^{m} \frac{X_{2j}Y_{2j}}{X_{1j}Y_{1j}} \frac{\frac{((1-\gamma)-\delta Y_{1j})}{\gamma^{1+\lfloor \frac{j}{3} \rfloor}} + \frac{\delta Y_{1j}}{\gamma^{2r+i+1+\lfloor \frac{j}{3} \rfloor}}}{((1-\gamma)-\delta Y_{2j}) + \frac{\delta Y_{2j}}{\gamma^{2r+i+1+\lfloor \frac{j}{3} \rfloor}}} \frac{1 + \frac{i+1+\frac{1}{\beta X_{1j}}}{2r}}{1 + \frac{i+1+\lfloor \frac{j-3}{3} \rfloor + \frac{1}{\beta X_{2j}}}{2r}} \\
&= \lim_{m\to\infty} y_{3i+j-6} \prod_{r=0}^{m} \frac{Y_{2j}}{Y_{1j}} \left(1 + \frac{i+1+\frac{1}{\beta X_{1j}}}{2r}\right)\left(1 - \frac{i+1+\lfloor \frac{j-3}{3} \rfloor + \frac{1}{\beta X_{2j}}}{2r} + O\left(\frac{1}{r^2}\right)\right)
\end{aligned}$$

$$= \lim_{m\to\infty} y_{3i+j-6} \prod_{r=0}^{m} \frac{Y_{2j}}{Y_{1j}} \left(1 + \frac{\tilde{N}_j}{2r} + O\left(\frac{1}{r^2}\right)\right), \tag{60}$$

for a large r. From (60), the results easily follow.
(c) If $Y_{1j} = \frac{1-\gamma}{\delta} \neq Y_{2j}$, from (23), we obtain

$$\lim_{m\to\infty} x_{6m+3i+j} = \lim_{m\to\infty} x_{3i+j-6} \prod_{r=0}^{m} Y_{2j} \frac{\delta}{\gamma^{2r+i+1+\lfloor \frac{j-3}{3} \rfloor}\left((1-\gamma) - \delta Y_{2j}\right) + \delta Y_{2j}} \left(1 + \frac{\hat{N}_j}{2r} + O\left(\frac{1}{r^2}\right)\right), \tag{61}$$

$$\lim_{m\to\infty} y_{6m+3i+j} = \lim_{m\to\infty} y_{3i+j-6} \prod_{r=0}^{m} Y_{2j} \frac{\delta}{\gamma^{2r+i+1+\lfloor \frac{j}{3} \rfloor}\left((1-\gamma) - \delta Y_{2j}\right) + \delta Y_{2j}} \left(1 + \frac{\tilde{N}_j}{2r} + O\left(\frac{1}{r^2}\right)\right), \tag{62}$$

for a large r. By using (61) and (62), the result easily follows.
(d) If $Y_{1j} \neq \frac{1-\gamma}{\delta} = Y_{2j}$, from (23), we have that

$$\lim_{m\to\infty} x_{6m+3i+j} = \lim_{m\to\infty} x_{3i+j-6} \prod_{r=0}^{m} \frac{1}{Y_{1j}} \frac{\gamma^{2r+i+1}\left((1-\gamma) - \delta Y_{1j}\right) + \delta Y_{1j}}{\delta} \left(1 + \frac{\hat{N}_j}{2r} + O\left(\frac{1}{r^2}\right)\right), \tag{63}$$

$$\lim_{m\to\infty} y_{6m+3i+j} = \lim_{m\to\infty} y_{3i+j-6} \prod_{r=0}^{m} \frac{1}{Y_{1j}} \frac{\gamma^{2r+i}\left((1-\gamma) - \delta Y_{1j}\right) + \delta Y_{1j}}{\delta} \left(1 + \frac{\tilde{N}_j}{2r} + O\left(\frac{1}{r^2}\right)\right), \tag{64}$$

for a large r. The results are easily obtained from (63) and (64).
(e) According to conditions, we have

$$\begin{aligned}
x_{6m+3i+j} &= x_{3i+j-6} \prod_{r=0}^{m} N_{2j} \frac{\left(1 + \frac{i + \frac{1}{\beta X_{1j}}}{2r}\right)}{\left(1 + \frac{i+1+\lfloor \frac{j}{3} \rfloor + \frac{1}{\beta X_{2j}}}{2r}\right)} \\
&= x_{3i+j-6} \prod_{r=0}^{m} \left(1 + \frac{i + \frac{1}{\beta X_{1j}}}{2r}\right)\left(1 - \frac{i + 1 + \lfloor \frac{j}{3} \rfloor + \frac{1}{\beta X_{2j}}}{2r} + O\left(\frac{1}{r^2}\right)\right) \\
&= x_{3i+j-6} C_1(m_0) \prod_{r=m_0+1}^{m} \left(1 + \frac{\hat{N}_j}{2r} + O\left(\frac{1}{r^2}\right)\right) \\
&= x_{3i+j-6} C_1(m_0) e^{\sum_{r=m_0+1}^{m}\left(\frac{\hat{N}_j}{2r} + O\left(\frac{1}{r^2}\right)\right)},
\end{aligned} \tag{65}$$

for a large r, $C_1(m_0) = \prod_{r=0}^{m_0}\left(\frac{\left(1+\frac{i+\frac{1}{\beta X_{1j}}}{2r}\right)}{\left(1+\frac{i+1+\lfloor \frac{j}{3} \rfloor+\frac{1}{\beta X_{2j}}}{2r}\right)}\right)$. From (65), the fact that $\sum_{r=m_0+1}^{m} \frac{1}{r} \to +\infty$, as $m \to \infty$, and the convergence of series $\sum_{r=m_0+1}^{+\infty} O\left(\frac{1}{r^2}\right)$, the statements can be seen clearly.

(f) According to conditions, we have

$$y_{6m+3i+j} = y_{3i+j-6} \prod_{r=0}^{m} N_{1j} \frac{\left(1 + \frac{i+1+\frac{1}{\beta X_{1j}}}{2r}\right)}{\left(1 + \frac{i+1+\lfloor \frac{j-3}{3} \rfloor + \frac{1}{\beta X_{2j}}}{2r}\right)}$$

$$= y_{3i+j-6}\prod_{r=0}^{m}\left(1+\frac{i+1+\frac{1}{\beta X_{1j}}}{2r}\right)\left(1-\frac{i+1+\lfloor\frac{j-3}{3}\rfloor+\frac{1}{\beta X_{2j}}}{2r}+O\left(\frac{1}{r^2}\right)\right)$$
$$= y_{3i+j-6}C_2(m_0)\prod_{r=m_0+1}^{m}\left(1+\frac{\tilde{N}_j}{2r}+O\left(\frac{1}{r^2}\right)\right)$$
$$= y_{3i+j-6}C_2(m_0)e^{\sum_{r=m_0+1}^{m}\left(\frac{\tilde{N}_j}{2r}+O\left(\frac{1}{r^2}\right)\right)}, \tag{66}$$

for a large r, $C_2(m_0)=\prod_{r=0}^{m_0}\frac{\left(1+\frac{i+1+\frac{1}{\beta X_{1j}}}{2r}\right)}{\left(1+\frac{i+1+\lfloor\frac{j-3}{3}\rfloor+\frac{1}{\beta X_{2j}}}{2r}\right)}$. From (66), the fact that $\sum_{r=m_0+1}^{m}\frac{1}{r}\to+\infty$, as $m\to\infty$, and the convergence of series $\sum_{r=m_0+1}^{+\infty}O\left(\frac{1}{r^2}\right)$, the statements can be seen clearly. □

Now, we investigate the cases $|\alpha|>1$ and $\gamma=1$ in the following theorem. Let
$\hat{M}_j:=\frac{1}{\delta Y_{1j}}-\frac{1}{\delta Y_{2j}}-\lfloor\frac{j-3}{3}\rfloor$, $\quad\tilde{M}_j:=\frac{1}{\delta Y_{1j}}-\frac{1}{\delta Y_{2j}}-1-\lfloor\frac{j}{3}\rfloor$.

Theorem 2.6. *Assume that $|\alpha|>1$, $\gamma=1$ and $(x_{6m+3i+j},y_{6m+3i+j})_{m\geq-1}$ for $i\in\{0,1\}$, $j=\overline{1,3}$ is a well-defined solutions of system* (19)*,then the following statements are true.*

(a) Suppose that $X_{1j}\neq\frac{1-\alpha}{\beta}\neq X_{2j}$ and $M_{2j}=1$,

- *(i) If $\hat{M}_j<0$, $\left|X_{2j}\right|<\left|X_{1j}\right|$, then $x_{6m+3i+j}\to0$ as $m\to\infty$.*
- *(ii) If $\hat{M}_j>0$, $\left|X_{2j}\right|>\left|X_{1j}\right|$, then $|x_{6m+3i+j}|\to\infty$ as $m\to\infty$.*
- *(iii) If $\hat{M}_j=0$, $X_{2j}=X_{1j}$, then $x_{6m+3i+j}$ is convergent.*

(b) Suppose that $X_{1j}\neq\frac{1-\alpha}{\beta}\neq X_{2j}$ and $M_{1j}=1$,

- *(i) If $\tilde{M}_j<0$, $\left|X_{2j}\right|<\left|X_{1j}\right|$, then $y_{6m+3i+j}\to0$ as $m\to\infty$.*
- *(ii) If $\tilde{M}_j>0$, $\left|X_{2j}\right|>\left|X_{1j}\right|$, then $|y_{6m+3i+j}|\to\infty$ as $m\to\infty$.*
- *(iii) If $\tilde{M}_j=0$, $X_{2j}=X_{1j}$, then $y_{6m+3i+j}$ is convergent.*

(c) If $X_{1j}=\frac{1-\alpha}{\beta}\neq X_{2j}$, then $x_{6m+3i+j}\to0$ and $y_{6m+3i+j}\to0$.

(d) If $X_{1j}\neq\frac{1-\alpha}{\beta}=X_{2j}$, then $|x_{6m+3i+j}|\to\infty$ and $|y_{6m+3i+j}|\to\infty$ as $m\to\infty$.

(e) If $X_{1j}=\frac{1-\alpha}{\beta}=X_{2j}$ and $M_{2j}=1$, then $x_{6m+3i+j}$ is convergent.

(f) If $X_{1j}=\frac{1-\alpha}{\beta}=X_{2j}$ and $M_{1j}=1$, then $y_{6m+3i+j}$ is convergent.

Proof. (a)
$(i),(ii),(iii)$ According to conditions, the first equation in (24) become

$$x_{6m+3i+j}=x_{3i+j-6}\prod_{r=0}^{m}\frac{X_{2j}Y_{2j}}{X_{1j}Y_{1j}}\frac{1+\delta(2r+i+1)Y_{1j}}{1+\delta(2r+i+1+\lfloor\frac{j-3}{3}\rfloor)Y_{2j}}\frac{\alpha^{2r+i}((1-\alpha)-\beta X_{1j})+\beta X_{1j}}{\alpha^{2r+i+1+\lfloor\frac{j}{3}\rfloor}((1-\alpha)-\beta X_{2j})+\beta X_{2j}}$$
$$=x_{3i+j-6}\prod_{r=0}^{m}\frac{X_{2j}}{X_{1j}}\frac{\left(1+\frac{i+1+\frac{1}{\delta Y_{1j}}}{2r}\right)}{\left(1+\frac{i+1+\lfloor\frac{j-3}{3}\rfloor+\frac{1}{\delta Y_{2j}}}{2r}\right)}\left(\frac{\frac{((1-\alpha)-\beta X_{1j})}{\alpha^{1+\lfloor\frac{j}{3}\rfloor}}+\frac{\beta X_{1j}}{\alpha^{2r+i+1+\lfloor\frac{j}{3}\rfloor}}}{(1-\alpha)-\beta X_{2j})+\frac{\beta X_{2j}}{\alpha^{2r+i+1+\lfloor\frac{j}{3}\rfloor}}}\right). \tag{67}$$

If $X_{1j} \neq \frac{1-\alpha}{\beta} \neq X_{2j}$ and $M_{2j} = 1$, by using (32), we have

$$\begin{aligned}\lim_{m\to\infty} x_{6m+3i+j} &= \lim_{m\to\infty} x_{3i+j-6} \prod_{r=0}^{m} \frac{X_{2j}}{X_{1j}} \frac{\left(1+\frac{i+1+\frac{1}{\delta Y_{1j}}}{2r}\right)}{\left(1+\frac{i+1+\lfloor\frac{j-3}{3}\rfloor+\frac{1}{\delta Y_{2j}}}{2r}\right)} \left(\frac{\frac{((1-\alpha)-\beta X_{1j})}{\alpha^{1+\lfloor\frac{j}{3}\rfloor}}+\frac{\beta X_{1j}}{\alpha^{2r+i+1+\lfloor\frac{j}{3}\rfloor}}}{(1-\alpha)-\beta X_{2j}+\frac{\beta X_{2j}}{\alpha^{2r+i+1+\lfloor\frac{j}{3}\rfloor}}}\right)\\ &= \lim_{m\to\infty} x_{3i+j-6} \prod_{r=0}^{m} \frac{X_{2j}}{X_{1j}} \left(1+\frac{i+1+\frac{1}{\delta Y_{1j}}}{2r}\right)\left(1-\frac{i+1+\lfloor\frac{j-3}{3}\rfloor+\frac{1}{\delta Y_{2j}}}{2r}+O\left(\frac{1}{r^2}\right)\right)\\ &= \lim_{m\to\infty} x_{3i+j-6} \prod_{r=0}^{m} \frac{X_{2j}}{X_{1j}} \left(1+\frac{\hat{M}_j}{2r}+O\left(\frac{1}{r^2}\right)\right). \end{aligned} \tag{68}$$

for a large r. From (68), the results easily follow.
(b)
(i), (ii), (iii) According to conditions, the second equation in (24) become

$$\begin{aligned} y_{6m+3i+j} &= y_{3i+j-6} \prod_{r=0}^{m} \frac{X_{2j}Y_{2j}}{X_{1j}Y_{1j}} \frac{\alpha^{2r+i+1}((1-\alpha)-\beta X_{1j})+\beta X_{1j}}{\alpha^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}((1-\alpha)-\beta X_{2j})+\beta X_{2j}} \frac{1+\delta(2r+i)Y_{1j}}{1+\delta(2r+i+1+\lfloor\frac{j}{3}\rfloor)Y_{2j}}\\ &= y_{3i+j-6} \prod_{r=0}^{m} \frac{X_{2j}}{X_{1j}} \left(\frac{1+\frac{i+\frac{1}{\delta Y_{1j}}}{2r}}{1+\frac{i+1+\lfloor\frac{j}{3}\rfloor+\frac{1}{\delta Y_{2j}}}{2r}}\right)\left(\frac{\frac{((1-\alpha)-\beta X_{1j})}{\alpha^{\lfloor\frac{j-3}{3}\rfloor}}+\frac{\beta X_{1j}}{\alpha^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}}}{(1-\alpha)-\beta X_{2j}+\frac{\beta X_{2j}}{\alpha^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}}}\right). \end{aligned} \tag{69}$$

If $X_{1j} \neq \frac{1-\alpha}{\beta} \neq X_{2j}$ and $M_{1j} = 1$, by using (32), we have

$$\begin{aligned}\lim_{m\to\infty} y_{6m+3i+j} &= \lim_{m\to\infty} y_{3i+j-6} \prod_{r=0}^{m} \frac{X_{2j}}{X_{1j}} \left(\frac{1+\frac{i+\frac{1}{\delta Y_{1j}}}{2r}}{1+\frac{i+1+\lfloor\frac{j}{3}\rfloor+\frac{1}{\delta Y_{2j}}}{2r}}\right)\left(\frac{\frac{((1-\alpha)-\beta X_{1j})}{\alpha^{\lfloor\frac{j-3}{3}\rfloor}}+\frac{\beta X_{1j}}{\alpha^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}}}{(1-\alpha)-\beta X_{2j}+\frac{\beta X_{2j}}{\alpha^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}}}\right)\\ &= \lim_{m\to\infty} y_{3i+j-6} \prod_{r=0}^{m} \frac{X_{2j}}{X_{1j}} \left(1+\frac{i+\frac{1}{\delta Y_{1j}}}{2r}\right)\left(1-\frac{i+1+\lfloor\frac{j}{3}\rfloor+\frac{1}{\delta Y_{2j}}}{2r}+O\left(\frac{1}{r^2}\right)\right)\\ &= \lim_{m\to\infty} y_{3i+j-6} \prod_{r=0}^{m} \frac{X_{2j}}{X_{1j}} \left(1+\frac{\tilde{M}_j}{2r}+O\left(\frac{1}{r^2}\right)\right). \end{aligned} \tag{70}$$

for a large r. From (70), the results easily follow.
(c) If $X_{1j} = \frac{1-\alpha}{\beta} \neq X_{2j}$, from (24), we have that

$$\lim_{m\to\infty} x_{6m+3i+j} = \lim_{m\to\infty} x_{3i+j-6} \prod_{r=0}^{m} X_{2j} \frac{\beta}{\alpha^{2r+i+1+\lfloor\frac{j}{3}\rfloor}\Big((1-\alpha)-\beta X_{2j}\Big)+\beta X_{2j}} \left(\frac{1+\delta(2r+i+1)Y_{1j}}{1+\delta(2r+i+1+\lfloor\frac{j-3}{3}\rfloor)Y_{2j}}\right), \tag{71}$$

$$\lim_{m\to\infty} y_{6m+3i+j} = \lim_{m\to\infty} y_{3i+j-6} \prod_{r=0}^{m} X_{2j} \frac{\beta}{\alpha^{2r+i+1+\lfloor\frac{j-3}{3}\rfloor}\Big((1-\alpha)-\beta X_{2j}\Big)+\beta X_{2j}} \left(\frac{1+\delta(2r+i)Y_{1j}}{1+\delta(2r+i+1+\lfloor\frac{j}{3}\rfloor)Y_{2j}}\right), \tag{72}$$

for a large r. By using (71) and (72), the result easily follows.
(d) If $X_{1j} \neq \frac{1-\alpha}{\beta} = X_{2j}$, from (24), we have that

$$\lim_{m\to\infty} x_{6m+3i+j} = \lim_{m\to\infty} x_{3i+j-6} \prod_{r=0}^{m} \frac{1}{X_{1j}} \frac{\alpha^{2r+i}\Big((1-\alpha)-\beta X_{1j}\Big)+\beta X_{1j}}{\beta} \left(\frac{1+\delta(2r+i+1)Y_{1j}}{1+\delta(2r+i+1+\lfloor\frac{j-3}{3}\rfloor)Y_{2j}}\right), \tag{73}$$

$$\lim_{m\to\infty} y_{6m+3i+j} = \lim_{m\to\infty} y_{3i+j-6} \prod_{r=0}^{m} \frac{1}{X_{1j}} \frac{\alpha^{2r+i+1}\Big((1-\alpha)-\beta X_{1j}\Big)+\beta X_{1j}}{\beta} \left(\frac{1+\delta(2r+i)Y_{1j}}{1+\delta(2r+i+1+\lfloor \frac{j}{3} \rfloor)Y_{2j}} \right), \tag{74}$$

for a large r. From (73) and (74), the result easily follows.
(e) According to initial conditions, we have that

$$\begin{aligned} x_{6m+3i+j} &= x_{3i+j-6} \prod_{r=0}^{m} \frac{1+\frac{i+1+\frac{1}{\delta Y_{1j}}}{2r}}{1+\frac{i+1+\lfloor \frac{j-3}{3} \rfloor+\frac{1}{\delta Y_{2j}}}{2r}} \\ &= x_{3i+j-6} C_3(m_0) \prod_{r=m_0+1}^{m} \left(1+\frac{\hat{M}_j}{2r}+O\Big(\frac{1}{r^2}\Big) \right) \\ &= x_{3i+j-6} C_3(m_0) e^{\sum_{r=m_0+1}^{m}\left(\frac{\hat{M}_j}{2r}+O\left(\frac{1}{r^2}\right)\right)}, \end{aligned} \tag{75}$$

for a large r, $C_3(m_0) = \prod_{r=0}^{m_0} \frac{1+\frac{i+1+\frac{1}{\delta Y_{1j}}}{2r}}{1+\frac{i+1+\lfloor \frac{j-3}{3} \rfloor+\frac{1}{\delta Y_{2j}}}{2r}}$. From (75), the fact that $\sum_{r=m_0+1}^{m} \frac{1}{r} \to +\infty$, as $m \to \infty$, and the convergence of series $\sum_{r=m_0+1}^{+\infty} O\left(\frac{1}{r^2}\right)$, the result is easily obtained.
(f) According to initial conditions, we have

$$\begin{aligned} y_{6m+3i+j} &= y_{3i+j-6} \prod_{r=0}^{m} \frac{\left(1+\frac{i+\frac{1}{\delta Y_{1j}}}{2r}\right)}{\left(1+\frac{i+1+\lfloor \frac{j}{3} \rfloor+\frac{1}{\delta Y_{2j}}}{2r}\right)} \\ &= y_{3i+j-6} C_4(m_0) \prod_{r=m_0+1}^{m} \left(1+\frac{\tilde{M}_j}{2r}+O\Big(\frac{1}{r^2}\Big) \right) \\ &= y_{3i+j-6} C_4(m_0) e^{\sum_{r=m_0+1}^{m}\left(\frac{\tilde{M}_j}{2r}+O\left(\frac{1}{r^2}\right)\right)}, \end{aligned} \tag{76}$$

for a large r, $C_4(m_0) = \prod_{r=0}^{m_0} \frac{\left(1+\frac{i+\frac{1}{\delta Y_{1j}}}{2r}\right)}{\left(1+\frac{i+1+\lfloor \frac{j}{3} \rfloor+\frac{1}{\delta Y_{2j}}}{2r}\right)}$. From (76), the fact that $\sum_{r=m_0+1}^{m} \frac{1}{r} \to +\infty$, as $m \to \infty$, and the convergence of series $\sum_{r=m_0+1}^{+\infty} O\left(\frac{1}{r^2}\right)$, the statements can be seen clearly. □

The case $\alpha = 1 = \gamma$ is considered in the following theorem.

Theorem 2.7. *Assume that $\alpha = \gamma = 1$ and $(x_{6m+3i+j}, y_{6m+3i+j})_{m\geq -1}$ for $i \in \{0,1\}$, $j = \overline{1,3}$ is a well-defined solutions of system* (19)*. Then the following statements are true.*

(a) If $\hat{M}_j + \hat{N}_j < 0$, then $x_{6m+3i+j} \to 0$ as $m \to \infty$.

(b) If $\hat{M}_j + \hat{N}_j > 0$, then $|x_{6m+3i+j}| \to \infty$ as $m \to \infty$.

(c) If $\hat{M}_j + \hat{N}_j = 0$, then $x_{6m+3i+j}$ is constant.

(d) If $\tilde{M}_j + \tilde{N}_j < 0$, then $y_{6m+3i+j} \to 0$ as $m \to \infty$.

(e) If $\tilde{M}_j + \tilde{N}_j > 0$, then $|y_{6m+3i+j}| \to \infty$ as $m \to \infty$.

(f) If $\tilde{M}_j + \tilde{N}_j = 0$*, then* $y_{6m+3i+j}$ *is constant.*

Proof. $(a),(b),(c)$ Let

$$A_r^i := \frac{1+\delta(2r+i+1)Y_{1j}}{1+\delta(2r+i+1+\lfloor\frac{j-3}{3}\rfloor)Y_{2j}}, \quad B_r^i := \frac{1+\beta(2r+i)X_{1j}}{1+\beta(2r+i+1+\lfloor\frac{j}{3}\rfloor)X_{2j}},\ i\in\{0,1\}. \tag{77}$$

From (77), we have

$$\begin{aligned} A_r^i &= \frac{1+\frac{i+1+\frac{1}{\delta Y_{1j}}}{2r}}{1+\frac{i+1+\lfloor\frac{j-3}{3}\rfloor+\frac{1}{\delta Y_{2j}}}{2r}} \\ &= 1+\frac{\hat{M}_j}{2r}+O\left(\frac{1}{r^2}\right). \end{aligned} \tag{78}$$

$$\begin{aligned} B_r^i &= \frac{1+\frac{i+\frac{1}{\beta X_{1j}}}{2r}}{1+\frac{i+1+\lfloor\frac{j}{3}\rfloor+\frac{1}{\beta X_{2j}}}{2r}} \\ &= 1+\frac{\hat{N}_j}{2r}+O\left(\frac{1}{r^2}\right). \end{aligned} \tag{79}$$

Then we have that

$$\begin{aligned} \lim_{m\to\infty} x_{6m+3i+j} &= \lim_{m\to\infty} x_{3i+j-6}\prod_{r=0}^{m}\left(1+\frac{\hat{M}_j}{2r}+O\left(\frac{1}{r^2}\right)\right)\left(1+\frac{\hat{N}_j}{2r}+O\left(\frac{1}{r^2}\right)\right) \\ &= \lim_{m\to\infty} x_{3i+j-6}\prod_{r=0}^{m}\left(1+\frac{\hat{M}_j+\hat{N}_j}{2r}+O\left(\frac{1}{r^2}\right)\right), \end{aligned} \tag{80}$$

for large enough r and $i\in\{0,1\}$, $j=\overline{1,3}$. From (78), (79) and (80), the results easily follow.
$(d),(e),(f)$

$$Q_r^i := \frac{1+\beta(2r+i+1)X_{1j}}{1+\beta(2r+i+1+\lfloor\frac{j-3}{3}\rfloor)X_{2j}}, \quad T_r^i := \frac{1+\delta(2r+i)Y_{1j}}{1+\delta(2r+i+1+\lfloor\frac{j}{3}\rfloor)Y_{2j}},\ i\in\{0,1\}. \tag{81}$$

From (81), we have

$$\begin{aligned} Q_r^i &= \frac{1+\frac{i+1+\frac{1}{\beta X_{1j}}}{2r}}{1+\frac{i+1+\lfloor\frac{j-3}{3}\rfloor+\frac{1}{\beta X_{2j}}}{2r}} \\ &= 1+\frac{\tilde{M}_j}{2r}+O\left(\frac{1}{r^2}\right). \end{aligned} \tag{82}$$

$$\begin{aligned} T_r^i &= \frac{1+\frac{i+\frac{1}{\delta Y_{1j}}}{2r}}{1+\frac{i+1+\lfloor\frac{j}{3}\rfloor+\frac{1}{\delta Y_{2j}}}{2r}} \\ &= 1+\frac{\tilde{N}_j}{2r}+O\left(\frac{1}{r^2}\right). \end{aligned} \tag{83}$$

Then we have that

$$\begin{aligned}\lim_{m\to\infty} y_{6m+3i+j} &= \lim_{m\to\infty} y_{3i+j-6}\prod_{r=0}^{m}\left(1+\frac{\tilde{M}_j}{2r}+O\left(\frac{1}{r^2}\right)\right)\left(1+\frac{\tilde{N}_j}{2r}+O\left(\frac{1}{r^2}\right)\right)\\ &= \lim_{m\to\infty} y_{3i+j-6}\prod_{r=0}^{m}\left(1+\frac{\tilde{M}_j+\tilde{N}_j}{2r}+O\left(\frac{1}{r^2}\right)\right),\end{aligned} \tag{84}$$

for large enough r and $i \in \{0, 1\}$, $j = \overline{1, 3}$. From (82), (83) and (84), the results easily follows. □

Now we consider the case $\alpha = 0$ and $\gamma = 0$.

Theorem 2.8. *Assume that $(x_n, y_n)_{n\geq -5}$ is a well-defined solutions of system (19). If $\alpha = \gamma = 0$, then $(x_n, y_n)_{n\geq -5}$ is eventually six periodic.*

Proof. If $\alpha = \gamma = 0$, then the system (19) becomes

$$\begin{cases} x_n = \frac{1}{\beta y_{n-1}x_{n-2}} = \frac{\delta y_{n-3}}{\beta} = x_{n-6}, \\ y_n = \frac{1}{\delta x_{n-1}y_{n-2}} = \frac{\beta x_{n-3}}{\delta} = y_{n-6},, \end{cases} n \geq 4. \tag{85}$$

From (85), this statements easily follows. □

Now we consider the case $\alpha \neq 0 \neq \gamma$, $\beta = 0 = \delta$. In this case, system (19) becomes

$$x_n = \frac{x_{n-3}y_{n-4}x_{n-5}}{\alpha y_{n-1}x_{n-2}} \quad y_n = \frac{y_{n-3}x_{n-4}y_{n-5}}{\gamma x_{n-1}y_{n-2}},\ n \in \mathbb{N}_0. \tag{86}$$

By using (22) for $\alpha \neq 0 \neq \gamma$, $\beta = 0 = \delta$, we get

$$\begin{cases} x_{6m+3i+j} = x_{3i+j-6}\left(\frac{X_{2j}Y_{2j}}{X_{1j}Y_{1j}}\right)^{m+1}\prod_{r=0}^{m}\frac{1}{\gamma^{\lfloor\frac{j-3}{3}\rfloor}\alpha^{1+\lfloor\frac{j}{3}\rfloor}} \\ y_{6m+3i+j} = y_{3i+j-6}\left(\frac{X_{2j}Y_{2j}}{X_{1j}Y_{1j}}\right)^{m+1}\prod_{r=0}^{m}\frac{1}{\alpha^{\lfloor\frac{j-3}{3}\rfloor}\gamma^{1+\lfloor\frac{j}{3}\rfloor}} \end{cases},\ m \in \mathbb{N}_0,\ i \in \{0, 1\},\ j = \overline{1, 3}. \tag{87}$$

Let

$$T_j := \frac{1}{\gamma^{\lfloor\frac{j-3}{3}\rfloor}\alpha^{1+\lfloor\frac{j}{3}\rfloor}}, \quad \hat{T}_j := \frac{1}{\alpha^{\lfloor\frac{j-3}{3}\rfloor}\gamma^{1+\lfloor\frac{j}{3}\rfloor}},\ j = \overline{1, 3}.$$

By using formulas (87) it is easy to see that the following results. We omit the proof.

Theorem 2.9. *Assume that $\alpha \neq 1 \neq \gamma$, $\beta = 0 = \delta$, $\frac{X_{2j}Y_{2j}}{X_{1j}Y_{1j}} = 1$ and $(x_{6m+3i+j}, y_{6m+3i+j})_{m\geq -1}$ for $i \in \{0, 1\}$, $j = \overline{1, 3}$ is a well-defined solutions of system (19). Then the following statements are true.*

(a) If $|T_j| < 1$, then $x_{6m+3i+j} \to 0$ as $m \to \infty$.

(b) If $|T_j| > 1$, then $|x_{6m+3i+j}| \to \infty$ as $m \to \infty$.

(c) If $T_j = 1$, then the sequence $x_{6m+3i+j}$ converges to six-period solutions.

(d) If $T_j = -1$, then the sequence $x_{6m+3i+j}$ converges to twelve-period sub-sequences.

(e) If $|\hat{T}_j| < 1$, then $y_{6m+3i+j} \to 0$ as $m \to \infty$.

(f) If $|\hat{T}_j| > 1$, then $|y_{6m+3i+j}| \to \infty$ as $m \to \infty$.

(g) If $\hat{T}_j = 1$, then the sequence $y_{6m+3i+j}$ converges to six-period solutions.

(h) *If $\hat{T}_j = -1$, then the sequence $y_{6m+3i+j}$ converges to twelve-period sub-sequences.*

If $\alpha = -1 = \gamma$, $\beta \neq 0 \neq \delta$, then from (22) we have

$$\begin{cases} x_{6m+3i+j} = x_{3i+j-6} \prod_{r=0}^{m} \frac{X_{2j}Y_{2j}}{X_{1j}Y_{1j}} \frac{(-1)^{2r+i+1}(2-\delta Y_{1j})+\delta Y_{1j}}{(-1)^{2r+i+1+\lfloor \frac{j-3}{3} \rfloor}(2-\delta Y_{2j})+\delta Y_{2j}} \frac{(-1)^{2r+i}(2-\beta X_{1j})+\beta X_{1j}}{(-1)^{2r+i+1+\lfloor \frac{j}{3} \rfloor}(2-\beta X_{2j})+\beta X_{2j}}, \\ y_{6m+3i+j} = y_{3i+j-6} \prod_{r=0}^{m} \frac{X_{2j}Y_{2j}}{X_{1j}Y_{1j}} \frac{(-1)^{2r+i+1}(2-\beta X_{1j})+\beta X_{1j}}{(-1)^{2r+i+1+\lfloor \frac{j-3}{3} \rfloor}(2-\beta X_{2j})+\beta X_{2j}} \frac{(-1)^{2r+i}(2-\delta Y_{1j})+\delta Y_{1j}}{(-1)^{2r+i+1+\lfloor \frac{j}{3} \rfloor}(2-\delta Y_{2j})+\delta Y_{2j}}, \end{cases} \tag{88}$$

for $m \in N_0$, $i \in \{0,1\}$ and $j = \overline{1,3}$. Using formulas (88) we have

$$x_{6m+j_1} = x_{j_1-6}\left(\frac{X_{2j_1}Y_{2j_1}}{X_{1j_1}Y_{1j_1}}\right)^{m+1} \frac{1}{\left(\frac{\beta X_{2j_1}-1}{\delta Y_{1j_1}-1}\right)^{m+1}},\ x_{6m+3} = x_{-3}\left(\frac{X_{23}Y_{23}}{X_{13}Y_{13}}\right)^{m+1} \frac{1}{\left(\frac{\delta Y_{23}-1}{\delta Y_{13}-1}\right)^{m+1}}, \tag{89}$$

$$x_{6m+j_1+3} = x_{j_1-3}\left(\frac{X_{2j_1}Y_{2j_1}}{X_{1j_1}Y_{1j_1}}\right)^{m+1} \frac{1}{\left(\frac{\delta Y_{2j_1}-1}{\beta X_{1j_1}-1}\right)^{m+1}},\ x_{6m+6} = x_{0}\left(\frac{X_{23}Y_{23}}{X_{13}Y_{13}}\right)^{m+1} \frac{1}{\left(\frac{\beta X_{23}-1}{\beta X_{13}-1}\right)^{m+1}}, \tag{90}$$

$$y_{6m+j_1} = y_{j_1-6}\left(\frac{X_{2j_1}Y_{2j_1}}{X_{1j_1}Y_{1j_1}}\right)^{m+1} \frac{1}{\left(\frac{\delta Y_{2j_1}-1}{\beta X_{1j_1}-1}\right)^{m+1}},\ y_{6m+3} = y_{-3}\left(\frac{X_{23}Y_{23}}{X_{13}Y_{13}}\right)^{m+1} \frac{1}{\left(\frac{\beta X_{23}-1}{\beta X_{13}-1}\right)^{m+1}}, \tag{91}$$

$$y_{6m+j_1+3} = y_{j_1-3}\left(\frac{X_{2j_1}Y_{2j_1}}{X_{1j_1}Y_{1j_1}}\right)^{m+1} \frac{1}{\left(\frac{\beta X_{2j_1}-1}{\delta Y_{1j_1}-1}\right)^{m+1}},\ y_{6m+6} = y_{0}\left(\frac{X_{23}Y_{23}}{X_{13}Y_{13}}\right)^{m+1} \frac{1}{\left(\frac{\delta Y_{23}-1}{\delta Y_{13}-1}\right)^{m+1}}, \tag{92}$$

for $j_1 \in \{1,2\}$. By using (16), we have the following equations

$$X_{11} = X_{23},\ X_{12} = X_{21},\ X_{13} = X_{22},\ Y_{11} = Y_{23},\ Y_{12} = Y_{21},\ Y_{13} = Y_{22}. \tag{93}$$

Using (89)- (92) and (93), we get

$$x_{6m+1} = x_{-5}\left(\frac{X_{21}Y_{21}}{X_{11}Y_{11}}\right)^{m+1} \frac{1}{\left(\frac{\beta X_{21}-1}{\delta Y_{11}-1}\right)^{m+1}},\ x_{6m+2} = x_{-4}\left(\frac{X_{22}Y_{22}}{X_{12}Y_{12}}\right)^{m+1} \frac{1}{\left(\frac{\beta X_{22}-1}{\delta Y_{12}-1}\right)^{m+1}}, \tag{94}$$

$$x_{6m+3} = x_{-3}\left(\frac{X_{11}Y_{11}}{X_{22}Y_{22}}\right)^{m+1} \frac{1}{\left(\frac{\delta Y_{11}-1}{\delta Y_{22}-1}\right)^{m+1}},\ x_{6m+4} = x_{-2}\left(\frac{X_{21}Y_{21}}{X_{11}Y_{11}}\right)^{m+1} \frac{1}{\left(\frac{\delta Y_{21}-1}{\beta X_{11}-1}\right)^{m+1}}, \tag{95}$$

$$x_{6m+5} = x_{-1}\left(\frac{X_{22}Y_{22}}{X_{12}Y_{12}}\right)^{m+1} \frac{1}{\left(\frac{\delta Y_{22}-1}{\beta X_{12}-1}\right)^{m+1}},\ x_{6m+6} = x_{0}\left(\frac{X_{11}Y_{11}}{X_{22}Y_{22}}\right)^{m+1} \frac{1}{\left(\frac{\beta X_{11}-1}{\beta X_{22}-1}\right)^{m+1}}, \tag{96}$$

$$y_{6m+1} = y_{-5}\left(\frac{X_{21}Y_{21}}{X_{11}Y_{11}}\right)^{m+1} \frac{1}{\left(\frac{\delta Y_{21}-1}{\beta X_{11}-1}\right)^{m+1}},\ y_{6m+2} = y_{-4}\left(\frac{X_{22}Y_{22}}{X_{12}Y_{12}}\right)^{m+1} \frac{1}{\left(\frac{\delta Y_{22}-1}{\beta X_{12}-1}\right)^{m+1}}, \tag{97}$$

$$y_{6m+3} = y_{-3}\left(\frac{X_{11}Y_{11}}{X_{22}Y_{22}}\right)^{m+1} \frac{1}{\left(\frac{\beta X_{11}-1}{\beta X_{22}-1}\right)^{m+1}},\ y_{6m+4} = y_{-2}\left(\frac{X_{21}Y_{21}}{X_{11}Y_{11}}\right)^{m+1} \frac{1}{\left(\frac{\beta X_{21}-1}{\delta Y_{11}-1}\right)^{m+1}}, \tag{98}$$

$$y_{6m+5} = y_{-1}\left(\frac{X_{22}Y_{22}}{X_{12}Y_{12}}\right)^{m+1} \frac{1}{\left(\frac{\beta X_{22}-1}{\delta Y_{12}-1}\right)^{m+1}},\ y_{6m+6} = y_{0}\left(\frac{X_{11}Y_{11}}{X_{22}Y_{22}}\right)^{m+1} \frac{1}{\left(\frac{\delta Y_{11}-1}{\delta Y_{22}-1}\right)^{m+1}}, \tag{99}$$

for $i \in \{0,1\}$, $j = \overline{1,3}$. From (94) - (99) it is not difficult to investigate the asymptotic behaviour of well-defined solutions of the system (88) for case $\alpha = -1 = \gamma$.
Let

$$P_{j_1} := \beta X_{1j_1} - 1,\ Q_{j_1} := \delta\, Y_{1j_1} - 1,\ R_{j_1} := \beta X_{2j_1} - 1,\ S_{j_1} := \delta Y_{2j_1} - 1,$$

for $j_1 \in \{1,2\}$.

Theorem 2.10. *Assume that $\alpha = -1 = \gamma$, $\beta \neq 0 \neq \delta$, $X_{2j_1}Y_{2j_1} = X_{1j_1}Y_{1j_1}$ for $j_1 \in \{1,2\}$ and $(x_n, y_n)_{n\geq -5}$ is a well-defined solutions of system of* (19).

(a) If $X_{11} = \frac{2}{\beta}$, then the following statements are true.

(i) If $|S_1| < 1$, then $|x_{6m+4}| \to \infty$ and $|y_{6m+1}| \to \infty$ as $m \to \infty$.

(ii) If $|S_1| > 1$, then $x_{6m+4} \to 0$ and $y_{6m+1} \to 0$ as $m \to \infty$.

(b) If $X_{11} = \frac{2}{\beta}$, then the following statements are true.

(i) If $|R_2| > 1$, then $|x_{6m+6}| \to \infty$ and $|y_{6m+3}| \to \infty$ as $m \to \infty$.

(ii) If $|R_2| < 1$, then $x_{6m+6} \to 0$ and $y_{6m+3} \to 0$ as $m \to \infty$.

(c) If $Y_{12} = \frac{2}{\delta}$, then the following statements are true.

(i) If $|R_2| < 1$, then $|x_{6m+2}| \to \infty$ and $|y_{6m+5}| \to \infty$ as $m \to \infty$.

(ii) If $|R_2| > 1$, then $x_{6m+2} \to 0$ and $y_{6m+5} \to 0$ as $m \to \infty$.

(d) If $Y_{21} = \frac{2}{\delta}$, then the following statements are true.

(i) If $|P_1| > 1$, then $|x_{6m+4}| \to \infty$ and $|y_{6m+1}| \to \infty$ as $m \to \infty$.

(ii) If $|P_1| < 1$, then $x_{6m+4} \to 0$ and $y_{6m+1} \to 0$ as $m \to \infty$.

(e) If $X_{12} = \frac{2}{\beta}$, then the following statements are true.

(i) If $|S_2| < 1$, then $|x_{6m+5}| \to \infty$ and $|y_{6m+2}| \to \infty$ as $m \to \infty$.

(ii) If $|S_2| > 1$, then $x_{6m+5} \to 0$ and $y_{6m+2} \to 0$ as $m \to \infty$.

(f) If $X_{21} = \frac{2}{\beta}$, then the following statements are true.

(i) If $|Q_1| > 1$, then $|x_{6m+1}| \to \infty$ and $|y_{6m+4}| \to \infty$ as $m \to \infty$.

(ii) If $|Q_1| < 1$, then $x_{6m+1} \to 0$ and $y_{6m+4} \to 0$ as $m \to \infty$.

(g) If $Y_{11} = \frac{2}{\delta}$, then the following statements are true.

(i) If $|R_1| < 1$, then $|x_{6m+1}| \to \infty$ and $|y_{6m+4}| \to \infty$ as $m \to \infty$.

(ii) If $|R_1| > 1$, then $x_{6m+1} \to 0$ and $y_{6m+4} \to 0$ as $m \to \infty$.

(h) If $Y_{11} = \frac{2}{\delta}$, then the following statements are true.

(i) If $|S_2| > 1$, then $|x_{6m+3}| \to \infty$ and $|y_{6m+6}| \to \infty$ as $m \to \infty$.

(ii) If $|S_2| < 1$, then $x_{6m+3} \to 0$ and $y_{6m+3} \to 0$ as $m \to \infty$.

(j) If $Y_{22} = \frac{2}{\delta}$, then the following statements are true.

(i) If $|Q_1| < 1$, then $|x_{6m+3}| \to \infty$ and $|y_{6m+6}| \to \infty$ as $m \to \infty$.

(ii) *If* $|Q_1| > 1$, *then* $x_{6m+3} \to 0$ *and* $y_{6m+6} \to 0$ *as* $m \to \infty$.

(k) *If* $Y_{22} = \frac{2}{\delta}$, *then the following statements are true.*

(i) *If* $|P_2| > 1$, *then* $|x_{6m+5}| \to \infty$ *and* $|y_{6m+2}| \to \infty$ *as* $m \to \infty$.

(ii) *If* $|P_2| < 1$, *then* $x_{6m+5} \to 0$ *and* $y_{6m+2} \to 0$ *as* $m \to \infty$.

(l) *If* $X_{22} = \frac{2}{\beta}$, *then the following statements are true.*

(i) *If* $|P_1| < 1$, *then* $|x_{6m+6}| \to \infty$ *and* $|y_{6m+3}| \to \infty$ *as* $m \to \infty$.

(ii) *If* $|P_1| > 1$, *then* $x_{6m+6} \to 0$ *and* $y_{6m+3} \to 0$ *as* $m \to \infty$.

(m) *If* $X_{22} = \frac{2}{\beta}$, *then the following statements are true.*

(i) *If* $|Q_2| > 1$, *then* $|x_{6m+2}| \to \infty$ *and* $|y_{6m+5}| \to \infty$ *as* $m \to \infty$.

(ii) *If* $|Q_2| < 1$, *then* $x_{6m+2} \to 0$ *and* $y_{6m+5} \to 0$ *as* $m \to \infty$.

3. Domain of undefinable solutions of system (3)

This section we proved that solutions of system (3) for which $x_{-j} = 0$, $y_{-j} = 0$ for some $j = \overline{1,5}$ are not defined. The forbidden set of the solutions of system (3) is defined with the following theorem. The next result characterizes the domain of undefinable solutions of system (3) for the cases $\alpha_n \neq 0$, $\beta_n \neq 0$, $\gamma_n \neq 0$, $\delta_n \neq 0$ for $n \in \mathbb{N}_0$.

Theorem 3.1. *Assume that* $\alpha_n \neq 0$, $\beta_n \neq 0$, $\gamma_n \neq 0$, $\delta_n \neq 0$, $n \in \mathbb{N}_0$. *Then the forbidden set of the solutions of system* (3) *is given by the set*

$$\begin{aligned}\Re = &\bigcup_{m\in\mathbb{N}_0}\bigcup_{k=0}^{2}\Big\{(x_{-5},...,x_{-1},y_{-5},...,y_{-1}) \in \mathbb{R}^{10} : x_{k-3}y_{k-4}x_{k-5} = \frac{1}{c_m}, y_{k-3}x_{k-4}y_{k-5} = \frac{1}{d_m},\\ &\text{when } c_m := -\sum_{s=0}^{m}\frac{\beta_{3s+k}}{\alpha_{3s+k}}\prod_{r=0}^{s-1}\frac{1}{\alpha_{3r+k}} \neq 0, d_m := -\sum_{s=0}^{m}\frac{\delta_{3s+k}}{\gamma_{3s+k}}\prod_{r=0}^{s-1}\frac{1}{\gamma_{3r+k}} \neq 0\Big\}\\ &\bigcup\bigcup_{j=1}^{5}\Big\{(x_{-5},...,x_{-1},y_{-5},...,y_{-1}) \in \mathbb{R}^{10} : x_{-j} = 0, y_{-j} = 0\Big\}.\end{aligned} \tag{100}$$

Proof. We have mentioned the set

$$\bigcup_{j=1}^{5}\Big\{(x_{-5},\dots,x_{-1},y_{-5},\dots,y_{-1}) \in \mathbb{R}^{10} : x_{-j} = 0, y_{-j} = 0\Big\}$$

belongs to the domain of undefinable solutions of (3).

Now we assume that $x_{-j} \neq 0$, $y_{-j} \neq 0$, $j = \overline{1,5}$. Note that system (3) is undefined if and only if

$$\alpha_n + \beta_n x_{n-3}y_{n-4}x_{n-5} = 0,\ \gamma_n + \delta_n y_{n-3}x_{n-4}y_{n-5} = 0,$$

that is,

$$x_{n-3}y_{n-4}x_{n-5} = -\frac{\alpha_n}{\beta_n},\ y_{n-3}x_{n-4}y_{n-5} = -\frac{\gamma_n}{\delta_n}, \tag{101}$$

for some $n \in \mathbb{N}_0$ (here we consider that $\beta_n \neq 0$ and $\delta_n \neq 0$, $n \in \mathbb{N}_0$). Since the change of variables (8) implies that system in (9) is transformed to the system (10), this along with (101) implies that the solution is not defined if

$$u_{3(m-1)+k} = -\frac{\beta_{3m+k}}{\alpha_{3m+k}},\ v_{3(m-1)+k} = -\frac{\delta_{3m+k}}{\gamma_{3m+k}}, \tag{102}$$

for some $m \in \mathbb{N}_0$ and $k = \overline{0,2}$.
Now we consider the following functions

$$f_{3m+k}(t) := \alpha_{3m+k}t + \beta_{3m+k}, \quad g_{3m+k}(t) := \gamma_{3m+k}t + \delta_{3m+k}, \tag{103}$$

for $m \in \mathbb{N}_0, k = \overline{0,2}$. Then

$$f^{-1}_{3m+k}(t) = \frac{t - \beta_{3m+k}}{\alpha_{3m+k}},\ g^{-1}_{3m+k}(t) = \frac{t - \delta_{3m+k}}{\gamma_{3m+k}}, \tag{104}$$

for $m \in \mathbb{N}_0$, $k = \overline{0,2}$, so that

$$f^{-1}_{3m+k}(0) = -\frac{\beta_{3m+k}}{\alpha_{3m+k}}, \quad g^{-1}_{3m+k}(0) = -\frac{\delta_{3m+k}}{\gamma_{3m+k}}, \tag{105}$$

for $m \in \mathbb{N}_0, k = \overline{0,2}$. Now we can write equations in (9) as

$$u_{3m+k} = f_{3m+k}(u_{3(m-1)+k}), \quad v_{3m+k} = g_{3m+k}(v_{3(m-1)+k}), \tag{106}$$

for $m \in \mathbb{N}_0, k = \overline{0,2}$. Then, we have

$$u_{3m+k} = f_{3m+k} \circ f_{3(m-1)+k} \circ \cdots \circ f_k(u_{k-3}), \quad v_{3m+k} = g_{3m+k} \circ g_{3(m-1)+k} \circ \cdots \circ g_k(v_{k-3}), \tag{107}$$

for $m \in \mathbb{N}_0$, $k = \overline{0,2}$. From (105) and (107) we have that (102) holds for some $m \in \mathbb{N}_0, k = \overline{0,2}$, if and only if

$$u_{k-3} = f_k^{-1} \circ \cdots \circ f^{-1}_{3m+k}(0),\ v_{k-3} = g_k^{-1} \circ \cdots \circ g^{-1}_{3m+k}(0),$$

that is,

$$u_{k-3} = -\sum_{s=0}^{m} \frac{\beta_{3s+k}}{\alpha_{3s+k}} \prod_{r=0}^{s-1} \frac{1}{\alpha_{3r+k}}, \quad v_{k-3} = -\sum_{s=0}^{m} \frac{\delta_{3s+k}}{\gamma_{3s+k}} \prod_{r=0}^{s-1} \frac{1}{\gamma_{3r+k}},$$

for some $m \in \mathbb{N}_0$, $k = \overline{0,2}$, which along with the relations

$$u_{k-3} = \frac{1}{x_{k-3}y_{k-4}x_{k-5}}, \quad v_{k-3} = \frac{1}{y_{k-3}x_{k-4}y_{k-5}}, k = \overline{0,2},$$

implies that the first union in Theorem 3.1 belongs to the domain of undefinable solutions too, as desired. □

4. Conclusion

In this study, the solutions of system (3) have obtained for the following particular cases:

(i) The general solutions of system (3) are given by formulas in (17) and (18).

(ii) If α_n, β_n, γ_n and δ_n are constant, then the general solutions of system (19) is given by formulas in (20) and (21).

(iii) If $\alpha \neq 1$, $\gamma \neq 1$, then the general solutions of system (19) is given by formulas in (22).

(iv) If $\alpha = 1$, $\gamma \neq 1$, then the general solutions of system (19) is given by formulas in (23).

(v) If $\alpha \neq 1$, $\gamma = 1$, then the general solutions of system (19) is given by formulas in (24).

(vi) If $\alpha = 1$, $\gamma = 1$, then the general solutions of system (19) is given by formulas in (25).

Moreover, the asymptotic behaviors of the solutions of system (19) have investigated. Finally, the forbidden set of the solutions of system (3) is obtained.

References

[1] Y. Akrour, M. Kara, N. Touafek, Y. Yazlık, *Solutions formulas for some general systems of difference equations*, Miskolc Math. Notes. **22**(2021), 529–555.

[2] Ö. Aktas, M. Kara, Y. Yazlık, *On a solvable difference equations system*, Filomat. **39**(2025), 2229-2267.

[3] M. M. Alzubaidi, E.M. Elsayed,*Analytical and solutions of fourth order difference equations*, Commun. in Adv. Math. Sci. **2**(2019), 9-21.

[4] S. Etemad, M.A. Ragusa, S. Rezapour, A. Zada, *Existence property of solutions for multi-order q-difference FBVPs based on condensing operators and end-point technique*, Fixed Point Theory. **25** (2024), 115–1421.

[5] A. Ghezal, *Note on a rational system of* $(4k + 4)-$ *order difference equations: Periodic solution and convergence*, J. Appl. Math. Comput. **69**(2023), 2207-2215.

[6] Y. Halim, M. Bayram, *On the solutions of a higher-order difference equation in terms of generalized Fibonacci sequences*, Math. Methods Appl. Sci. **39**(2016), 2974.

[7] M. Kara, Y. Yazlik, *Solvability of a system of nonlinear difference equations of higher Order*, Turkish J. Math. **43**(2019), 1533-1565.

[8] M. Kara, Y. Yazlik, N. Touafek, Y. Akrour, *On a three-dimensional system of difference equations with variable coefficients*, J. Appl. Math. Infor. **39**(2021), 381-403.

[9] M. Kara, Y. Yazlik, *On a solvable system of non-linear difference equations with variable coefficients*, J. Sci. Arts. **1**(2021), 145-162.

[10] M. Kara, *Solvability of a three-dimensional system of nonlinear difference equations*, Math. Sci. and Appl. E-Notes. **10**(2022), 1-15.

[11] D. Karakaya, Y. Yazlik, M. Kara, *On a solvable system of difference equations of sixth- order*, Miskolc Math. Notes. **24**(2023), 1405-1426.

[12] N. Song, *Normal forms for nonautonomous nonlinear difference systems under nonuniform dichotomy spectrum*, J. Funct. Spaces. **2024** (2024), 6656183.

[13] S. Stević, J. Diblik, B. Iricanin, Z. Smarda, *On a solvable system of rational difference equations*, J. Difference Equ. Appl. **20**(2014), 811-825.

[14] S. Stević, J. Diblik, B. Iricanin, Z. Smarda, *On a fifth-order difference equation*, J. Comput. Anal. Appl. **20**(2016), 1214-1227.

[15] N. Taskara, D. Tollu, Y. Yazlik, *Solutions of rational difference system of order three in terms of Padovan numbers*, J. Adv. Res. Appl. Math. **7**(2015), 18–29.

[16] N. Taskara, K. Uslu, D. T. Tollu, *The periodicity and solutions of the rational difference equation with periodic coefficients*, Comput. Math. Appl. **62**(2011), 1807-1813.

[17] D. T. Tollu, Y. Yazlik, N. Taşkara, *On a solvable nonlinear difference equation of higher order*, Turkish J. Math. **42**(2017), 1765-1778.

[18] D.T. Tollu, I. Yalcinkaya, *On solvability of a three-dimensional system of nonlinear difference equations*, Dyn. Contin. Discrete Impuls. Syst. Ser. B Appl. Algorithms. **29**(2022), 35-47.

[19] N. Touafek, *On a general system of difference equations defined by homogeneous functions*, Math. Slovaca. **71**(2021), 697-720.

[20] N. Touafek, E. M. Elsayed, *On the periodicity of some systems of nonlinear difference equations*, Bull. Math. Soc. Sci. Math. Roumanie. **55**(2012), 217–224.

[21] I. Yalcinkaya, H. Ahmad, D. T. Tollu, Y. M. Li, *On a system of k-difference equations of order three*, Math. Probl. Eng. **2020**(2020), 1-11.

[22] I. Yalcinkaya, H. El-Metwally, A. E. Hamza, *Periodic solutions for some systems of difference equations*, Konuralp J. Math. **8**(2020), 114-121.

[23] Y. Yazlik, *On the solutions and behavior of rational difference equations*, J. Comput. Anal. Appl. **17**(2014), 584-594.

[24] Y. Yazlik, M. Güngör, *On the solvable nonlinear difference equation of sixth-order*, Journal of Science and Arts. **47**(2019), 399-414.

[25] Y. Yazlik, M. C. Fidancı, M. K. Hassani, *Stability analysis of a three-dimensional system of difference equations with quadratic terms*, J. Appl. Math. Comput. **70**(2024), 2521-2539.