Filomat 39:29 (2025), 10461–10475
https://doi.org/10.2298/FIL2529461N

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Existence and multiplicity of solutions for nonlinear second-order differential equations with instantaneous and non-instantaneous impulses via variational methods

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Abstract. This paper explores a class of nonlinear second-order differential equations characterized by both instantaneous and non-instantaneous impulsive effects. We employ variational methods and critical point theory to investigate the existence and multiplicity of weak solutions. Specifically, we reformulate the problem as the minimization of an energy functional within an appropriate function space. Critical points of this energy function correspond to weak solutions of the impulsive problem under consideration. By imposing distinct growth conditions on the nonlinearities and impulsive functions, we rigorously establish the existence of at least one solution and infinitely many solutions for the considered problem. We present some examples to illustrate our results.

1. Introduction

Impulsive differential equations are a powerful tool for modeling dynamic systems that experience abrupt changes. These changes can represent real-world phenomena like sudden population shifts, power outages, the firing patterns of neurons, or economic jolts from policy interventions. Within this framework, two main types of impulses are studied. The first type deals with instantaneous impulses, where the state of the system changes rapidly compared to the overall timescale of the process. These have been extensively studied, with detailed discussions readily available in various reference works [6, 10, 11, 17, 18, 22, 23, 29]. However, instantaneous impulses have limitations. They cannot accurately capture events with a finite duration, such as earthquakes or tsunamis. To address this shortcoming, Hernández and O'Regan introduced the concept of non-instantaneous impulses in [16]. These impulses represent an effect that begins at a specific time and unfolds over a finite time interval, providing a more realistic description of certain real-world phenomena. This innovation has significantly enriched the field, as evidenced by subsequent work like the monograph [1].

2020 *Mathematics Subject Classification*. Primary 34A37; Secondary 34K45, 35J20.

Keywords. Impulsive differential equations, instantaneous impulses, non-instantaneous impulses, fractional equations, variational methods, energy functional, critical point theory.

Received: 16 August 2025; Revised: 01 September 2025; Accepted: 04 September 2025

Communicated by Maria Alessandra Ragusa

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In our research, we focus on a specific category of nonlinear second-order differential equations. These equations exhibit two distinct types of impulses: instantaneous and non-instantaneous. Our investigation centers around the following problem

$$\begin{cases} -x''(t)+\Lambda_i(t)x(t) &= \sigma_i(t,x(t)), \quad t\in(e_i,s_{i+1}],\ i=\overline{0,m},\\ x'(s_i^+)-x'(s_i^-) &= \Theta_i(x(s_i)), \quad i=\overline{1,m},\\ x'(t) &= x'(s_i^+), \quad t\in(s_i,e_i],\ i=\overline{1,m},\\ x'(e_i^+) &= x'(e_i^-), \quad i=\overline{1,m},\\ x(0) &= x(T)\ =\ 0. \end{cases} \tag{1}$$

Where $0=e_0<s_1<e_1<s_2<e_2<\ldots<s_m<e_m<s_{m+1}=T$, $\Lambda_i\in L^\infty(e_i,s_{i+1}]$, σ_i are Carathéodory functions defined on $(e_i,s_{i+1}]\times\mathbb{R}$, $x'(s_i^\pm)=\lim_{t\to s_i^\pm}x'(t)$, $\Theta_i\in C(\mathbb{R})$. In these equations, instantaneous impulses occur at points s_i, whereas non-instantaneous impulses persist over intervals $(s_i,e_i]$ while maintaining a constant derivative.

The existence of solutions to impulsive problems has been explored using various classical tools, including fixed point theory, the theory of analytic semigroups, the comparison method, topological degree theory, and variational methods. A plethora of research has investigated this area, as evidenced by works such as [5, 7, 8, 21, 27, 32, 33]. Notably, the first researchers to use variational methods to solve boundary value problems for impulsive differential equations were Tian and Ge [30] and Nieto and O'Regan [26].

Bai and Nieto [4] were the first to delve into the variational structure of general non-instantaneous impulsive problems. They introduced the concept of weak solutions for linear equations involving non-instantaneous impulses

$$\begin{cases} -x''(t) &= \sigma_i(t), \quad t\in(e_i,s_{i+1}],\ i=\overline{0,m},\\ x'(t) &= \alpha_i, \quad t\in(s_i,e_i],\ i=\overline{1,m},\\ x'(e_i^+) &= x'(e_i^-), \quad i=\overline{1,m},\\ x'(0) &= \alpha_0,\\ x(0) &= x(T)\ =\ 0. \end{cases} \tag{2}$$

In this formulation, impulses commence abruptly at points s_i, maintaining a constant derivative over finite intervals $(s_i,e_i]$. Here, $\sigma_i\in L^2(e_i,s_{i+1})$, and α_i are predefined constants. Utilizing the classical Lax-Milgram theorem, the authors unveiled the variational nature of the problem. Through this approach, they established the existence and uniqueness of weak solutions as critical points.

The previous study allowed dealing with the corresponding nonlinear problems. The same authors with Wang [5] considered a class of nonlinear differential equations with non-instantaneous impulses

$$\begin{cases} -x''(t) &= \partial_x F_i\Big(t,x(t)-x(t_{i+1})\Big), \quad t\in(e_i,s_{i+1}],\ i=\overline{0,m},\\ x'(t) &= \alpha_i, \quad t\in(s_i,e_i],\ i=\overline{1,m},\\ x'(e_i^+) &= x'(e_i^-), \quad i=\overline{1,m},\\ x'(0) &= \alpha_0,\\ x(0) &= x(T)\ =\ 0. \end{cases} \tag{3}$$

The nonlinear functions $\partial_x F_i(t,x)$ represent the derivatives of $F_i(t,x)$ with respect to x. The authors employ variational methods and critical point theory to establish a criterion ensuring that problem (3) possesses at least two distinct nonzero bounded weak solutions.

In contrast, based on the study conducted on problem (2), Tian and Zhang [31] incorporated instantaneous impulses into non-instantaneous impulsive differential equations. They expanded linear terms into nonlinear ones to address the following second-order differential equations featuring simultaneous instantaneous and non-instantaneous impulses

$$\begin{cases} -x''(t) = \sigma_i(t, x(t)), & t \in (e_i, s_{i+1}],\ i = \overline{0, m}, \\ x'(s_i^+) - x'(s_i^-) = \Theta_i(x(s_i)), & i = \overline{1, m}, \\ x'(t) = x'(s_i^+), & t \in (s_i, e_i],\ i = \overline{1, m}, \\ x'(e_i^+) = x'(e_i^-), & i = \overline{1, m}, \\ x(0) = x(T) = 0. \end{cases} \tag{4}$$

In this scenario, instantaneous impulses take place at the points s_i, while non-instantaneous impulses persist over the intervals $(s_i, e_i]$. Through the application of Ekeland's variational principle, they established the existence of classical solutions to the formulated problem.

Building upon insights gained from the referenced works ((2), (3), and (4)), and inspired by the aforementioned facts, our objective is to extend the scope of problem (4). We intend to accomplish this by introducing perturbation terms, thereby establishing and addressing the problem articulated in (1). We demonstrate that the quest to ascertain the existence and multiplicity of solutions of problem (1) can effectively be reframed as an equivalent problem of minimizing some energy functional within a suitable function space. In this context, the critical points of this energy function correspond to solutions of the impulsive problem under consideration. Assuming distinct growth conditions for the nonlinearities and impulsive functions, we establish the existence of at least one solution and infinitely many solutions for (1).

In concluding this section, we highlight the investigations conducted by Zhang and Liu [38] and Zhou *et al* [40]. They explored fractional differential equations with both instantaneous and non-instantaneous impulses, expanding upon the findings of [31], and utilized variational methods to derive solutions. For further recent works, interested readers can refer to, for instance, [2, 3, 20, 35, 37, 39], and explore the references therein. Additionally, those interested in examining coupled systems of differential equations with impulsive effects may find relevant insights in works such as [25].

The remainder of the paper is structured as follows: Section two outlines the necessary assumptions and presents the principal results. Section three covers essential preliminaries to provide background information. Section four details the framework of our problem, and the final section presents the proof of the principal results. Additionally, two examples are provided to illustrate our main findings.

2. Assumptions and main results

Throughout the paper, we make the following set of assumptions.

(A) Assuming that

$$\forall i = \overline{0, m} :\ \nu_i := \operatorname{ess\,inf}_{t \in (e_i, s_{i+1}]} \Lambda_i(t) > -\lambda_1,$$

where $\lambda_1 := \frac{\pi^2}{T^2}$ is the first eigenvalue of the Dirichlet problem

$$\begin{cases} -x''(t) = \lambda x(t),\ t \in (0, T), \\ x(0) = x(T) = 0. \end{cases} \tag{5}$$

(B_1) There exist $a_i, b_i > 0$, and $\mu_i \in [0, 1)$ for $i = \overline{0, m}$, such that for all $(t, x) \in (e_i, s_{i+1}] \times \mathbb{R}$:

$$|\sigma_i(t, x)| \leq a_i + b_i |x|^{\mu_i}.$$

(B_2) There exist $c_i, d_i > 0$, and $\eta_i \in [0, 1), i = \overline{1, m}$, such that for all $x \in \mathbb{R}$:

$$|\Theta_i(x)| \leq c_i + d_i |x|^{\eta_i}.$$

(C_1) (I) There exists $\alpha > 2$ such that for all $i = \overline{1, m}$ and for all $x \in \mathbb{R}$:

$$0 \leq \Theta_i(x) x \leq \alpha \int_0^x \Theta_i(t) dt.$$

(II) There exist $\delta_i > 0$ for $i = \overline{1,m}$, such that for all $x \in \mathbb{R}$:

$$\int_0^x \Theta_i(t)dt \leq \delta_i |x|^\alpha.$$

(C_2) For all $i = \overline{0,m}$, we define $P_i(t,x) = \int_0^x \sigma_i(t,s)ds$. We assume that the functions P_i and σ_i satisfy the following conditions

(i) There exist $R > 0, K > 0$, and $2 < \alpha < \beta$ such that for a.e. $t \in (e_i, s_{i+1}]$ and for all $x \in \mathbb{R}$:

$$|x| \geq R \Longrightarrow P_i(t,x) \geq K|x|^\beta.$$

(ii) For all $t \in (e_i, s_{i+1}]$ and $x \in \mathbb{R}$, we have

$$\alpha P_i(t,x) \leq x\sigma_i(t,x).$$

(iii) For all $t \in (e_i, s_{i+1}]$ and $x \in \mathbb{R}$, then

$$\sigma_i(t,x) = o(|x|) \text{ as } x \longrightarrow 0.$$

(D_1) For all $i = \overline{0,m}$, $\sigma_i(t,x)$ is an odd function with respect to the variable x.

(D_2) For all $i = \overline{1,m}$, the function Θ_i is odd.

(E) For all $i = \overline{1,m}$, Θ_i is non-decreasing.

The following theorems summarize the main findings of this study.

Theorem 2.1. *Assume that the conditions (A), (B_1) and (B_2) are fulfilled. Under these assumptions, the problem (1) possesses at least one weak solution.*

Theorem 2.2. *Let (A) and $(C_1), (C_2)$ be assumed true. In such a scenario, the problem (1) possesses at least one weak solution.*

The subsequent results relate to the existence of an infinite number of solutions.

Theorem 2.3. *Assume that the conditions (A), $(C_1), (C_2)$ and $(D_1), (D_2)$ are satisfied. Under these assumptions, the problem (1) admits an infinite number of weak solutions.*

Theorem 2.4. *Under the assumptions (A), (B_2), (C_2), (D_1), (D_2) and (E), the problem (1) exhibits an infinite set of weak solutions.*

3. Prerequisites

This section provides a concise overview of key tools in differential calculus and critical point theory that will be utilized throughout this paper. For a more comprehensive review, we recommend referring to the sources [24, 28].

We use the symbol X to represent a real Banach space.

Definition 3.1. *(Weakly lower semicontinuous) A functional $F : X \longrightarrow \mathbb{R}$ is said to be weakly lower semicontinuous if, for any sequence $(x_j) \subset X$ such that $x_j \rightharpoonup x$ weakly in X, we have*

$$\liminf_{j \longrightarrow \infty} F(x_j) \geq F(x).$$

Remark 3.2. *[24, Th. 1.2.] If F is continuous and convex on X, then F is weakly lower semicontinuous. As an illustrative application, consider the norm, which is a weakly lower semicontinuous function. Specifically, for any sequence $(x_j) \subset X$ converging weakly to x, the following inequality holds*

$$\liminf_{j \longrightarrow \infty} \|x_j\|_X \geq \|x\|_X.$$

Definition 3.3. *(Minimizing sequence) A sequence $(x_j) \subset X$ is termed a minimizing sequence for the functional $F : X \longrightarrow \mathbb{R}$ if it satisfies the following condition*

$$\inf_{x \in X} F(x) = \lim_{j \to \infty} F(x_j).$$

Definition 3.4 (Coercive). *A functional $F : X \longrightarrow \mathbb{R}$ is said to be coercive if for all $x \in X$,*

$$F(x) \longrightarrow +\infty \quad \text{if} \quad \|x\|_X \longrightarrow +\infty.$$

The following theorem represents a classical result in the calculus of variations, ensuring the existence of a critical point for a given functional F defined on a reflexive Banach space.

Theorem 3.5. *[24, Th. 1.1.] In a reflexive Banach space X, if a function F is weakly lower semicontinuous and possesses a sequence that minimizes it and is bounded, then F reaches its minimum on X.*

Remark 3.6. *[24, Page 4] The existence of a bounded minimizing sequence is particularly guaranteed when F is coercive.*

The mountain pass theorem is a fundamental result in the calculus of variations, establishing the existence of a critical point for a functional F defined on a Banach space X, subject to certain conditions. Let B_r denote the open ball in X with a radius r centered at the origin, and let ∂B_r denote its boundary. The Palais-Smale condition (a compactness condition) an essential concept for the theorem, is defined as follows

Definition 3.7 (Palais-Smale condition (PS)). *[24, Def. 6.3.] The Palais-Smale condition is satisfied if, for any sequence (x_j) in X with $\big(F(x_j)\big)$ bounded and $F'(x_j)$ (the derivative of F at x_j) tending to zero, there exists a convergent subsequence.*

Theorem 3.8 (Mountain Pass Theorem). *[28, Page 4] Assume that F satisfies the following conditions*

1. *$F \in C^1(X, \mathbb{R})$ with $F(0) = 0$.*
2. *F satisfies the Palais-Smale condition (PS).*
3. *There exists $r, \rho > 0$ such that $F_{|\partial B_r} \geq \rho$.*
4. *There is an $x_0 \in X \backslash B_r$ such that $F(x_0) \leq 0$.*

Then F possesses a critical value.

Theorem 3.9 (Symmetric Mountain Pass Theorem). *[28, Page 5] Assume that F is defined on an infinite-dimensional real Banach space X, with the following conditions*

1. *$F \in C^1(X, \mathbb{R})$ with $F(0) = 0$.*
2. *F satisfies (PS).*
3. *F is an even functional.*
4. *Assume that $X = Y \bigoplus Z$, with Y being of finite dimension.*
5. *There exists $r, \rho > 0$ such that $F_{|\partial B_r \cap Z} \geq \rho$.*
6. *Whenever W is a finite-dimensional subspace of X, there exists a specific constant $\varsigma = \varsigma(W)$ such that $F(x) \leq 0$ everywhere on $W \backslash B_\varsigma$.*

Then, F attains a sequence of critical values that is unbounded.

4. Functional spaces and norms

We begin by defining the functional space relevant to our problem. We will use the Sobolev space denoted by $X = H_0^1(0, T)$ equipped with inner products

$$
\begin{aligned}
(x, y)_1 &= \int_0^T \Big[x(t)y(t) + x'(t)y'(t)\Big]dt,\\
(x, y)_2 &= \int_0^T x'(t)y'(t)\,dt,
\end{aligned}
$$

which naturally leads to the associated norms

$$
\begin{aligned}
\|x\|_1 &= \left(\int_0^T \Big[|x(t)|^2 + \big|x'(t)\big|^2\Big]dt\right)^{\frac{1}{2}},\\
\|x\|_2 &= \left(\int_0^T \big|x'(t)\big|^2\,dt\right)^{\frac{1}{2}}.
\end{aligned}
$$

We introduce Poincaré's inequality, expressed as

$$
\forall x \in H_0^1(0, T): \ \lambda_1 \int_0^T x^2(t)dt \leq \int_0^T \big|x'(t)\big|^2 dt, \tag{6}
$$

where $\lambda_1 = \frac{\pi^2}{T^2}$ represents the first eigenvalue associated with problem (5). This inequality readily implies the equivalence of the norms $\|\cdot\|_1$ and $\|\cdot\|_2$.

We now define a new norm, denoted by $\|\cdot\|_X$, that incorporates additional information specific to our problem. This norm is given by

$$
\|x\|_X \ = \ \left[\int_0^T \big|x'(t)\big|^2 dt + \sum_{i=0}^{m} \int_{e_i}^{s_{i+1}} \Lambda_i(t)x^2(t)dt\right]^{\frac{1}{2}}.
$$

This norm is well-defined. For any $x \in H_0^1(0, T)$, we can utilize assumption (A) along with Poincaré's inequality (6) to observe the following

$$
\begin{aligned}
\int_0^T \big|x'(t)\big|^2 dt + \sum_{i=0}^{m} \int_{e_i}^{s_{i+1}} \Lambda_i(t)x^2(t)dt &\geq \int_0^T \big|x'(t)\big|^2 dt + \sum_{i=0}^{m} \nu_i \int_{e_i}^{s_{i+1}} x^2(t)dt\\
&\geq \int_0^T \big|x'(t)\big|^2 dt - \lambda_1 \sum_{i=0}^{m} \int_{e_i}^{s_{i+1}} x^2(t)dt\\
&\geq \int_0^T \big|x'(t)\big|^2 dt - \lambda_1 \int_0^T x^2(t)dt\\
&\geq \int_0^T \big|x'(t)\big|^2 dt - \int_0^T \big|x'(t)\big|^2 dt\\
&= 0.
\end{aligned}
$$

This demonstrates the non-negativity of the expression.

Proposition 4.1. *If assumption (A) is satisfied, the norms $\|\cdot\|_2$ and $\|\cdot\|_X$ in the Sobolev space X are equivalent.*

Proof. Assuming (A) is satisfied, we have $\nu_i > -\lambda_1$, allowing us to find $\zeta_i \in (0,1)$ such that $\nu_i \geq -\lambda_1(1-\zeta_i)$. This implies $\nu_i \geq -\lambda_1(1-\zeta)$ for $i = \overline{0,m}$, where $\zeta = \min\{\zeta_i,\ i = \overline{0,m}\}$. Thus, for all $x \in X$, we get

$$\begin{aligned} \|x\|_X^2 &\geq \int_0^T \left|x'(t)\right|^2 dt + \sum_{i=0}^m \nu_i \int_{e_i}^{s_{i+1}} x^2(t)dt \\ &\geq \int_0^T \left|x'(t)\right|^2 dt - \lambda_1(1-\zeta) \sum_{i=0}^m \int_{e_i}^{s_{i+1}} x^2(t)dt \\ &\geq \int_0^T \left|x'(t)\right|^2 dt - \lambda_1(1-\zeta) \int_0^T x^2(t)dt. \end{aligned}$$

Applying Poincaré's inequality (6), we obtain

$$\|x\|_X^2 \geq \int_0^T \left|x'(t)\right|^2 dt - (1-\zeta) \int_0^T \left|x'(t)\right|^2 dt = \zeta \int_0^T \left|x'(t)\right|^2 dt,$$

leading to

$$\|x\|_X^2 \geq \zeta \|x\|_2^2.$$

On the other hand, considering $|\Lambda|_\infty = \max\left\{\|\Lambda_i\|_\infty,\ i = \overline{0,m}\right\}$. This implies

$$\begin{aligned} \|x\|_X^2 &\leq \int_0^T \left|x'(t)\right|^2 dt + |\Lambda|_\infty \sum_{i=0}^m \int_{e_i}^{s_{i+1}} x^2(t)dt \\ &\leq \int_0^T \left|x'(t)\right|^2 dt + |\Lambda|_\infty \int_0^T x^2(t)dt. \end{aligned}$$

Referring to (6), we additionally derive

$$\begin{aligned} \|x\|_X^2 &\leq \int_0^T \left|x'(t)\right|^2 dt + \frac{|\Lambda|_\infty}{\lambda_1} \int_0^T \left|x'(t)\right|^2 dt \\ &= \left(1 + \frac{|\Lambda|_\infty}{\lambda_1}\right) \|x\|_2^2. \end{aligned}$$

Thus, the norms $\|\cdot\|_2$ and $\|\cdot\|_X$ are equivalent. □

We conclude this section by defining $C[0,T]$ as the set of all continuous functions on $[0,T]$ with the natural norm

$$\|x\|_\infty = \max_{t\in[0,T]} |x(t)|.$$

Lemma 4.2. *There exists a positive constant γ such that for any $x \in H$, the following inequality holds*

$$\|x\|_\infty \leq \gamma \left\|x\right\|_X.$$

Proof. Utilizing the continuity of the injection from $H_0^1(0,T)$ into $C[0,T]$, and incorporating Proposition 4.1, we establish the desired result. □

5. Proof of the main results

Below, we prove the core results established earlier in this paper.

5.1. *Variational formula*

Building upon the ideas from the variational approach, for each $y \in X$. First, we have

$$\begin{aligned} -\int_0^T x''(t)y(t)dt &= -\sum_{i=0}^{m}\int_{e_i}^{s_{i+1}} x''(t)y(t)dt - \sum_{i=1}^{m}\int_{s_i}^{e_i} x''(t)y(t)dt \\ &= \int_0^T x'(t)y'(t)dt + \sum_{i=1}^{m}\Big(x'(s_i^+) - x'(s_i^-)\Big)y(s_i) + \sum_{i=1}^{m}\Big(x'(e_i^+) - x'(e_i^-)\Big)y(e_i) \\ &= \int_0^T x'(t)y'(t)dt + \sum_{i=1}^{m}\Theta_i\Big(x(s_i)\Big)y(s_i). \end{aligned} \tag{7}$$

On another note,

$$\begin{aligned} -\int_0^T x''(t)y(t)\,dt &= -\sum_{i=0}^{m}\int_{e_i}^{s_{i+1}} x''(t)y(t)dt - \sum_{i=1}^{m}\int_{s_i}^{e_i} x''(t)y(t)dt \\ &= \sum_{i=0}^{m}\int_{e_i}^{s_{i+1}} \sigma_i\Big(t, x(t)\Big)y(t)dt - \sum_{i=0}^{m}\int_{e_i}^{s_{i+1}} \Lambda_i(t)x(t)y(t)dt - \sum_{i=1}^{m}\int_{s_i}^{e_i} \frac{d}{dt}\Big(x'(s_i^+)\Big)y(t)dt \\ &= \sum_{i=0}^{m}\int_{e_i}^{s_{i+1}} \sigma_i\Big(t, x(t)\Big)y(t)dt - \sum_{i=0}^{m}\int_{e_i}^{s_{i+1}} \Lambda_i(t)x(t)y(t)dt. \end{aligned} \tag{8}$$

Consequently, combining (7) and (8), we derive

$$\int_0^T x'(t)y'(t)dt + \sum_{i=1}^{m}\Theta_i\Big(x(s_i)\Big)y(s_i) = \sum_{i=0}^{m}\int_{e_i}^{s_{i+1}} \sigma_i\Big(t, x(t)\Big)y(t)dt - \sum_{i=0}^{m}\int_{e_i}^{s_{i+1}} \Lambda_i(t)x(t)y(t)dt. \tag{9}$$

The equality (9) leads us to the introduction of a weak solution for problem (1).

Definition 5.1. *We say $x \in X$ is a weak solution to problem* (1) *if equation (9) is satisfied for every $y \in X$.*

Now, considering the energy functional $\Phi : X \longrightarrow \mathbb{R}$ corresponding to problem (1), defined as

$$\Phi(x) = \frac{1}{2}\int_0^T |x'(t)|^2 dt + \sum_{i=1}^{m}\int_0^{x(s_i)} \Theta_i(s)ds - \sum_{i=0}^{m}\int_{e_i}^{s_{i+1}} P_i\Big(t, x(t)\Big)dt + \frac{1}{2}\sum_{i=0}^{m}\int_{e_i}^{s_{i+1}} \Lambda_i(t)x^2(t)dt,$$

where $P_i\Big(t, x(t)\Big) = \int_0^{x(t)} \sigma_i(t,s)ds$. Alternatively, we have

$$\Phi(x) = \frac{1}{2}\|x\|_X^2 + \sum_{i=1}^{m}\int_0^{x(s_i)} \Theta_i(s)ds - \sum_{i=0}^{m}\int_{e_i}^{s_{i+1}} P_i\Big(t, x(t)\Big)dt. \tag{10}$$

Given the continuity of σ_i and Θ_i, we easily deduce that $\Phi \in C^1(X, \mathbb{R})$. The derivative of Φ at $x \in X$ is, for any $y \in X$, expressed as

$$\Phi'(x)(y) = \int_0^T x'(t)y'(t)dt + \sum_{i=1}^{m}\Theta_i\Big(x(s_i)\Big)y(s_i) - \sum_{i=0}^{m}\int_{e_i}^{s_{i+1}} \sigma_i\Big(t, x(t)\Big)y(t)dt + \sum_{i=0}^{m}\int_{e_i}^{s_{i+1}} \Lambda_i(t)x(t)y(t)dt. \tag{11}$$

This confirms that the weak solutions of problem (1) correspond to critical points of Φ, meaning that $x \in X$ is a weak solution of problem (1) if and only if $\Phi'(x)(y) = 0$ for all $y \in X$.

5.2. Proof of Theorem 2.1

Lemma 5.2. *The functional Φ defined by (10) is weakly lower semicontinuous and coercive.*

Proof. To demonstrate the weak lower semicontinuity of Φ, consider a sequence $(x_j) \subset X$ such that $x_j \rightharpoonup x$. By leveraging the compactness of the injection from $H_0^1(0,T)$ into $C[0,T]$, we observe uniform convergence of (x_j) to x over the interval $[0,T]$. Given that $\liminf_{j\to\infty} \|x_j\|_X \geq \|x\|_X$ (see Remark 3.2), we can conclude that

$$\begin{aligned}
\liminf_{j\longrightarrow\infty} \Phi(x_j) &= \liminf_{j\longrightarrow\infty}\Big(\frac{1}{2}\|x_j\|_X^2 + \sum_{i=1}^{m}\int_0^{x_j(s_i)}\Theta_i(s)ds - \sum_{i=0}^{m}\int_{e_i}^{s_{i+1}} P_i\big(t,x_j(t)\big)dt\Big)\\
&\geq \frac{1}{2}\|x\|_X^2 + \sum_{i=1}^{m}\int_0^{x(s_i)}\Theta_i(s)ds - \sum_{i=0}^{m}\int_{e_i}^{s_{i+1}} P_i\big(t,x(t)\big)dt\\
&= \Phi(x).
\end{aligned}$$

This implies that the functional Φ is weakly lower semicontinuous.

On the other hand, for any $x \in X$, by (B_1), (B_2) and Lemma 4.2, we have

$$\begin{aligned}
\Phi(x) &\geq \frac{1}{2}\|x\|_X^2 - \sum_{i=1}^{m}\int_0^{x(s_i)}\big(c_i + d_i|s|^{\eta_i}\big)ds - \sum_{i=0}^{m}\int_{e_i}^{s_{i+1}}\big(a_i|x(t)| + b_i|x(t)|^{\mu_i+1}\big)dt\\
&\geq \frac{1}{2}\|x\|_X^2 - m\max_i\{c_i\}\|x\|_\infty - \max_i\{d_i\}\sum_{i=1}^{m}\|x\|_\infty^{\eta_i+1} - \max_i\{a_i\}T\|x\|_\infty - \max_i\big\{b_i\big\}T\sum_{i=0}^{m}\|x\|_\infty^{\mu_i+1}\\
&\geq \frac{1}{2}\|x\|_X^2 - m\max_i\{c_i\}\gamma\|x\|_X - \max_i\{d_i\}\sum_{i=1}^{m}\gamma^{\eta_i+1}\|x\|_X^{\eta_i+1} - \max_i\{a_i\}T\gamma\|x\|_X\\
&- \max_i\big\{b_i\big\}T\sum_{i=0}^{m}\gamma^{\mu_i+1}\|x\|_X^{\mu_i+1}.
\end{aligned}$$

Since $\mu_i + 1, \eta_i + 1 < 2$, it follows that $\lim_{\|x\|_X\to\infty}\Phi(x) = \infty$. In simpler terms, Φ exhibits coercive behavior. □

Using Lemma 5.2 and Remark 3.6, it follows that the functional Φ fulfills all the requirements outlined in Theorem 3.5. Consequently, Φ possesses a minimum within H, which constitutes a critical point of Φ. Thus, problem (1) admits at least one solution.

Directly thereafter, we have the following corollary.

Corollary 5.3. *Under the fulfillment of assumption (A) and the boundedness of σ_i, $i = \overline{0,m}$, and Θ_i, $i = \overline{1,m}$, problem* (1) *possesses at least one solution.*

5.3. Proof of Theorem 2.2

The proof is structured across four segments. Firstly, by direct replacement we get easily $\Phi(0) = 0$. To show that $\Phi \in C^1(X,\mathbb{R})$, we need, for any fixed $x \in X$, to write the increment $\Phi(x+y) - \Phi(x)$ as the sum of a linear continuous form in $y \in X$, denoted $\Phi'(x)(y)$, plus a remainder $o(y)$, with

$$\lim_{\|y\|_X\to 0}\frac{|o(y)|}{\|y\|_X} = 0.$$

We compute

$$\begin{aligned}
\Phi(x+y) - \Phi(x) &= \frac{1}{2}\int_0^T\big(|x'(t)+y'(t)|^2 - |x'(t)|^2\big)dt + \sum_{i=1}^{m}\Big(\int_0^{x(s_i)+y(s_i)}\Theta_i(s)ds - \int_0^{x(s_i)}\Theta_i(s)ds\Big)\\
&-\sum_{i=0}^{m}\int_{e_i}^{s_{i+1}}\Big(P_i\big(t,x(t)+y(t)\big) - P_i\big(t,x(t)\big)\Big)dt + \frac{1}{2}\sum_{i=0}^{m}\int_{e_i}^{s_{i+1}}\Big(\Lambda_i(t)\big((x(t)+y(t))^2 - x^2(t)\big)\Big)dt.
\end{aligned}$$

Expanding the first integral gives

$$|x'(t)+y'(t)|^2-|x'(t)|^2=2x'(t)y'(t)+|y'(t)|^2,$$

so the linear contribution is $\int_0^T x'(t)y'(t)dt$, while the remainder is $\frac{1}{2}\int_0^T |y'(t)|^2dt$.
For the impulse terms we use the mean value theorem

$$\int_0^{x(s_i)+y(s_i)}\Theta_i(s)ds-\int_0^{x(s_i)}\Theta_i(s)ds=\Theta_i\Big(x(s_i)\Big)y(s_i)+o(|y(s_i)|).$$

Since $P_i\Big(t,x(t)\Big)=\int_0^{x(t)}\sigma_i(t,s)ds$, we obtain

$$P_i(t,x(t)+y(t))-P_i(t,x(t))=\sigma_i\Big(t,x(t)\Big)y(t)+o(|y(t)|).$$

Expanding the quadratic term yields

$$(x(t)+y(t))^2-x^2(t)=2x(t)y(t)+|y(t)|^2,$$

which gives the linear contribution $\int_{e_i}^{s_{i+1}}\Lambda_i(t)x(t)y(t)dt$.
Collecting all the linear parts we arrive at

$$\Phi'(x)(y)=\int_0^T x'(t)y'(t)dt+\sum_{i=1}^m\Theta_i\Big(x(s_i)\Big)y(s_i)-\sum_{i=0}^m\int_{e_i}^{s_{i+1}}\sigma_i\Big(t,x(t)\Big)y(t)dt+\sum_{i=0}^m\int_{e_i}^{s_{i+1}}\Lambda_i(t)x(t)y(t)dt,$$

and the remaining terms are of order $o(\|y\|_X)$.

Secondly, we establish that Φ fulfills the (PS) condition. Take a sequence $(x_j)\subset X$ with $\Big(\Phi(x_j)\Big)$ bounded and $\Phi'(x_j)\to 0$. Utilizing (10), (11), (C_1)-(I), and (C_2)-(ii) we conclude that

$$\begin{aligned}
\alpha\Phi\left(x_j\right)-\Phi'\left(x_j\right)(xj)=&\frac{\alpha}{2}\left\|x_j\right\|_X^2+\alpha\sum_{i=1}^m\int_0^{x_j(s_i)}\Theta_i(s)ds-\alpha\sum_{i=0}^m\int_{e_i}^{s_{i+1}}P_i\left(t,x_j\right)dt-\int_0^T\left|x_j'(t)\right|^2dt\\
&-\sum_{i=1}^m\Theta_i\left(x_j\left(s_i\right)\right)x_j\left(s_i\right)+\sum_{i=0}^m\int_{e_i}^{s_{i+1}}\sigma_i\left(t,x_j\right)x_jdt-\sum_{i=0}^m\int_{e_i}^{s_{i+1}}\Lambda_i(t)x_j^2(t)dt\\
=&\left(\frac{\alpha}{2}-1\right)\left\|x_j\right\|_X^2+\left(\alpha\sum_{i=1}^m\int_0^{x_j(s_i)}\Theta_i(s)ds-\sum_{i=1}^m\Theta_i\left(x_j\left(s_i\right)\right)x_j\left(s_i\right)\right)\\
&+\left(\sum_{i=0}^m\int_{e_i}^{s_{i+1}}\sigma_i\left(t,x_j\right)x_jdt-\alpha\sum_{i=0}^m\int_{e_i}^{s_{i+1}}P_i\left(t,x_j\right)dt\right)\\
\geq&\left(\frac{\alpha}{2}-1\right)\left\|x_j\right\|_X^2.
\end{aligned}$$

Assuming $\alpha>2$, we deduce that the sequence (x_j) is bounded in X. Furthermore, leveraging the reflexivity of $H_0^1(0,T)$ and the compact injection from $H_0^1(0,T)$ into $C[0,T]$, and if required, by passing to a subsequence, we can establish the existence of $(x_j)\in X$ such that

$$\begin{cases} x_j\rightharpoonup x & \text{in } X,\\ x_j\to x & \text{uniformly in } C[0,T],\\ \text{as} & j\to+\infty.\end{cases}$$

Therefore, as $j \to +\infty$, we acquire

$$\begin{aligned}&\sum_{i=1}^{m}\Big(\Theta_i\big(x_j(s_i)\big)-\Theta_i\big(x(s_i)\big)\Big)\Big(x_j(s_i)-x(s_i)\Big)\to 0,\\&\sum_{i=0}^{m}\int_{e_i}^{s_{i+1}}\Big(\sigma_i(t,x_j)-\sigma_i(t,x)\Big)(x_j-x)dt\to 0.\end{aligned}\tag{12}$$

Moreover, by (11), we have

$$\begin{aligned}\Big(\Phi'\left(x_j\right)-\Phi'(x)\Big)\left(x_j-x\right)=\left\|x_j-x\right\|_X^2+&\sum_{i=1}^{m}\Big(\Theta_i\big(x_j(s_i)\big)-\Theta_i\big(x(s_i)\big)\Big)\Big(x_j(s_i)-x(s_i)\Big)\\&-\sum_{i=0}^{m}\int_{e_i}^{s_{i+1}}\Big(\sigma_i(t,x_j)-\sigma_i(t,x)\Big)\Big(x_j-x\Big)dt.\end{aligned}\tag{13}$$

Given that $\Phi'\left(x_j\right)\to 0$ and $x_j \rightharpoonup x$, we can infer that

$$\Big(\Phi'(x_j)-\Phi'(x)\Big)(x_j-x)\to 0,\quad \text{as } j\to+\infty.\tag{14}$$

Hence, (12), (13), and (14) lead to $\left\|x_j-x\right\|_X \to 0$ as $j \to +\infty$. In other words, $x_j \to x$ in X, indicating the satisfaction of the (PS) condition by Φ.

Thirdly, we confirm that Φ meets the requirement stated in assumption 3 of Theorem 3.8. The Sobolev embedding theorem asserts the existence of a positive constant $\kappa > 0$ such that, for any $x \in X$, the following inequality holds

$$\|x\|_{L^2}^2 \le \kappa\|x\|_X^2.\tag{15}$$

Utilizing condition (C_2)-(iii), it follows that

$$P_i(t,x)=o\left(x^2\right),\quad \text{as } x\to 0,$$

for the proof, please refer to Appendix.
Let $\varepsilon=\frac{1}{4\kappa}$, then there exists $\tau > 0$ such that for any $x \in X$, the inequality

$$|P_i(t,x)|\le \frac{1}{4\kappa}x^2,\tag{16}$$

holds whenever $|x| < \tau$.
Furthermore, based on condition (C_1)-(I), we deduce

$$\int_0^{x(s_i)}\Theta_i(s)ds\ge 0.\tag{17}$$

It is evident that the condition $\|x\|_X \le \frac{\tau}{\gamma}$, with γ as defined in Lemma 4.2, implies $\|x\|_\infty < \tau$. Consequently, this leads to $|x(t)| < \tau$ for all $t \in [0,T]$. Hence, employing (10), (15), (16), and (17), we can conclude that

$$\begin{aligned}\Phi(x)=&\frac{1}{2}\|x\|_X^2+\sum_{i=1}^{m}\int_0^{x(s_i)}\Theta_i(s)ds-\sum_{i=0}^{m}\int_{e_i}^{s_{i+1}}P_i\Big(t,x(t)\Big)dt\\\ge&\frac{1}{2}\|x\|_X^2-\frac{1}{4\kappa}\sum_{i=0}^{m}\int_{e_i}^{s_{i+1}}x^2(t)dt\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{1}{2}\|x\|_X^2 - \frac{1}{4\kappa}\int_0^T x^2(t)dt \\
&= \frac{1}{2}\|x\|_X^2 - \frac{1}{4\kappa}\|x\|_{L^2}^2 \\
&\geq \frac{1}{4}\|x\|_X^2.
\end{aligned}
$$

We set $r = \frac{\tau}{\gamma}$ and $\rho = \frac{\tau^2}{4\gamma^2}$. With this selection, we can ensure that $\Phi(x) \geq \rho > 0$ for any $x \in \partial B_r$.

Finally, we demonstrate the satisfaction of assumption 4 of Theorem 3.8. Consider any $x \in X\backslash\{0\}$, we define $\chi_i = \Big\{t \in (e_i, s_{i+1}] \;/\; |x(t)| \neq 0\Big\}$, $\chi_i^c = \Big\{t \in (e_i, s_{i+1}] \;/\; |x(t)| = 0\Big\}$. Let Υ be a positive real number considered large enough. Utilizing (C_2)-(i), we obtain

$$
\begin{aligned}
\int_{e_i}^{s_{i+1}} P_i\Big(t, \Upsilon x(t)\Big)dt &= \int_{\chi_i} P_i\Big(t, \Upsilon x(t)\Big)dt + \int_{\chi_i^c} P_i(t, 0)dt \\
&= \int_{\chi_i} P_i\Big(t, \Upsilon x(t)\Big)dt \\
&\geq K\int_{\chi_i} |\Upsilon x(t)|^\beta dt. \qquad (18)
\end{aligned}
$$

Next, using (C_1)-(II) and Lemma 4.2, and taking (18) into account, we get

$$
\begin{aligned}
\Phi(\Upsilon x) &= \frac{1}{2}\|\Upsilon x\|_X^2 + \sum_{i=1}^m \int_0^{\Upsilon x(s_i)} \Theta_i(s)ds - \sum_{i=0}^m \int_{e_i}^{s_{i+1}} P_i\Big(t, \Upsilon x(t)\Big)dt \\
&\leq \frac{\Upsilon^2}{2}\|x\|_X^2 + \Upsilon^\alpha \sum_{i=1}^m \delta_i\,|x(s_i)|^\alpha - K\Upsilon^\beta \sum_{i=0}^m \int_{\chi_i} |x(t)|^\beta dt \\
&\leq \frac{\Upsilon^2}{2}\|x\|_X^2 + \|x\|_\infty^\alpha \Upsilon^\alpha \sum_{i=1}^m \delta_i - K\Upsilon^\beta \sum_{i=0}^m \int_{\chi_i} |x(t)|^\beta dt \\
&\leq \frac{\Upsilon^2}{2}\|x\|_X^2 + \gamma^\alpha\|x\|_X^\alpha \Upsilon^\alpha \sum_{i=1}^m \delta_i - K\Upsilon^\beta \sum_{i=0}^m \int_{\chi_i} |x(t)|^\beta dt. \qquad (19)
\end{aligned}
$$

Now, consider any element $x \in X$ where $\|x\|_X = \sqrt{2}$. Applying (19), we have

$$
\Phi(\Upsilon x) \leq \Upsilon^2 + \left((\sqrt{2}\gamma)^\alpha \sum_{i=1}^m \delta_i\right)\Upsilon^\alpha - \left(K\sum_{i=0}^m \int_{\chi_i} |x(t)|^\beta dt\right)\Upsilon^\beta,
$$

given that $2 < \alpha < \beta$, this inequality implies that $\lim_{\Upsilon\to+\infty}\Phi(\Upsilon x) = -\infty$. Consequently, there exists $\Upsilon_0 \in \mathbb{R}_+^\star$ with $\Upsilon_0 > r$ such that $\Phi(\Upsilon_0 x) \leq 0$. According to Theorem 3.8, there is at least one solution for problem (1).

5.4. Proof of Theorem 2.3

To establish this result, we utilize Theorem 3.9. Drawing from the proof of Theorem 2.2, we conclude that $\Phi \in C^1(X, \mathbb{R})$ and $\Phi(0) = 0$, fulfilling the (PS) condition. Moreover, the conditions (D_1) and (D_2) together signify that Φ exhibits even symmetry.

The equation (5) yields eigenvalues expressed as a sequence of positive numbers $\lambda_n = \left(\frac{n\pi}{T}\right)^2$ for $n = 1, 2, \ldots$. Denoting E_n as the feature space associated with λ_n. Consequently, we have $X = H_0^1(0,T) = \overline{\bigoplus_{n=1}^{+\infty} E_n}$.

Now, let $Y = \bigoplus_{n=1}^{2} E_n$ and $Z = \overline{\bigoplus_{n=3}^{+\infty} E_n}$. This implies $X = Y \bigoplus Z$, where Y is of finite dimension. As observed in the proof of Theorem 2.2, there exist constants r and $\rho > 0$ such that $\Phi(x) \geq \rho$ for any $x \in \partial B_r \cap Z$.

Additionally, following a similar argument as in the proof of Theorem 2.2, for any x in a finite-dimensional subspace $W \subset X$, we can deduce that $\lim_{\Upsilon\to+\infty} \Phi(\Upsilon x) = -\infty$. Consequently, there exists $\varsigma = \varsigma(W)$ such that $\Phi(\Upsilon x) \leq 0$ on $W \backslash B_\varsigma$.

By applying Theorem 3.9, we can assert that problem (1) admits an infinite number of solutions.

5.5. Proof of Theorem 2.4

We begin by noting that $\Phi \in C^1(X, \mathbb{R})$ with $\Phi(0) = 0$ is an even function. Our first task is to establish that Φ fulfills the (PS) condition. Drawing from the proof of Theorem 2.2 and utilizing (10), (11), (B_2), (C_2)-(ii), and Lemma 4.2, we can express the inequality

$$\begin{aligned}
\alpha\Phi\left(x_j\right) - \Phi'\left(x_j\right)\left(x_j\right) &= \left(\frac{\alpha}{2} - 1\right)\left\|x_j\right\|_X^2 + \alpha\sum_{i=1}^{m}\int_0^{x_j(s_i)}\Theta_i(s)ds - \sum_{i=1}^{m}\Theta_i\!\left(x_j\left(s_i\right)\right)x_j\left(s_i\right) \\
&\quad + \left(\sum_{i=0}^{m}\int_{e_i}^{s_{i+1}}\sigma_i\left(t, x_j\right)x_j dt - \alpha\sum_{i=0}^{m}\int_{e_i}^{s_{i+1}}P_i\left(t, x_j\right)dt\right) \\
&\geq \left(\frac{\alpha}{2} - 1\right)\left\|x_j\right\|_X^2 - \alpha\sum_{i=1}^{m}\left(c_i\left\|x_j\right\|_\infty + d_i\left\|x_j\right\|_\infty^{\eta_i+1}\right) - \sum_{i=1}^{m}\left(c_i\left\|x_j\right\|_\infty + d_i\left\|x_j\right\|_\infty^{\eta_i+1}\right) \\
&\geq \left(\frac{\alpha}{2} - 1\right)\left\|x_j\right\|_X^2 - (\alpha+1)\left(\sum_{i=1}^{m}c_i\gamma\left\|x_j\right\|_X + \sum_{i=1}^{m}d_i\gamma^{\eta_i+1}\left\|x_j\right\|_X^{\eta_i+1}\right).
\end{aligned}$$

This confirms that (x_j) is bounded in X. The remainder of the proof, demonstrating the satisfaction of the (PS) condition, closely parallels that in Theorem 2.2.

Secondly, considering the odd and non-decreasing nature of Θ_i, $i = \overline{1, m}$, it follows that $\int_0^{x(s_i)}\Theta_i(s)ds \geq 0$. By employing similar reasoning as in the proofs of Theorems 2.2 and 2.3, we can readily confirm the satisfaction of condition 5 in Theorem 3.9.

Finally, the proof of condition 6 of Theorem 3.9 mirrors that in Theorem 2.3, with assumption (B_2) replacing (C_1)-(II). Thus, according to Theorem 3.9, problem (1) possesses infinitely many solutions.

Examples

Following are two examples demonstrating the practical use of our findings.

Example 1.

Examine the problem defined for $T = 1$ involving both instantaneous and non-instantaneous impulses

$$\begin{cases}
-x''(t) + \Lambda_i(t)x(t) = \sigma_i(t, x(t)), & t \in (e_i, s_{i+1}],\ i = 0, 1, \\
x'(s_1^+) - x'(s_1^-) = \Theta_1(x(s_1)), & \\
x'(t) = x'(s_1^+), & t \in (s_1, e_1], \\
x'(e_1^+) = x'(e_1^-), & \\
x(0) = x(1) = 0, &
\end{cases} \tag{20}$$

where,

$$\begin{aligned}
&\Lambda_i(t) = (t - e_i) + (t - e_i)^2, \\
&\sigma_i(t, x) = e^{-t^2} - \sin(x) + |x|^{\frac{1}{2}}, \\
&\Theta_1(x) = 3\sin(x) + |x|^{\frac{2}{3}}.
\end{aligned}$$

Firstly, it is evident that $\nu_i = 0 > -\lambda_1$, thus satisfying assumption (A). Secondly, for $a_i = 2$, $b_i = 1$, and $\mu_i = \frac{1}{2}$, condition (B_1) is met. Finally, with $c_1 = 3$, $d_1 = 1$, and $\eta_1 = \frac{2}{3}$, we satisfy (B_2). Therefore, by Theorem 2.1, the impulsive problem (20) has at least one solution.

Example 2.

Consider problem (20) with the following parameter definitions

$$\begin{aligned}\Lambda_i(t) &= (t-e_i)+(t-e_i)^2,\\ \sigma_i(t,x) &= x^5,\\ \Theta_1(x) &= x^3.\end{aligned}$$

Clearly, condition (A) holds. By selecting $\alpha = 4$ and $\delta_i = \frac{1}{4}$, condition (C_1) is satisfied. Since $P_i(t,x) = \frac{x^6}{6}$, taking $K = \frac{1}{6}$ and $\beta = 6$ ensures condition (C_2) is met. Hence, problem (20) has at least one solution by Theorem 2.2. Furthermore, conditions (D_1) and (D_2) are fulfilled. Applying Theorem 2.3, it can be concluded that problem (20) possesses an infinite number of solutions.

Appendix

Lemma 5.4. *Under the condition* (C_2)*-(iii), we have*

$$P_i(t,x) = o\left(x^2\right), \quad \text{as } x \to 0.$$

Proof. Consider any $x \in \mathbb{R}^\star$. Using the variable change $s = xh$, along with the assumption (C_2)-(iii), we obtain

$$\begin{aligned}P_i(t,x) &= \int_0^x \sigma_i(t,s)ds\\ &= x\int_0^1 \sigma_i(t,xh)dh\\ &= x\int_0^1 o(|x|)dh\\ &= xo(|x|)\\ &= o(x^2).\end{aligned}$$

Thus, we have shown that $P_i(t,x)$ is indeed $o(x^2)$ as $x \to 0$. The lemma is thereby proven. □

References

[1] R.P. Agarwal, S. Hristova, D. O'Regan, *Non-instantaneous Impulses in Differential Equations*, Springer, 2017.

[2] H.M. Ahmed, A.M.S. Ahmed, M.A. Ragusa, *On some non-instantaneous impulsive differential equations with fractional brownian motion and Poisson jumps*, TWMS Journal of Pure and Applied Mathematics, **14** (1) (2023), 125–140.

[3] S. Aslan, A.O. Akdemir, New estimations for quasi-convex functions and $(h,m)-$convex functions with the help of Caputo-Fabrizio fractional integral operators, Electronic Journal of Applied Mathematics, **1**, (2023).

[4] L. Bai, J.J. Nieto, *Variational approach to differential equations with not instantaneous impulses*, Appl. Math. Lett. **73** (2017), 44–48.

[5] L. Bai, J.J. Nieto, X. Wang, *Variational approach to non-instantaneous impulsive nonlinear equations*. J. Nonlinear Sci. Appl. **10** (2017), 2440–2448.

[6] M. Benchohra, J. Henderson, S.K. Ntouyas, *Impulsive Differential Equations and Inclusions*, Hindawi Publishing Corporation, New York, 2006.

[7] M. Benchohra, J.J. Nieto, A. Ouahab, *Impulsive differential inclusions via variational method*, Georgian Math. J. **24** (2017), 313–323.

[8] V. Colao, L. Muglia, H.K. Xu, *Existence of solutions for a second-order differential equation with non-instantaneous impulses and delay*, Ann. Mat. Pura Appl. **195** (2016), 697–716.

[9] Y. Chu, Y. Liu, *Approximation controllability for a class of instantanous and non-instantanous impulsive semilinear system with finite time delay*, Evolution Equations and Control Theory, **12** (2023), 1193-1207.

[10] S. Djebali, L. Górniewicz, A. Ouahab, *Existence and Structure of solution Sets for Impulsive Differential Inclusions: A Survey*, Lecture Notes in Nonlinear Analysis 13. Toruń: Nicolaus Copernicus University, Juliusz Schauder Center for Nonlinear Studies, 2012.

[11] J.R. Graef, J. Henderson, A. Ouahab, *Impulsive Differential Inclusions A Fixed Point Approach*, De Gruyter Series in Nonlinear Analysis and Applications, **20**, De Gruyter, Berlin, 2013.

[12] H.A. Hammad, H. Aydi, H. Isik, M. De la Sen, *Existence and stability results for a coupled system of impulsive fractional differential equations with Hadamard fractional derivatives*, AIMS Mathematics, **8** (2023), no. 3, 6913–6941.

[13] S. Heidarkhani, A. Cabada,G.A. Afrouzi, S. Moradi, G. Caristi, *A variational approach to perturbed impulsive fractional differential equations*, J. Comput. Appl. Math. **341** (2018), 42–60.
[14] S. Heidarkhani, A. Salari, *Existence of three solutions for impulsive fractional differential systems through variational methods*, TWMS J. Appl. Eng. Math. **9** (2019), no. 3, 646–657.
[15] S. Heidarkhani, A. Salari, *Nontrivial solutions for impulsive fractional differential systems through variational methods*, Math. Methods Appl. Sci. **43** (2020), no. 10, 6529–6541.
[16] E. Hernández, D. O'Regan, *On a new class of abstract impulsive differential equations*, Proc. Amer. Math. Soc. **141** (2013), no. 5, 1641–1649.
[17] S. Hristova, *Qualitative Investigations and Approximate Methods for Impulsive Equations*, Nova Science, New York, 2009.
[18] V. Lakshmikantham, D.D. Bainov, P.S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989.
[19] D.P. Li, F.Q. Chen, Y.K. An, *The existence of solutions for an impulsive fractional coupled system of* (p, q)*-Laplacian type without the Ambrosetti-Rabinowitz condition*, Math. Methods Appl. Sci. **42** (2019), no. 5, 1449–1464.
[20] D. Li, Y. Li, F. Chen, X. Feng, *Instantaneous and Non-Instantaneous Impulsive Boundary Value Problem Involving the Generalized* Ψ*-Caputo Fractional Derivative*, Fractal Fract. **7** (2023), no. 3, 206.
[21] J.L. Li, J.J. Nieto, *Existence of positive solutions for multipoint boundary value problem on the half-line with impulses*, Bound. Value Probl. **2009** (2009), no. 834158, 1.
[22] R. Liu, J.R. Wang, D. O'Regan, *Existence of solutions to nonlinear impulsive fuzzy differential equations*, Filomat, **37** (4) (2023), 1223–1240.
[23] S.M.A. Maqbol, R.S. Jain, B.S. Reddy, *Existence results of random impulsive integrodifferential inclusions with time-varying delays*, Journal of Function Spaces, (2024).
[24] J. Mawhin, M. Willem, *Critical Point Theory and Hamiltonian Systems, Applied Mathematical Sciences*, Springer-Verlag, New York, 1989.
[25] R. Nesraoui, D. Abdelkader, J.J. Nieto, A. Ouahab, *Variational approach to non-instantaneous impulsive system of differential equations*, Nonlinear Stud. **28** (2021), no. 2, 563–573.
[26] J.J. Nieto, D. O'Regan, *Variational approach to impulsive differential equations*, Nonlinear Anal. Real Word Appl. **10** (2009), 680–690.
[27] M. Pierri, D. O'Regan, V. Rolnik,*Existence of solutions for semi-linear abstract differential equations with not instantaneous impulses*, Appl. Math. Comput. **219** (2013), 6743–6749.
[28] P.H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, CBMS Regional Conference Series in Mathematics, Am. Math. Soc., Providence, 1986.
[29] Samoilenko, A.M.; Perestyuk, N.A.; *Impulsive Differential Equations*. World Scientific, Singapore, 1995.
[30] Y. Tian, W. Ge, *Application of variational methods to boundary value problem for impulsive differential equations*, Pro. Edinb. Math. Soc. **51** (2008), 509–527.
[31] Y. Tian, M. Zhang, *Variational method to differential equations with instantaneous and non-instantaneous impulses*, Appl. Math. Lett. **94** (2019), 160–165.
[32] Y. Tian, Y. Zhang, *The existence of solution and dependence on functional parameter for BVP of fractional differential equation*, J. Appl. Anal. Comput. **12** (2022), no. 2, 591–608.
[33] Y.F. Wei, S.M. Shang, Z.B. Bai, *Applications of variational methods to some three-point boundary value problems with instantaneous and noninstantaneous impulses*, Nonlinear Anal., Model. Control, **27** (2022), no. 3, 466–478.
[34] L. Wu, Z. Xiang, *A study of integrated pest management models with instantaneous and non-instantaneous impulse effects*, Mathematical Biosciences and Engineering, **21** (2024), pp. 3063–3094.
[35] M. Xia, X. Zhang, J. Xie, *Existence and multiplicity of solutions for a fourth-order differential system with instantaneous and non-instantaneous impulses*, Open Mathematics, **21** (2023), no. 1, 20220553.
[36] W. Zhang, Z. Wang, J. Ni, *Variational method to the fractional impulsive equation with Neumann boundary condtions*, Journal of Applied Analysis and Computation, **14** (2024), pp. 2890–2902.
[37] W. Yao, *Existence and multiplicity of solutions for three-point boundary value problems with instantaneous and noninstantaneous impulses*, Bound. Value Probl. **2023** (2023), no. 15, 1.
[38] W. Zhang, W.B. Liu, *Variational approach to fractional Dirichlet problem with instantaneous and non-instantaneous impulses*, Appl. Math. Lett. **99** (2020), 105993.
[39] H. Zhang, W. Yao, *Three solutions for a three-point boundary value problem with instantaneous and non-instantaneous impulses*, AIMS Mathematics, **8** (2023), no. 9, 21312–21328.
[40] J.W. Zhou, Y.M. Deng, Y.N. Wang,*Variational approach to p-Laplacian fractional differential equations with instantaneous and non-instantaneous impulses*, Appl. Math. Lett. **104** (2020), 106251.