Filomat 39:29 (2025), 10477–10490
https://doi.org/10.2298/FIL2529477J

Published by Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Periodic non-negative solutions for anisotropic $p(z)$-Laplacian parabolic equations with strong nonlinearity

Hamza Jourhmane[a], Abderrazak Kassidi[a]

[a] *Laboratory of Applied Mathematics and Scientific Computing, Sultan Moulay Slimane University, Beni Mellal, Morocco*

Abstract. In this paper, we investigate a degenerate parabolic equation involving an anisotropic $p(z)$-Laplacian operator and a strongly nonlinear source term, subject to Dirichlet-type boundary conditions. Our main objective is to establish the existence of a nonnegative periodic solution. The proof relies on the Leray–Schauder topological degree, whose application in this framework requires a careful treatment due to the degeneracy of the operator and the nonlinear nature of the source.

1. Introduction

Let Ω be a bounded convex domain in $\mathbb{R}^N$ $\left(N > 1\right)$ with smooth boundary. The aim of this work is to prove the existence of periodic non-negative solution for the following anisotropic parabolic equation

$$\begin{cases} \partial_t f = \displaystyle\sum_{k=1}^{N} \partial_k\Big(|\partial_k f|^{p_k(z)-2}\partial_k f\Big) + |f|^{p_0(z)-2}f + a(z,t)f^{m(z)} & \text{in } D_\tau, \\ f(z,t) = 0, & z,t) \in \partial\Omega \times (0,\tau), \\ f(z,t+\tau) = f(z,t), & (z,t) \in \Omega \times \mathbb{R}, \end{cases} \tag{1}$$

where $2 \leq p_1(z), \ldots, p_N(z) \leq \infty$, and we denote $p_+ = \max\{p_{1+}, \ldots, p_{N+}\}$ and $p_- = \min\{p_{1-}, \ldots, p_{N-}\}$, where $p_{k+} = \max\{p_k(z), z \in \Omega\}$ and $p_{k-} = \min\{p_k(z), z \in \Omega\}$ for all $k = 1, \ldots, N$. We assume that $p_0(z) \geq p_+$, $\tau > 0$, $a(z,t)$ is continuous in $\overline{\Omega} \times \mathbb{R}$, time periodic with period τ and positive on $D_\tau = \Omega \times (0,\tau)$, and that $p_+ - 1 < p_- - 1 + \frac{p_-}{N}$ so that m verifies $p_+ - 1 < m(z) < p_- - 1 + \frac{p_-}{N}$.
Mathematical modelling of natural phenomena most frequently end up using differential equations to understand, use or solve problems. It may be about economics [37], epidemiology and various domains. The interest in a periodic solution for periodic problems comes from many domains, one of which are relativistic physics [4, 12], radiative gas [31], and microbiology [15]. Different methods might be referred to in order study this type of problem, namely, in [29] authors use the Leray-Schauder fixed point theorem, where the sub- and super-solution method are used by [10], and for further details see [1–3, 36] and the

2020 *Mathematics Subject Classification*. Primary 35B10; Secondary 35K55, 35K65, 47H11.
Keywords. Anisotropic $p(z)$-Laplacian, Dirichlet conditions; Generalized Sobolev spaces; Non-negative solution; Periodic solution; Topological degree
Received: 17 August 2025; Accepted: 01 September 2025
Communicated by Maria Alessandra Ragusa
Email addresses: hamzajourh@gmail.com (Hamza Jourhmane), a.kassidi@usms.ma (Abderrazak Kassidi)
ORCID iD: https://orcid.org/0000-0002-9105-1123 (Abderrazak Kassidi)

references therein.
In the last decades, mathematicians like Antontsev and Shmarev [6], and other scientists from vatious fields are interested in anisotropic elliptic and parabolic equations. One can argue that the principal reason is their appearance in physical and mechanical processes in continuous anisotropic medium. One can check out [5, 7, 30], and the references therein for more on this subject.
A much simpler example of a semi-linear equation is studied in [10]

$$\partial_t f = \Delta f + u(z,t,f), \tag{2}$$

such that u is periodic with respect to t. If $u(z,t,f) = v(z,t)f^m$, with $v(z,t)$ being a time periodic positive map, M. J. Esteban [17] proved that the equation (2) has indeed a solution that is nonegative and periodic, provided that $1 < m < \frac{3N+8}{3N-4}$, and enhanced the result of [18] by establishing the same results with the more general condition $1 < m < \frac{N}{N-2}$.
Charkaoui et al. [10] were interested in the equation

$$\begin{cases} \partial_t f = \Delta f + u(z,t) - H(z,t,\nabla u) & (z,t) \in D_\tau, \\ f(z,t) = 0 & (z,t) \in \partial\Omega \times (0,\tau), \\ f(z,0) = f(z,\tau) & z \in \Omega, \end{cases}$$

where H is a Carathéodory function and $u \in L^1(D_\tau)$ is nonnegative and time periodic. Such that $u = (u_1, \ldots, u_M)$, $f = (f_1, \ldots, f_M)$ and $H = (H_1, \ldots, H_M)$.
It was a biological problem denoting $f(z,t)$ as the density of an object at time t and position z that motivated R.Huang et al. [27] to study the degenerate problem

$$\begin{cases} \partial_t f - \Delta f^m = (a - \Psi[f])f & (z,t) \in D_\tau, \\ f(z,t) = 0 & (z,t) \in \partial\Omega \times (0,\tau), \\ f(z,0) = f(z,\tau) & z \in \Omega, \end{cases}$$

such that Δf^m represents the way the studied species tend to avoid clutter, m greater than 1, $a(z,t)$ is the highest possible value of the species' increasing rate at time t and position z, and $\Psi[f] : L^2(\Omega)^+ \to \mathbb{R}^+$ being a continuous bounded functional.
The above cited references are what motivated us to investigate the existence of a nontrivial nonnegative periodic solution for the equation (1), after utilizing the topological degree of Leray-Schauder as well as the idea of scaling [17, 18], within [22–24] the same authors discussed the same kind of equation but with simpler forms.

After presenting the problem and giving a brief historical of the operators and methods used, we will define in section 2 the form of the solution then announce the main theorem. Sections 3, 4 and 5, we will introduce the lemmas needed as well as their respective proofs whilst section 6 will use these lemmas to show the veracity of the main theorem.

2. Main result

Let us first start by introducing the spaces we work with

Definition 2.1. *If Ω is a bounded and open subset in $\mathbb{R}^N (N \geq 2)$, a real-valued continuous function p is said to be log-Hölder continuous on Ω when there is a positive number C verifying*

$$|p(z) - p(y)| \leq \frac{C}{\left|\log|z-y|\right|} \quad \textit{for any } z, y \in \overline{\Omega} \quad \textit{provided that} \quad |z-y| < \frac{1}{2}.$$

With a log-Hölder continuous function p we denote

$$p^- = \min_{z\in\overline{\Omega}} p(z) \quad \text{and} \quad p^+ = \max_{z\in\overline{\Omega}} p(z),$$

and the $C_+(\overline{\Omega})$ the space of log-Hölder continuous functions where $1 < p^- \leq p^+ < N$

Definition 2.2. *The generalized Lebesgue space $L^{p(z)}(\Omega)$ is the set of all measurable maps $f : \Omega \mapsto \mathbb{R}$ that allows the integral:*

$$\varrho_{p(z)}(f) := \int_{\Omega} |f|^{p(z)} dz$$

to be bounded, this integral is named the convex modular and

$$\|f\|_{L^{p(z)}} = \inf\left\{ \nu > 0 \ \text{verifying} \ \varrho_{p(z)}(\frac{f}{\nu}) \leq 1 \right\}$$

is a well defined norm on $L^{p(z)}(\Omega)$ called the Luxemburg norm.

One should notice that the space $\left(L^{p(z)}(\Omega), \|\cdot\|_{L^{p(z)}}\right)$ is a Banach space. Moreover, it is separable and reflexive with the topological dual $L^{p'(z)}(\Omega)$ such that $\dfrac{1}{p(z)} + \dfrac{1}{p'(z)} = 1$.
Following the approach proposed in [8], we extend the variable exponent $p : \overline{\Omega} \to (1, +\infty)$ to $\overline{D_\tau} = [0, T] \times \overline{\Omega}$ by setting $p(t, z) := p(z)$ for all $(t, z) \in \overline{D_\tau}$. Consequently, the variable exponent Lebesgue space $L^{p(z)}(D_\tau)$ is defined as follows

$$L^{p(z)}(D_\tau) = \Big\{f : D_\tau \to \mathbb{R}, \text{ measurable such that} \int_{D_\tau} |f(t, z)|^{p(z)}\, dzdt < \infty\Big\}.$$

The associated norm on $L^{p(z)}(D_\tau)$ is given by

$$\|f\|_{p(z)} = \inf\Big\{\alpha > 0 \ : \ \int_{D_\tau} \Big|\frac{f(t, z)}{\alpha}\Big|^{p(z)} dzdt \leq 1\Big\}.$$

With this norm, the space $\left(L^{p(z)}(D_\tau), \|\cdot\|_{p(z)}\right)$ is a reflexive and separable Banach space.
We now introduce the anisotropic Sobolev space with a variable exponent. For this purpose, we consider the vectorial function $\vec{p} : \bar{\Omega} \to \mathbb{R}^N$ as follows

$$\vec{p} := \vec{p}(z) = (p_1(z), \cdots, p_N(z)),$$

where

$$p_k \in C_+(\bar{\Omega}) \text{ for any } k \in \{1, \cdots, N\}.$$

The anisotropic variable exponent Sobolev space $W^{1,\vec{p}(\cdot)}(\Omega)$ is then defined as

$$W^{1,\vec{p}}(\Omega) = \Big\{f \in W^{1,1}(\Omega) : \partial_k f \in L^{p_k(z)}(\Omega) \text{ for } k = 1, \cdots, N\Big\},$$

note that

$$W_0^{1,\vec{p}}(\Omega) = W^{1,\vec{p}}(\Omega) \cap W_0^{1,1}(\Omega)$$

The anisotropic variable exponent Sobolev space $W_0^{1,\vec{p}}(\Omega)$ can also be defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,\vec{p}}(\Omega)$ with respect to the norm

$$\|f\|_{W_0^{1,\vec{p}}(\Omega)} := \sum_{k=1}^{N} \left\|\partial_k f\right\|_{p_k(z)}.$$

These spaces are separable and reflexive Banach spaces [20, 33].
In what follows, we introduce the proposed framework to solve the problem (1). Let $0 < \tau < \infty$, we define the space

$$L^{\vec{p}(z)}\Big(0, \tau; W_0^{1,\vec{p}(z)}(\Omega)\Big) = \Big\{f \in L^{p(z)}(D_\tau) : \sum_{k=1}^{N} \int_0^\tau \left\|\partial_k f\right\|_{p_k(z)}^{p_{k-}} dt < \infty\Big\},$$

equipped with the norm

$$\|u\|_{L^{\vec{p}(z)}} = \sum_{k=1}^{N} \Big(\int_0^{\tau} \left\|\partial_k f\right\|_{p_k(z)} dt \Big)^{\frac{1}{p_{k-}}}.$$

Considering the fact that the equation studied is degenerate, the problem (1) generally lacks a classical solution. Therefore, we focus on its weak solutions, defined as follows.

Definition 2.3. *We denote by $C_\tau(\overline{D_\tau})$ the maps from $C(\Omega \times \mathbb{R})$ τ-periodic by the variable t and take h in $C_\tau(\overline{D_\tau})$. A map $f \in L^{\vec{p}(z)}\Big(0,\tau; W_0^{1,\vec{p}(z)}(\Omega)\Big) \cap C_\tau(\overline{D_\tau})$ is said to be a solution of the problem* (1) *if it verifies*

$$\int_{D_\tau} f \partial_t \psi dz dt + \sum_{k=1}^{N} \int_{D_\tau} \left|\partial_k f\right|^{p_k(z)-2} \partial_k f \partial_k \psi dz dt + \int_{D_\tau} \Big(|f|^{p_0(z)-2} f + a(z,t) f^{m(z)}\Big) \psi dz dt = 0,$$

for all $\psi \in C^1(\overline{D_\tau})$ subject to $\psi(.,0) = \psi(.,\tau)$ and $\psi(z,t) = 0$ in case (z,t) is in $\partial\Omega \times (0,\tau)$.

Theorem 2.4. *Equation* (1) *admits a nonnegative solution that is nontrivial*

$$f \in C\Big(0,\tau; W_0^{1,\vec{p}(z)}(\Omega)\Big) \cap C_\tau(\overline{D_\tau}) \text{ with } \quad \partial_t f \in L^2(D_\tau),$$

for which $N > 1$ and $p_+(z) - 1 < m(z) < p_-(z) - 1 + \frac{p_-(z)}{N}$.

To prove this we will regularize the equation and prove the following

Proposition 2.5. *Under all assumptions on Theorem 2.4, this problem admits a nonnegative solution f_σ*

$$\begin{cases} \partial_t f_\sigma = \sum_{k=1}^{N} \partial_k \Big(\left|\partial_k f_\sigma\right|^2 + \sigma \Big)^{\frac{p_k(z)-2}{2}} \partial_k f_\sigma + |u_\sigma|^{p_0(z)-2} f_\sigma + a(z,t) f_\sigma^{m(z)}, & (t,z) \in \Omega \times \mathbb{R}, \\ f_\sigma(z,t) = 0, & (t,z) \in \partial\Omega \times (0,T), \\ f_\sigma(z,0) = u_\sigma(z,T), & z \in \Omega, \end{cases}$$

and we have the existence of r and R positive subject to,

$$r \leq \max_{\overline{D_\tau}} u_\sigma(z,t) \leq R.$$

For the proof of this proposition, the theory of topological degree is employed. Specifically, let's analyze a one-parameter equation that connects the semi-linear operator to a simpler operator, the Laplacian:

$$\partial_t f = \sum_{k=1}^{N} \partial_k \Big(\Big(\left|\nu \partial_k f\right|^2 + \sigma \Big)^{\frac{p_k(z)-2}{2}} \partial_k f \Big) + v(z,t), \tag{3}$$

where $v \in C_\tau(\overline{D_\tau})$ and $\lambda \in [0,1]$. Section 3 holds the proof that for all v in $C_\tau(\overline{D_\tau})$ and $0 \leq \nu \leq 1$, equation (3) admits exactly one solution $f \in C_\tau(\overline{D_\tau})$. Additionally, the functional $\mathcal{M} : [0,1] \times C_\tau(\overline{D_\tau}) \to C_\tau(\overline{D_\tau})$ defined as $f = \mathcal{M}(\nu, v)$ is compact as well as the functional $\mathcal{M}(\nu, \Psi(f))$ such that $\Psi(f) = |f_+|^{p_0(z)-2} u_+ + a(z,t) f_+^{m(z)}$. The constant $\lambda \in [0,1]$ complicates proving the compactness. To address this, let's first establish what follows

$$\deg(I - \mathcal{M}(1, \Psi(\cdot)), B_R(0) \setminus B_r(0), 0) \neq 0,$$

where $B_\rho(0)$ represents the ball in $C_\tau(Q_\tau)$ that has ρ as radius and centered at the origin In Section 4, the existence of a constant $r > 0$ is shown, independent of σ, with

$$\deg(I - \mathcal{M}(1, \Psi(\cdot)), B_r(0), 0) = 1.$$

Next, in Section 5, we will demonstrate that

$$\deg(I - \mathcal{M}(1, \Psi(\cdot)), B_R(0), 0) = 0$$

with a sufficiently big $R > r$, also independent of σ. Once these results are established, proving the proposition will reduce to deriving a supremum to our solution. As stated before, an upper bound will be obtained using the blow-up (scaling) argument, a technique extensively utilized in [17, 18], among others. Finally, in Section 6, Theorem 2.4 will be proven as a consequence of this proposition.

3. Characteristics of $\mathcal{M}$

To keep it simple, suppose $a(z,t)$ is continuous in the Hölder sense. This assumption does not restrict generality, as it can be eliminated through an approximation process.

Lemma 3.1. *For all $0 \le \nu \le 1$ and v in $C_\tau(\overline{D_\tau})$, equation to (3) admits exactly one solution $f \in C\Big(0,\tau; W_0^{1,\vec{p}(z)}(\Omega)\Big) \cap C_\tau(\overline{D_\tau}), \partial_t f \in L^2(D_\tau)$, and it satisfies*

$$\|f\|_\infty = \|\mathcal{M}(\nu, v)\|_\infty \le C\Big(\frac{\|v\|_\infty}{\delta}\Big)^{\delta+1} \quad \text{for all} \quad \delta \in]0,1[, \tag{4}$$

$$\Big\|\partial_t f\Big\|_2 \le C\Big\|v\Big\|_2, \tag{5}$$

for which C depends only upon N, σ, and $\vec{p}(z)$.

Proof. If $\nu = 0$ the references [17, 18] have the solution. Here and below $\lambda \neq 0$ is supposed. By [38], equation (3) admits a unique solution $f \in L^{\vec{p}}\Big(0,\tau; W_0^{1,\vec{p}(z)}(\Omega)\Big)$ for all v in $C_\tau(\overline{D_\tau})$ and $0 \le \nu \le 1$. One can check [13], to see that we have $f \in C^m(\overline{D_\tau})$, and $\nabla u \in C^m(\overline{D_\tau})$.
By substituting a different test function into the equation associated to f, to $|f|^{r_i(z)} f$, one gets

$$\frac{d}{dt}\|f(t)\|_{r_i(z)+2}^{r_i^-+2} + C(\sigma, p_+)\Big\|\sum_{k=1}^N \partial_k\Big(|f|^{\frac{r_i(z)}{2}} f\Big)\Big\|_2^2 \le \|v\|_\infty \|f\|_{r_i(z)+2}^{r_i^-+1},$$

such that $r_1^- = 1$ and for any $k \ge 1$, $r_k^- = 2r_{k-1}^- + 2 = 2^k - 2$, $f(t) = f(\cdot, t)$. Let $w_k = |f|^{\frac{r_k(z)}{2}} f$, we get

$$\partial_t\Big\|w_i\Big\|_2^2 + C(\sigma, p_+)\Big\|\sum_{k=1}^N \partial_k w_k\Big\|_2^2 \le \|v\|_z \Big\|w_i\Big\|_2^{\frac{2(r_i^-+1)}{r_i^-+2}}.$$

The inequality (4) is established by utilixing the technique of Moser iteration , one can look at [34].
Finally, by simply using $\partial_t u$ as a test function, equation (5) can be easily derived. □

Lemma 3.2. *The map $\mathcal{M} : [0,1] \times C_\tau(\overline{D_\tau}) \to C_\tau(\overline{D_\tau})$ is well defined and compact.*

Proof. We begin by proving that $u = \mathcal{M}(\nu, g) \in C_\tau(\overline{D_\tau})$ for any $0 \le \nu \le 1$ and g in $C_\tau(\overline{D_\tau})$. In case $\nu = 0$, from Theorem 10.1 of [28] and the periodicity of u, one obtains

$$\Big|f\Big(z_1, t_1\Big) - f\Big(z_2, t_2\Big)\Big| \le \gamma\Big(\Big|z_1 - z_2\Big| + \Big|t_1 - t_2\Big|^{\frac{1}{2}}\Big)^\beta, \tag{6}$$

such that γ and β are positive constants that depend upon $N, \sigma, p_1, \ldots, p_N$, $\|f\|_\infty$ and, with Lemma 3.1 $\|v\|_\infty$.
In case $\lambda \neq 0$, by (3), $w = \nu^{\frac{1}{2}} f$ verifies

$$\partial_t w = \sum_{i=k}^N \partial_k\Big(\Big|\partial_k w\Big|^2 + \sigma\Big)^{\frac{p_k(z)-2}{2}} \partial_k w + \nu^{\frac{1}{2}} w(z,t).$$

Observing the time periodicity of w and applying the result from [14], we conclude that w is Hölder continuous in D_τ. Moreover, by applying Theorem 10.1 in [28] to w, and then returning to f to obtain an inequality similar to (6), we can use the Arzelà-Ascoli theorem to show that the image of any bounded set in $[0,1]\times C_\tau(Q_\tau)$ under the map $\mathcal{M}$ is a compact subset of $C_\tau(Q_\tau)$.
So we can show the continuity of $\mathcal{M}$, let us take $\nu_n \to \nu, v_n \to v$ as $n\to\infty$ and $f_n = \mathcal{M}\Big(\nu_n, v_n\Big)$. Using both inequalities (4) and (6) we get the existence of $f \in C_\tau(\overline{D_\tau})$ such that

$$f_k(z,t) \to f(z,t) \quad \text{uniformly in } D_\tau, \tag{7}$$

If necessary, f_n may represent its own subsequence. To prove that $f = \mathcal{M}(\nu, v)$, we follow the same approach as in [43].
It is sufficient to multiply (3) by f_n, and integrate over D_τ, to obtain that for any $1 \le k \le N$

$$\int_{D_\tau} \Big(\nu_k\Big|\partial_k f_k\Big|^2 + \sigma\Big)^{\frac{p_k(z)-2}{2}} \Big(\partial_k f_k\Big)^2 dzdt \le C,$$

and hence

$$\int_{D_\tau} \nu_n^{\frac{p_k(z)-2}{2}} \Big|\partial_k f_n\Big|^{p_k(z)} dzdt \le C, \tag{8}$$

$$\int_{D_\tau} \sigma^{\frac{p_k(z)-2}{2}} \Big|\partial_k f_n\Big|^2 dzdt \le C. \tag{9}$$

Notice that C represents a constant number that may have different values. We have

$$\begin{aligned}\Big|\Big(\nu_n\Big|\partial_k f_n\Big|^2 + \sigma\Big)^{\frac{p_{k-}-2}{2}} \partial_k f_n\Big|^{\frac{p_{k-}}{p_{k-}-1}} &\le \Big(\nu_n\Big|\partial_k f_n\Big|^2 + \sigma\Big)^{\frac{p_{k-}(p_{k-}-2)}{2(p_{k-}-1)}} \Big|\partial_k f_n\Big|^{p_{k-}(p_{k-}-1)} \\ &\le C\Big(\nu_n^{\frac{p_{k-}(p_{k-}-2)}{2(p_{k-}-1)}} \Big|\partial_k f_n\Big|^{p_{k-}} + \sigma^{\frac{p_{k-}(p_{k-}-2)}{2(p_{k-}-1)}} \Big|\partial_k f_n\Big|^{\frac{p_{k-}}{p_{k-}-1}}\Big),\end{aligned}$$

the inequalities (8) and (9) allow us to get

$$\begin{aligned}\int_{D_\tau} \Big|\Big(\nu_n\Big|\partial_k f_n\Big|^2 + \sigma\Big)^{\frac{p_k(z)-2}{2}} \partial_k f_n\Big|^{\frac{p_k(z)}{p_k(z)-1}} dzdt &\le C\nu_n^{\frac{p_{k-}(p_{k-}-2)}{2(p_{k-}-1)}} \int_{D_\tau} \Big|\partial_k f_n\Big|^{p_k(z)} dzdt + C\sigma^{\frac{p_{k-}(p_{k-}-2)}{2(p_{k-}-1)}} \int_{D_\tau} \Big|\partial_k f_n\Big|^{\frac{p_k(z)}{p_k(z)-1}} dzdt \\ &\le C\nu_n^{\frac{p_{k-}-2}{2(p_{k-}-1)}} \int_{D_\tau} \nu_k^{\frac{p_k(z)-2}{2}} \Big|\partial_k f_n\Big|^2 dzdt + C\sigma^{\frac{p_{k-}(p_{k-}-2)}{2(p_{k-}-1)}} \Big(\frac{C}{\sigma^{\frac{p_{k-}-1}{2}}}\Big)^{\frac{p_{k-}}{2(p_{k-}-1)}} \le C,\end{aligned}$$

guaranteeing the existence of $\xi_k \in L^{\frac{p_k(z)}{p_k(z)-1}}(D_\tau)$ where

$$\Big(\Big(\nu_n\Big|\partial_k f_n\Big|^2 + \sigma\Big)^{\frac{p_k(z)-2}{2}}\Big)\partial_k f_n \rightharpoonup \xi_k \quad \text{weakly in } L^{\frac{p_k(z)}{p_k(z)-1}}(D_\tau),$$

subsequences have the exact notation of their original. Hence, we have

$$\int_{D_\tau} f\partial_t\psi dzdt = \int_{D_\tau} \xi\nabla\psi dzdt - \int_{D_\tau} v\psi dzdt, \tag{10}$$

for any $\psi \in C_0^\infty(D_\tau)$, such that $\xi = \Big(\xi_1, \ldots, \xi_N\Big)$. To finish the proof, we have got to demonstrate the following

$$\int_{D_\tau} \xi\nabla\psi dzdt = \sum_{k=1}^{N} \int_{D_\tau} \Big(\nu\Big|\partial_k f_k\Big|^2 + \sigma\Big)^{\frac{p_k(z)-2}{2}} \partial_k f_n \partial_k psidzdt. \tag{11}$$

To start with, the following integral is non negative

$$\int_{D_\tau} \Big[\Big(\Big(\nu_n\Big|\partial_k f_n\Big|^2+\sigma\Big)^{\frac{p_k(z)-2}{2}}\Big)\partial_k\Big(\nu_n^{\frac{1}{2}} f_n\Big)-\Big(\Big|\partial_k g\Big|^2+\sigma\Big)^{\frac{p_k(z)-2}{2}}\partial_k g\Big]\Big[\partial_k\Big(\nu_n^{\frac{1}{2}} f_n\Big)-\partial_k g\Big]dzdt, \tag{12}$$

for all $g \in L^{\vec{p}(z)}\Big(0,\tau;W_0^{1,\vec{p}(z)}(\Omega)\Big)$. Indeed, take $R(Z)=\Big(|Z|^2+\sigma\Big)^{\frac{p_k(z)-2}{2}}Z$; one can see that

$$R'(z)=\Big(|Z|^2+\sigma\Big)^{\frac{p_k(z)-2}{2}}\mathcal{I}+(p_k(z)-2)\Big(|Z|^2+\sigma\Big)^{\frac{p_k(z)-4}{2}}ZZ^T,$$

is a positive definite matrix, so that we have

$$\Big(R(\partial_k\Big(\nu_k^{\frac{1}{2}} f_k\Big))-R(\partial_k g)\Big)\Big(\partial_k\Big(\nu_n^{\frac{1}{2}} f_n\Big)-\partial_k g\Big)\geq 0,$$

and (12) is proven. By the periodicity of f_n as well as the equations it verifies, we get

$$\sum_{k=1}^{N}\int_{D_\tau}\Big(\nu_n\Big|\partial_k f_n\Big|^2+\sigma\Big)^{\frac{p_k(z)-2}{2}}\Big|\partial_k f_n\Big|^2 dzdt=\int_{D_\tau} v_k f_k dzdt,$$

combined with (12) derive

$$\int_{D_\tau} v_n f_n dzdt \geq \sum_{k=1}^{N}\int_{D_\tau}\Big(\nu_k\Big|\partial_k f_n\Big|^2+\sigma\Big)^{\frac{p_k(z)-2}{2}}\partial_k f_n\partial_k g dzdt+\sum_{k=1}^{N}\int_{D_\tau}\Big(\nu_n\Big|\partial_k v\Big|^2+\sigma\Big)^{\frac{p_k(z)-2}{2}}\partial_k g\partial_k(f_n-g)dzdt.$$

Letting $n\to\infty$, we have

$$\int_{D_\tau} vf dzdt \geq \int_{D_\tau}\xi\nabla g dzdt+\sum_{k=1}^{N}\int_{D_\tau}\Big(\nu_k\Big|\partial_k g\Big|^2+\sigma\Big)^{\frac{p_k(z)-2}{2}}\partial_k g\partial_k(f_n-g)dzdt. \tag{13}$$

Additionally, taking $\psi=f$ in (10) yields

$$\int_{D_\tau}\xi\nabla f dzdt=\int_{D_\tau} vf dzdt. \tag{14}$$

Together with (13) and (14) give

$$\int_{D_\tau}\Big(\xi_k-\Big(\nu\Big|\partial_k g\Big|^2+\sigma\Big)^{\frac{p_k(z)-2}{2}}\partial_k g\Big)\Big(\partial_k f-\partial_k g\Big)dzdt\geq 0.$$

Letting $g=f-\nu\psi$ with $\nu>0,\ \psi\in C_0^\infty(D_\tau)$, we get

$$\int_{D_\tau}\Big(\xi_k-\Big(\nu\Big|\partial_k(f-\nu\psi)\Big|^2+\sigma\Big)^{\frac{p_k(z)-2}{2}}\partial_k(f-\nu\psi)\Big)\partial_k\psi dzdt\geq 0.$$

Taking $\nu\to 0$ yields

$$\int_{D_\tau}\Big(\xi_k-\Big(\nu\Big|\partial_k f\Big|^2+\sigma\Big)^{\frac{p_k(z)-2}{2}}\Big)\partial_k\partial_k\psi dzdt\geq 0.$$

Following a similar approach, one is able to easily show that the inverse inequality holds as well, which means (11) is true. □

4. Topological degree on a small ball $B_r(0)$

Lemma 4.1. *Under the assumptions of Theorem 2.4 , we have*
$\deg\Big(\mathcal{I} - \mathcal{M}(1, \Phi(\cdot)), B_r(0), 0\Big) = 1$ *with an $r > 0$ that is independent on σ.*

Proof. Observe that the map $\mathcal{M}(1, \lambda\Phi(f))$ is compact, as $\mathcal{M}$ is compact and Φ is continuous. Using the homotopy invariance of degree

$$\deg\Big(\mathcal{I} - \mathcal{M}(1, \Phi(\cdot)), B_r(0), 0\Big) = \deg\Big(I, B_r(0), 0\Big) = 1,$$

assuming that

$$\mathcal{M}(1, \lambda\Phi(f)) \neq f \quad \text{for } \lambda \in [0, 1], \quad f \in \partial B_r(0). \tag{15}$$

Which we will prove by taking

$$r = \Big(\frac{1}{AC_0^{p_+}|\Omega|^{1-p_+/q}}\Big)^{\frac{1}{m_+-q+1}},$$

where $q = \frac{Np_+}{N-p_+}$ in case $p_+ < N$, $q = p_+ + 1$ and if $p_+ \geq N$, $A = \max_{\overline{D_\tau}} a(z, t)$, We use f_ν to denote the periodic solution to

$$\partial_t f = \sum_{k=1}^{N} \partial_k\Big(\Big(\big|\partial_k f\big|^2 + \sigma\Big)^{\frac{p_k(z)-2}{2}} \partial_k f\Big) + \lambda(|f|^{p_0(z)-2} f + a(z, t) f^{m(z)}), \tag{16}$$

Applying the maximum principle and the fact that f_ν is continuous, one obtains $f_\nu(z, t) \geq 0$. By multiplying equation (16) by f_ν and integrate within D_τ one has

$$K := \lambda \int_{D_\tau} |f_\nu|^{p_0(z)} + a(z, t) f_\nu^{m(z)+1} dz dt - \sum_{k=1}^{N} \int_{D_\tau} \Big(\big|\partial_k f_\nu\big|^2 + \sigma\Big)^{\frac{p_k(z)-2}{2}} \big|\partial_k f_\nu\big|^2 dz dt = 0, \tag{17}$$

In the following, the embedding theorem will be used

$$\big\|f_\nu\big\|_q \leq C_0 \big\|f_\nu\big\|_{L^{\vec{p}(z)}}.$$

In case $p_+ < N$, by (17) we have

$$K \leq A \int_{D_\tau} f_\nu^{m(z)+1} dz dt - \frac{1}{C_0^{p_+}} \int_0^\tau \big\|f_\nu\big\|_q^{p_+} dt. \tag{18}$$

If $m_+ + 1 \geq q$, one gets

$$K \leq A \max_{\overline{D_\tau}} f_\nu^{m_++1-q} \int_0^\tau \|f\|_q^q dt - \frac{1}{C_0^{p_+}} \int_0^\tau \|f_\nu\|_q^{p_+} dt \leq \int_0^\tau \big\|f_\nu\big\|_q^{p_+} \Big(A|\Omega|^{\frac{q-p_+}{q}} \max_{\overline{D_\tau}} f_\nu^{m_++1-q} - \frac{1}{C_0^{p_+}}\Big) dt. \tag{19}$$

If (15) were not true, then we would have $f_\nu \in \partial B_r(0)$. Therefore

$$\max_{\overline{D_\tau}} f_\nu(z, t) = r = \Big(\frac{1}{M|\Omega|^{1-\frac{p_+}{q}}}\Big)^{\frac{1}{m_++1-p_+}},$$

and the last integral in (19) equals $-\int_0^\tau \big\|f_\nu\big\|_q^{p_+} dt < 0$, contradicting with the equality (17).
If $p_+ < m_+ + 1 < q$, by applying the Hölder inequality for the integral on the right hand side in (18)

$$\int_\Omega f_\nu^{m_++1} dz \leq \big\|f_\nu\big\|_q^{m_++1} |\Omega|^{\frac{q-1-m_+}{q}},$$

then

$$K \leq A|\Omega|^{\frac{q-1-m_+}{q}} \int_0^\tau \left\|f_\nu\right\|_q^{m_++1} dt - \frac{1}{C_0^{p_+}} \int_0^\tau \left\|f_\nu\right\|_q^{p_+} dt = \int_0^\tau \left(A|\Omega|^{\frac{q-p_+}{q}} \left\|f_\nu\right\|_q^{m_++1-p_+} - \frac{1}{C_0^{p_+}}\right)\left\|f_\nu\right\|_q^{p_+} dt. \tag{20}$$

Suppose (15) is not true, then $f_\nu \in \partial B_r(0)$, so we get

$$\max_{\overline{D_\tau}} f_\nu(z,t) = r = \left(\frac{1}{A|\Omega|^{1-p_+/q}}\right)^{\frac{1}{m_++1-p_+}},$$

making the final integral on (20) equal to $-\int_0^\tau \left\|f_\nu\right\|_q^{p_+} dt < 0$. This inconsistency implies $f_\nu \notin \partial B_r(0)$.
If $p_+ \geq N$, we take $r = \left(\frac{1}{A|\Omega|^{1-\frac{p_+}{q}}}\right)^{\frac{1}{m_+-p_++1}}$ with $q = p_+ + 1$. □

5. Topological degree on a large ball $B_R(0)$

From here onwards, ν will denote an eigenvalue of $-\Delta$ and ψ_ν a positive eigenfunction associated with ν.

Lemma 5.1. *Let f_ν be a nonnegative periodic solution of*

$$\partial_t f = \sum_{k=1}^N \partial_k\left(\left(\nu\left|\partial_k f\right|^2 + \sigma\right)^{\frac{p_k(z)-2}{2}} \partial_k f\right) + a(z,t)u^{m(z)} + (1-\nu)\left(\nu\sigma^{\frac{p_+-2}{2}} f + 1\right), \tag{21}$$

with the Dirichlet boundary value condition of problem (1)*, where* $0 \leq \lambda \leq 1$. *Assuming that the conditions of Theorem 2.4 hold, there is a constant $L > 0$ unrelated to ν, and*

$$\|f\|_\infty \leq L.$$

If $\nu = 1$, L would be independent of σ too.

Proof. Let $0 < \sigma \leq 1$. Assume that f_ν has no bound. Meaning there will be sequences $\left(\nu_n\right)_{n\geq 0} \subset [0,1]$ and $\left(f_n\right)_{n\geq 0}$, where

$$M_n = \max_{\overline{D_\tau}} f_n(z,t) = f_n\left(z_n, t_n\right) \to \infty \quad \text{as } n \to \infty.$$

We suppose that $\nu_n \to \nu_\infty$ and $\left(z_n, t_n\right) \to \left(z_\infty, t_\infty\right)$ as $n \to \infty$.
We start by showing that $\nu_n \neq 0$ for any n. Indeed, if $\nu_n = 0$ for an n, then (21) turns into

$$\partial_t f = \sigma^{\frac{p_+-2}{2}} \Delta f + a(z,t) f^{m(z)} + \nu\sigma^{\frac{p_+-2}{2}} f + 1. \tag{22}$$

Multiplied by ψ_ν, integrated over D_τ, and considering the periodicity of f as well, we get

$$\begin{aligned}0 = \int_{D_\tau} \partial_t \psi_\nu f dz dt &= \int_{D_\tau} \sigma^{\frac{p_+-2}{2}} \psi_\nu \Delta f dz dt + \int_{D_\tau} a(z,t) f^{m(z)} \psi_\nu dz dt \\ &+ \nu \int_{D_\tau} \sigma^{\frac{p_+-2}{2}} f \psi_\nu dz dt + \tau \int_\Omega \psi_\nu dz = \int_{D_\tau} a(z,t) f^{m(z)} \psi_\nu dz dt + \tau \int_\Omega \psi_\nu dz > 0,\end{aligned}$$

which can't be true.
For each n, we define μ_n, z, s, and v_n as

$$\mu_n^{\frac{p_+}{m_+-p_++1}} M_n = 1,\ z = \frac{z - z_n}{\mu_n},\ s = \frac{t - t_n}{\mu_n^{(m_+-1)p_+/(m_+-p_++1)}},$$

$$g_n(z,s) = \mu_n^{\frac{p_+}{m_+-p_++1}} f_n(z,t).$$

Based on the convexity of Ω, we have $\delta_0 > 0$ so that we get $\text{dist}\Big(z_n, \partial\Omega\Big) \geq \delta_0$ from [26] and [32]. Hence, the map $g_n(z,s)$ has proper definition on

$$D_{n,\delta_0} = D\Big(\frac{\delta_0}{2\mu_n}\Big)\Big(\frac{-\tau}{\mu_n^{(m_+-1)p_+/(m_+-p_++1)}\nu_n^{(p_+-2)/2}}, \frac{\tau}{\mu_n^{(m_+-1)p_+/(m_+-p_++1)}\nu_n^{(p_+-2)/2}}\Big),$$

such that $D(l)$ is the ball of $\mathbb{R}^N$ with radius l and center 0. In D_{n,δ_0}, the map $h_n(z,s) = \nu_n^{\frac{1}{2}} g_n(z,s)$ verifies

$$\partial_s h_n = \mu_n^{\frac{p_+^2(p_+-2)(1-m_+)}{(m_+-p_++1)^2}} \sum_{k=1}^{N} \partial_k\Big(\Big(\Big|\partial_k w_n\Big|^2 + \mu_n^{\frac{2p_+}{m_+-p_++1}}\sigma\Big)^{\frac{p_k(z)-2}{2}}\partial_k w_n\Big)$$
$$+ h\Big(z_n + \mu_n z, t_n + s\mu_n^{\frac{(m_+-1)p_+}{m_+-p_++1}}\Big) g_n^{m(z)-1} h_n + \Big(1-\nu_n\Big)\Big(\nu\sigma^{\frac{p_+-2}{2}}\mu_n^{\frac{p_+(m_+-1)}{m-p_++1}} h_n + \nu_n^{\frac{1}{2}}\mu_n^{\frac{m_+p_+}{m_+-p_++1}}\Big).$$

Since $\Big\|g_n\Big\|_\infty = g_n(0,0) = 1$, we have $\Big\|h_n\Big\|_\infty = h_n(0,0) = \nu_n^{\frac{1}{2}}$. For any given $\delta > 0$, let

$$S_1 = D(2\delta) \times \Big(\frac{-2d}{\nu_n^{(p_+-2)/2}}, \frac{2d}{\nu_n^{(p_+-2)/2}}\Big), \quad \text{and} \quad S_2 = D(\delta) \times \Big(\frac{-d}{\nu_n^{(p_+-2)/2}}, \frac{d}{\nu_n^{(p_+-2)/2}}\Big).$$

Considering $\mu_n \to 0$ when $n \to \infty$, one can see that $S_2 \subset S_1 \subset D_{n,\delta_0}$. Using Theorem 1.1 in [13] knowing that $N > 1$, we have

$$\Big|h_n\Big(z_1,s_1\Big) - h_n\Big(z_2,s_2\Big)\Big| \leq \gamma\Big(\Big|z_1 - z_2\Big| + \Big|s_1 - s_2\Big|^{\frac{1}{2}}\Big)^{\beta},$$

implying the existence of a function $g \in C\Big(\mathbb{R}^N \times \mathbb{R}\Big)$ such that

$$g_n(z,s) \to g(z,s) \quad \text{in } C_{\text{loc}}\Big(\mathbb{R}^N \times \mathbb{R}\Big), \tag{23}$$

and a subset $Q \subset S_2$ that contains $(0,0)$, where for all $(z,s) \in Q$

$$g_n(z,s) \geq \frac{1}{2}. \tag{24}$$

Set $\kappa \in C_0^\infty\Big(\mathbb{R}^N \times \mathbb{R}\Big)$ be a smooth cut-off function such that

$$\kappa(z,t) = 1 \text{ in } D(r) \times \Big(T_1, T_2\Big), \quad |D\kappa| \leq \frac{C}{r} \quad \text{and} \quad \Big|\frac{\partial\kappa}{\partial s}\Big| \leq \frac{C}{T_2 - T_1}.$$

If we multiply, by $h_n\kappa^\theta$ ($\theta > p_+$), the equation verified by h_n, and integrate over D_n, σ_0, we get

$$\frac{1}{2}\int_{D_n,\delta_0} \partial_s h_n^2 \kappa^\theta dzds + \int_{D_{n,\delta_0}} \mu_n^{\frac{p_+^2(p_+-2)(1-m_+)}{(m_+-p_++1)^2}} \sum_{k=1}^{N}\Big(\Big|\partial_k h_n\Big|^2 + \mu_n^{\frac{2p_+}{m_+-p_++1}}\sigma\Big)^{\frac{p_k(z)-2}{2}} \partial_k h_n \partial_k\Big(h_n\kappa^\theta\Big)dzds$$
$$= \int_{A_{n,\delta_0}} \Big[h g_n^{m(z)-1} h_n + \Big(1-\nu_n\Big)\Big(\nu\sigma^{\frac{p_+-2}{2}}\mu_n^{\frac{p_+(m_+-1)}{m_+-p_++1}} h_n + \nu_n^{\frac{1}{2}}\mu_n^{\frac{m_+p_+}{m_+-p_++1}}\Big)\Big] h_n\kappa^\theta dzds, \tag{25}$$

provided n is sufficiently large so that D_{n,δ_0} contains $D(2r) \times \Big(2T_1 - T_2, 2T_2 - T_1\Big)$.
Notice that

$$\Big|\int_{D_{n,\delta_0}} \partial_s h_n^2 \kappa^\theta dzds\Big| = \Big|\int_{D_{n,\delta_0}} \Big(\partial_s\Big(h_n^2\kappa^\theta\Big) - r h_n^2 \kappa^{\theta-1}\partial_s\kappa\Big)dzds\Big| = \Big|\int_{D_{n,\delta_0}} \theta h_n^2 \kappa^{\theta-1}\partial_s\kappa dzds\Big|$$
$$\leq \nu_n \frac{C}{T_2 - T_1} \text{ meas}\Big(D(2r) \times \Big(2T_1 - T_2, 2T_2 - T_1\Big)\Big) = C\nu_n r^N, \tag{26}$$

On the other hand, we obtain

$$\int_{D_{n,\delta_0}} \mu_n^{\frac{p_+^2(p_+-2)(1-m_+)}{(m_+-p_++1)^2}} \sum_{k=1}^{N} \left(\left|\partial_k h_n\right|^2 + \mu_n^{\frac{2p_+}{m_+-p_++1}}\sigma\right)^{\frac{p_k(z)-2}{2}} \partial_k h_n \partial_k\left(w_n \kappa^\theta\right) dzds$$
$$\leq \int_{D_{n,\delta_0}} \sum_{k=1}^{N} \left(\left|\partial_k h_n\right|^2 + \mu_n^{\frac{2p_+}{m_+-p_++1}}\sigma\right)^{\frac{p_k(z)-2}{2}} \partial_k h_n\left(\partial_k h_n \kappa^\theta + r\kappa^{\theta-1} h_n \partial_k \kappa\right) dzds$$
$$\leq \frac{1}{2}\int_{D_{n,\delta_0}} \kappa^\theta \left|\nabla h_n\right|^{p_+} dzds + \theta \int_{D_n\cdot\delta_0} \sum_{k=1}^{N} \left(\left|\partial_k h_n\right|^2 + \mu_n^{\frac{2p_+}{m_+-p_++1}}\sigma\right)^{\frac{p_k(z)-2}{2}} \partial_k h_n \kappa^{\theta-1} h_n \partial_k \kappa dzds, \quad (27)$$

and

$$\int_{D_{n,\delta_0}} \left[a g_n^{m(z)-1} h_n + \left(1-\nu_n\right)\left(\nu\sigma^{\frac{p_+-2}{2}} \mu_n^{\frac{p_+(m_+-1)}{m_+-p_++1}} h_n + \nu_n^{\frac{1}{2}} \mu_n^{\frac{m+p_+}{m_+-p_++1}}\right)\right] h_n \kappa^\theta dzds \leq C\nu_n \int_{D_{n,\delta_0}} \kappa^\theta dzds \leq C\nu_n r^N \left(T_2 - T_1\right). \quad (28)$$

Furthermore, and using Young's inequality

$$\left|\theta \int_{D_{n,\delta_0}} \sum_{k=1}^{N} \left(\left|\partial_k h_n\right|^2 + \mu_n^{\frac{2p_+}{m_+-p_++1}}\sigma\right)^{\frac{p_k(z)-2}{2}} \partial_k h_n \kappa^{\theta-1} w_n \partial_k \kappa dyds\right|$$
$$\leq C \int_{D_{n,\delta_0}} \sum_{k=1}^{N} \left(\left|\partial_k h_n\right|^{p_k(z)-2} + \mu_n^{\frac{p_+(p_+-2)}{m_+-p_++1}} \sigma^{\frac{p_k-2}{2}}\right) \kappa^{\theta-1} w_n \left|\partial_k h_n\right| \left|\partial_k \kappa\right| dzds$$
$$\leq \frac{1}{4} \sum_{k=1}^{N} \int_{D_{n,\delta_0}} \kappa^\theta \left|\partial_k h_n\right|^{p_k(z)} dzds + C \sum_{k=1}^{N} \int_{D_{n,\delta_0}} w_n^{p_k(z)} \kappa^{\theta-p_k(z)} \left|\partial_k h_n\right|^{p_k(z)} dyds$$
$$+ \frac{1}{4} \sum_{k=1}^{N} \int_{D_{n,\delta_0}} \mu_n^{\frac{p_+(p_+-2)}{m_+-p_++1}} \sigma^{\frac{p_+-2}{2}} \left(\kappa^\theta \left|\partial_k h_n\right|^2 + C w_n^2 \kappa^{\theta-2} \left|\partial_k \kappa\right|^2\right) dzds, \quad (29)$$

Combining the inequalities (25)-(29) yields

$$\int_{D_{n,\delta_0}} \sum_{k=1}^{N} \kappa^\theta \left|\nabla h_n\right|^{p_+} dzds + \sum_{k=1}^{N} \int_{D_{n,\delta_0}} \mu_n^{\frac{p_+(p_+-2)}{m_+-p_++1}} \sigma^{\frac{p_+-2}{2}} \kappa^\theta \left|\nabla h_n\right|^2 dzds$$
$$\leq C\nu_n r^N + C\nu_n s^N \left(T_2 - T_1\right) + C\nu_n^{\frac{p_+}{2}} r^N \left(T_2 - T_1\right)\left(\frac{C}{r}\right)^{p_+} + C\nu_n \sigma^{\frac{p_+-2}{2}} r^N (T_2 - T_1)\left(\frac{C}{r}\right)^2 \mu_n^{\frac{p_+(p_+-2)}{m_+-p_++1}} = C_1 \nu_n,$$

such that the constant C_1 relate only to r and $T_1 - T_2$, so we get for all $r > 0$, and $T_2 > T_1$

$$\nu_n^{\frac{p_+-2}{2}} \sum_{k=1}^{N} \int_{T_1}^{T_2} \int_{B_r} \left|\partial_k g_n\right|^{p_k(z)} dzds \leq C \text{ and } \mu_n^{\frac{p_+(p_+-2)}{m_+-p_++1}} \sigma^{\frac{p_+-2}{2}} \sum_{k=1}^{N} \int_{T_1}^{T_2} \int_{B_r} \left|\partial_k g_n\right|^2 dzds \leq C, \quad (30)$$

In case $\nu_\infty = 0$, then for all $\psi \in C_0^\infty\left(\mathbb{R}^N \times \mathbb{R}\right)$

$$\left|\int_{D_{n,\delta_0}} \sum_{k=1}^{N} \left(\nu_n \left|\partial_k g_n\right|^2 + \mu_n^{\frac{2p_+}{m_+-p_++1}}\sigma\right)^{\frac{p_k(z)-2}{2}} \partial_k g_n \partial_k \psi dzds\right| \leq C \int_{D_{n,\delta_0}} \left(\nu_n^{\frac{p_+-2}{2}} \left|\partial_k g_n\right|^{p_+-2} + \mu_n^{\frac{p_+(p_+-2)}{m_+-p_++1}} \sigma^{\frac{p_+-2}{2}}\right) \left|\partial_k g_n\right| \left|\partial_k \psi\right| dzds$$
$$\leq C\left(\int_{\text{supp}\,\psi} \nu_n^{\frac{p_+-2}{2}} \left|\partial_k g_n\right|^{p_+} dyds\right)^{\frac{p_+-1}{p_+}} \left(\int_{\text{supp}\,\psi} \nu_n^{\frac{p_+-2}{2}} \left|\partial_k \psi\right|^{p_+} dzds\right)^{\frac{1}{p_+}}$$
$$+ C\left(\int_{\text{supp}\,\psi} \mu_n^{\frac{p_+(p_+-2)}{m_+-p_++1}} \sigma^{\frac{p_+-2}{2}} \left|\partial_k g_n\right|^2 dzds\right)^{\frac{1}{2}} \left(\int_{\text{supp}\,\psi} \mu_n^{\frac{p_+(p_+-2)}{m_+-p_++1}} \sigma^{\frac{p_+-2}{2}} \left|\partial_k \psi\right|^2 dzds\right)^{\frac{1}{2}} \leq C_\psi \nu_n^{\frac{p_+-2}{2}} + C_\psi \sigma^{\frac{p_+-2}{4}} \mu_n^{\frac{p_+(p_+-2)}{m_+-p_++1}},$$

As a result

$$\sum_{k=1}^{N}\int_{D_{n,\delta_0}}\left(\nu_n\left|\partial_k g_n\right|^2+\mu_n^{\frac{4p_+}{m_+-p_++1}}\sigma\right)^{\frac{p_k-2}{2}}\partial_k g_n\partial_k\psi dzds\to 0,$$

when $n\to\infty$.
Applying Lebesgue's theorem and (23), we have

$$\int g\partial_s\psi dzds+a\Big(z_\infty,t_\infty\Big)\int g^{m(z)}\psi dzds=0\quad\text{ for all }\psi\text{ in }C_0^\infty\Big(\mathbb{R}^N\times\mathbb{R}\Big),$$

which gives us

$$\partial_s g\Big(z_0,s\Big)=a\Big(z_\infty,t_\infty\Big)g^{m(z)}\Big(z_0,s\Big),\tag{31}$$

for almost all points. Since v is continuous, we demonstrate that the equality (31) holds for all $s\in(-\infty,\infty)$. However, from (23) and (24) we know that there exists of z_0 such that $v\Big(z_0,0\Big)>0$ implying the conclusion that a global solution to (31) cannot exist when $m>p_+-1\geq 1$. This Inconsistency indicates the equality $\nu_\infty=0$ to not be possible. Therefore, since $\nu_\infty\neq 0$ and by using (30), that

$$\sum_{k=1}^{N}\left(\nu_n\left|\partial_k g_n\right|^2+\mu_n^{\frac{4p_+}{m-p_++1}}\sigma\right)^{\frac{p_k(z)-2}{2}}\partial_k g_n\rightharpoonup\xi_k,$$

in $L_{loc}^{\frac{p_+}{p_+-1}}\Big(\mathbb{R}^N\times\mathbb{R}\Big)$. By applying a similar reasoning to that in section 3, we can prove that

$$\xi_k=\nu_\infty^{\frac{p_+-2}{2}}|\partial_k g|^{p_+-2}\partial_k g.$$

Knowing that $\mu_n\to 0$ as $n\to\infty$ as well as the inequality

$$\int\left(\left(\nu_n\left|\partial_k g_n\right|^2+\mu_n^{\frac{4p_+}{m-p_++1}}\sigma\right)^{\frac{p_+-2}{2}}\partial_k g_n-\left(\nu_n\left|\partial_k h_n\right|^2+\mu_n^{\frac{4p_+}{m-p_++1}}\sigma\right)^{\frac{p_+-2}{2}}\partial_k h_n\right)\Big(\partial_k g_n-\partial_k h_n\Big)dzds\geq 0,$$

is fulfilled, we derive that $g\in C\Big(\mathbb{R}^N\times\mathbb{R}\Big)\cap L_{\text{loc}}^{p_+}\Big(\mathbb{R};W_{\text{loc}}^{1,\vec{p}(z)}\Big(\mathbb{R}^N\Big)\Big)$ and satisfy the following equation

$$\partial_s g=s_\infty^{\frac{p_+-2}{2}}\sum_{k=1}^{N}\text{div}\left(|\partial_k g|^{p_k(z)-2}\partial_k g\right)+a\Big(z_\infty,t_\infty\Big)g^{m(z)},\quad(z,s)\in\mathbb{R}^N\times\mathbb{R}.\tag{32}$$

However, based on [21, 25], we deduce that the Cauchy problem for (32) cannot have a global nontrivial nonnegative solution when $p_+-1<m<p_--1+\frac{p_-}{N}$. This contradiction implies that the first statement of our lemma holds. The second statement of the lemma can be proven in a similar manner. □

Lemma 5.2. *Assuming the conditions of Theorem 2.4 hold, there exists $R>r$ where* $\deg\left(\mathcal{I}-\mathcal{M}(1,\Psi(\cdot)),B_R(0),0\right)=0$.

Proof. Let $\hat{\Psi}(u)(z,t)=a(z,t)f_+^m+\nu f_++1$ for $f\in C_\tau(\overline{D_\tau})$ and define $G(\nu,f)=\mathcal{M}(\nu,\lambda\Phi(u)+(1-\nu)\hat{\Psi}(f))$. According to Lemma 3.1 and Lemma 3.2, $G(\nu,\cdot)$ is well defined from $C_\tau(\overline{D_\tau})$ to $C_\tau(\overline{D_\tau})$ and is compact. Hence, by applying the homotopy invariance of the degree, we get

$$\deg\left(\mathcal{I}-\mathcal{M}(1,\Psi(\cdot)),B_R(0),0\right)=\deg\left(\mathcal{I}-\mathcal{M}(0,\hat{\Phi}(\cdot)),B_R(0),0\right),$$

provided that

$$G(\nu,f)\neq f\quad\text{ for any }f\in\partial B_R(0),\ \lambda\in[0,1].$$

Actually, Lemma 5.1 shows that this equality holds for $R > \max\{L, r\}$.
Additionally, following a close approach to the demonstration of Lemma 5.1, there is no nonnegative periodic solution for (22). Therefore, we conclude that $\deg\left(\mathcal{I} - \mathcal{M}(0, \hat{\Psi}(\cdot)), B_R(0), 0\right) = 0$ which indicates

$$\deg(\mathcal{I} - \mathcal{M}(1, \Psi(\cdot)), B_R(0), 0\Big) = 0.$$

□

6. Conclusion

To summarize everything, we get the result of Theorem 2.4 based on the previous lemmas and the proposition. In fact, after combining Lemma 4.1 and Lemma 5.2, one achieves that

$$\deg\left(\mathcal{I} - \mathcal{M}(1, \Phi(\cdot)), B_R(0) \backslash B_r(0), 0\right) = -1,$$

translating to the existence of a periodic solution f_σ to (3), satisfying

$$r \leq \max_{\overline{D_\tau}} u_\sigma(z, t) \leq R \quad \text{for any } 0 < \sigma < 1.$$

This proves of the previously claimed proposition. Theorem 3 in [35] and the uniform boundedness of f_σ suffice to get the uniform Hölder continuity of f_σ allowing the application of the Arzelà-Ascoli theorem in order to infer the existence of a map $f \in C_\tau(\overline{D_\tau})$ and a subsequence of $\left\{f_\sigma\right\}$, which we will also denote by $\left\{f_\sigma\right\}$ without loss of generality, such that

$$f_\sigma \to f \quad \text{uniformly in } \overline{D_\tau}.$$

Recall that $r \leq \max_{\overline{D_\tau}} f(z, t) \leq R$. Furthermore, based on Lemma 3.1 and the proof of Lemma 3.2, the following inequalities hold true

$$\left\|\partial_t f_\sigma\right\|_2 \leq C_1, \quad \int_{D_\tau} \sum_{k=1}^{N} \left(\nu\left|\partial_t f_\sigma\right|^2 + \sigma\right)^{\frac{p_k(z)-2}{2}} \left|\partial_t f_\sigma\right|^2 dz dt \leq C_2,$$

whith C_1 and C_2 being two constants independent of σ. In an equivalent way to section 3 we can show that f is actually a periodic solution to the equation (1). Which leads us to the conclusion of Theorem 2.4.

References

[1] A. Abbassi, C. Allalou, A. Kassidi, *Existence of weak solutions for nonlinear p-elliptic problem by topological degree*, Nonlinear Dyn. Syst. Theory **20** (2020), 229–241.

[2] A. Abbassi, C. Allalou, A. Kassidi, *Existence results for some nonlinear elliptic equations via topological degree methods*, J. Elliptic Parabolic Equ. **7** (2021), 121–136.

[3] A. Abbassi, C. Allalou, A. Kassidi, *Topological degree methods for a Neumann problem governed by nonlinear elliptic equation*, Moroccan J. Pure Appl. Anal. **6** (2020), 231–242.

[4] A. Aberqi, J. Bennouna, O. Benslimane, M. A. Ragusa, *Existence results for double phase problem in Sobolev–Orlicz spaces with variable exponents in complete manifold*, Mediterr. J. Math. **19** (2022), 158.

[5] S. Antontsev, M. Chipot, *Anisotropic equations: uniqueness and existence results*, Differ. Integral Equ. **6** (2008), 401–419.

[6] S. Antontsev, S. Shmarev, *Evolution PDEs with nonstandard growth conditions*, Atlantis Studies in Differential Equations **4** (2015).

[7] M. Bendahmane, K. H. Karlsen, *Anisotropic nonlinear elliptic systems with measure data and anisotropic harmonic maps into spheres*, Electron. J. Differ. Equ. **46** (2006), 1–30.

[8] M. Bendahmane, P. Wittbold, A. Zimmermann, *Renormalized solutions for a nonlinear parabolic equation with variable exponents and L^1-data*, J. Differ. Equ. **148** (2010), 1483–1515.

[9] J. Berkovits, *Extension of the Leray–Schauder degree for abstract Hammerstein type mappings*, J. Differ. Equ. **234** (2007), 289–310.

[10] A. Charkaoui, G. Kouadri, O. Selt, N. E. Alaa, *Existence results of weak periodic solution for some quasilinear parabolic problem with L^1 data*, Ann. Univ. Craiova Math. Comput. Sci. Ser. **46** (2019), 66–77.

[11] Y. Chen, S. Levine, M. Rao, *Variable exponent linear growth functionals in image restoration*, SIAM J. Appl. Math. **66** (2006), 1383–1406.

[12] J. Chen, Z. Zhang, G. Chang, J. Zhao, *Periodic solutions to Klein–Gordon systems with linear couplings*, Adv. Nonlinear Stud. **21** (2021), 633–660.

[13] E. Di Benedetto, *Degenerate parabolic equations*, Springer-Verlag, New York, 1993.

[14] E. Di Benedetto, *On the local behavior of solutions of degenerate parabolic equations with measurable coefficients*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (4) **13** (1986), 487–535.

[15] M. Di Francesco, A. Lorz, P. Markowich, *Chemotaxis-fluid coupled model for swimming bacteria with nonlinear diffusion: global existence and asymptotic behavior*, Discrete Contin. Dyn. Syst. **28** (2010), 1437–1453.

[16] L. Diening, P. Harjulehto, P. Hästö, M. Růžička, *Lebesgue and Sobolev spaces with variable exponents*, Springer, 2011.

[17] M. Esteban, *A remark on the existence of positive periodic solutions of superlinear parabolic problems*, Proc. Am. Math. Soc. **102** (1988), 131–136.

[18] M. Esteban, *On periodic solutions of superlinear parabolic problems*, Trans. Am. Math. Soc. **293** (1986), 171–189.

[19] X. L. Fan, D. Zhao, *On the spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$*, Anal. Appl. **263** (2001), 424–446.

[20] X. Fan, *Anisotropic variable exponent Sobolev spaces and $\vec{p}(x)$-Laplacian equations*, Complex Var. Elliptic Equ. **56** (2011), 623–642.

[21] H. Fujita, *On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$*, J. Fac. Sci. Univ. Tokyo Sect. I A Math. **16** (1966), 105–113.

[22] H. Jourhmane, A. Kassidi, M. Elomari, K. Hilal, *Existence of periodic solution for double-phase parabolic problems with strongly nonlinear source*, Filomat **37** (2023), 9357–9370.

[23] H. Jourhmane, A. Kassidi, K. Hilal, M. Elomari, *Periodic solutions for a degenerate double-phase parabolic equation with variable growth*, Adv. Oper. Theory **8** (2023), 71.

[24] H. Jourhmane, A. Kassidi, M. Elomari, K. Hilal, *Existence of non-negative periodic solutions for a degenerate anisotropic parabolic equation with strongly nonlinear source*, Analysis (2023).

[25] Z. Junning, *On the Cauchy problem and initial traces for the evolution p-Laplacian equations with strongly nonlinear sources*, J. Differ. Equ. **121** (1995), 329–383.

[26] B. Gidas, W. M. Ni, L. Nirenberg, *Symmetry and related properties via the maximum principle*, Commun. Math. Phys. **68** (1979), 203–243.

[27] R. Huang, Y. Wang, Y. Ke, *Existence of non-trivial non-negative periodic solutions for a class of degenerate parabolic equations with non-local terms*, Discrete Contin. Dyn. Syst. B **5** (2005), 1005–1014.

[28] O. A. Ladyzhenskaja, V. A. Solonnikov, N. N. Ural'tzeva, *Linear and quasilinear equations of parabolic type*, Transl. Math. Monogr., Vol. 23, Amer. Math. Soc., Providence, 1968.

[29] A. C. Lazer, P. J. McKenna, *Large-amplitude periodic oscillations in suspension bridges: some new connections with nonlinear analysis*, SIAM Rev. **32** (1990), 537–578.

[30] F. Q. Li, *Anisotropic elliptic equations in L^m*, J. Convex Anal. **2** (2001), 417–422.

[31] K. Li, L. Ruan, A. Yang, *Periodic entropy solution to a conservation law with nonlocal source arising in radiative gas*, J. Math. Anal. Appl. **512** (2022), 117–126.

[32] N. Mizoguchi, *Periodic solutions for degenerate diffusion equations*, Indiana Univ. Math. J. **44** (1995), 413–432.

[33] M. Mihăilescu, P. Pucci, V. Rădulescu, *Eigenvalue problems for anisotropic quasilinear elliptic equations with variable exponent*, J. Math. Anal. Appl. **340** (2008), 687–698.

[34] M. Nakao, *Periodic solutions of some nonlinear degenerate parabolic equations*, J. Math. Anal. Appl. **104** (1984), 554–567.

[35] Y. Ohara, *L^∞-estimates of solutions of some nonlinear degenerate parabolic equations*, Nonlinear Anal. **18** (1992), 413–426.

[36] M. A. Ragusa, A. Tachikawa, *Regularity for minimizers for functionals of double phase with variable exponents*, Adv. Nonlinear Anal. **9** (2020), 710–728.

[37] E. Scharfenaker, *Statistical equilibrium methods in analytical political economy*, J. Econ. Surv. **36** (2022), 276–309.

[38] T. I. Seidman, *Periodic solutions of a nonlinear parabolic equation*, J. Differ. Equ. **19** (1975), 242–257.

[39] M. Troisi, *Teoremi di inclusione per spazi di Sobolev non isotropi*, Ricerche Mat. **18** (1969), 3–24.

[40] V. V. Zhikov, *On Lavrentiev's phenomenon*, Russ. J. Math. Phys. **3** (1995), 249–269.

[41] V. V. Zhikov, *On some variational problems*, Russ. J. Math. Phys. **5** (1997), 105–116.

[42] V. V. Zhikov, S. M. Kozlov, O. A. Oleinik, *Homogenization of differential operators and integral functionals*, Springer-Verlag, Berlin, 1994.

[43] W. Zhuoqun, Z. Junning, Y. Jingxue, L. Huilai, *Nonlinear diffusion equations*, Jilin Univ. Press, Changcun, 1996.