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Hermite-Hadamard inequalities for exponential convexity classes with applications

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Abstract. This paper presents a comprehensive analysis of (s, m)-exponential-type functions which are convex, emphasizing their foundational algebraic properties. Building upon this framework, we drive class of latest results of Hermite-Hadamard-type inequalities by using Caputo-Fabrizio fractional integral operator. This study introduces improved integral inequalities for functions exhibiting (s, m)-exponential convexity in their powered absolute derivatives. The efficacy of these results is demonstrated through refined estimations of special means and tighter error bounds in trapezoidal and midpoint integration schemes.

1. Introduction

Convexity is a fundamental concept in mathematical analysis, particularly valuable in the derivation of error bounds across a range of analytical and computational methods. In numerical integration, for instance, convex functions are instrumental in estimating the accuracy of approximation techniques such as the trapezoidal rule. Their influence extends beyond numerical analysis to nonlinear optimization problems and inequalities involving special means. Since the seminal work of Jensen on convex inequalities, the theoretical framework surrounding convex functions has seen substantial growth, with widespread applications in areas including optimization, probability theory, and functional analysis.

Significant strides have been made in the theory of convex inequalities, particularly within numerical integration (see [1–4]). These advances have led to generalized convexity models that enable the construction of sharper inequalities for complex analytical tasks. The integration of convex analysis and inequality frameworks has produced essential results such as the Hermite–Hadamard, Simpson, and Ostrowski inequalities. These classical forms have since been generalized through the use of extended convexity types

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and fractional operators, including those from the Riemann–Liouville and Caputo–Fabrizio formulations, as well as k-fractional systems.

Convexity plays a crucial role in estimating error bounds for different mathematical problems. For instance, numerical integration methods, such as the trapezoidal rule, utilize convexity to assess error estimates. Similarly, nonlinear programming techniques have been applied to special means, further demonstrating the versatility of convexity. Since Jensen's introduction of the first convex inequality, extensive research has been dedicated to convex inequalities, which have found applications in optimization, probability theory, and functional analysis. For further references on quadrature methods, see [1–4] and the works cited therein. Researchers employ a wide range of methods and innovative approaches to extend and refine the theory of convexity. Over the years, numerous generalizations have emerged, allowing for the resolution of complex problems in both theoretical and applied sciences. The growing connection between convexity and inequalities has significantly broadened its scope, leading to various fundamental results in mathematical analysis. Many inequalities documented in the literature stem from the application of convexity in disciplines such as numerical analysis, optimization, and probability theory. Many authors employ various fractional operators to obtain error estimates for different types of inequalities [30–32].

The concept of generalized convexity enables a broader class of inequalities, including Hermite–Hadamard, Simpson, Ostrowski, Opial, and Bullen inequalities. These inequalities have been formulated using diverse convexity classes and different integral operators. The Caputo fractional derivative, introduced by Michele Caputo in 1967, is particularly significant due to its non-singular kernel, which allows transformation into an integral using the Laplace transform. This operator is frequently applied in physical models, where it provides clear and precise interpretations. Recent studies have explored the relationship between integral inequalities and fractional operators, leading to the development of new inequalities using various convexity classes [5–8]. Researchers have employed the Caputo–Fabrizio operator to establish inequalities through exponentially convex mappings and s-convex functions. Additionally, generalizations of Hermite–Hadamard-type inequalities have been proposed for functions exhibiting different forms of convexity. Several works have introduced novel bounds and refined inequalities using the Caputo–Fabrizio fractional integral operator [9–11]. For a comprehensive review of these developments, see Refs. [12–16]. This property provides both iff conditions for function will be convex and serves as a foundation for the classical Hermite–Hadamard inequality as follows:

$$\Omega\left(\frac{\theta_a+\theta_b}{2}\right)\leq\frac{1}{\theta_b-\theta_a}\int_{\theta_a}^{\theta_b}\Omega\left(\varrho\right)d\varrho\leq\frac{\Omega\left(\theta_a\right)+\Omega\left(\theta_b\right)}{2}.$$

Definition 1.1. *[17] A function Ω: $[0,+\infty)\to\mathbb{R}$ is said to be s-convex for some real $s\in(0,1]$ or equivalently, Ω is a member of the class K_s^2, if it satisfies the following condition:*

$$\Omega\left(\varrho\theta_a+(1-\varrho)\theta_b\right)\leq\varrho^s\Omega\left(\theta_a\right)+(1-\varrho)^s\,\Omega\left(\theta_b\right).$$

G. Toader invent the idea of m-convexity in [18] as a parameter-based extension of classical convexity. This concept has since become an active area of research in both analysis and optimization.

Definition 1.2. $\Omega:[0,a_n)\to\mathbb{R}$*, $a_n>0$, is said to be m-convex where $m\in(0,1]$, if*

$$\Omega\left(\varrho\theta_a+m\left(1-\varrho\right)\theta_b\right)\leq\varrho\Omega\left(\theta_a\right)+m\left(1-\varrho\right)\Omega\left(\theta_b\right).$$

The inequality is required to hold for all $\theta_a,\theta_b\in[0,a_n)$ and $\varrho\in[0,1]$. A function Ω is classified as m-convex if this condition is satisfied throughout the specified domain. On the other hand, is referred to as m-concave when the negative of the function, meets the criteria for m-convexity. Building on this framework, Eftekhari [19] recently proposed the notion of (s,m)-convexity in the second sense, characterized as follows:

Definition 1.3. *Given function $\Omega:[0,+\infty)\to\mathbb{R}$ is referred to as (s,m)-convex in the second sense for a given real number $s\in(0,1]$, if it satisfies the following condition:*

$$\Omega\left(\varrho\theta_a+m\left(1-\varrho\right)\theta_b\right)\leq\varrho^s\Omega\left(\theta_a\right)+m\left(1-\varrho\right)^s\Omega\left(\theta_b\right).$$

holds for all $\theta_a, \theta_b \in [0, +\infty)$ and $\varrho \in [0, 1]$.

In recent years, fractional calculus has emerged as a rapidly advancing field, offering generalized definitions of derivatives and integrals of non-integer order. Its conceptual roots trace back to a 1695 correspondence between Leibniz and L Hôpital. Modern interest in fractional calculus has grown substantially due to the limitations of classical (integer-order) methods in capturing complex real-world phenomena. This branch of analysis extends traditional techniques, allowing differentiation and integration to be defined for arbitrary (real or complex) orders, thus enhancing precision in various mathematical frameworks.

Fractional differential and integral equations have proven to be highly effective in modeling diverse systems. Their flexibility, stemming from the use of fractional-order terms, allows for more nuanced and accurate representations than those offered by standard integer-order models. Importantly, fractional integral inequalities play a crucial role in analyzing the stability and uniqueness of solutions to these equations. Today, fractional calculus is widely applied across numerous scientific and engineering domains, such as electrical and mechanical systems, control theory, viscoelastic materials, rheology, optics, and nonlinear physical models highlighting its broad interdisciplinary relevance [20–25].

2. Analytical foundations and Algebraic Behavior of (s, m)–Exponential Type Convex Functions

In this section, we introduce a new concept known as the convexity of (s, m)-exponential function and explore its fundamental algebraic properties.

Definition 2.1. *Function which defined on only nonnegative values Ω: $A \to \mathbb{R}$ is (s, m)-exponential type convex for $s, m \in (0, 1]$, if*

$$\Omega(\varrho\theta_a + m(1-\varrho)\theta_b) \leq e^{s\varrho}\Omega(\theta_a) + m\left(e^{(1-\varrho)^s} - 1\right)\Omega(\theta_b),$$

for all verify $\theta_a, \theta_b \in A$ and $\varrho \in [0, 1]$.

Remark 2.2. *When $m = s = 1$ the structure aligns with the exponential-type convexity framework established by Iscan in [26].*

Remark 2.3. *The collection of (s, m)–exponential convex functions, where $m \in (0, 1]$ and $s \in [ln2.5, 1]$ is defined over the interval $[0, +\infty)$.*

Proof. Consider an arbitrary $t \in A$, with fixed parameters $m \in (0, 1]$ and $s \in [ln2.5, 1]$. Utilizing Definition 2.1 the case $u = 1$, we derive following result:

$$\Omega(u) \leq (e^s - 1)\Omega(u) \Rightarrow (e^s - 2)\Omega(u) \geq 0 \Rightarrow \Omega(u) \geq 0.$$

□

Lemma 2.4. *For every $\varrho \in (0, 1]$, given fixed parameters $m \in (0, 1]$ and $s \in [ln2.5, 1]$, following holds:*

$$(e^{s\varrho} - 1) \geq \varrho^s \text{ and } \left(e^{(1-\varrho)^s} - 1\right) \geq (1-\varrho)^s.$$

Proof. The proof follows in a standard manner. □

Proposition 2.5. *Any nonnegative function that is (s, m)-convex is also satisfies the conditions of (s, m)-exponential type convexity for specific values of $m \in (0, 1]$ and $s \in [ln2.5, 1]$.*

Proof. By using of Lemma (2.4) under the assumption that $m \in (0, 1]$ and $s \in [ln2.5, 1]$ are fixed, we arrive at the following conclusion: □

$$\begin{aligned}\Omega(\varrho\theta_a + m(1-\varrho)\theta_b) &\leq \varrho^s\Omega(\theta_a) + m(1-\varrho)^s\Omega(\theta_b) \\ &\leq (e^{s\varrho} - 1)\Omega(\theta_a) + m\left(e^{(1-\varrho)^s} - 1\right)\Omega(\theta_b).\end{aligned}$$

Theorem 2.6. *Let $\Omega, \phi : [a_1, a_2] \to \mathbb{R}$. If Ω and ϕ are (s, m)-exponential convexity for few fixed $s, m \in (0, 1]$, then have:*

1. $\Omega + \phi$ are (s, m)-exponential function which fulfill convexity.

2. For real number (nonnegative) b, $b\Omega$ is (s, m)-exponential function which fulfill convexity.

Proof. Based on definition (2.1) and assuming fixed values $s, m \in (0, 1]$, the conclusion follows immediately. □

Definition 2.7. *[27, 28] Consider $H^1(a_1, a_2)$, the Sobolev space of first order, defined as follows:*

$$H^1(a_1, a_2) = \left\{ g \in L^2(a_1, a_2) : g' \in L^2(a_1, a_2) \right\},$$

where

$$L^2(a_1, a_2) = \left\{ g(j) : \left(\int_{a_1}^{a_2} g^2(j)\, dj \right)^{\frac{1}{2}} < \infty \right\}.$$

Let $f \in H^1(a_1, a_2)$, $a_1 < a_2$, $\sigma \in [0, 1]$, then the notion of left derivative in the sense of Caputo-Fabrizio is defined as:

$$\left({}_{a_1}^{CFD}D^{\xi} f \right)(j) = \frac{\beta(\sigma)}{1-\sigma} \int_{a_1}^{j} f'(t)\, e^{\frac{-\sigma(j-u)^{\sigma}}{1-\sigma}}\, du,$$

$j > \sigma$ and the associated integral operator is:

$$\left({}_{a_1}^{CF}I^{\sigma} f \right)(j) = \frac{1-\sigma}{\beta(\sigma)} f(j) + \frac{\sigma}{\beta(\sigma)} \int_{a_1}^{j} f(t)\, du,$$

where $\beta(\sigma) > 0$ is the normalization function satisfying $\beta(0) = \beta(1) = 1$. For $\sigma = 0$, $\sigma = 1$, the definition of the right derivative is given below.

$$\begin{aligned} \left({}_{a_1}^{CFD}D^{0} f \right)(j) &= f'(z) \\ \left({}_{a_1}^{CFD}D^{1} f \right)(j) &= f(j) - f(a_1). \end{aligned}$$

For the right derivative operator:

$$\left({}_{a_2}^{CFD}D^{\sigma} f \right)(j) = \frac{\beta(\sigma)}{1-\sigma} \int_{j}^{b} f'(t)\, e^{\frac{-\sigma(t-j)^{\sigma}}{1-\sigma}}\, du,$$

$j < a_2$ and the associated integral operator is:

$$\left({}^{CF}I_{a_2}^{\sigma} f \right)(j) = \frac{1-\sigma}{\beta(\sigma)} f(j) + \frac{\sigma}{\beta(\sigma)} \int_{j}^{b} f(u)\, du,$$

here, $\beta(\sigma) > 0$ denotes a normalization function with the property that $\beta(0) = \beta(1) = 1$.

To the best of our knowledge, this work is the first to establish Hermite-Hadamard inequalities for (s, m)-exponential type convex functions via the Caputo-Fabrizio fractional integral framework, marking a significant advancement in the field. These convexity classes are particularly significant because they unify and generalize various known types of convexity through appropriate parameter choices. The results also demonstrate how our approach extends several existing inequalities in the literature. Additionally,This work develops improved Hermite-Hadamard-type inequalities under the assumption that the absolute value of the first derivative, elevated to a fixed exponent, satisfies (s, m)-exponential-type convexity. Additionally, we provide applications in deriving novel bounds for certain means and in analyzing error estimates for the trapezoidal and midpoint quadrature formulas. Following a review of related literature, Section 2 introduces the concept of (s,m)-exponential-type convexity and investigates its key algebraic characteristics. Section 3 presents generalized Hermite-Hadamard type inequalities for such convex functions, including inequalities involving the product of two (s,m)-exponentially convex functions. In Section 4, we provide refined versions of these inequalities by incorporating the Caputo-Fabrizio fractional integral operator and analyzing functions whose derivatives meet specific convexity conditions. Section 5 offers practical applications, focusing on bounds for special means and deriving error estimates for numerical rules. The paper concludes in Section 6 with a concise summary of the main theoretical and applied outcomes.

3. Refined Hermite-Hadamard Type Inequalities

We focus on developing sharpened forms of the Hermite-Hadamard inequality under the assumption that the powered absolute value of the first derivative adheres to (s,m)-exponential-type convexity, employing the Caputo-Fabrizio fractional integral framework.

Lemma 3.1. *Let $0 < k \leq 1$ and consider a function $\Omega : \left[a_1, \frac{a_2}{k}\right] \to \mathbb{R}$ is differentiable on its domain $\left(a_1, \frac{a_2}{k}\right)$ with $0 < a_1 < a_2$. If $\Omega' \in L_1\left[a_1, \frac{a_2}{k}\right]$, the following equality is satisfied:*

$$\begin{aligned}
&\frac{\Omega(a_1) + \Omega\left(\frac{a_2}{k}\right)}{2} - \frac{\beta(\sigma)k}{\sigma(a_2 - ka_1)}\left[\left({}^{CF}_{a_1}I^{\sigma} f\right)(u) + \left({}^{CF}I^{\sigma}_{\frac{a_2}{k}} f\right)(u)\right] - \frac{2(1-\sigma)}{\sigma(a_2 - ka_1)} \\
=\ &\left(\frac{a_2 - ka_1}{2k}\right)\int_0^1 (1-2u)\,\Omega'\left(ua_1 + (1-u)\frac{a_2}{k}\right)dt.
\end{aligned}$$

Proof. *Utilizing the method of integration by parts, we derive the following result:*

$$\begin{aligned}
&\left(\frac{a_2 - ka_1}{2k}\right)\int_0^1 (1-2u)\,\Omega'\left(ua_1 + (1-u)\frac{a_2}{k}\right)du \\
=\ &\left(\frac{a_2 - ka_1}{2k}\right)\left[\left.\frac{(1-2u)\,\Omega\left(ua_1 + (1-u)\frac{a_2}{k}\right)}{a_1 - \frac{a_2}{k}}\right|_0^1 - \int_0^1 \frac{\Omega\left(ua_1 + (1-u)\frac{a_2}{k}\right)}{a_1 - \frac{a_2}{k}}(-2)\,du\right] \\
=\ &\left(\frac{a_2 - ka_1}{2k}\right)\left[\frac{-\Omega(a_1) - \Omega\left(\frac{a_2}{k}\right)}{a_1 - \frac{a_2}{k}} + \frac{2k}{a_2 - ka_1}\int_0^1 \Omega\left(ua_1 + (1-u)\frac{a_2}{k}\right)du\right] \\
=\ &\left(\frac{a_2 - ka_1}{2k}\right)\left[\frac{k\left(\Omega(a_1) + \Omega\left(\frac{a_2}{k}\right)\right)}{a_1 - \frac{a_2}{k}} - \frac{2k}{a_2 - ka_1}\int_0^1 \Omega\left(ua_1 + (1-u)\frac{a_2}{k}\right)du\right] \\
=\ &\frac{\Omega(a_1) + \Omega\left(\frac{a_2}{k}\right)}{2} - \frac{k}{a_2 - ka_1}\int_{a_1}^{\frac{a_2}{k}} \Omega(\theta)\,d\theta.
\end{aligned}$$

Adding both sides $\frac{2(1-\sigma)}{\beta(\sigma)(a_2 - ka_1)}$, we have

$$\begin{aligned}
&\left(\frac{a_2 - ka_1}{2k}\right)\int_0^1 (1-2u)\,\Omega'\left(ua_1 + (1-u)\frac{a_2}{k}\right)du + \frac{2(1-\sigma)}{\beta(\sigma)(a_2 - ka_1)} \\
=\ &\frac{\Omega(a_1) + \Omega\left(\frac{a_2}{k}\right)}{2} - \frac{k}{a_2 - ka_1}\int_{a_1}^{\frac{a_2}{k}} \Omega(\theta)\,d\theta + \frac{2(1-\sigma)}{\beta(\sigma)(a_2 - ka_1)} \\
=\ &\frac{\Omega(a_1) + \Omega\left(\frac{a_2}{k}\right)}{2} - \frac{k}{\beta(\sigma)(a_2 - ka_1)}\left[\int_{a_1}^{u} f(u)\,du + \frac{(1-\sigma)}{\beta(\sigma)} + \int_t^{\frac{a_2}{k}} f(u)\,du + \frac{(1-\sigma)}{\beta(\sigma)}\right]
\end{aligned}$$

$$= \frac{\Omega(a_1)+\Omega\left(\frac{a_2}{k}\right)}{2} - \frac{\beta(\sigma)k}{\sigma(a_2-ka_1)}\left[\left({}^{CF}_{a_1}I^{\sigma}f\right)(u)+\left({}^{CF}I^{\sigma}_{\frac{a_2}{k}}f\right)(u)\right].$$

As a result, we obtain the following expression

$$\frac{\Omega(a_1)+\Omega\left(\frac{a_2}{k}\right)}{2} - \frac{\beta(\sigma)k}{\sigma(a_2-ka_1)}\left[\left({}^{CF}_{a_1}I^{\sigma}f\right)(u)+\left({}^{CF}I^{\sigma}_{\frac{a_2}{k}}f\right)(u)\right] - \frac{2(1-\sigma)}{\sigma(a_2-ka_1)}$$
$$= \left(\frac{a_2-ka_1}{2k}\right)\int_0^1 (1-2u)\,\Omega'\left(ua_1+(1-u)\frac{a_2}{k}\right)du.$$

This concludes the proof of Lemma 3.1. □

Remark 3.2. *If we put $k=1$ in our Lemma 3.1 then we get Lemma (2.3) in [15].*

Lemma 3.3. *Let $0<k\le 1$, and consider a function $\Omega:[ka_1,a_2]\to\mathbb{R}$ is differentiable on the open interval (ka_1,a_2) where $0<a_1<a_2$. If the derivative Ω' belongs to the space $L_1[ka_1,a_2]$, then the following identity holds:*

$$\frac{\Omega(ka_1)+\Omega(a_2)}{2} - \frac{\beta(\sigma)}{\sigma(a_2-ka_1)}\left[\left({}^{CF}_{ka_1}I^{\sigma}f\right)(u)+\left({}^{CF}I^{\sigma}_{a_2}f\right)(u)\right] - \frac{2(1-\sigma)}{\sigma(a_2-ka_1)}$$
$$= \left(\frac{a_2-ka_1}{2}\right)\int_0^1 (2u-1)\,\Omega'\left(k(1-u)a_1+ua_2\right)du.$$

Proof. Utilizing the method of integration by parts, we derive the following result:

$$\left(\frac{a_2-ka_1}{2}\right)\int_0^1 (2u-1)\,\Omega'\left(k(1-u)a_1+ua_2\right)du$$
$$= \left(\frac{a_2-ka_1}{2}\right)\left[\frac{(2u-1)\Omega(k(1-u)a_1+ua_2)}{a_2-ka_1}\Bigg|_0^1 - \int_0^1 \frac{\Omega(k(1-u)a_1+ta_2)}{a_2-ka_1}(2)\,du\right]$$
$$= \left(\frac{a_2-ka_1}{2}\right)\left[\frac{\Omega(a_2)+\Omega(ka_1)}{a_2-ka_1} - \frac{2}{a_2-ka_1}\int_0^1 \Omega(k(1-u)a_1+ua_2)\,du\right]$$
$$= \left(\frac{a_2-ka_1}{2}\right)\left[\frac{\Omega(a_2)+\Omega(ka_1)}{a_2-ka_1} - \frac{2}{a_2-ka_1}\int_0^1 \Omega(k(1-u)a_1+ua_2)\,du\right]$$
$$= \frac{\Omega(ka_1)+\Omega(a_2)}{2} - \frac{1}{a_2-ka_1}\int_{ka_1}^{a_2}\Omega(\theta)\,d\theta.$$

Adding both sides $\frac{2(1-\sigma)}{\beta(\sigma)(a_2-ka_1)}$, we have

$$\left(\frac{a_2-ka_1}{2}\right)\int_0^1 (2u-1)\,\Omega'\left(k(1-u)a_1+ua_2\right)du + \frac{2(1-\sigma)}{\beta(\sigma)(a_2-ka_1)}$$
$$= \frac{\Omega(ka_1)+\Omega(a_2)}{2} - \frac{1}{a_2-ka_1}\int_{ka_1}^{a_2}\Omega(\theta)\,d\theta + \frac{2(1-\sigma)}{\beta(\sigma)(a_2-ka_1)}$$
$$= \frac{\Omega(ka_1)+\Omega(a_2)}{2} - \frac{1}{\beta(\sigma)(a_2-ka_1)}\left[\int_{ka_1}^{u} f(u)\,du + \frac{(1-\sigma)}{\beta(\sigma)} + \int_u^{a_2} f(u)\,du + \frac{(1-\sigma)}{\beta(\sigma)}\right]$$
$$= \frac{\Omega(ka_1)+\Omega(a_2)}{2} - \frac{\beta(\sigma)}{\sigma(a_2-ka_1)}\left[\left({}^{CF}_{ka_1}I^{\sigma}f\right)(u)+\left({}^{CF}I^{\sigma}_{a_2}f\right)(u)\right].$$

As a result, we obtain the following expression

$$\frac{\Omega(ka_1)+\Omega(a_2)}{2} - \frac{\beta(\sigma)}{\sigma(a_2-ka_1)}\left[\left({}^{CF}_{ka_1}I^{\sigma}f\right)(u)+\left({}^{CF}I^{\sigma}_{a_2}f\right)(u)\right] - \frac{2(1-\sigma)}{\sigma(a_2-ka_1)}$$

$$= \left(\frac{a_2 - ka_1}{2}\right)\int_0^1 (2u-1)\,\Omega'\ (k(1-u)a_1 + ua_2)\,du.$$

This concludes the proof of Lemma 3.3. □

Lemma 3.4. *Let* $0 < k \le 1$, *and consider a function* $\Omega : [ka_1, a_2] \to \mathbb{R}$ *is differentiable on the open interval* (ka_1, a_2) *where* $0 < a_1 < a_2$. *If the derivative* Ω' *belongs to the space* $L_1[ka_1, a_2]$, *then the following identity holds:*

$$\frac{\Omega(ka_1) + \Omega(a_2)}{k+1} - \frac{\beta(\sigma)}{\sigma(a_2 - ka_1)}\left[\left({}^{CF}_{ka_1}I^{\sigma} f\right)(u) + \left({}^{CF}I^{\sigma}_{a_2} f\right)(u)\right] - \frac{2(1-\sigma)}{\sigma(a_2 - ka_1)}$$
$$= \left(\frac{a_2 - ka_1}{k+1}\right)\int_0^1 (2u-1)\,\Omega'\ (k(1-u)a_1 + ua_2)\,du.$$

Proof. Utilizing the method of integration by parts, we derive the following result:

$$\left(\frac{a_2 - ka_1}{k+1}\right)\int_0^1 (2u-1)\,\Omega'\ (k(1-u)a_1 + ua_2)\,du$$
$$= \left(\frac{a_2 - ka_1}{k+1}\right)\left[\left.\frac{(2u-1)\,\Omega(k(1-u)a_1 + ua_2)}{a_2 - ka_1}\right|_0^1 - \int_0^1 \frac{\Omega(k(1-u)a_1 + ua_2)}{a_2 - ka_1}(2)\,du\right]$$
$$= \left(\frac{a_2 - ka_1}{k+1}\right)\left[\frac{\Omega(a_2) + \Omega(ka_1)}{a_2 - ka_1} - \frac{2}{a_2 - ka_1}\int_0^1 \Omega(k(1-u)a_1 + ua_2)\,du\right]$$
$$= \frac{\Omega(a_2) + \Omega(ka_1)}{k+1} - \frac{2}{k+1}\int_0^1 \Omega(k(1-u)a_1 + ta_2)\,du$$
$$= \frac{\Omega(ka_1) + \Omega(a_2)}{k+1} - \frac{1}{(k+1)(a_2 - ka_1)}\int_{ka_1}^{a_2} \Omega(\theta)\,d\theta.$$

Adding both sides $\frac{2(1-\sigma)}{\beta(\sigma)(a_2 - ka_1)}$, we have

$$\left(\frac{a_2 - ka_1}{k+1}\right)\int_0^1 (2u-1)\,\Omega'\ (k(1-u)a_1 + ua_2)\,du + \frac{2(1-\sigma)}{\beta(\sigma)(a_2 - ka_1)}$$
$$= \frac{\Omega(ka_1) + \Omega(a_2)}{k+1} - \frac{1}{(k+1)(a_2 - ka_1)}\int_{ka_1}^{a_2} \Omega(\theta)\,d\theta + \frac{2(1-\sigma)}{\beta(\sigma)(a_2 - ka_1)}$$
$$= \frac{\Omega(ka_1) + \Omega(a_2)}{k+1} - \frac{1}{\beta(\sigma)(k+1)}\left[\int_{ka_1}^{u} f(u)\,du + \frac{(1-\sigma)}{\beta(\sigma)} + \int_{u}^{a_2} f(u)\,du + \frac{(1-\sigma)}{\beta(\sigma)}\right]$$
$$= \frac{\Omega(ka_1) + \Omega(a_2)}{k+1} - \frac{\beta(\sigma)}{\sigma(k+1)}\left[\left({}^{CF}_{ka_1}I^{\sigma} f\right)(u) + \left({}^{CF}I^{\sigma}_{a_2} f\right)(u)\right].$$

As a result, we obtain the following expression

$$\frac{\Omega(ka_1) + \Omega(a_2)}{k+1} - \frac{\beta(\sigma)}{\sigma(k+1)}\left[\left({}^{CF}_{ka_1}I^{\sigma} f\right)(u) + \left({}^{CF}I^{\sigma}_{a_2} f\right)(u)\right] - \frac{2(1-\sigma)}{\sigma(a_2 - ka_1)}$$
$$= \left(\frac{a_2 - ka_1}{k+1}\right)\int_0^1 (2u-1)\,\Omega'\ (k(1-u)a_1 + ua_2)\,du.$$

This concludes the proof of Lemma 3.4. □

Lemma 3.5. *Let* $0 < k \le 1$, *and consider a function* $\Omega : \left[a_1, \frac{a_2}{k}\right] \to \mathbb{R}$ *where* $0 < a_1 < a_2$, *is differentiable on open interval* $\left(a_1, \frac{a_2}{k}\right)$ *that* $\Omega' \in L_1\left[a_1, \frac{a_2}{k}\right]$. *Then, the following identity holds:*

$$\frac{\beta(\sigma)\,k}{\sigma(a_2 - ka_1)}\left[\left({}^{CF}_{a_1}I^{\sigma} f\right)(u) + \left({}^{CF}I^{\sigma}_{\frac{a_2}{k}} f\right)(u)\right] - \Omega\left(\frac{a_1 + a_2}{2k}\right) - \frac{2(1-\sigma)}{\sigma(a_2 - ka_1)}$$

$$= \frac{a_2 - ka_1}{k}\left[\int_0^1 t\Omega'\left(ua_1 + (1-u)\frac{a_2}{k}\right)du - \int_{\frac{1}{2}}^1 \Omega'\left(ua_1 + (1-u)\frac{a_2}{k}\right)du\right].$$

Proof. Utilizing the method of integration by parts, we derive the following result:

$$\begin{aligned}
&\frac{a_2 - ka_1}{k}\left[\int_0^1 u\Omega'\left(ua_1 + (1-u)\frac{a_2}{k}\right)du - \int_{\frac{1}{2}}^1 \Omega'\left(ua_1 + (1-u)\frac{a_2}{k}\right)du\right] \\
=\;& \frac{a_2 - ka_1}{k}\left[\left.\frac{u\Omega\left(ua_1 + (1-u)\frac{a_2}{k}\right)}{a_1 - \frac{a_2}{k}}\right|_0^1 - \int_0^1 \frac{\Omega\left(ua_1 + (1-u)\frac{a_2}{k}\right)}{a_1 - \frac{a_2}{k}}du - \left.\frac{\Omega\left(ua_1 + (1-u)\frac{a_2}{k}\right)}{a_1 - \frac{a_2}{k}}\right|_{\frac{1}{2}}^1\right] \\
=\;& \frac{a_2 - ka_1}{k}\left[\frac{k\Omega(a_1)}{ka_1 - a_2} - \frac{k}{ka_1 - a_2}\int_0^1 \Omega\left(ua_1 + (1-u)\frac{a_2}{k}\right)du - \frac{k}{ka_1 - a_2}\int_0^1 \Omega(a_1) - \Omega\left(\frac{a_1 + a_2}{2k}\right)du\right] \\
=\;& \frac{k}{a_2 - ka_1}\int_{a_1}^{\frac{a_2}{k}} \Omega(\theta)\,d\theta - \Omega\left(\frac{a_1 + a_2}{2k}\right).
\end{aligned}$$

Adding both sides $\frac{2(1-\sigma)}{\beta(\sigma)(a_2 - ka_1)}$, we have

$$\begin{aligned}
&\frac{a_2 - ka_1}{k}\left[\int_0^1 u\Omega'\left(ua_1 + (1-u)\frac{a_2}{k}\right)du - \int_{\frac{1}{2}}^1 \Omega'\left(ua_1 + (1-u)\frac{a_2}{k}\right)du\right] + \frac{2(1-\sigma)}{\beta(\sigma)(a_2 - ka_1)} \\
=\;& \frac{k}{a_2 - ka_1}\int_{a_1}^{\frac{a_2}{k}} \Omega(\theta)\,d\theta - \Omega\left(\frac{a_1 + a_2}{2k}\right) + \frac{2(1-\sigma)}{\beta(\sigma)(a_2 - ka_1)} \\
=\;& -\Omega\left(\frac{a_1 + a_2}{2k}\right) + \frac{k}{\beta(\sigma)(a_2 - ka_1)}\left[\int_{a_1}^u f(u)\,du + \frac{(1-\sigma)}{\beta(\sigma)} + \int_u^{\frac{a_2}{k}} f(u)\,du + \frac{(1-\sigma)}{\beta(\sigma)}\right] \\
=\;& -\Omega\left(\frac{a_1 + a_2}{2k}\right) + \frac{\beta(\sigma)k}{\sigma(a_2 - ka_1)}\left[\left({}^{CF}_{a_1}I^{\sigma} f\right)(u) + \left({}^{CF}I^{\sigma}_{\frac{a_2}{k}} f\right)(u)\right].
\end{aligned}$$

As a result, we obtain the following expression

$$\begin{aligned}
&-\Omega\left(\frac{a_1 + a_2}{2k}\right) + \frac{\beta(\sigma)k}{\sigma(a_2 - ka_1)}\left[\left({}^{CF}_{a_1}I^{\sigma} f\right)(u) + \left({}^{CF}I^{\sigma}_{\frac{a_2}{k}} f\right)(u)\right] - \frac{2(1-\sigma)}{\sigma(a_2 - ka_1)} \\
=\;& \frac{a_2 - ka_1}{k}\left[\int_0^1 u\Omega'\left(ua_1 + (1-u)\frac{a_2}{k}\right)du - \int_{\frac{1}{2}}^1 \Omega'\left(ua_1 + (1-u)\frac{a_2}{k}\right)du\right].
\end{aligned}$$

This concludes the proof of Lemma 3.5. □

Lemma 3.6. *Let $0 < k \leq 1$, and consider a function $\Omega : [ka_1, a_2] \to \mathbb{R}$ is differentiable on the open interval (ka_1, a_2) where $0 < a_1 < a_2$. If the derivative Ω' belongs to the space $L_1[ka_1, a_2]$, then the following identity holds:*

$$\begin{aligned}
&\frac{\beta(\sigma)}{\sigma(a_2 - ka_1)}\left[\left({}^{CF}_{ka_1}I^{\sigma} f\right)(u) + \left({}^{CF}I^{\sigma}_{a_2} f\right)(u)\right] - \Omega\left(\frac{ka_1 + a_2}{2}\right) - \frac{2(1-\sigma)}{\sigma(a_2 - ka_1)} \\
=\;& (a_2 - ka_1)\left[-\int_0^1 u\Omega'(k(1-u)a_1 + ua_2)\,du + \int_{\frac{1}{2}}^1 \Omega'(k(1-u)a_1 + ua_2)\,du\right].
\end{aligned}$$

Proof. Utilizing the method of integration by parts, we derive the following result:

$$\begin{aligned}
&(a_2 - ka_1)\left[-\int_0^1 u\Omega'(k(1-u)a_1 + ua_2)\,du + \int_{\frac{1}{2}}^1 \Omega'(k(1-u)a_1 + ua_2)\,du\right] \\
=\;& (a_2 - ka_1)\left[\left.\frac{-u\Omega(k(1-u)a_1 + ua_2)}{a_2 - ka_1}\right|_0^1 - \int_0^1 \frac{\Omega(k(1-u)a_1 + ua_2)}{a_2 - ka_1}du + \left.\frac{\Omega(k(1-u)a_1 + ua_2)}{a_2 - ka_1}\right|_{\frac{1}{2}}^1\right]
\end{aligned}$$

$$
\begin{aligned}
&= (a_2-ka_1)\left[\frac{-\Omega(a_2)}{a_2-ka_1}+\frac{1}{a_2-ka_1}\int_0^1\Omega(k(1-u)a_1+ua_2)\,du+\frac{\Omega(a_2)-\Omega\left(\frac{ka_1+a_2}{2}\right)}{a_2-ka_1}\right]\\
&= \frac{1}{a_2-ka_1}\int_{ka_1}^{a_2}\Omega(\theta)\,d\theta-\Omega\left(\frac{ka_1+a_2}{2}\right).
\end{aligned}
$$

Adding both sides $\frac{2(1-\sigma)}{\beta(\sigma)(a_2-ka_1)}$, we have

$$
\begin{aligned}
&(a_2-ka_1)\left[-\int_0^1 u\Omega'(k(1-u)a_1+ua_2)\,du+\int_{\frac{1}{2}}^1\Omega'(k(1-u)a_1+ua_2)\,du\right]+\frac{2(1-\sigma)}{\beta(\sigma)(a_2-ka_1)}\\
&= \frac{1}{a_2-ka_1}\int_{ka_1}^{a_2}\Omega(\theta)\,d\theta-\Omega\left(\frac{ka_1+a_2}{2}\right)+\frac{2(1-\sigma)}{\beta(\sigma)(a_2-ka_1)}\\
&= -\Omega\left(\frac{ka_1+a_2}{2}\right)+\frac{1}{\beta(\sigma)(a_2-ka_1)}\left[\int_{ka_1}^{u}f(u)\,du+\frac{(1-\sigma)}{\beta(\sigma)}+\int_u^{a_2}f(u)\,du+\frac{(1-\sigma)}{\beta(\sigma)}\right]\\
&= -\Omega\left(\frac{ka_1+a_2}{2}\right)-\frac{\beta(\sigma)}{\sigma(a_2-ka_1)}\left[\left({}^{CF}_{ka_1}I^{\sigma}f\right)(u)+\left({}^{CF}I^{\sigma}_{a_2}f\right)(u)\right].
\end{aligned}
$$

As a result, we obtain the following expression

$$
\begin{aligned}
&\frac{\beta(\sigma)}{\sigma(a_2-ka_1)}\left[\left({}^{CF}_{ka_1}I^{\sigma}f\right)(u)+\left({}^{CF}I^{\sigma}_{a_2}f\right)(u)\right]-\Omega\left(\frac{ka_1+a_2}{2}\right)-\frac{2(1-\sigma)}{\sigma(a_2-ka_1)}\\
&= (a_2-ka_1)\left[-\int_0^1 u\Omega'(k(1-u)a_1+ua_2)\,du+\int_{\frac{1}{2}}^1\Omega'(k(1-u)a_1+ua_2)\,du\right].
\end{aligned}
$$

This concludes the proof of Lemma 3.6. □

Theorem 3.7. *Let $0<k\le 1$ and a mapping $\Omega:\left(0,\frac{a_2}{m}\right]\to\mathbb{R}$ is differentiable on $\left(0,\frac{a_2}{m}\right)$ where $0<a_1<a_2$. If $|\Omega'|^q$ is (s,m)–exponential type convex on $\left(0,\frac{a_2}{mk}\right]$ for $q>1$ and $q^{-1}+p^{-1}=1$, then for some fixed $s,m\in(0,1]$, the following inequality is satisfied:*

$$
\begin{aligned}
&\left|\frac{\Omega(a_1)+\Omega\left(\frac{a_2}{k}\right)}{2}-\frac{\beta(\sigma)k}{\sigma(a_2-ka_1)}\left[\left({}^{CF}_{a_1}I^{\sigma}f\right)(t)+\left({}^{CF}I^{\sigma}_{\frac{a_2}{k}}f\right)(u)\right]-\frac{2(1-\sigma)}{\sigma(a_2-ka_1)}\right|\\
&\le \left(\frac{a_2-ka_1}{2k}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left\{\left(\frac{e^s-s-1}{s}\right)\left(|\Omega'(a_1)|^q+m\left|\Omega'\left(\frac{a_2}{mk}\right)\right|^q\right)\right\}^{\frac{1}{q}}.
\end{aligned}
$$

Proof. Using Lemma 3.1, Hölder's inequality and (s,m)-exponential type convexity of $|\Omega'|^q$, we have

$$
\begin{aligned}
&\left|\frac{\Omega(a_1)+\Omega\left(\frac{a_2}{k}\right)}{2}-\frac{\beta(\sigma)k}{\sigma(a_2-ka_1)}\left[\left({}^{CF}_{a_1}I^{\sigma}f\right)(u)+\left({}^{CF}I^{\sigma}_{\frac{a_2}{k}}f\right)(u)\right]-\frac{2(1-\sigma)}{\sigma(a_2-ka_1)}\right|\\
&\le \left(\frac{a_2-ka_1}{2k}\right)\left(\int_0^1|1-2\varrho|^p\,d\varrho\right)^{\frac{1}{p}}\left\{\int_0^1\left|\Omega'\left(\varrho a_1+(1-\varrho)\frac{a_2}{k}\right)\right|^q d\varrho\right\}^{\frac{1}{q}}\\
&\le \left(\frac{a_2-ka_1}{2k}\right)\left(\int_0^1|1-2\varrho|^p\,d\varrho\right)^{\frac{1}{p}}\left\{\int_0^1\left[(e^{s\varrho}-1)|\Omega'(a_1)|^q+m\left(e^{(1-\varrho)s}-1\right)\left|\Omega'\left(\frac{a_2}{mk}\right)\right|^q\right]d\varrho\right\}^{\frac{1}{q}}\\
&= \left(\frac{a_2-ka_1}{2k}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\times\left\{\left(\frac{e^s-s-1}{s}\right)\left(|\Omega'(a_1)|^q+m\left|\Omega'\left(\frac{a_2}{mk}\right)\right|^q\right)\right\}^{\frac{1}{q}}.
\end{aligned}
$$

This concludes the proof. □

Example 3.8. *In above theorem if we take a growing function $\Omega(x) = \log(1 + 0.5x)$. is used to illustrate the under exponential-type convexity. It shows that the L.H.S remains below the R.H.S, confirming the validity of the inequality, the behavior of this has been shown graphically in figure 1.*

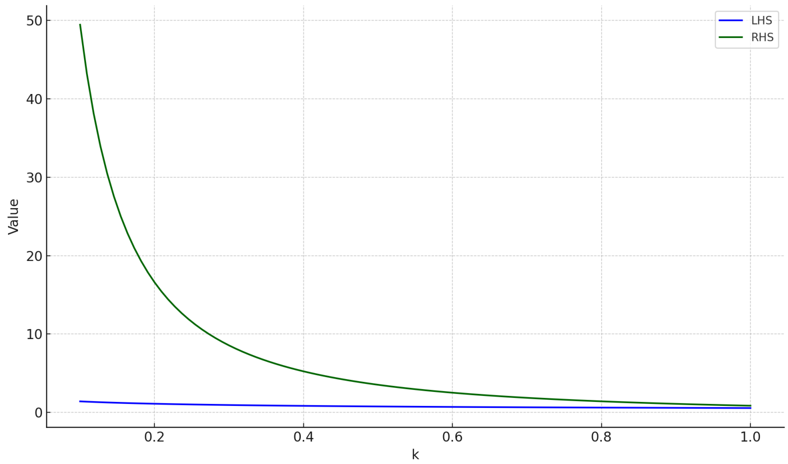

Figure 1:

Theorem 3.9. *Suppose that Let $0 < k \leq 1$ and a mapping $\Omega : \left(0, \frac{a_2}{mk}\right] \to \mathbb{R}$ is differentiable on $\left(0, \frac{a_2}{mk}\right)$ with $0 < a_1 < a_2$. If $|\Omega'|^q$ is (s, m)–exponential type convex on $\left(0, \frac{a_2}{mk}\right]$ for $q > 1$, then for some fixed $s, m \in (0, 1]$, the following inequality is satisfied:*

$$\left| \frac{\Omega(a_1) + \Omega\left(\frac{a_2}{k}\right)}{2} - \frac{\beta(\sigma) k}{\sigma(a_2 - ka_1)} \left[\left({}^{CF}_{a_1}I^{\sigma} f \right)(u) + \left({}^{CF}I^{\sigma}_{\frac{a_2}{k}} f \right)(u) \right] - \frac{2(1-\sigma)}{\sigma(a_2 - ka_1)} \right|$$
$$\leq \left(\frac{a_2 - ka_1}{2k} \right) \left(\frac{1}{2} \right)^{1-\frac{1}{q}} \left\{ \left(\frac{2(s-2)e^s + 8e^{\frac{s}{2}} - s^2 - 2s - 4}{2s^2} \right) \left(|\Omega'(a_1)|^q + m \left| \Omega'\left(\frac{a_2}{mk} \right) \right|^q \right) \right\}^{\frac{1}{q}}.$$

Proof. Using Lemma 3.1, power mean inequality and (s, m)-exponential type convexity of $|\Omega'|^q$, we have

$$\left| \frac{\Omega(a_1) + \Omega\left(\frac{a_2}{k}\right)}{2} - \frac{\beta(\sigma) k}{\sigma(a_2 - ka_1)} \left[\left({}^{CF}_{a_1}I^{\sigma} f \right)(u) + \left({}^{CF}I^{\sigma}_{\frac{a_2}{k}} f \right)(u) \right] - \frac{2(1-\sigma)}{\sigma(a_2 - ka_1)} \right|$$
$$\leq \left(\frac{a_2 - ka_1}{2k} \right) \left(\int_0^1 |1 - 2\varrho| \, d\varrho \right)^{1-\frac{1}{q}} \left\{ \int_0^1 |1 - 2\varrho| \left| \Omega'\left(\varrho a_1 + (1-\varrho) \frac{a_2}{k} \right) \right| d\varrho \right\}^{\frac{1}{q}}$$
$$\leq \left(\frac{a_2 - ka_1}{2k} \right) \left(\int_0^1 |1 - 2\varrho| \, d\varrho \right)^{1-\frac{1}{q}} \left\{ \int_0^1 |1 - 2\varrho| \left[(e^{s\varrho} - 1) |\Omega'(a_1)|^q + m \left(e^{(1-\varrho)s} - 1 \right) \left| \Omega'\left(\frac{a_2}{mk} \right) \right|^q \right] d\varrho \right\}^{\frac{1}{q}}$$
$$= \left(\frac{a_2 - ka_1}{2k} \right) \left(\frac{1}{2} \right)^{1-\frac{1}{q}} \left\{ \left(\frac{2(s-2)e^s + 8e^{\frac{s}{2}} - s^2 - 2s - 4}{2s^2} \right) \left(|\Omega'(a_1)|^q + m \left| \Omega'\left(\frac{a_2}{mk} \right) \right|^q \right) \right\}^{\frac{1}{q}}.$$

This concludes the proof. □

Theorem 3.10. *Let Let $0 < k \leq 1$ and a mapping $\Omega : \left(0, \frac{a_2}{m}\right] \to \mathbb{R}$ is differentiable on $\left(0, \frac{a_2}{m}\right)$ where $0 < a_1 < a_2$. If $|\Omega'|^q$ is (s,m)–exponential type convex on $\left(0, \frac{a_2}{mk}\right]$ for $q > 1$ and $q^{-1} + p^{-1} = 1$, then for some fixed $s, m \in (0,1]$, the following inequality is satisfied:*

$$\left|\frac{\Omega(ka_1) + \Omega(a_2)}{2} - \frac{\beta(\sigma)}{\sigma(a_2 - ka_1)}\left[\left({}_{ka_1}^{CF}I^{\sigma} f\right)(u) + \left({}^{CF}I_{a_2}^{\sigma} f\right)(u)\right] - \frac{2(1-\sigma)}{\sigma(a_2 - ka_1)}\right|$$
$$\leq \left(\frac{a_2 - ka_1}{2}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \times \left\{\left(\frac{e^s - s - 1}{s}\right)\left(|\Omega'(ka_1)|^q + m\left|\Omega'\left(\frac{a_2}{m}\right)\right|^q\right)\right\}^{\frac{1}{q}}.$$

Proof. By using of Lemma 3.3, Hölder's inequality and (s,m)-exponential type convexity of $|\Omega'|^q$, we have

$$\left|\frac{\Omega(ka_1) + \Omega(a_2)}{2} - \frac{\beta(\sigma)}{\sigma(a_2 - ka_1)}\left[\left({}_{ka_1}^{CF}I^{\sigma} f\right)(u) + \left({}^{CF}I_{a_2}^{\sigma} f\right)(u)\right] - \frac{2(1-\sigma)}{\sigma(a_2 - ka_1)}\right|$$
$$\leq \left(\frac{a_2 - ka_1}{2}\right)\left(\int_0^1 |2\varrho - 1|^p \, d\varrho\right)^{\frac{1}{p}} \left\{\int_0^1 |\Omega'(k(1-\varrho)a_1) + \varrho a_2|^q \, d\varrho\right\}^{\frac{1}{q}}$$
$$\leq \left(\frac{a_2 - ka_1}{2}\right)\left(\int_0^1 |2\varrho - 1|^p \, d\varrho\right)^{\frac{1}{p}} \left\{\int_0^1 \left[m\left(e^{s\varrho} - 1\right)\left|\Omega'\left(\frac{a_2}{m}\right)\right|^q + \left(e^{(1-\varrho)s} - 1\right)|\Omega'(ka_1)|^q\right] d\varrho\right\}^{\frac{1}{q}}$$
$$= \left(\frac{a_2 - ka_1}{2}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}} \times \left\{\left(\frac{e^s - s - 1}{s}\right)\left(|\Omega'(ka_1)|^q + m\left|\Omega'\left(\frac{a_2}{m}\right)\right|^q\right)\right\}^{\frac{1}{q}}.$$

This concludes the proof. □

Example 3.11. *In above theorem if we take an increasing function $\Omega(x) = \log(1 + 0.3x)$ is used to illustrate the under exponential-type convexity. It shows that the L.H.S remains below the R.H.S, confirming the validity of the inequality, the behavior of this has been shown graphically in figure 3.*

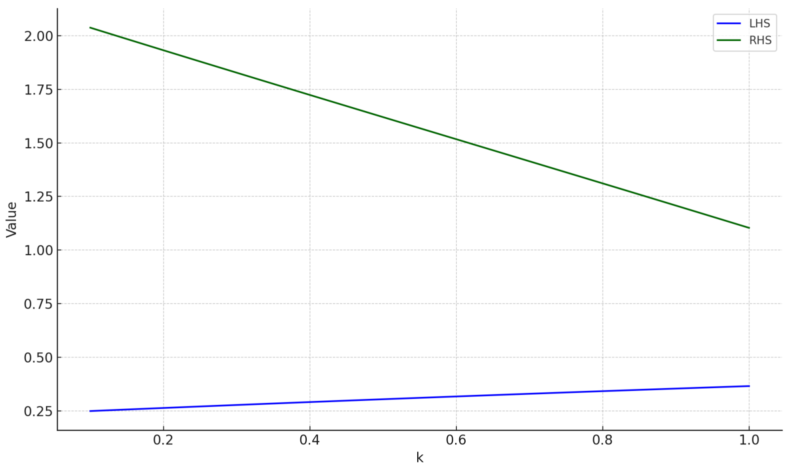

Figure 2:

Theorem 3.12. *Suppose that $0 < k \leq 1$ and a mapping $\Omega : \left(0, \frac{a_2}{mk}\right] \to \mathbb{R}$ is differentiable on $\left(0, \frac{a_2}{mk}\right)$ with $0 < a_1 < a_2$. If $|\Omega'|^q$ (s,m)–exponential type convex on $\left(0, \frac{a_2}{mk}\right]$ for $q > 1$, then for some fixed $s, m \in (0,1]$, the following inequality* *is satisfied:*

$$\begin{aligned}&\left|\frac{\Omega(ka_1)+\Omega(a_2)}{2}-\frac{\beta(\sigma)}{\sigma(a_2-ka_1)}\left[\left({}^{CF}_{ka_1}I^{\sigma}f\right)(u)+\left({}^{CF}I^{\sigma}_{a_2}f\right)(u)\right]-\frac{2(1-\sigma)}{\sigma(a_2-ka_1)}\right|\\ \leq\ &\left(\frac{a_2-ka_1}{2}\right)\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left\{\left(\frac{2(s-2)e^s+8e^{\frac{s}{2}}-s^2-2s-4}{2s^2}\right)\left(|\Omega'(ka_1)|^q+m\left|\Omega'\left(\frac{a_2}{m}\right)\right|^q\right)\right\}^{\frac{1}{q}}.\end{aligned}$$

Proof. By using of Lemma 3.3, power mean inequality and (s,m)-exponential type convexity of $|\Omega'|^q$, we have

$$\begin{aligned}&\left|\frac{\Omega(ka_1)+\Omega(a_2)}{2}-\frac{\beta(\sigma)}{\sigma(a_2-ka_1)}\left[\left({}^{CF}_{ka_1}I^{\sigma}f\right)(u)+\left({}^{CF}I^{\sigma}_{a_2}f\right)(u)\right]-\frac{2(1-\sigma)}{\sigma(a_2-ka_1)}\right|\\ \leq\ &\left(\frac{a_2-ka_1}{2}\right)\left\{\int_0^1|2\varrho-1|\left|\Omega'(k(1-\varrho)a_1+\varrho a_2)\right|d\varrho\right\}\\ \leq\ &\left(\frac{a_2-ka_1}{2}\right)\left(\int_0^1|2\varrho-1|d\varrho\right)^{1-\frac{1}{q}}\left\{\int_0^1|2\varrho-1|\left|\Omega'(k(1-\varrho)a_1+\varrho a_2)\right|^q d\varrho\right\}^{\frac{1}{q}}\\ \leq\ &\left(\frac{a_2-ka_1}{2}\right)\left(\int_0^1|2\varrho-1|d\varrho\right)^{1-\frac{1}{q}}\left\{\int_0^1|2\varrho-1|\left[\left(e^{(1-\varrho)s}-1\right)|\Omega'(ka_1)|^q+m\left(e^{s\varrho}-1\right)\left|\Omega'\left(\frac{a_2}{m}\right)\right|^q\right]d\varrho\right\}^{\frac{1}{q}}\\ =\ &\left(\frac{a_2-ka_1}{2}\right)\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left\{\left(\frac{2(s-2)e^s+8e^{\frac{s}{2}}-s^2-2s-4}{2s^2}\right)\left(|\Omega'(ka_1)|^q+m\left|\Omega'\left(\frac{a_2}{m}\right)\right|^q\right)\right\}^{\frac{1}{q}}.\end{aligned}$$

This concludes the proof. □

Theorem 3.13. *Let $0<k\leq 1$, and consider a function $\Omega:\left(0,\frac{a_2}{m}\right]\to\Re$ that is differentiable on the interval $\left(0,\frac{a_2}{m}\right)$, where $0<a_1<a_2$. Assume that the function $|\Omega'|^q$ is (s,m)–exponential type convex over the domain $\left(0,\frac{a_2}{mk}\right]$ for some $q>1$ and let p be such that $q^{-1}+p^{-1}=1$. Then, for fixed values $s,m\in(0,1]$, the inequality given below holds:*

$$\begin{aligned}&\left|\frac{\Omega(ka_1)+\Omega(a_2)}{k+1}-\frac{\beta(\sigma)}{\sigma(a_2-ka_1)}\left[\left({}^{CF}_{ka_1}I^{\sigma}f\right)(u)+\left({}^{CF}I^{\sigma}_{a_2}f\right)(u)\right]-\frac{2(1-\sigma)}{\sigma(a_2-ka_1)}\right|\\ \leq\ &\left(\frac{a_2-ka_1}{k+1}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left\{\left(\frac{e^s-s-1}{s}\right)\left(|\Omega'(ka_1)|^q+m\left|\Omega'\left(\frac{a_2}{m}\right)\right|^q\right)\right\}^{\frac{1}{q}}.\end{aligned}$$

Proof. Using Lemma 3.4, Hölder's inequality and (s,m)-exponential type convexity of $|\Omega'|^q$, we have

$$\begin{aligned}&\left|\frac{\Omega(ka_1)+\Omega(a_2)}{k+1}-\frac{\beta(\sigma)}{\sigma(a_2-ka_1)}\left[\left({}^{CF}_{ka_1}I^{\sigma}f\right)(u)+\left({}^{CF}I^{\sigma}_{a_2}f\right)(u)\right]-\frac{2(1-\sigma)}{\sigma(a_2-ka_1)}\right|\\ \leq\ &\left(\frac{a_2-ka_1}{k+1}\right)\left(\int_0^1|2\varrho-1|^p d\varrho\right)^{\frac{1}{p}}\left\{\int_0^1\left|\Omega'(k(1-\varrho)a_1)+\varrho a_2\right|^q d\varrho\right\}^{\frac{1}{q}}\\ \leq\ &\left(\frac{a_2-ka_1}{k+1}\right)\left(\int_0^1|2\varrho-1|^p d\varrho\right)^{\frac{1}{p}}\left\{\int_0^1\left[\left(e^{(1-\varrho)s}-1\right)|\Omega'(ka_1)|^q+m\left(e^{s\varrho}-1\right)\left|\Omega'\left(\frac{a_2}{m}\right)\right|^q\right]d\varrho\right\}^{\frac{1}{q}}\\ =\ &\left(\frac{a_2-ka_1}{k+1}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\times\left\{\left(\frac{e^s-s-1}{s}\right)\left(|\Omega'(ka_1)|^q+m\left|\Omega'\left(\frac{a_2}{m}\right)\right|^q\right)\right\}^{\frac{1}{q}}.\end{aligned}$$

This concludes the proof. □

Theorem 3.14. *Let $0 < k \leq 1$, and consider a function $\Omega : \left(0, \frac{a_2}{mk}\right] \to \mathbb{R}$ is differentiable on the interval $\left(0, \frac{a_2}{mk}\right)$, where a_1 and a_2 satisfy $0 < a_1 < a_2$. Suppose that $|\Omega'|^q$ is (s,m)–exponential type convex over $\left(0, \frac{a_2}{mk}\right]$ for some $q > 1$. Then, for fixed values of $s, m \in (0,1]$, the following inequality is holds:*

$$\left|\frac{\Omega(ka_1) + \Omega(a_2)}{k+1} - \frac{\beta(\sigma)}{\sigma(a_2 - ka_1)}\left[\left({}^{CF}_{ka_1}I^{\sigma} f\right)(u) + \left({}^{CF}I^{\sigma}_{a_2} f\right)(u)\right] - \frac{2(1-\sigma)}{\sigma(a_2 - ka_1)}\right|$$
$$\leq \left(\frac{a_2 - ka_1}{k+1}\right)\left(\frac{1}{2}\right)^{1-\frac{1}{q}} \times \left\{\left(\frac{2(s-2)e^s + 8e^{\frac{s}{2}} - s^2 - 2s - 4}{2s^2}\right)\left(|\Omega'(ka_1)|^q + m\left|\Omega'\left(\frac{a_2}{m}\right)\right|^q\right)\right\}^{\frac{1}{q}}.$$

Proof. Using Lemma 3.4, applying the power mean inequality in combination with the (s,m)-exponential convexity property of $|\Omega'|^q$, we obtain

$$\left|\frac{\Omega(ka_1) + \Omega(a_2)}{k+1} - \frac{\beta(\sigma)}{\sigma(a_2 - ka_1)}\left[\left({}^{CF}_{ka_1}I^{\sigma} f\right)(u) + \left({}^{CF}I^{\sigma}_{a_2} f\right)(u)\right] - \frac{2(1-\sigma)}{\sigma(a_2 - ka_1)}\right|$$
$$\leq \left(\frac{a_2 - ka_1}{k+1}\right)\left\{\int_0^1 |2\varrho - 1|\left|\Omega'(k(1-\varrho)a_1 + \varrho a_2)\right| d\varrho\right\}$$
$$\leq \left(\frac{a_2 - ka_1}{k+1}\right)\left(\int_0^1 |2\varrho - 1| d\varrho\right)^{1-\frac{1}{q}}\left\{\int_0^1 |2\varrho - 1|\left|\Omega'(k(1-\varrho)a_1 + \varrho a_2)\right|^q d\varrho\right\}^{\frac{1}{q}}$$
$$\leq \left(\frac{a_2 - ka_1}{k+1}\right)\left(\int_0^1 |2\varrho - 1| d\varrho\right)^{1-\frac{1}{q}}\left\{\int_0^1 |2\varrho - 1|\left[\left(e^{(1-\varrho)s} - 1\right)|\Omega'(ka_1)|^q + m\left(e^{s\varrho} - 1\right)\left|\Omega'\left(\frac{a_2}{m}\right)\right|^q\right] d\varrho\right\}^{\frac{1}{q}}$$
$$= \left(\frac{a_2 - ka_1}{k+1}\right)\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left\{\left(\frac{2(s-2)e^s + 8e^{\frac{s}{2}} - s^2 - 2s - 4}{2s^2}\right)\left(|\Omega'(ka_1)|^q + m\left|\Omega'\left(\frac{a_2}{m}\right)\right|^q\right)\right\}^{\frac{1}{q}}.$$

This concludes the proof. □

Theorem 3.15. *Let $0 < k \leq 1$, and consider a function $\Omega : \left(0, \frac{a_2}{m}\right] \to \Re$ that is differentiable on the interval $\left(0, \frac{a_2}{m}\right)$, where $0 < a_1 < a_2$. Assume that the function $|\Omega'|^q$ is (s,m)–exponential type convex over the domain $\left(0, \frac{a_2}{mk}\right]$ for some $q > 1$ and let p be such that $q^{-1} + p^{-1} = 1$. Then, for fixed values $s, m \in (0,1]$, the inequality given below holds:*

$$\left|\frac{\beta(\sigma)k}{\sigma(a_2 - ka_1)}\left[\left({}^{CF}_{a_1}I^{\sigma} f\right)(u) + \left({}^{CF}I^{\sigma}_{\frac{a_2}{k}} f\right)(u)\right] - \Omega\left(\frac{a_1 + a_2}{2k}\right) - \frac{2(1-\sigma)}{\sigma(a_2 - ka_1)}\right|$$
$$\leq \left(\frac{a_2 - ka_1}{2}\right)\left[\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left\{\left(\frac{e^s - s - 1}{s}\right)\left(|\Omega'(a_1)|^q + m\left|\Omega'\left(\frac{a_2}{km}\right)\right|^q\right)\right\}^{\frac{1}{q}}\right. \tag{1}$$
$$\left. + \left(\frac{1}{2}\right)^{\frac{1}{p}}\left\{|\Omega'(a_1)|^q\left(\frac{2e^s - 2e^{\frac{s}{2}} - s}{2s}\right) + m\left|\Omega'\left(\frac{a_2}{km}\right)\left(\frac{2e^{\frac{s}{2}} - s - 2}{2s}\right)\right|^q\right\}^{\frac{1}{q}}\right].$$

Proof. By using of Lemma 3.5, together with Hölder's inequality and assumption that $|\Omega'|^q$ is convex in the (s,m) -exponential sense, we deduce the following inequality:

$$\left|\frac{\beta(\sigma)k}{\sigma(a_2 - ka_1)}\left[\left({}^{CF}_{a_1}I^{\sigma} f\right)(t) + \left({}^{CF}I^{\sigma}_{\frac{a_2}{k}} f\right)(u)\right] - \Omega\left(\frac{a_1 + a_2}{2k}\right) - \frac{2(1-\sigma)}{\sigma(a_2 - ka_1)}\right|$$
$$\leq \left(\frac{a_2 - ka_1}{k}\right)\left[\left(\int_0^1 \varrho^p d\varrho\right)^{\frac{1}{p}}\left(\int_0^1 \left|\Omega'\left(\varrho a_1 + (1-\varrho)\frac{a_2}{k}\right)\right|^q d\varrho\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\int_0^1 1 d\varrho\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1 \left|\Omega'\left(\varrho a_1 + (1-\varrho)\frac{a_2}{k}\right)\right|^q d\varrho\right)^{\frac{1}{q}}\right]$$

$$\leq \left(\frac{a_2-ka_1}{k}\right)\left[\left(\int_0^1 \varrho^p d\varrho\right)^{\frac{1}{p}}\left(\int_0^1\left(\left(e^{y\varrho}-1\right)|\Omega'(a_1)|^q+m\left(e^{(1-\varrho)s}-1\right)\left|\Omega'\left(\frac{a_2}{km}\right)\right|^q\right)d\varrho\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1 1d\varrho\right)^{\frac{1}{p}}\left(\int_0^1\left(\left(e^{s\varrho}-1\right)|\Omega'(a_1)|^q+m\left(e^{(1-\varrho)s}-1\right)\left|\Omega'\left(\frac{a_2}{km}\right)\right|^q\right)d\varrho\right)^{\frac{1}{q}}\right]$$
$$=\left(\frac{a_2-ka_1}{k}\right)\left[\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left\{\left(\frac{e^s-s-1}{s}\right)\left(|\Omega'(a_1)|^q+m\left|\Omega'\left(\frac{a_2}{km}\right)\right|^q\right)\right\}^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{1}{2}\right)^{\frac{1}{p}}\left\{|\Omega'(a_1)|^q\left(\frac{2e^s-2e^{\frac{s}{2}}-s}{2s}\right)+m\left|\Omega'\left(\frac{a_2}{km}\right)\left(\frac{2e^{\frac{s}{2}}-s-2}{2s}\right)\right|^q\right\}^{\frac{1}{q}}\right].$$

This concludes the proof. □

Theorem 3.16. *Let k be a real number such that $0 < k \leq 1$, and consider a function $\Omega : \left(0, \frac{a_2}{m}\right] \to \mathbb{R}$ which is differentiable on interval $\left(0, \frac{a_2}{m}\right)$, where $0 < a_1 < a_2$. Assume that $|\Omega'|^q$ is (s,m)–exponential type convex on $\left(0, \frac{a_2}{mk}\right]$ for $q > 1$. Then, for any fixed values $s, m \in (0,1]$, the inequality below holds:*

$$\left|\frac{\beta(\sigma)k}{\sigma(a_2-ka_1)}\left[\left({}^{CF}_{a_1}I^{\sigma}f\right)(t)+\left({}^{CF}I^{\sigma}_{\frac{a_2}{k}}f\right)(u)\right]-\Omega\left(\frac{a_1+a_2}{2k}\right)-\frac{2(1-\sigma)}{\sigma(a_2-ka_1)}\right|$$
$$\leq\left(\frac{a_2-ka_1}{2}\right)\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left(|\Omega'(a_1)|^q\left(\frac{2(s-1)e^s-s^2+2}{2s^2}\right)+m\left|\Omega'\left(\frac{a_2}{km}\right)\right|^q\left(\frac{2e^s-s^2-2s-2}{2s^2}\right)\right)^{\frac{1}{q}}\right.$$
$$\left.+\left\{|\Omega'(a_1)|^q\left(\frac{e^s-e^{\frac{s}{2}}}{s}-\frac{1}{2}\right)+m\left|\Omega'\left(\frac{a_2}{km}\right)\right|^q\left(\frac{2e^{\frac{s}{2}}-s-2}{2s}\right)\right\}^{\frac{1}{q}}\right]. \quad (2)$$

Proof. By applying Lemma 3.5 in conjunction with the power mean inequality and considering that $|\Omega'|^q$ satisfies (s,m)-exponential type convexity, we derive the following result:

$$\left|\frac{\beta(\sigma)k}{\sigma(a_2-ka_1)}\left[\left({}^{CF}_{a_1}I^{\sigma}f\right)(t)+\left({}^{CF}I^{\sigma}_{\frac{a_2}{k}}f\right)(u)\right]-\Omega\left(\frac{a_1+a_2}{2k}\right)-\frac{2(1-\sigma)}{\sigma(a_2-ka_1)}\right|$$
$$\leq\left(\frac{a_2-ka_1}{k}\right)\left\{\left(\int_0^1\varrho\left|\Omega'\left(\varrho a_1+(1-\varrho)\frac{a_2}{k}\right)\right|d\varrho\right)+\int_{\frac{1}{2}}^1\left|\Omega'\left(\varrho a_1+(1-\varrho)\frac{a_2}{k}\right)\right|d\varrho\right\}$$
$$\leq\left(\frac{a_2-ka_1}{k}\right)\left\{\left(\int_0^1\varrho d\varrho\right)^{1-\frac{1}{q}}\left(\int_0^1\varrho\left|\Omega'\left(\varrho a_1+(1-\varrho)\frac{a_2}{k}\right)\right|^q d\varrho\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1 1d\varrho\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1\left|\Omega'\left(\varrho a_1+(1-\varrho)\frac{a_2}{k}\right)\right|^q d\varrho\right)^{\frac{1}{q}}\right\}$$
$$\leq\left(\frac{a_2-ka_1}{k}\right)\left[\left(\int_0^1\varrho d\varrho\right)^{1-\frac{1}{q}}\left(\int_0^1\varrho\left\{\left(e^{y\varrho}-1\right)|\Omega'(a_1)|^q+m\left(e^{(1-\varrho)s}-1\right)\left|\Omega'\left(\frac{a_2}{km}\right)\right|^q\right\}d\varrho\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\frac{1}{2}}^1 1d\varrho\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1\left\{\left(e^{s\varrho}-1\right)|\Omega'(a_1)|^q+m\left(e^{(1-\varrho)s}-1\right)\left|\Omega'\left(\frac{a_2}{km}\right)\right|^q\right\}d\varrho\right)^{\frac{1}{q}}\right]$$
$$=\left(\frac{a_2-ka_1}{k}\right)\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left\{|\Omega'(a_1)|^q\left(\frac{2(s-1)e^s-s^2+2}{2s^2}\right)+m\left|\Omega'\left(\frac{a_2}{km}\right)\right|^q\left(\frac{2e^s-s^2-2s-2}{2s^2}\right)\right\}^{\frac{1}{q}}\right.$$

$$+\left\{|\Omega'\ (a_1)|^q\left(\frac{e^s-s^{\frac{s}{2}}}{s}-\frac{1}{2}\right)+m\left|\Omega'\ \left(\frac{a_2}{km}\right)\right|^q\left(\frac{2s^{\frac{s}{2}}-s-2}{2s}\right)\right\}^{\frac{1}{q}}\Bigg].$$

This concludes the proof. □

Theorem 3.17. *Let $0<k\le 1$ and consider a differentiable function $\Omega:\left(0,\frac{a_2}{m}\right]\to\mathbb{R}$, where $0<a_1<a_2$. Suppose that the function $|\Omega'|^q$ is (s,m)–exponential type convex on the interval $\left(0,\frac{a_2}{mk}\right]$ for $q>1$, with the conjugate exponent relationship $q^{-1}+p^{-1}=1$. Then, for fixed parameters $s,m\in(0,1]$, the inequality below holds:*

$$\begin{aligned}&\left|\frac{\beta(\sigma)}{\sigma(a_2-ka_1)}\left[\left({}^{CF}_{ka_1}I^{\sigma}f\right)(t)+\left({}^{CF}I^{\sigma}_{a_2}f\right)(u)\right]-\Omega\left(\frac{ka_1+a_2}{2}\right)-\frac{2(1-\sigma)}{\sigma(a_2-ka_1)}\right|\\ \le\ &(a_2-ka_1)\left[\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left\{\left(\frac{e^s-s-1}{s}\right)\left(|\Omega'\ (a_1)|^q+m\left|\Omega'\ \left(\frac{a_2}{km}\right)\right|^q\right)\right\}^{\frac{1}{q}}\right.\\ &\left.+\left(\frac{1}{2}\right)^{\frac{1}{p}}\left\{|\Omega'\ (a_1)|^q\left(\frac{2e^s-2e^{\frac{s}{2}}-s}{2s}\right)+m\left|\Omega'\ \left(\frac{a_2}{km}\right)\left(\frac{2e^{\frac{s}{2}}-s-2}{2s}\right)\right|^q\right\}^{\frac{1}{q}}\right].\end{aligned}\tag{3}$$

Proof. By applying Lemma 3.6 in combination with Hölder's inequality and leveraging the (y,m)-exponential type convexity of $|\Omega'|^q$, the following result can be obtained:

$$\begin{aligned}&\left|\frac{\beta(\sigma)}{\sigma(a_2-ka_1)}\left[\left({}^{CF}_{ka_1}I^{\sigma}f\right)(t)+\left({}^{CF}I^{\sigma}_{a_2}f\right)(u)\right]-\Omega\left(\frac{ka_1+a_2}{2}\right)-\frac{2(1-\sigma)}{\sigma(a_2-ka_1)}\right|\\ \le\ &(a_2-ka_1)\left[\left(\int_0^1\varrho^p d\varrho\right)^{\frac{1}{p}}\left(\int_0^1\left|\Omega'\ (k(1-\varrho)a_1+(1-\varrho)a_2)\right|^q d\varrho\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\int_0^1 1d\varrho\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|\Omega'\ (k(1-\varrho)a_1+(1-\varrho)a_2)\right|^q d\varrho\right)^{\frac{1}{q}}\right]\\ \le\ &(a_2-ka_1)\left[\left(\int_0^1\varrho^p d\varrho\right)^{\frac{1}{p}}\left(\int_0^1\left(\left(e^{(1-\varrho)s}-1\right)|\Omega'\ (ka_1)|^q+m\left(e^{s\varrho}-1\right)\left|\Omega'\ \left(\frac{a_2}{m}\right)\right|^q\right)d\varrho\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\int_0^1 1d\varrho\right)^{\frac{1}{p}}\left(\int_0^1\left(\left(e^{(1-\varrho)s}-1\right)|\Omega'\ (ka_1)|^q+m\left(e^{s\varrho}-1\right)\left|\Omega'\ \left(\frac{a_2}{m}\right)\right|^q\right)d\varrho\right)^{\frac{1}{q}}\right]\\ =\ &(a_2-ka_1)\left[\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left\{\left(\frac{e^s-s-1}{s}\right)\left(|\Omega'\ (a_1)|^q+m\left|\Omega'\ \left(\frac{a_2}{km}\right)\right|^q\right)\right\}^{\frac{1}{q}}\right.\\ &\left.+\left(\frac{1}{2}\right)^{\frac{1}{p}}\left\{|\Omega'\ (a_1)|^q\left(\frac{2e^s-2s^{\frac{s}{2}}-s}{2s}\right)+m\left|\Omega'\ \left(\frac{a_2}{km}\right)\left(\frac{2s^{\frac{s}{2}}-s-2}{2s}\right)\right|^q\right\}^{\frac{1}{q}}\right].\end{aligned}$$

This concludes the proof. □

Theorem 3.18. *Suppose $0<k\le 1$, and let $\Omega:\left(0,\frac{a_2}{m}\right]\to\mathbb{R}$ be differentiable over its domain, where $0<a_1<a_2$. If $|\Omega'|^q$ (s,m)–exponential type convex on the interval $\left(0,\frac{a_2}{mk}\right]$ for $q>1$, then the inequality below holds for fixed $s,m\in(0,1]$:*

$$\left|\frac{\beta(\sigma)}{\sigma(a_2-ka_1)}\left[\left({}^{CF}_{ka_1}I^{\sigma}f\right)(t)+\left({}^{CF}I^{\sigma}_{a_2}f\right)(u)\right]-\Omega\left(\frac{ka_1+a_2}{2}\right)-\frac{2(1-\sigma)}{\sigma(a_2-ka_1)}\right|$$

$$\leq (a_2 - ka_1)\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left(\left(\frac{2e^s - s^2 - 2s - 2}{2s^2}\right)|\Omega'(ka_1)|^q + m\left(\frac{2(s-1)e^s - s^2 + 2}{2s^2}\right)\left|\Omega'\left(\frac{a_2}{km}\right)\right|^q\right)^{\frac{1}{q}} + \left\{\left(\frac{2e^{\frac{s}{2}} - s - 2}{2s}\right)|\Omega'(ka_1)|^q + m\left(\frac{2e^s - 2e^{\frac{s}{2}} - s}{2s}\right)\left|\Omega'\left(\frac{a_2}{km}\right)\right|^q\right\}^{\frac{1}{q}}\right]. \quad (4)$$

Proof. By using of Lemma 3.6 together with the power mean inequality, and utilizing the fact that $|\Omega'|^q$ adheres to the conditions of (s, m)-exponential type convexity, we deduce the following result:

$$\left|\frac{\beta(\sigma)}{\sigma(a_2 - ka_1)}\left[\left({}^{CF}_{ka_1}I^{\sigma} f\right)(u) + \left({}^{CF}I^{\sigma}_{a_2} f\right)(u)\right] - \Omega\left(\frac{ka_1 + a_2}{2}\right) - \frac{2(1-\sigma)}{\sigma(a_2 - ka_1)}\right|$$

$$\leq (a_2 - ka_1)\left\{\left(\int_0^1 |-\varrho|\left|\Omega'(k(1-\varrho)a_1 + (1-\varrho)a_2)\right| d\varrho\right) + \int_{\frac{1}{2}}^1 \left|\Omega'(k(1-\varrho)a_1 + (1-\varrho)a_2)\right| d\varrho\right\}$$

$$\leq (a_2 - ka_1)\left\{\left(\int_0^1 \varrho d\varrho\right)^{1-\frac{1}{q}}\left(\int_0^1 \varrho\left|\Omega'(k(1-\varrho)a_1 + (1-\varrho)a_2)\right|^q d\varrho\right)^{\frac{1}{q}} + \left(\int_0^1 1 d\varrho\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1 \left|\Omega'(k(1-\varrho)a_1 + (1-\varrho)a_2)\right|^q d\varrho\right)^{\frac{1}{q}}\right\}$$

$$\leq (a_2 - ka_1)\left[\left(\int_0^1 \varrho d\varrho\right)^{1-\frac{1}{q}}\left(\int_0^1 \varrho\left\{\left(e^{(1-\varrho)s} - 1\right)|\Omega'(ka_1)|^q + m\left(e^{s\varrho} - 1\right)\left|\Omega'\left(\frac{a_2}{m}\right)\right|^q\right\} d\varrho\right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^1 1 d\varrho\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1 \left\{\left(e^{(1-\varrho)s} - 1\right)|\Omega'(ka_1)|^q + m\left(e^{s\varrho} - 1\right)\left|\Omega'\left(\frac{a_2}{m}\right)\right|^q\right\} d\varrho\right)^{\frac{1}{q}}\right]$$

$$= (a_2 - ka_1)\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left\{\left(\frac{2e^s - s^2 - 2s - 2}{2s^2}\right)|\Omega'(a_1)|^q + m\left(\frac{2(s-1)e^s - s^2 + 2}{2s^2}\right)\left|\Omega'\left(\frac{a_2}{m}\right)\right|^q\right\}^{\frac{1}{q}} + \left\{\left(\frac{2s^{\frac{s}{2}} - s - 2}{2s}\right)|\Omega'(a_1)|^q + m\left(\frac{2e^s - 2e^{\frac{s}{2}} - s}{2s}\right)\left|\Omega'\left(\frac{a_2}{m}\right)\right|^q\right\}^{\frac{1}{q}}\right].$$

This concludes the proof. □

4. Conclusion

This paper develops the notion of (s, m)-exponential-type convexity and examines its principal algebraic properties. Based on this new class of functions, we derive extended Hermite-Hadamard-type inequalities by using Caputo-Fabrizio integral operator. In particular, we obtain refined estimates for functions whose 1st order derivatives, in modulus value and raised to a given exponent, fulfill the criteria of (s, m)-exponential convexity. The study extends beyond theoretical development by applying the results to obtain new inequalities for special means and to analyze the associated errors in numerical integration methods, specifically the trapezoidal and midpoint rules. These contributions are expected to provide a foundation for future research in convex analysis and fractional calculus.

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