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Diminished Sombor index and its relationship with topological indices

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Abstract. In this paper, we investigate the Diminished Sombor index (DSO), a recently introduced degree-based topological index for a simple graph G, defined by

$$DSO(G) = \sum_{uv\in E} \frac{\sqrt{d_u^2 + d_v^2}}{d_u + d_v},$$

where d_u denotes the degree of a vertex $u \in V$. We establish several sharp bounds for this index in terms of classical topological indices such as the Zagreb index, the Albertson index, the Harmonic index, the Randić index, and the geometric-arithmetic index. Furthermore, the relationships between the DSO index and these indices are thoroughly analyzed, with characterizations of extremal graphs achieving equality conditions.

1. Introduction

Let G be a simple graph with vertex set V and edge set E, where $|V|$ and $|E|$ denote the order (number of vertices) and size (number of edges) of G, respectively. The degree of a vertex u, denoted by d_u, refers to the number of vertices adjacent to u. The maximum and minimum degrees of G are denoted by Δ and δ, respectively. An edge connecting two adjacent vertices u and v is denoted as $uv \in E$. The complement of G, denoted by $\bar{G}$, is a graph with the same vertex set V, in which two vertices are adjacent if and only if they are not adjacent in G. Any additional graph-theoretic terminology used in this paper but not defined here can be found in [9].

In mathematical chemistry, topological indices, numerical descriptors derived from molecular graphs, have emerged as indispensable tools. These indices quantitatively capture structural features of molecules and play a key role in uncovering correlations between molecular structure and various physicochemical properties. Their predictive power, which eliminates the need for experimental synthesis, makes them highly effective for virtual screening in the search for potential drug candidates or materials with desired characteristics [3]. For a detailed exploration of degree-based topological indices, readers are referred to the comprehensive survey in [10].

The first Zagreb index ($M_1(G)$) of a graph G is formulated by Gutman and Trinajstić [11] as follows

$$M_1(G) = \sum_{uv\in E} (d_u + d_v) = \sum_{u\in V} d_u^2.$$

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In 1975, Milan Randić [26] introduced the Randić index which is defined as follows

$$R(G) = \sum_{uv\in E} \frac{1}{\sqrt{d_u d_v}}.$$

In [1], the Albertson index which is sometimes referred to as the third Zagreb index [6], is defined as

$$Alb(G) = \sum_{uv\in E} |d_u - d_v|.$$

The geometric–arithmetic index (GA index), introduced by Vukiĉeviĉ and Furtula in [30], is defined as

$$GA(G) = \sum_{uv\in E} \frac{2\sqrt{d_u d_v}}{d_u + d_v}.$$

The Harmonic index is defined in [5] as

$$H(G) = \sum_{uv\in E} \frac{2}{d_u + d_v},$$

and in [29] the inverse sum indeg index is defined as

$$ISI(G) = \sum_{uv\in E} \frac{d_u d_v}{d_u + d_v}.$$

Kulli in [15] introduced the geometric-arithmetic F-index of a graph G as

$$GAF(G) = \sum_{uv\in E} \frac{2d_u d_v}{d_u^2 + d_v^2}.$$

The sum–connectivity index ($\chi(G)$) was established in [33] and can be expressed as

$$\chi(G) = \sum_{uv\in E} \frac{1}{\sqrt{d_u + d_v}}.$$

A generalization of the $\chi(G)$ index, known as the general sum-connectivity index, was introduced in [34]. It is formally defined as

$$\chi_\alpha(G) = \sum_{uv\in E} (d_u + d_v)^\alpha,$$

where $\alpha \in \mathbb{R}$ is an arbitrary number. The forgotten index ($F(G)$) [7] and the multiplicative forgotten index ($\Pi_F(G)$) [8] are defined as

$$F(G) = \sum_{uv\in E} (d_u^2 + d_v^2), \qquad \Pi_F(G) = \prod_{uv\in E} (d_u^2 + d_v^2).$$

The sum-connectivity F–index, proposed by Kulli in [13], is defined as

$$SF(G) = \sum_{uv\in E} \frac{1}{\sqrt{d_u^2 + d_v^2}}.$$

In 2021, Kulli introduced the first Banhatti–Sombor index for a connected graph G in [14]. It is defined as

$$BSO(G) = \sum_{uv\in E} \sqrt{\frac{1}{d_u^2} + \frac{1}{d_v^2}}.$$

The Adriatic indices were introduced and studied over a series of publications several years ago [29]. Among them, only a limited number have shown potential utility in modeling physicochemical properties of molecular structures. One such index is the so-called symmetric division degree index ($SDD(G)$), which is defined as

$$SDD(G) = \sum_{uv\in E} \frac{d_u^2 + d_v^2}{d_u d_v}.$$

The Sombor index, a prominent topological index in graph theory, is proposed in [12]. It is defined as $SO(G) = \sum_{uv\in E} \sqrt{d_u^2 + d_v^2}$ and has garnered significant attention for its applications in quantitative structure–property/activity relationships (QSPR/QSAR) studies [12, 16, 17, 21, 22, 24, 28, 32]. Recently, a variant of the Sombor index was introduced in [27] and is called the Diminished Sombor index in [23]. This index is defined as follows

$$DSO(G) = \sum_{uv\in E} \frac{\sqrt{d_u^2 + d_v^2}}{d_u + d_v}.$$

Movahedi et al. [23] derived bounds for the Diminished Sombor (DSO) index, identified extremal graphs, and established Nordhaus–Gaddum-type inequalities. They also presented numerical analyses regarding the structure–dependence of the DSO index and its potential applications in chemistry. In [20], the tricyclic graph of a specified order that attains the maximum DSO is identified, and its distinctive structural characteristics are examined.

In this paper, we aim to advance the study of the DSO index by deriving new bounds in terms of fundamental graph parameters and exploring its connections with several classical topological indices. Our objective is to offer a deeper understanding of the Diminished Sombor index and to encourage its further investigation and application in the field of chemical graph theory.

2. Preliminaries

In this section, we present several known inequalities and results that will be used in the proofs of our main theorems.

Lemma 2.1. [19] *Let $a = (a_i)_{i=1}^n$ and $b = (b_i)_{i=1}^n$ be sequences of real numbers where $a_i \geq 0$ and $b_i > 0$. Then, for any $t \leq 0$ or $t \geq 1$, we have*

$$\Big(\sum_{i=1}^n a_i b_i\Big)^t \leq \Big(\sum_{i=1}^n a_i b_i^t\Big)\Big(\sum_{i=1}^n a_i\Big)^{t-1}.$$

Equality holds if and only if $t = 0$ or $t = 1$, or $b_1 = \cdots = b_n$, or there exists $1 \leq r \leq n-1$ such that $a_1 = \cdots = a_r$ and $b_{r+1} = \cdots = b_n$.

Lemma 2.2. [25] *Let $x = (x_i)_{i=1}^n$, and $a = (a_i)_{i=1}^n$ be two sequences of positive real numbers. For any $r > 0$,*

$$\frac{\left(\sum_{i=1}^n x_i\right)^{r+1}}{\left(\sum_{i=1}^n a_i\right)^r} \leq \sum_{i=1}^n \frac{x_i^{r+1}}{a_i^r}.$$

Equality holds if and only if either $r = 0$ or $\frac{x_1}{a_1} = \cdots = \frac{x_n}{a_n}$.

Lemma 2.3. [2] *Let G be a connected graph with n vertices and m edges. Then*

$$M_1(G) \leq m(m+1),$$

with equality for $n > 3$ if and only if $G \simeq K_3$ or $G \simeq S_n$.

Lemma 2.4. [18] *If $a_i, b_i \geq 0$ and $Ab_i \leq a_i \leq Bb_i$ for $1 \leq i \leq n$, then*

$$\Big(\sum_{i=1}^{n} a_i^2\Big)\Big(\sum_{i=1}^{n} b_i^2\Big) \leq \frac{(A+B)^2}{4AB}\Big(\sum_{i=1}^{n} a_i b_i\Big)^2.$$

The equality holds if and only if $A = B$ and $a_i = Ab_i$.

Lemma 2.5. [31] *Let G be a simple graph with m edges. Then*

$$H(G) \geq \frac{2m^2}{M_1(G)}.$$

Lemma 2.6. [4] *Let $\mathbf{a} = (a_i)_{i=1}^n$ and $\mathbf{b} = (b_i)_{i=1}^n$ be two nonnegative sequences arranged in decreasing order, such that $a_1 \neq 0$ and $b_1 \neq 0$, and let $\mathbf{w} = (w_i)_{i=1}^n$ be a nonnegative sequence. Then the following inequality holds*

$$\left(\sum_{i=1}^{n} w_i a_i^2\right)\left(\sum_{i=1}^{n} w_i b_i^2\right) \leq \max\left(b_1 \sum_{i=1}^{n} w_i a_i,\ a_1 \sum_{i=1}^{n} w_i b_i\right)\sum_{i=1}^{n} w_i a_i b_i.$$

Lemma 2.7. [35] *Let $a = (a_i)_{i=1}^n$, be positive real number sequence. Then*

$$\left(\sum_{i=1}^{n} \sqrt{a_i}\right)^2 \geq \sum_{i=1}^{n} a_i + n(n-1)\left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}}.$$

3. Main Results

In this section, we obtain several bounds for the Diminished Sombor index in terms of some graph parameters.

Theorem 3.1. *For a simple graph G of order $n \geq 2$, then*

$$\frac{\sqrt{2}}{4\Delta}Alb(G) + \frac{1}{2}GA(G) \leq DSO(G) \leq \frac{1}{2\delta}Alb(G) + \frac{\sqrt{2}}{2}GA(G).$$

Equality on the left-hand side holds if and only if $G \cong \overline{K_n}$ and in the right-hand side holds if and only if G consists of components, each of which is a regular graph.

Proof. For any two real numbers $a, b \geq 0$, we have

$$\frac{\sqrt{2}}{2}\Big(\sqrt{a} + \sqrt{b}\Big) \leq \sqrt{a+b}, \tag{1}$$

and the equality holds if and only if $a = b$.

Let $a = (d_u - d_v)^2$ and $b = 2d_ud_v$. Therefore, we get

$$\frac{\sqrt{2}}{2}\Big(\sqrt{(d_u - d_v)^2} + \sqrt{2d_ud_v}\Big) \leq \sqrt{d_u^2 + d_v^2}.$$

So

$$\frac{\sqrt{2}}{2}\Big(\frac{|d_u - d_v|}{d_u + d_v} + \sqrt{2}\frac{\sqrt{d_ud_v}}{d_u + d_v}\Big) \leq \frac{\sqrt{d_u^2 + d_v^2}}{d_u + d_v}.$$

Since $2\delta \leq d_u + d_v \leq 2\Delta$ for any $u, v \in V$, we obtain

$$
\begin{aligned}
DSO(G) &= \sum_{uv\in E} \frac{\sqrt{d_u^2 + d_v^2}}{d_u + d_v} \\
&\geq \frac{\sqrt{2}}{2} \sum_{uv\in E} \frac{|d_u - d_v|}{d_u + d_v} + \frac{1}{2} \sum_{uv\in E} \frac{2\sqrt{d_u d_v}}{d_u + d_v} \\
&\geq \frac{1}{2\Delta}\Big(\frac{\sqrt{2}}{2} \sum_{uv\in E} |d_u - d_v|\Big) + \frac{1}{2} \sum_{uv\in E} \frac{2\sqrt{d_u d_v}}{d_u + d_v} \\
&= \frac{\sqrt{2}}{4\Delta} Alb(G) + \frac{1}{2} GA(G).
\end{aligned}
$$

Equality in the above inequalities holds if and only if $G \simeq \overline{K}_n$.

For proving the upper bound, we use this fact that $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ and the equality holds if and only if $a = 0$ or $b = 0$. Therefore, by substituting $a = (d_u - d_v)^2$ and $b = 2d_u d_v$, we obtain

$$
\sqrt{d_u^2 + d_v^2} \leq \sqrt{(d_u - d_v)^2} + \sqrt{2d_u d_v} = |d_u - d_v| + \sqrt{2}\sqrt{d_u d_v}.
$$

Since $2\delta \leq d_u + d_v \leq 2\Delta$, we have

$$
\begin{aligned}
DSO(G) &= \sum_{uv\in E} \frac{\sqrt{d_u^2 + d_v^2}}{d_u + d_v} \\
&\leq \sum_{uv\in E} \frac{|d_u - d_v|}{d_u + d_v} + \sqrt{2} \sum_{uv\in E} \frac{\sqrt{d_u d_v}}{d_u + d_v} \\
&\leq \frac{1}{2\delta} \sum_{uv\in E} |d_u - d_v| + \frac{\sqrt{2}}{2} \sum_{uv\in E} \frac{2\sqrt{d_u d_v}}{d_u + d_v} \\
&= \frac{1}{2\delta} Alb(G) + \frac{\sqrt{2}}{2} GA(G).
\end{aligned}
$$

Equality holds if and only if $d_u = d_v$ for any $uv \in E$. □

Theorem 3.2. *Let G be a simple graph of size m. Then*

$$
DSO(G) \leq \frac{\sqrt{2}}{2}(Alb(G) + m).
$$

Equality holds if and only if $d_u = d_v$ for every edge $uv \in E$; equivalently, each component of G is a regular graph (not necessarily with the same degree).

Proof. For any edge $uv \in E$, set

$$
a = \frac{1}{2}(d_u - d_v)^2, \qquad b = \frac{1}{2}(d_u + d_v)^2.
$$

By the inequality $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$, with equality if and only if $a = 0$ or $b = 0$, we have

$$
\sqrt{\frac{1}{2}(d_u - d_v)^2 + \frac{1}{2}(d_u + d_v)^2} \leq \frac{\sqrt{2}}{2}\Big(\sqrt{(d_u - d_v)^2} + \sqrt{(d_u + d_v)^2}\Big).
$$

So

$$
\frac{\sqrt{d_u^2 + d_v^2}}{d_u + d_v} \leq \frac{\sqrt{2}}{2}\Big(\frac{|d_u - d_v|}{d_u + d_v} + \frac{d_u + d_v}{d_u + d_v}\Big).
$$

Therefore, we get

$$\begin{aligned} DSO(G) &= \sum_{uv\in E} \frac{\sqrt{d_u^2+d_v^2}}{d_u+d_v} \le \frac{\sqrt{2}}{2}\sum_{uv\in E}\Big(\frac{|d_u-d_v|}{d_u+d_v}+\frac{d_u+d_v}{d_u+d_v}\Big) \\ &\le \frac{\sqrt{2}}{2}\left[\sum_{uv\in E}|d_u-d_v| + \sum_{uv\in E}\frac{d_u+d_v}{d_u+d_v}\right] \\ &= \frac{\sqrt{2}}{2}\Big(Alb(G)+m\Big). \end{aligned}$$

For the equality condition, note that $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$ holds if and only if $a = 0$ or $b = 0$. With $a = \frac{1}{2}(d_u - d_v)^2$ and $b = \frac{1}{2}(d_u + d_v)^2$, we must have $(d_u - d_v)^2 = 0$ for every edge $uv \in E$ (since $d_u + d_v > 0$ on any edge), i.e., $d_u = d_v$ for every $uv \in E$. This is equivalent to saying that each connected component of G is a regular graph (though components may have different degrees). This completes the proof. □

Corollary 3.3. *Let G be a connected graph of size m. Then*

$$DSO(G) \le \frac{\sqrt{2}}{2} m(\Delta - \delta + 1).$$

The equality holds if and only if G is a regular graph.

Proof. Since $\delta \le d_u \le \Delta$ for any vertex $u \in V$, based on the definition of the $Alb(G)$ index, we have

$$Alb(G) = \sum_{uv\in E} |d_u - d_v| \le \sum_{uv\in E} |\Delta - \delta| = m(\Delta - \delta).$$

By applying Theorem 3.2, we obtain

$$\begin{aligned} DSO(G) &\le \frac{\sqrt{2}}{2}(Alb(G)+m) \\ &\le \frac{\sqrt{2}}{2}(m(\Delta-\delta)+m) \\ &= \frac{\sqrt{2}}{2}m(\Delta-\delta+1). \end{aligned}$$

For equality, we need equality in both bounds: in Theorem 3.2 this holds if and only if $d_u = d_v$ for every edge $uv \in E$, and in $\mathrm{Alb}(G) \le m(\Delta - \delta)$ it holds if and only if $\Delta = \delta$. Thus the equality holds if and only if G is a regular graph. □

Theorem 3.4. *For a simple graph G,*

$$DSO(G) \le \frac{\sqrt{2}}{2}\left(\sqrt{H(G)\Big(M_1(G) - H(G)\Big)}\right).$$

Equality holds if and only if sum of the squares of the degrees of adjacent vertices is a contrast.

Proof. By substituting $a_i = \frac{1}{d_u+d_v}$ and $b_i = \sqrt{d_u^2+d_v^2}$ into Lemma 2.1, we have

$$\begin{aligned}\left(DSO(G)\right)^2 &= \Big(\sum_{uv\in E}\frac{\sqrt{d_u^2+d_v^2}}{d_u+d_v}\Big)^2\\ &\le \Big(\sum_{uv\in E}\frac{d_u^2+d_v^2}{d_u+d_v}\Big)\Big(\sum_{uv\in E}\frac{1}{d_u+d_v}\Big)\\ &\le \Big(\sum_{uv\in E}\frac{(d_u+d_v)^2}{d_u+d_v}-\sum_{uv\in E}\frac{2}{d_u+d_v}\Big)\Big(\frac{1}{2}\sum_{uv\in E}\frac{2}{d_u+d_v}\Big)\\ &= \frac{1}{2}\left(M_1(G)-H(G)\right)H(G).\end{aligned}$$

Consequently, we have

$$DSO(G) \le \frac{\sqrt{2}}{2}\left(\sqrt{H(G)\big(M_1(G)-H(G)\big)}\right).$$

Equality holds if and only if $\sqrt{d_u^2+d_v^2}$ is a constant for any $uv \in E$. This implies that the equality holds if and only if $d_u^2+d_v^2$ is a constant for any two adjacent vertices. □

Theorem 3.5. *Let G be a simple graph of size m. Then*

$$DSO(G) \le \frac{m}{m+1}\sqrt{(m+1)^2-2}.$$

Proof. We consider $f(x) = \sqrt{x(\alpha-x)}$ and thus, $f'(x) = \frac{\alpha-2x}{2\sqrt{x\alpha-x^2}} < 0$ for $\frac{\alpha}{2} \le x \le \alpha$ and $f'(x) > 0$ for $x \le \frac{\alpha}{2}$. Since $H(G) \le \frac{M_1(G)}{2}$ and using Lemma 2.5 we have $H(G) \ge \frac{2m^2}{M_1(G)}$, thus $f(H(G)) \le f\left(\frac{2m^2}{M_1(G)}\right)$. Therefore using Theorem 3.4, we have

$$\begin{aligned}DSO(G) &\le \frac{\sqrt{2}}{2}\left(\sqrt{H(G)\big(M_1(G)-H(G)\big)}\right)\\ &\le \frac{\sqrt{2}}{2}\left(\sqrt{\frac{2m^2}{M_1(G)}\Big(M_1(G)-\frac{2m^2}{M_1(G)}\Big)}\right)\\ &= m\sqrt{\frac{M_1(G)}{M_1(G)}-\frac{2m^2}{M_1(G)^2}}\\ &= m\sqrt{1-\frac{2m^2}{M_1(G)^2}}.\end{aligned}$$

On the other hand, we suppose that $g(x) = \sqrt{1-\frac{2m^2}{x^2}}$ and $g'(x) = \frac{2m^2x^{-3}}{\sqrt{1-\frac{2m^2}{x^2}}} > 0$. Therefore, using Lemma 2.3, we get

$$DSO(G) \le m\sqrt{1-\frac{2m^2}{M_1(G)^2}} \le \frac{m}{m+1}\sqrt{(m+1)^2-2}.$$

□

Theorem 3.6. *Let G be a graph of size m. Then*

$$DSO(G) \ge \sqrt{\alpha m\,(m-GAF(G))},$$

where $\alpha = \frac{4\sqrt{2}(\Delta+\delta)(\sqrt{\Delta^2+\delta^2})}{\left(\sqrt{2(\Delta^2+\delta^2)}+(\Delta+\delta)\right)^2}$. *Equality holds if and only if each component of G is a regular graph (not necessarily with the same degree).*

Proof. We first establish the following bounds for any $u, v \in V$.

$$\frac{1}{\sqrt{2}} \leq \frac{\sqrt{d_u^2 + d_v^2}}{d_u + d_v} \leq \frac{\sqrt{\Delta^2 + \delta^2}}{\Delta + \delta}. \tag{2}$$

Consider the function $f(x) = \frac{\sqrt{1+x^2}}{1+x}$, whose derivative is $f'(x) = \frac{x-1}{(1+x)^2\sqrt{1+x^2}}$. Therefore, $f(x)$ is increasing for $x \geq 1$. We know that $0 < \delta \leq d_u \leq \Delta$ for any $u \in V$, and consequently, $\frac{\delta}{\Delta} \leq \frac{d_u}{d_v} \leq \frac{\Delta}{\delta}$. Since $f(x)$ is the increasing function, we have $f\left(\frac{d_u}{d_v}\right) \leq f\left(\frac{\Delta}{\delta}\right)$. Therefore,

$$\frac{\sqrt{d_u^2 + d_v^2}}{d_u + d_v} = \frac{\sqrt{1 + \frac{d_u^2}{d_v^2}}}{1 + \frac{d_u}{d_v}} \leq \frac{\sqrt{1 + \frac{\Delta^2}{\delta^2}}}{1 + \frac{\Delta}{\delta}} = \frac{\sqrt{\Delta^2 + \delta^2}}{\Delta + \delta}.$$

On the other hand, since $(d_u - d_v)^2 \geq 0$, we have $\frac{1}{\sqrt{2}} \leq \frac{\sqrt{d_u^2+d_v^2}}{d_u+d_v}$.

By applying Lemma 2.4 with $a_i = \frac{\sqrt{d_u^2+d_v^2}}{d_u+d_v}$, $b_i = 1$, $A = \frac{1}{\sqrt{2}}$, and $B = \frac{\sqrt{\Delta^2+\delta^2}}{\Delta+\delta}$, we have

$$\Big(\sum_{uv\in E} \frac{d_u^2 + d_v^2}{(d_u + d_v)^2}\Big)\Big(\sum_{uv\in E} 1\Big) \leq \frac{(\frac{1}{\sqrt{2}} + \frac{\sqrt{\Delta^2+\delta^2}}{\Delta+\delta})^2}{\frac{4}{\sqrt{2}} \times \frac{\sqrt{\Delta^2+\delta^2}}{\Delta+\delta}}\Big(\sum_{uv\in E} \frac{\sqrt{d_u^2 + d_v^2}}{d_u + d_v}\Big)^2.$$

So

$$m\Big(\sum_{uv\in E} 1 - \sum_{uv\in E} \frac{2d_ud_v}{(d_u + d_v)^2}\Big) \leq \frac{\left(\sqrt{2(\Delta^2 + \delta^2)} + \Delta + \delta\right)^2}{4\sqrt{2}(\Delta + \delta)(\sqrt{\Delta^2 + \delta^2})}\Big(DSO(G)\Big)^2.$$

Since $GAF(G) = \sum_{uv\in E} \frac{2d_ud_v}{d_u^2+d_v^2} \geq \sum_{uv\in E} \frac{2d_ud_v}{(d_u+d_v)^2}$, we have

$$m\,(m - GAF(G))\,\frac{4\sqrt{2}(\Delta + \delta)(\sqrt{\Delta^2 + \delta^2})}{\left(\sqrt{2(\Delta^2 + \delta^2)} + \Delta + \delta\right)^2} \leq (DSO(G))^2.$$

With considering $\alpha = \frac{4\sqrt{2}(\Delta+\delta)(\sqrt{\Delta^2+\delta^2})}{\left(\sqrt{2(\Delta^2+\delta^2)}+(\Delta+\delta)\right)^2}$,

$$DSO(G) \geq \sqrt{\alpha m\,(m - GAF(G))}.$$

Equality holds if and only if $\frac{1}{\sqrt{2}} = \frac{\sqrt{\Delta^2+\delta^2}}{\Delta+\delta} = \frac{\sqrt{d_u^2+d_v^2}}{d_u+d_v}$ for any $uv \in E$, i.e., for any $uv \in E$, $\delta = d_u = d_v = \Delta$. □

Theorem 3.7. *Let G be a simple connected graph. Then*

$$DSO(G) \geq \frac{M_1(G) - 2ISI(G) + \delta\Delta H(G)}{\sqrt{2}(\Delta + \delta)}.$$

Equality holds if and only if G is a regular graph.

Proof. For any $u, v \in V$, we have $\sqrt{2}\delta \leq \sqrt{d_u^2 + d_v^2} \leq \sqrt{2}\Delta$. Hence

$$\left(\sqrt{d_u^2 + d_v^2} - \sqrt{2}\delta\right)\left(\sqrt{2}\Delta - \sqrt{d_u^2 + d_v^2}\right) \geq 0,$$

which implies

$$\sqrt{2}(\Delta+\delta)\sqrt{d_u^2+d_v^2} \geq d_u^2+d_v^2+2\delta\Delta.$$

Using the definition of the DSO index, we get

$$\begin{aligned} DSO(G) &= \sum_{uv\in E}\frac{\sqrt{d_u^2+d_v^2}}{d_u+d_v} \\ &\geq \frac{1}{\sqrt{2}(\Delta+\delta)}\Big(\sum_{uv\in E}\frac{d_u^2+d_v^2+2\delta\Delta}{d_u+d_v}\Big) \\ &= \frac{1}{\sqrt{2}(\Delta+\delta)}\Big(\sum_{uv\in E}\frac{(d_u+d_v)^2-2d_ud_v}{d_u+d_v}+\delta\Delta\sum_{uv\in E}\frac{2}{d_u+d_v}\Big) \\ &= \frac{1}{\sqrt{2}(\Delta+\delta)}\Big(\sum_{uv\in E}(d_u+d_v)-2\sum_{uv\in E}\frac{d_ud_v}{d_u+d_v}+\delta\Delta\sum_{uv\in E}\frac{2}{d_u+d_v}\Big) \\ &= \frac{1}{\sqrt{2}(\Delta+\delta)}\Big(M_1(G)-2ISI(G)+\delta\Delta H(G)\Big). \end{aligned}$$

Equality holds if and only if $d_u = d_v$ for every $uv \in E$ and $\delta = \Delta$, i.e., G is a regular graph. □

Theorem 3.8. *Let G be a simple graph. Then*

$$\frac{(\chi(G))^2}{SF(G)} \leq DSO(G) \leq \sqrt{F(G)\chi_{-2}(G)}$$

Equality holds if and only if each component of G is a regular graph (not necessarily with the same degree).

Proof. By the Cauchy–Schwarz inequality,

$$\begin{aligned} (DSO(G))^2 &= \Big(\sum_{uv\in E}\frac{\sqrt{d_u^2+d_v^2}}{d_u+d_v}\Big)^2 \\ &\leq \Big(\sum_{uv\in E}\Big(\sqrt{d_u^2+d_v^2}\Big)^2\Big)\Big(\sum_{uv\in E}\frac{1}{(d_u+d_v)^2}\Big) \\ &= \Big(\sum_{uv\in E}\big(d_u^2+d_v^2\big)\Big)\Big(\sum_{uv\in E}(d_u+d_v)^{-2}\Big) \\ &= F(G)\chi_{-2}(G). \end{aligned}$$

Hence

$$DSO(G) \leq \sqrt{F(G)\chi_{-2}(G)}.$$

Equality in the upper bound holds if and only if the sequences $\sqrt{d_u^2+d_v^2}$ and $\frac{1}{(d_u+d_v)}$ are proportional over all edges, which is equivalent to $d_u = d_v$ on every edge; this forces each component to be regular.

For the lower bound, apply Lemma 2.2 with $x_i = \frac{1}{\sqrt{d_u+d_v}}$, $a_i = \frac{1}{\sqrt{d_u^2+d_v^2}}$, and $r = 1$. Then

$$\begin{aligned} DSO(G) &= \sum_{uv\in E}\frac{\sqrt{d_u^2+d_v^2}}{d_u+d_v} = \sum_{uv\in E}\frac{\frac{1}{(\sqrt{d_u+d_v})^2}}{\frac{1}{\sqrt{d_u^2+d_v^2}}} \\ &\geq \frac{\Big(\sum_{uv\in E}\frac{1}{\sqrt{d_u+d_v}}\Big)^2}{\sum_{uv\in E}\frac{1}{\sqrt{d_u^2+d_v^2}}} = \frac{(\chi(G))^2}{SF(G)}. \end{aligned}$$

Equality in the lower bound requires $\frac{x_i}{a_i}$ to be constant over edges, i.e., $d_u = d_v$ on every edge. Therefore, equality throughout holds if and only if each component of G is regular (not necessarily with the same degree across components). □

Theorem 3.9. *Let G be a connected graph. Then*

$$DSO(G) \geq \frac{(R(G))^2}{BSO(G)}.$$

Proof. By substituting $t = 2$, $a_i = \frac{\sqrt{d_u^2+d_v^2}}{d_u+d_v}$ and $b_i = \sqrt{\frac{d_u+d_v}{d_ud_v}}$ into Lemma 2.1, we have

$$\begin{aligned}
BSO(G) &= \sum_{uv\in E} \frac{\sqrt{d_u^2+d_v^2}}{d_ud_v} \\
&= \sum_{uv\in E} \left(\frac{\sqrt{d_u^2+d_v^2}}{d_u+d_v}\right)\left(\sqrt{\frac{d_u+d_v}{d_ud_v}}\right)^2 \\
&\geq \frac{\left(\sum_{uv\in E}\left(\frac{\sqrt{d_u^2+d_v^2}}{d_u+d_v}\right)\left(\frac{\sqrt{d_u+d_v}}{\sqrt{d_ud_v}}\right)\right)^2}{\sum_{uv\in E}\frac{\sqrt{d_u^2+d_v^2}}{d_u+d_v}} \\
&= \frac{\left(\sum_{uv\in E}\left(\frac{1}{\sqrt{d_ud_v}}\right)\left(\frac{\sqrt{(d_u^2+d_v^2)(d_u+d_v)}}{d_u+d_v}\right)\right)^2}{DSO(G)} \\
&\geq \frac{\left(\sum_{uv\in E}\left(\frac{1}{\sqrt{d_ud_v}}\right)\left(\frac{\sqrt{(d_u+d_v)^2}}{d_u+d_v}\right)\right)^2}{DSO(G)} \\
&= \frac{(R(G))^2}{DSO(G)}.
\end{aligned}$$

Rearranging yields $DSO(G) \geq \frac{(R(G))^2}{BSO(G)}$, as claimed. □

Theorem 3.10. *Let G be a connected graph of size m and the minimum degree $\delta \geq 2$. Then*

$$\frac{\sqrt{2}}{2\Delta}SDD(G) \leq DSO(G) \leq \left(\frac{\Delta}{2\delta}\right)\sqrt{mSDD(G)}.$$

Equality holds if and only if G is a regular graph.

Proof. By substituting $r = 1$, $x_i = \frac{\sqrt{d_u^2+d_v^2}}{d_u+d_v}$ and $a_i = \frac{d_ud_v}{(d_u+d_v)^2}$ into Lemma 2.2, we have

$$\begin{aligned}
SDD(G) &= \sum_{uv\in E}\frac{d_u^2+d_v^2}{d_ud_v} = \sum_{uv\in E}\left(\frac{\sqrt{d_u^2+d_v^2}}{d_u+d_v}\right)^2\left(\frac{(d_u+d_v)^2}{d_ud_v}\right) \\
&= \sum_{uv\in E}\frac{\left(\frac{\sqrt{d_u^2+d_v^2}}{d_u+d_v}\right)^2}{\frac{d_ud_v}{(d_u+d_v)^2}} \geq \frac{\left(\sum_{uv\in E}\frac{\sqrt{d_u^2+d_v^2}}{d_u+d_v}\right)^2}{\sum_{uv\in E}\frac{d_ud_v}{(d_u+d_v)^2}} \\
&= \frac{(DSO(G))^2}{\sum_{uv\in E}\frac{d_ud_v}{(d_u+d_v)^2}}
\end{aligned}$$

Since for any $u \in V$, $\delta \leq d_u \leq \Delta$, we have $\frac{\delta^2}{4\Delta^2} \leq \frac{d_u d_v}{(d_u+d_v)^2} \leq \frac{\Delta^2}{4\delta^2}$. The equality holds if and only if $\delta = \Delta$. Therefore we get

$$SDD(G) \geq \frac{(DSO(G))^2}{\sum_{uv\in E} \frac{d_u d_v}{(d_u+d_v)^2}} \geq \frac{(DSO(G))^2}{\sum_{uv\in E} \frac{\Delta^2}{4\delta^2}} = \frac{(DSO(G))^2}{\left(\frac{\Delta}{2\delta}\right)^2 m}.$$

Consequently,

$$DSO(G) \leq \sqrt{\left(\frac{\Delta}{2\delta}\right)^2 mSDD(G)} = \left(\frac{\Delta}{2\delta}\right)\sqrt{mSDD(G)}.$$

Equality here holds if and only if $\delta = \Delta$, hence G is a regular graph.

For the lower bound, by substituting $a_i = \sqrt{d_u^2 + d_v^2} = b_i$ and $w_i = \frac{1}{d_u+d_v}$ into Lemma 2.6, we get

$$\left(\sum_{uv\in E} \frac{d_u^2 + d_v^2}{d_u + d_v}\right)\left(\sum_{uv\in E} \frac{d_u^2 + d_v^2}{d_u + d_v}\right) \leq \max\left(\sqrt{2}\Delta \sum_{uv\in E} \frac{\sqrt{d_u^2 + d_v^2}}{d_u + d_v},\ \sqrt{2}\Delta \sum_{uv\in E} \frac{\sqrt{d_u^2 + d_v^2}}{d_u + d_v}\right) \sum_{uv\in E} \frac{d_u^2 + d_v^2}{d_u + d_v}.$$

Hence,

$$\sum_{uv\in E} \frac{d_u^2 + d_v^2}{d_u + d_v} \leq \sqrt{2}\Delta \sum_{uv\in E} \frac{\sqrt{d_u^2 + d_v^2}}{d_u + d_v}.$$

Therefore, since $d_u \geq 2$ for any $u \in V$

$$\begin{aligned}\sqrt{2}\Delta DSO(G) = \sqrt{2}\Delta \sum_{uv\in E} \frac{\sqrt{d_u^2 + d_v^2}}{d_u + d_v} &\geq \sum_{uv\in E} \frac{d_u^2 + d_v^2}{d_u + d_v}\\ &\geq \sum_{uv\in E} \frac{d_u^2 + d_v^2}{d_u d_v} = SDD(G),\end{aligned}$$

which yields

$$DSO(G) \geq \frac{\sqrt{2}}{2\Delta} SDD(G).$$

The equality in these steps holds if and only if $d_u = d_v$ for every $uv \in E$, i.e., G is a regular graph. This completes the proof. □

Theorem 3.11. *Let G be a connected graph. Then*

$$\frac{SO(G)}{2\Delta} \leq DSO(G) \leq \frac{SO(G)}{2\delta}.$$

Equality holds if and only if G is a regular graph.

Proof. For any edge $uv \in E$, we have $2\delta \leq d_u + d_v \leq 2\Delta$. Hence

$$\frac{1}{2\Delta} \sum_{uv\in E} \sqrt{d_u^2 + d_v^2} \leq \sum_{uv\in E} \sqrt{d_u^2 + d_v^2} \times \frac{1}{d_u + d_v} \leq \frac{1}{2\delta} \sum_{uv\in E} \sqrt{d_u^2 + d_v^2}.$$

So

$$\frac{SO(G)}{2\Delta} \leq DSO(G) \leq \frac{SO(G)}{2\delta}.$$

The equality holds if and only if $\delta = \Delta$, i.e., G is a regular graph. □

Theorem 3.12. *Let G be a simple graph of size m. Then*

$$DSO(G) \geq \frac{1}{2\Delta}\sqrt{F(G) + m(m-1)\left(\Pi_F(G)\right)^{\frac{1}{m}}}.$$

Proof. For $a_i = d_u^2 + d_v^2$ in Lemma 2.7, we get

$$\begin{aligned}(2\Delta DSO(G))^2 &\geq \left(\sum_{uv\in E}(d_u+d_v)\,\frac{\sqrt{d_u^2+d_v^2}}{d_u+d_v}\right)^2\\ &= \left(\sum_{uv\in E}\sqrt{d_u^2+d_v^2}\right)^2\\ &\geq \sum_{uv\in E}(d_u^2+d_v^2) + m(m-1)\left(\prod_{uv\in E}(d_u^2+d_v^2)\right)^{\frac{1}{m}}\\ &= F(G) + m(m-1)\left(\Pi_F(G)\right)^{\frac{1}{m}}.\end{aligned}$$

Therefore, we have

$$DSO(G) \geq \frac{1}{2\Delta}\sqrt{F(G) + m(m-1)\left(\Pi_F(G)\right)^{\frac{1}{m}}}.$$

□

Theorem 3.13. *Let G be a graph of order n and size m. Then*

$$\frac{1}{2}BSO(G) \leq DSO(G) \leq \frac{n}{4}BSO(G).$$

Equality on the left-hand side holds if and only if $G \cong mK_2$ and in the right-hand side holds if and only if each component of G is a regular graph (not necessarily with the same degree).

Proof. For each $uv \in E$, by Jensen's inequality applied to the convex function $f(x) = 1/x$,

$$\frac{2}{d_u+d_v} \leq \frac{1}{2}\left(\frac{1}{d_u}+\frac{1}{d_v}\right) = \frac{1}{2}\left(\frac{d_u+d_v}{d_ud_v}\right). \tag{3}$$

Therefore, using 3 and the fact that $d_u + d_v \leq n$ for any $uv \in E$, we obtain

$$\begin{aligned}DSO(G) &= \sum_{uv\in E}\frac{\sqrt{d_u^2+d_v^2}}{d_u+d_v} = \frac{1}{2}\sum_{uv\in E}\left(\sqrt{d_u^2+d_v^2}\right)\left(\frac{2}{d_u+d_v}\right)\\ &\leq \frac{1}{4}\sum_{uv\in E}\left(\sqrt{d_u^2+d_v^2}\right)\left(\frac{d_u+d_v}{d_ud_v}\right) = \frac{1}{4}\sum_{uv\in E}(d_u+d_v)\left(\frac{\sqrt{d_u^2+d_v^2}}{d_ud_v}\right)\\ &\leq \frac{n}{4}\sum_{uv\in E}\frac{\sqrt{d_u^2+d_v^2}}{d_ud_v} = \frac{n}{4}BSO(G).\end{aligned}$$

Equality in the inequalities above holds if and only if $d_u = d_v$ for every $uv \in E$ in G.

Next, since $0 < \frac{1}{d_u} \leq 1$ for any $u \in V$, we have $\frac{1}{d_u} + \frac{1}{d_v} \leq 2$, and hence

$$\frac{2}{d_u+d_v} \geq \frac{1}{d_ud_v}.$$

Thus,

$$DSO(G) = \sum_{uv\in E} \frac{\sqrt{d_u^2 + d_v^2}}{d_u + d_v} = \frac{1}{2}\sum_{uv\in E} \left(\sqrt{d_u^2 + d_v^2}\right)\left(\frac{2}{d_u + d_v}\right)$$
$$\geq \frac{1}{2}\sum_{uv\in E} \frac{\sqrt{d_u^2 + d_v^2}}{d_u d_v} = \frac{1}{2}BSO(G).$$

Equality here holds if and only if $d_u = 1$ for every $u \in V$, i.e., $G \cong mK_2$. □

4. Conclusion

In this paper, we conducted a comprehensive investigation of the diminished Sombor index and its relationships with several classical topological indices. By establishing new bounds and characterizing extremal graphs, our results contribute to a deeper understanding of the mathematical properties and potential applications of the DSO index, particularly in chemical graph theory. The connections established between DSO and well-known indices such as the Zagreb, Albertson, Harmonic, Randić, and geometric-arithmetic indices highlight the versatility and relevance of this new invariant.

Future research directions include extending the theoretical framework of DSO to broader graph classes, investigating algorithmic and computational aspects, and assessing its practical performance in chemical and interdisciplinary settings (e.g., QSPR/QSAR). Such studies may yield further insights and expand the utility of DSO in both mathematical analysis and applied modeling.

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