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# Construction of 5-helix systems of index 2

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**Abstract.** A 5-*helix* $S^{(3)}(1,5)$ of *centre* $\{c\}$ is a 3-uniform hypergraph, with 5 hyperedges, all having in common exactly the *centre* $\{c\}$ of degree 5, and the remaining vertices of degree 1. In this paper we determine the spectrum of $S^{(3)}(1,5)$-*designs* of index 2 with pairwise distinct blocks.

## 1. Introduction

Let $K_v^{(h)} = (X, \mathcal{E})$ be the complete hypergraph, uniform of rank $h$, defined on the vertex set $X = \{x_1, x_2, \ldots, x_v\}$, that is, $\mathcal{E}$ is the collection of all the subsets of $X$ whose cardinality is $h$; we will call a set of cardinality $h$ an $h$-subset.

If $H^{(h)}$ is a subhypergraph of $K_v^{(h)}$, then an $H^{(h)}$-*design*, having *order* $v$ and index $\lambda$, is a pair $\Sigma = (X, \mathcal{B})$, where $X$ is a finite set of cardinality $v$, whose elements are called *vertices*, and $\mathcal{B}$ is a collection of hypergraphs over $X$, called *blocks*, all isomorphic to $H^{(h)}$, under the condition that every $h$-subset of $X$ is an hyperedge of exactly $\lambda$ hypergraphs of the collection $\mathcal{B}$. An $H^{(h)}$-*design*, of order $v$ and index $\lambda$, is also called an $H^{(h)}$-*decomposition* of $\lambda K_v^{(h)}$,

The study of hypergraph designs has become an important research area of combinatorial design (see, for example, [1–9]).

In [7], we started the study of special hypergraph designs called hyperstars. Specifically, a *hyperstar* $S^{(h)}(r,s)$ is the $h$-uniform hypergraph having $s$ hyperedges and order $(h-r)s+r$, such that all the edges have in common exactly the same $r$ vertices, which form its *centre*; and all the vertices of the centre have degree $s$.

In this paper we focus on the hyperstar $S^{(3)}(1,5)$, that is, on the 3-uniform hypergraph having 5 hyperedges and order 11, such that all the edges have in common exactly the same vertex, which forms its *centre*, and all the vertices of the centre have degree 5. The hyperedges of $S^{(3)}(1,5)$ are $\{\{0,1,2\},\{0,3,4\},\{0,5,6\},\{0,7,8\},\{0,9,10\}\}$, the border is the set of the pairs $\{\{1,2\},\{3,4\},\{5,6\},\{7,8\},\{9,10\}\}$. We call it a 5-*helix system*. Figure 1 shows a block of a 5-helix system.

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We determine the spectrum of a 5-helix system for index 2 with pairwise distinct blocks. Since a system of index 2 is constructed from the union of two disjoint systems of index 1, the existence of a system of index 1 is immediate from the proofs.

In the following we will make use of the following notation.

**Notation 1.1.** *Let $X$ be a finite set of cardinality $v$. We denote by $\binom{X}{k} = \{A \subseteq X \mid \#A = k\}$ for the collection of $k$-subsets of $X$, i.e., it is the set of the subsets of $X$ of cardinality $k$.*

**Notation 1.2.** *Given the cyclic group $X = (\{1, 2, \ldots v\}, +)$, a subset $Y \subseteq X$ and $a \in X$, we denote by $Y + a := \{y + a \mid y \in Y\}$, and call it a "translate"of $Y$.*

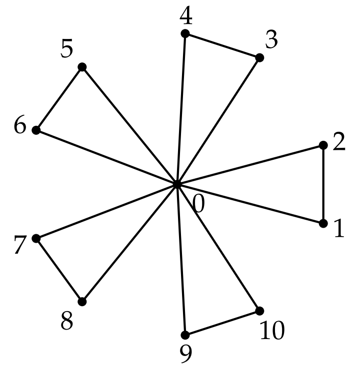


Figure 1: A 5-helix block

## 2. Preliminary lemmas

This section is devoted to show some lemmas that will be useful to determine the spectrum of a 5-helix system for index 2 with pairwise distinct blocks. In the sequel, to shorten the notation, by 5-helix system for index 2 we mean a 5-helix system for index 2 with pairwise distinct blocks.

The following result gives us a numerical condition for the existence of a 5-helix.

**Lemma 2.1.** *If there exists a 5-helix of index 2 and order $v$, then $v \geq 11$ and $v \equiv 0, 1, 2 \mod 5$.*

*Proof.* Note that the block of a 5-helix has 11 vertices, then it is $v \geq 11$. Given a 5-helix $(X, \mathcal{B})$ of order $v$, the number of triples of $X$ is $\binom{v}{3}$ and, if each triple has multiplicity $\lambda = 2$ then there are $2\binom{v}{3}$ triples; since each block contains 5 triples it follows that $5|\mathcal{B}| = 2\binom{v}{3}$, hence $v \equiv 0, 1, 2 \mod 5$. □

**Definition 2.2.** *Let $X$ be a set of $v$ elements. We call a $\Delta(0, 2, 5)$-block the set*

$$\Delta = \Big\{\{x_1, y_1\}, \{x_2, y_2\}, \{x_3, y_3\}, \{x_4, y_4\}, \{x_5, y_5\}\Big\}$$

*with $x_h, y_k \in X$ such that $\{x_i, y_i\} \cap \{x_j, y_j\} = \emptyset$ if $i \neq j$. A partition of $\binom{X}{2}$ in $\Delta(0, 2, 5)$-blocks is a $\Delta(0, 2, 5)$-decomposition over $X$.*

We start our study with the case $v = 11$.

**Lemma 2.3.** *If $v = 11$ then there exist a $\Delta(0, 2, 5)$-decomposition of index 2.*

*Proof.* Define

$$\Delta = \{\{10,11\},\{7,9\},\{2,5\},\{4,8\},\{1,6\}\}$$

and

$$\Delta' = \{\{9,10\},\{1,3\},\{4,7\},\{2,6\},\{11,5\}\}\,.$$

If

$$\mathcal{D} = \{\Delta + a \mid 0 \le a < 11\}$$
$$\mathcal{D}' = \{\Delta' + a \mid 0 \le a < 11\}$$

denote the translates of $\Delta$ and $\Delta'$, it is straightforward to check that $(X,\mathcal{D})$ and $(X,\mathcal{D}')$ are two disjoint $\Delta(0,2,5)$-decompositions over $X$. Thus, $(X,\mathcal{D}\cup\mathcal{D}')$ is a $\Delta(0,2,5)$-decomposition over $X$ of index 2. □

**Remark 2.4.** *From now on, a system of index 2 is always obtained by joining two disjoint systems of index 1. Therefore, the existence of a system of index 1 is immediate from the proofs.*

**Lemma 2.5.** *Let $X$ be a set of $v$ elements with $v \equiv 0,1 \mod 5$ and $v \ge 15$. Then there exist two $\Delta(0,2,5)$-decompositions $\mathcal{M}$ and $\mathcal{N}$ with no blocks in common, i.e., $\mathcal{M}\cap\mathcal{N}=\emptyset$.*

*Proof.* Set $X := \{1,2,\dots,v\}$ and consider the cyclic group $(X,+)$ of order $v$. We observe that the set $\binom{X}{2}$ can be decomposed in $\lfloor \frac{v}{2} \rfloor$ sets of the following form

$$X_\phi = \{\{i+k, j+k\} \mid 0 \le k < v, j-i=\phi\}$$

for $1 \le \phi \le \lfloor \frac{v}{2} \rfloor$.

We have

$$|X_\phi| = \begin{cases} v & \text{if } v \text{ is odd and for every } \phi \\ v & \text{if } v \text{ is even and for every } \phi \neq \lfloor \frac{v}{2} \rfloor \\ \frac{v}{2} & \text{if } \phi = \lfloor \frac{v}{2} \rfloor. \end{cases}$$

We consider three cases:

case i) $v \equiv 0 \mod 5$.

If $v \equiv 0 \mod 5$ then $|X_\phi|$ is multiple of 5. Given $i \in X$ we define the following subsets of $X_\phi$ in the following way: for $1 \le \phi \le 4$ we set:

$$\Delta_1 = \Big\{\{i,i+1\},\{i+2,i+3\},\{i+4,i+5\},\{i+6,i+7\},\{i+8,i+9\}\Big\}$$
$$\Delta_2 = \Big\{\{i,i+2\},\{i+1,i+3\},\{i+4,i+6\},\{i+7,i+9\},\{i+10,i+12\}\Big\}$$
$$\Delta_3 = \Big\{\{i,i+3\},\{i+1,i+4\},\{i+2,i+5\},\{i+8,i+11\},\{i+9,i+12\}\Big\}$$
$$\Delta_4 = \Big\{\{i,i+4\},\{i+1,i+5\},\{i+2,i+6\},\{i+3,i+8\},\{i+9,i+13\}\Big\}.$$

For $5 \le \phi \le \lfloor \frac{v}{2} \rfloor$ we set

$$\Delta_\phi = \Big\{\{i,i+\phi\},\{i+1,i+\phi+1\},\{i+2,i+\phi+2\},\{i+3,i+\phi+3\},\{i+4,i+\phi+4\}\Big\}.$$

Define a $\Delta(0,2,5)$-decomposition over $X$ by setting

$$\Delta_\phi + 5a = \Big\{\{x+5a, y+5a\} \mid \{x,y\} \in \Delta_\phi\Big\}.$$

Then it is easy to check that if $1 \le a < b \le \frac{v}{5}$ then $(\Delta_\phi + 5a) \cap (\Delta_\phi + 5b) = \emptyset$. Hence each $X_\phi$ can be partitioned as $X_\phi = \cup_a(\Delta_\phi + 5a)$. It is worthwhile to remark that, if $v$ is even, $X_{v/2}$ is a matching of $\binom{X}{2}$ and one can choose a partition of $X_{v/2}$ without any restriction.

case ii) $v \equiv 1 \mod 5$ and $v$ odd.
If $v \equiv 1 \mod 5$ and $v$ odd, then $\lfloor \frac{v}{2} \rfloor \equiv 0 \mod 5$. Hence $\binom{X}{2}$ can be decomposed as $\binom{X}{2} = \cup_\phi X_\phi$ where $|X_\phi| = v$ and $1 \leq \phi \leq \lfloor \frac{v}{2} \rfloor$. Given $k \in \{0, \dots, \frac{1}{5}\lfloor \frac{v}{2} \rfloor - 1\}$, define a $\Delta(0,2,5)$-decomposition over $X$ by setting

$$\Gamma_k := \{\{i, i+1+5k\}, \{i+2, i+4+5k\}, \{i+3, i+6+5k\},$$

$$\{i+5, i+9+5k\}, \{i+7, i+12+5k\}\}$$

as base blocks and considering $\binom{X}{2} = \cup_{k,a}(\Gamma_k + a)$ with $0 \leq a < v$.

case iii) $v \equiv 1 \mod 5$ and $v$ even.
If $v \equiv 1 \mod 5$ and $v$ is even, then $\lfloor \frac{v}{2} \rfloor \equiv 3 \mod 5$. In this case we have $\binom{X}{2} = \cup_{\phi=1}^{\lfloor \frac{v}{2} \rfloor} X_\phi$. Decompose $X_1 \cup X_2 \cup X_{\frac{v}{2}}$ using as base block

$$\Xi := \left\{\{i, i+1\}, \{i+1+\frac{v}{2}, i+2+\frac{v}{2}\}, \{i+2, i+4\}, \{i+2+\frac{v}{2}, i+4+\frac{v}{2}\}, \{i+3, i+3+\frac{v}{2}\}\right\}.$$

For $a = 0, \dots, \frac{v}{2}$, we consider the other blocks as translates $\Xi + a$, hence $X_1 \cup X_2 \cup X_{\frac{v}{2}} = \cup_a(\Xi + a)$ with $\Xi + a$ pairwise disjoint $\Delta(0,2,5)$'s; note that $|X_\phi| = v$ if $1 \leq \phi < \lfloor \frac{v}{2} \rfloor$ and $|X_\phi| = \frac{v}{2}$ if $\phi = \lfloor \frac{v}{2} \rfloor = \frac{v}{2}$.
If $3 \leq \phi < \frac{v}{2}$, the number of remaining sets $X_\phi$ is $\frac{v}{2} - 3$ which is a multiple of 5. Define

$$\Xi_k = \{\{i, i+3+5k\}, \{i+1, i+5+5k\}, \{i+2, i+7+5k\}, \{i+4, i+10+5k\}, \{i+6, i+13+5k\}\}$$

for $0 \leq k < \bar{k}$ where $\bar{k} = \frac{v-6}{10}$. The translates di $\Xi$ and $\Xi_k$, i.e., $\Xi + a$, $\Xi_k + a$, give a $\Delta(0,2,5)$-decomposition over $X$.

In summary, if $v \equiv 0,1 \mod 5$, there exists a $\Delta(0,2,5)$-decomposition of $X$ which depends on $i \in X$; it is straightforward to check that for $i = 1,2$ we get two different $\Delta(0,2,5)$-decompositions $(X, \mathcal{D})$ and $(X, \mathcal{D}')$ which are disjoint, i.e. if $\Delta \in \mathcal{D}$ and $\Delta' \in \mathcal{D}'$ then $\Delta \neq \Delta'$. □

## 3. Main results

**Theorem 3.1.** *There exist two disjoint $\Delta(0,2,5)$-decompositions of order $v$ over $X$ if and only if $v \geq 11$ and $v \equiv 0,1 \mod 5$.*

*Proof.* If $v \geq 11$ such that $v \equiv 0,1 \mod 5$ then we get two $\Delta(0,2,5)$-decompositions over $X$ that are disjoint from lemmas 2.3 and 2.5. If there exist two disjoint $\Delta(0,2,5)$-decompositions over $X$ then $v \geq 11$ with $v \equiv 0,1 \mod 5$. □

We now give a couple of lemmas that describe a recursive construction for 5-helix designs.

**Lemma 3.2.** *If there exists a 5-helix of order $v \equiv 0,1 \mod 5$ and index 2, then there exists a 5-helix of order $v+1$ and index 2.*

*Proof.* Let $(X, \mathcal{E})$ be a 5-helix design of order $v \geq 11$ such that $v \equiv 0,1 \mod 5$ and index 2. From lemmas 2.3, 2.5 there exist $(X, \mathcal{D})$ and $(X, \mathcal{D}')$ two disjoint $\Delta(0,2,5)$-decompositions of order $v$. Given $* \notin X$ and $\alpha \in \mathcal{D}$, let us call $\alpha_*$ the hypergraph with centre $\{*\}$ and border $\alpha$, i.e., $\alpha_* = \left\{\{*\} \cup \{x,y\} \mid \{x,y\} \in \alpha\right\}$, and define $\mathcal{D}_* = \{\alpha_* \mid \alpha \in \mathcal{D}\}$. It is easily verified that $(X \cup \{*\}, \mathcal{E} \cup \mathcal{D}_*)$ and $(X \cup \{*\}, \mathcal{E}' \cup \mathcal{D}'_*)$ are disjoint 5-helix designs over $X \cup \{*\}$. □

Another recursive construction is described in the following

**Lemma 3.3.** *Suppose $v \equiv 0,1 \mod 5$ and $u \equiv 0,1 \mod 5$ and that there exist 5-helix systems of index 2 and order $v+1$ and $u+1$, then there exists a 5-helix of order $v+u$ and index 2.*

*Proof.* Let $X$ and $Y$ be sets of cardinality $v$ and $u$ as in hypothesis, choose $\overline{x} \in X$ and consider a 5-helix $(Y \cup \{\overline{x}\}, \mathcal{E})$ of order $u+1$, choose $\overline{y} \in Y$ and consider a 5-helix $(X \cup \{\overline{y}\}, \mathcal{E}')$ of order $v+1$ of index 2. Given $(X, \mathcal{D})$ a $\Delta(0,2,5)$-decomposition of $X$, which exists by lemmas 2.3,2.5, let us call $\mathcal{D}_y$ the set of 5-helix blocks $\alpha_y = \Big\{\{y\} \cup \{a,b\} \mid \{a,b\} \in \alpha \in \mathcal{D}\Big\}$ for every $y \in Y \setminus \{\overline{y}\}$. Analogously, given $(Y, \mathcal{D}')$ a $\Delta(0,2,5)$-decomposition of $Y$, consider $\mathcal{D}'_x$ the set of blocks $\alpha_x = \Big\{\{x\} \cup \{a,b\} \mid \{a,b\} \in \alpha \in \mathcal{D}'\Big\}$ for every $x \in X \setminus \{\overline{x}\}$. It is straightforward that $(X \cup Y, \mathcal{E} \cup \mathcal{E}' \cup \mathcal{D}_y \cup \mathcal{D}'_x)$ is a 5-helix, since if $\{a,b,c\} \in \binom{X \cup Y}{3}$ then $a,b,c \in X \cup \{\overline{y}\}$ or $a,b,c \in Y \cup \{\overline{x}\}$ or $a,b \in X$ and $c \in Y \setminus \{\overline{y}\}$ or $a,b \in Y$ and $c \in X \setminus \{\overline{x}\}$. Besides

$$\binom{u+v}{3} = \binom{u+1}{3} + \binom{v+1}{3} + \binom{u}{2}(v-1) + \binom{v}{2}(u-1)$$

for every $u, v \in \mathbb{N}$, hence each triple $\{a,b,c\} \in \binom{X \cup Y}{3}$ appears once and only once in the system $(X \cup Y, \mathcal{E} \cup \mathcal{E}' \cup \mathcal{D}_y \cup \mathcal{D}'_x)$; since we can consider disjoint systems in the construction of $\mathcal{D}, \mathcal{D}', \mathcal{E}, \mathcal{E}'$ it follows that, given $u, v$ which satisfy the hypothesis, there exists a 5-helix of order $u+v$ and index 2. □

In order to exploit the previous constructions we need existence results for 5-helix systems of order $v = 11$ and $v = 15$.

**Theorem 3.4.** *There exists a 5-helix of order 11 and index 2.*

*Proof.* The number of blocks of 5-helix of order 11 is 33, hence to construct a 5-helix of order 11 and index 2 we need 66 distinct blocks isomorphic to an $S^3(1,5)$. Let $(X, +)$ be the cyclic group of order 11 whose elements are $\{0, 1, \ldots, 10\}$, and consider the following 3 blocks:

$$\alpha = \Big\{\{0,1,10\}, \{0,2,3\}, \{0,8,9\}, \{0,4,7\}, \{0,5,6\}\Big\}$$
$$\beta = \Big\{\{0,1,7\}, \{0,2,4\}, \{0,6,8\}, \{0,5,9\}, \{0,3,10\}\Big\}$$
$$\gamma = \Big\{\{0,1,5\}, \{0,2,8\}, \{0,3,6\}, \{0,4,9\}, \{0,7,10\}\Big\}.$$

The 33 blocks $\alpha + a, \beta + a, \gamma + a$ for every $a \in \mathbb{Z}/(11)$ give a 5-helix $H_{11}$ of order 11.

In order to obtain another 5-helix $H'_{11}$ disjoint from the previous one, one can perform a permutation on the vertices of every block of $H_{11}$ in such a way that the blocks obtained after the permutation do not belong to $H_{11}$. For example, the cyclic permutation $\sigma$ su $X$, decomposed into disjoint cycles and written in cyclic notation, is $\sigma = (1,2)(5,6)$ and it gives a second 5-helix $H'_{11}$ disjoint from $H_{11}$. □

And finally, we construct a 5-helix system of order $v = 15$ and index 2 in the following

**Theorem 3.5.** *There exists a 5-helix of order 15 and index 2.*

*Proof.* Let us consider 5-helix blocks with vertices in $\mathbb{Z}/(13) \cup \{\alpha, \beta\}$, where $\mathbb{Z}/(13)$ is the cyclic group of order 13 and $\alpha, \beta \notin \mathbb{Z}/(13)$. In particular consider the following blocks as base blocks

$$\{\{0,4,11\}, \{0,7,8\}, \{0,3,10\}, \{0,2,9\}, \{0,5,\beta\}\}$$
$$\{\{0,1,8\}, \{0,6,12\}, \{0,5,11\}, \{0,4,10\}, \{0,3,\beta\}\}$$
$$\{\{0,4,7\}, \{0,6,11\}, \{0,1,5\}, \{0,8,10\}, \{0,2,\beta\}\}$$
$$\{\{0,2,10\}, \{0,8,12\}, \{0,5,9\}, \{0,1,4\}, \{0,6,\beta\}\}$$
$$\{\{0,5,8\}, \{0,2,11\}, \{0,3,4\}, \{0,12,\beta\}, \{0,1,\alpha\}\}$$
$$\{\{0,10,11\}, \{0,2,3\}, \{0,1,12\}, \{0,9,\beta\}, \{0,4,\alpha\}\}$$
$$\{\{\alpha,0,2\}, \{\alpha,3,6\}, \{\alpha,4,12\}, \{\alpha,5,11\}, \{\alpha,1,\beta\}\}$$

and, translation is performed utilizing the following rule

$$x + a :- \begin{cases} x + a & \text{if } x \in \mathbb{Z}/(13) \\ \alpha & \text{if } x = \alpha \\ \beta & \text{if } x = \beta. \end{cases}$$

Thus, the 91 blocks obtained form a 5-helix system $H_{15}$ and, a second disjoint 5-helix system can be achieved by swapping $\alpha$ with $\beta$ in every block of $H_{15}$. □

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