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A novel mathematical model on human feelings via fractional operator with non-singular kernel

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Abstract. Mathematics is a set of tools that try to describe human life and the universe which include symmetrical and asymmetrical structures by using various methods, provide analyzes and make predictions for the solution of problems by reaching syntheses. Considering the chaotic nature of life and human feelings, mathematical models that offer descriptions within a certain symmetrical order gain importance. Fractional derivative and integral operators introduced in fractional analysis are very effective and useful concepts that serve these purposes. In this study, based on the analysis of a literary work, an equation system that models human feelings is discussed with the help of the Caputo-Fabrizio fractional operator, which has a non-singular kernel. The existence and uniqueness of the solution of the system of equations included in this model and the compatibility of the numerical solutions with the analysis of the literary work according to different parameter values with the help of simulations are discussed.

1. Introduction

Mathematics, like literature, tries to explain the foundations and nature of life by focusing on human and human needs. Then, by contributing to applied sciences, life sciences and other disciplines, it ultimately becomes a source for innovations offered to people in industry, technology, industry and many similar fields. Various number systems, formulas explaining concepts, systems of equations that reveal real-world problems, and almost all geometric concepts contain a unique aesthetic structure, which can be considered as an indication of the ability of mathematics to describe nature that has spread to symbols and numbers. Examining variability, determining validity, making inferences, estimating, making decisions and making generalizations are the problems that form the basis of scientific research. The transition from the told to the understood is actually the essential part of these problems, and this transition becomes meaningful with estimation and sheds light on a wider sample. Science can be seen as processes of producing useful

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knowledge for human life and nature. It is a tool that provides an algebraic basis for mathematical problems in these processes, conveys abstract concepts to understandable and explainable forms on this ground, and reveals methods for solving the problem after the definition of the problem. Moreover, mathematics, which illuminates the processes of producing useful knowledge by revealing the existence, uniqueness, stability and error-freeness of the solutions; It provides the transition from the known to the targeted information. Expressing physical phenomena, real-world problems, and the dynamics of life is the first step towards a solution, and the basic principle of mathematical expression is to clarify conditions. The process of examining variability with the clarification of conditions; Initial conditions and boundary conditions and functions and parameters that cause variability are brought together in a certain systematic form. This attained form is called a mathematical model of a physical phenomenon, real-world problems, and the dynamics of life. Mathematical modeling is a process consisting of the steps of examining the variability, obtaining the solution, existence-uniqueness, stability and numerical solutions, and is completed by interpreting the results of the model with the help of graphs, simulations and figures. The theory-practice harmony is ensured by the suitable analysis methods, derivative-integral operators, parameters, boundary conditions, solution methods and findings for the nature of the problem symbolized by the model. Fractional derivative and integral operators have become a subject of interest in recent years in many applied sciences such as physics, engineering, mathematical biology and statistics, as well as in all branches of mathematics. For some basic studies in which different mathematical models are developed using fractional derivative and integral operators, the following studies are presented to attract the attention of the readers. In [8], the authors analyzed an epidemiological mathematical model via fractional derivative operators for computer viruses. Further in [9], a chaotic finance mathematical model in the sense of Atangana-Baleanu fractional derivative operator was discussed. Moreover, there are important recent studies on fractional derivative and integral operators, see the papers [1, 2, 7, 12, 17] and [18–27]. The mathematical model considered in this study was simulated by using some numerical solution methods.

Literature is an artistic endeavor that uses the forms of communication it has established to engage readers in a way that addresses the agenda of human existence, which it sets at the center of the work, and provides an answer to the inner urge for narration. What is said and how it is told—a problem that also illuminates literature—is the fundamental standard sought in the works produced within the context of this activity. Those who consider what constitutes beautiful poetry have frequently recognized or understood a poem's attractiveness but have found it challenging to articulate why poetry is lovely [15]. One of the oldest forms of literary expression, poetry, relies on feeling and intuition to communicate events, circumstances, or ideas. Poetry is "the revelation of an event from the heart in language; it is the abrupt birth of language and its remaining in language" [4]. With the addition of metaphorical components or language devices, poetry has greater potential for expression and transmission, increasing its capacity to influence readers, shape or extend their imaginations, and arouse aesthetic pleasure. This genre, which is important for disclosing deep meanings, provides the door to various interpretations and assessments thanks to the development of a meta-language with poetic signs. Poetry is almost one enormous word, not a collection of small words put together piecemeal. If the reader is able to recognize himself in the poet's soul's mirror and recognize certain states that are similar to his own, he can draw conclusions that will help him make sense of his own existence in the face of this one, enormous word [14].

Poetry, which expresses a lot with few words, turns into a being that gains literariness and eternity not only with what is told, but also with the system of meanings that the text refers to. Poetry, with the immortality it has achieved, grants its poet the ability to navigate languages, hearts and minds independently of time and place in the world. Shahriar (Seyid Muhammed Hüseyin Behçet-Tabrizi) can be shown among the poets who achieved this success in the Turkish territory of the 20th century. Shahriar, who writes poems that transform words into words and contribute to the magical identity of the word with the intensity of emotion, texts the theme of love in his poem Yar Qasidi in a form that "constitutes and completes the individual with its creative and revitalizing referents that create self-consciousness"[10]. The poem reveals the dimensions of the love of the subject for the lover he cannot meet, on the plane that covers the processes before and after the marriage of the beloved woman. In the poem, although it seems to be handled after the wedding (which is told), it is also pointed out before the wedding with the guidance of the concepts (it seems). The determinant of the basic dynamics of both stages is the power of the emotions. Therefore, it is imperative for an acceptable analysis to determine the variability of feelings through what is explained and understood by words and concepts. Because the main purpose of text interpretation is to reach objective judgments that show the artist's intention and to be able to confirm their validity, instead of subjective evaluations that develop at the points where the reader's cultural level leaves them missing or adds excess.

Shahriar prepared the formal aspect of his poetry by utilizing a variety of harmonies. With diverse word and sound repetitions, he established a mathematical rhythm. He then continued using natural expressions, including those between couplets and outside of them, to produce harmonies that represent cross-mathematical order or a spiral of harmony [16].

Yar Qasidi

You are my messenger, sit down, I ordered some tea
I've been tingling and whimpering so much that she threw her thoughts my way.

Oh, what a night I could not sleep and I dreamed of you with a lullaby
And you slept and I told my eyes to count the stars

Everyone thought of you as a star, I called you the moon
After you left, life would have been bitter, even if it was sweet.

From every beautiful face, I was looking for a sign from you to see where or what it looks like
Your departure was like the setting of the sun, but I saw it like the setting of the moon

Now I call summer, winter. Whereas before I used to say even winters are summers
Sometimes I remember the memory of your wedding and I cheer like crazy

Then I mourn again and cry with a sigh.
My lap is filled like the sea with tears and the tears in my eyes are all flowing like a river

And I live such a miserable life, just saying oh and oh!

Shahriar

Based on the analysis of the *Yar Qasidi* poem, which was brought to the literature by Shahriyar, we will try to examine the variability of the feelings of the protagonists of the poem by acting on the axis of what is told-understood with the help of mathematical modeling. The work is organized as follows: In Section 2, the importance and characteristics of literature and mathematics in human life and the qualities of poetry as a literary genre are explained. By mentioning Shahriyar's artistic characteristics, an attention was drawn to the contents and the order of his poem, *Yar Qasidi,* and its possible connections to mathematics. The definitions of the Caputo–Fabirizo fractional derivative and integral operator are introduced. Further, various properties of these operators are given along with stating some basic theorems, lemmas and remarks. In Section 3, a mathematical model is created based on the analysis of the poem *Yar Qasidi* written by Shahriar. Moreover, the mathematical model has been extended to the Caputo–Fabirizio fractional derivative operator. Afterwards, the existence and uniqueness of the solution of the model is demonstrated by appropriate methods. In Section 4, the stability of the mathematical model is examined with the Hyers-Ulam stability method. In Section 5, numerical method used for the solution of the model is introduced and its solution is made. The simulations for different values of the parameter contained in the fractional derivative operator of the mathematical model of the *Yar Qasidi* poem are shown and interpreted in Section 6. The last section is reserved for the conclusion part.

2. Material and Methods

M. Caputo and M. Fabrizio defined the fractional derivative and integral operator with an exponential kernel without a singular kernel in 2015. This section contains general information about the operator.

Detailed information about the definitions, theorems and lemmas can be found in [5, 6, 13] studies.

The well-known Caputo fractional derivative operator [5] was defined by M. Caputo in 1967.

Definition 2.1. *The Caputo fractional time derivative are defined as follows [5],*

$$ {}_{a}^{C}D_{t}^{\vartheta} f(t) = \frac{1}{\Gamma(1-\vartheta)} \int_a^t \frac{f'(r)}{(t-r)^{\vartheta}} dr, \tag{1}$$

where $\vartheta \in [0,1], a \in (-\infty, t), f \in H^1[a,b], b > a$.

The Caputo-Fabrizio time fractional derivative can also be applied to functions that do not belong to $H^1(a,b)$. Indeed, the definition 2.1 can be formulated also for $f \in L^1(-\infty, c)$ for any $\vartheta \in [0,1]$. The definition for this is given below.

Definition 2.2. *The Caputo-Fabrizio fractional time derivative are defined as follows [6],*

$$ {}_{a}^{CF}\mathfrak{D}_{t}^{\vartheta} f(t) = \frac{M(\vartheta)}{1-\vartheta} \int_a^t \frac{df(x)}{dx} exp\Big[-\frac{\vartheta}{1-\vartheta}(t-x)\Big] dx, \tag{2}$$

where $f \in H^1(a,b), b > a, \vartheta \in (0,1)$. $M(\vartheta)$ *is a normalization function such that* $M(0) = M(1) = 1$.

As it is known, the fractional derivatives of Caputo and Caputo-Fabrizio give zero for the constant function. On the other hand, unlike the Caputo operator, the Caputo-Fabrizio kernel does not have singularity for $t = x$ which uses the exponential kernel. In addition, the Caputo-Fabrizio operator can be applied within the function that does not belong to the $H^1(a,b)$ space with the help of the following definition.

$$ {}^{CF}\mathfrak{D}_{t}^{\vartheta}(f(t)) = \frac{\vartheta M(\vartheta)}{1-\vartheta} \int_{-\infty}^t \big(f(t) - f(x)\big) exp\Big[-\frac{\vartheta}{1-\vartheta}(t-x)\Big] dx, \tag{3}$$

where $f \in L^1(-\infty, b), \vartheta \in [0,1]$.

Remark 2.3. *If* $\eta = \frac{1-\vartheta}{\vartheta} \in (0,\infty), \vartheta = \frac{1}{1+\eta} \in [0,1]$, *the following equation can be written easily.*

$$ \mathfrak{D}_{t}^{\eta}(f(t)) = \frac{N(\eta)}{\eta} \int_a^t \frac{df(x)}{dx} exp\Big[-\frac{t-x}{\eta}\Big] dx, \tag{4}$$

where $\eta \in [0,\infty)$ *and* $N(\eta)$ *is the corresponding normalization term of* $M(\vartheta)$ *such that* $N(0) = N(\infty) = 1$.

$$ \lim_{\vartheta\to 0} \frac{1}{\vartheta} exp\Big[-\frac{t-x}{\vartheta}\Big] = \delta(x-t), \tag{5}$$

and $\vartheta \to 1$, we have $\eta \to 0$. Then,

$$ \begin{aligned} \lim_{\vartheta\to 1} \mathfrak{D}_{t}^{\vartheta} f(t) &= \lim_{\vartheta\to 1} \frac{M(\vartheta)}{1-\vartheta} \int_a^t \frac{df(x)}{dx} exp\Big[-\frac{\vartheta}{1-\vartheta}(t-x)\Big] dx \\ &= \lim_{\eta\to 0} \frac{N(\eta)}{\eta} \int_a^t \frac{df(x)}{dx} exp\Big[-\frac{t-x}{\eta}\Big] dx = f'(t). \end{aligned} \tag{6}$$

On the other hand when $\vartheta \to 0$, then $\eta \to \infty$. Hence,

$$ \begin{aligned} \lim_{\vartheta\to 0} \mathfrak{D}_{t}^{\vartheta} f(t) &= \lim_{\vartheta\to 0} \frac{M(\vartheta)}{1-\vartheta} \int_a^t \frac{df(x)}{dx} exp\Big[-\frac{\vartheta}{1-\vartheta}(t-x)\Big] dx \\ &= \lim_{\eta\to \infty} \frac{N(\eta)}{\eta} \int_a^t \frac{df(x)}{dx} exp\Big[-\frac{t-x}{\eta}\Big] dx = f(t) - f(a). \end{aligned} \tag{7}$$

Theorem 2.4. *For Caputo-Fabrizio time fractional time derivative, if the function $f(t)$ is such that*

$$f^{(s)}(a) = 0, \quad s = 1, 2, \ldots, n, \tag{8}$$

then, we have

$$\mathfrak{D}_t^{(n)}\left(\mathfrak{D}_t^{(\vartheta)} f(t)\right) = \mathfrak{D}_t^{(\vartheta)}\left(\mathfrak{D}_t^{(n)} f(t)\right). \tag{9}$$

Proof. Considering $n = 1$ for $\mathfrak{D}_t^{(\vartheta+1)} f(t)$, we get

$$\mathfrak{D}_t^{(\vartheta)}\left(\mathfrak{D}_t^{(1)} f(t)\right) = \frac{M(\vartheta)}{1-\vartheta}\int_a^t f'(\tau) exp\left[-\frac{\vartheta(t-\tau)}{1-\vartheta}\right] d\tau. \tag{10}$$

So, assuming $f'(a) = 0$, we obtain,

$$\begin{aligned}
\mathfrak{D}_t^{(\vartheta)}\left(\mathfrak{D}_t^{(1)} f(t)\right) &= \frac{M(\vartheta)}{1-\vartheta}\int_a^t \left(\frac{d}{d\tau} f'(\tau)\right) exp\left[-\frac{\vartheta(t-\tau)}{1-\vartheta}\right] d\tau \\
&= \frac{M(\vartheta)}{1-\vartheta}\left[\int_a^t \frac{d}{d\tau}(f'(\tau)) exp\left(-\frac{\vartheta(t-\tau)}{1-\vartheta}\right) d\tau \right. \\
&\quad \left. - \frac{\vartheta}{1-\vartheta}\int_a^t f'(\tau) exp\left(-\frac{\vartheta(t-\tau)}{1-\vartheta}\right) d\tau\right] \\
&= \frac{M(\vartheta)}{1-\vartheta}\left[f'(t) - \frac{\vartheta}{1-\vartheta}\int_a^t f'(\tau) exp\left(-\frac{\vartheta(t-\tau)}{1-\vartheta}\right) d\tau\right],
\end{aligned} \tag{11}$$

otherwise

$$\begin{aligned}
\mathfrak{D}_t^{(1)}\left(\mathfrak{D}_t^{(\vartheta)} f(t)\right) &= \frac{d}{dt}\left(\frac{M(\vartheta)}{1-\vartheta}\int_a^t f'(\tau)) exp\left[-\frac{\vartheta(t-\tau)}{1-\vartheta}\right] d\tau\right) \\
&= \frac{M(\vartheta)}{1-\vartheta}\left[f'(t) - \frac{\vartheta}{1-\vartheta}\int_a^t f'(\tau) exp\left(-\frac{\vartheta(t-\tau)}{1-\vartheta}\right) d\tau\right].
\end{aligned} \tag{12}$$

For $n = 1$, the theorem has been proven. Similarly, the proof can be generalized for any $n > 1$ [6]. □

The Laplace transform plays an important role in defining fractional derivative operators and finding integral operators. For this reason, the Laplace transform of the Caputo-Fabrizio fractional derivative operator is given below [6]. Let $a = 0$ in equation (4).

$$LT\left[\mathfrak{D}_t^{(\vartheta)} f(t)\right] = \frac{1}{(1-\vartheta)}\int_0^\infty exp(-pt)\int_0^t f'(x) exp\left(-\frac{\vartheta(t-x)}{1-\vartheta}\right) dx dt. \tag{13}$$

Hence, from the property of Laplace transform of a convolution, we have

$$\begin{aligned}
LT\left[\mathfrak{D}_t^{(\vartheta)} f(t)\right] &= \frac{1}{(1-\vartheta)} LT\left(f'(t)\right) LT\left(exp - \left(\frac{\vartheta t}{1-\vartheta}\right)\right) \\
&= \frac{pLT\left(f(t) - f(0)\right)}{p + \vartheta(1-p)}.
\end{aligned} \tag{14}$$

Similarly,

$$LT\left[\mathfrak{D}_t^{(\vartheta+n)}f(t)\right] = \frac{1}{(1-\vartheta)}LT\left(f^{(n+1)}(t)\right)LT\left(exp-\left(\frac{\vartheta t}{1-\vartheta}\right)\right)$$
$$= \frac{p^{n+1}LT[f(t)]-p^n f(0)-p^{n-1}f'(0)-\cdots-f^{(n)(0)}}{p+\vartheta(1-p)}. \tag{15}$$

Also, the Caputo-Fabrizio fractional derivative operator can be written as follows for $0<\vartheta<1$.

Definition 2.5. *The Caputo-Fabrizio fractional derivative of order ϑ is as follows [13],*

$$^{CF}\mathfrak{D}_{\star}^{\vartheta}(f(t)) = \frac{1}{1-\vartheta}\int_0^t f'(x)exp\Big[-\vartheta\frac{t-x}{1-\vartheta}\Big]dx. \tag{16}$$

In addition, the Caputo-Fabrizio fractional integral operator is also defined in the [13].

Definition 2.6. *Let $0<\vartheta<1$. The fractional integral of order ϑ of a function f is defined by [13],*

$$^{CF}I^{\vartheta}f(t) = \frac{2(1-\vartheta)}{(2-\vartheta)M(\vartheta)}u(t) + \frac{2\vartheta}{(2-\vartheta)M(\vartheta)}\int_0^t u(s)ds, \qquad t\geq 0. \tag{17}$$

3. Results

There are thousands of important literary works such as poetry, stories and novels in the literature. These works differ from each other in terms of protagonists, subjects and content. In this study, a mathematical model has been developed by analyzing the poetry *Yar Qasidi* written by Shahriar. This mathematical model is as follows.

$$\begin{aligned}\frac{dA}{dt} &= (\alpha+\beta)A(t)+(\gamma+\delta)B(t)+L,\\ \frac{dB}{dt} &= (\theta+\rho)B(t)+(\sigma+\eta)A(t).\end{aligned} \tag{18}$$

Here, A denotes the lover man, while B expresses the beloved woman. According to the analysis of the poetry, an important point was taken into consideration while making the mathematical modeling and this situation changes when the beloved woman marries a different man. The parameters $\alpha,\gamma,\theta,\sigma>0$ are assumed to be positive because these parameters represent the time elapsed before the beloved woman marries with another man. On the other hand, the parameters $\beta,\delta,\rho,\eta<0$ are assumed to be negative because these parameters represent the period after the beloved woman marries another man. The L parameter represents the lover man's strong love for the beloved woman. Therefore, it is clear that $L>0$. Considering the lover man's pre-marriage period, α shows the internal feeling indicators that guide the lover positively, γ is the external (from the lover man's side) feeling indicators that guide the lover positively, θ is the internal feeling indicators that guide the lover positively and σ refers to the external (in beloved woman's side) feeling indicators that direct the lover positively. On the other hand, considering the lover man's post-marriage process, β shows the inner feeling indicators that direct the lover negatively, δ the external (from the beloved woman's side) feelings indicators that direct the lover negatively, ρ the internal feelings that direct the lover negatively, η denotes indicators of external (from the beloved woman's side) feelings that negatively affect the lover.

The definition of the Caputo-Fabrizio fractional derivative operator was given in (2). When the system (18) is extended to the CF operator with the help of this definition, the following system of equations is obtained.

$$\begin{aligned}{}^{CF}_{0}\mathfrak{D}_t^{\vartheta}A(t) &= (\alpha+\beta)A(t)+(\gamma+\delta)B(t)+L,\\ {}^{CF}_{0}\mathfrak{D}_t^{\vartheta}B(t) &= (\theta+\rho)B(t)+(\sigma+\eta)A(t).\end{aligned} \tag{19}$$

The initial values of the extended mathematical model are as follows,

$$A_0(t) = A(0), \quad B_0(t) = B(0).$$

3.1. Existence of solution for mathematical model

In the previous section, a fractional mathematical model was obtained by analyzing the poetry *Yar Qasidi* written by Shahriar. In this section, the existence solution of the mathematical model will be done with the fixed point theorem. First, when the equation system (19) is arranged with the Caputo-Fabrizio integral operator, the following system is obtained.

$$\begin{aligned} A(t) - A(0) &= \frac{2(1-\vartheta)}{M(\vartheta)(2-\vartheta)}\Big((\alpha+\beta)A(t) + (\gamma+\delta)B(t) + L\Big) \\ &\quad + \frac{2\vartheta}{M(\vartheta)(2-\vartheta)}\int_0^t \Big((\alpha+\beta)A(x) + (\gamma+\delta)B(x) + L\Big)dx, \\ B(t) - B(0) &= \frac{2(1-\vartheta)}{M(\vartheta)(2-\vartheta)}\Big((\theta+\rho)B(t) + (\sigma+\eta)A(t)\Big) \\ &\quad + \frac{2\vartheta}{M(\vartheta)(2-\vartheta)}\int_0^t \Big((\theta+\rho)B(x) + (\sigma+\eta)A(x)\Big)dx. \end{aligned} \tag{20}$$

It would be useful to write the kernels as follows to avoid the processing intensity,

$$\begin{aligned} \mathfrak{B}_1(t,A) &= (\alpha+\beta)A(t) + (\gamma+\delta)B(t) + L, \\ \mathfrak{B}_2(t,B) &= (\theta+\rho)B(t) + (\sigma+\eta)A(t). \end{aligned} \tag{21}$$

It can be assumed that the following assumption C: $A(t), B(t), A^\star(t), B^\star(t)$ are continuous functions, so that $\|A(t)\| \leq \mathfrak{S}_g, \|B(t)\| \leq \mathfrak{S}$.

Theorem 3.1. *The kernels $\mathfrak{B}_g, \mathfrak{B}$ are satisfying the Lipschitz condition if the assumption C is true and are contractions provided that $\Xi_1, \Xi_2 < 1$.*

Proof. First of all, we will begin to prove that $\mathfrak{B}_g(t,A)$ satisfies the Lipschitz condition. Let $A(t)$ and $A^\star(t)$ are two functions, then

$$\begin{aligned} \|\mathfrak{B}_g(t,A) - \mathfrak{B}_g(t,A^\star)\| &= \|\Big((\alpha+\beta)A(t) + (\gamma+\delta)B(t) + L\Big) - \Big((\alpha+\beta)A^\star(t) + (\gamma+\delta)B(t) + L\Big) \\ &\leq \Big(\alpha+\beta\Big)\|A - A^\star\| \\ &= \Xi_1\|A - A^\star\|. \end{aligned} \tag{22}$$

Next, we will prove that $\mathfrak{B}(t,B)$ satisfies the Lipschitz condition. Let $B(t)$ and $B^\star(t)$ are two functions, then

$$\begin{aligned} \|\mathfrak{B}(t,B) - \mathfrak{B}(t,B^\star)\| &= \|\Big((\theta+\rho)B(t) + (\sigma+\eta)A(t)\Big) - \Big((\theta+\rho)B^\star(t) + (\sigma+\eta)A(t)\Big) \\ &\leq \Big(\theta+\rho\Big)\|B - B^\star\| \\ &= \Xi_2\|B - B^\star\|. \end{aligned} \tag{23}$$

All kernels which $\mathfrak{B}_g, \mathfrak{B}$ are satisfying the conditions, so that they are contractions with $\Xi_1, \Xi_2 < 1$. As a result, this completes the proof. □

By using the kernels $\mathfrak{B}_g, \mathfrak{B}$ and initial conditions $A(0) = B(0) = 0$, when the system of equations (20) is rewritten in this state

$$
\begin{aligned}
A(t) &= \frac{2(1-\vartheta)}{M(\vartheta)(2-\vartheta)}\Big((\alpha+\beta)A(t)+(\gamma+\delta)B(t)+L\Big)\\
&\quad+\frac{2\vartheta}{M(\vartheta)(2-\vartheta)}\int_0^t\Big((\alpha+\beta)A(x)+(\gamma+\delta)B(x)+L\Big)dx,\\
B(t) &= \frac{2(1-\vartheta)}{M(\vartheta)(2-\vartheta)}\Big((\theta+\rho)B(t)+(\sigma+\eta)A(t)\Big)\\
&\quad+\frac{2\vartheta}{M(\vartheta)(2-\vartheta)}\int_0^t\Big((\theta+\rho)B(x)+(\sigma+\eta)A(x)\Big)dx.
\end{aligned}
\tag{24}
$$

Then the following system of equations can be defined with the help of a recursive formula.

$$
\begin{aligned}
A_n(t) &= \frac{2(1-\vartheta)}{M(\vartheta)(2-\vartheta)}\Big((\alpha+\beta)A_{n-1}(t)+(\gamma+\delta)B(t)+L\Big)\\
&\quad+\frac{2\vartheta}{M(\vartheta)(2-\vartheta)}\int_0^t\Big((\alpha+\beta)A_{n-1}(x)+(\gamma+\delta)B(x)+L\Big)dx,\\
B_n(t) &= \frac{2(1-\vartheta)}{M(\vartheta)(2-\vartheta)}\Big((\theta+\rho)B_{n-1}(t)+(\sigma+\eta)A(t)\Big)\\
&\quad+\frac{2\vartheta}{M(\vartheta)(2-\vartheta)}\int_0^t\Big((\theta+\rho)B_{n-1}(x)+(\sigma+\eta)A(x)\Big)dx.
\end{aligned}
\tag{25}
$$

Furthermore, we consider the differences $(A_{n+1}-A_n)(t)$ and $(B_{n+1}-B_n)(t)$ as follows:

$$
\begin{aligned}
(A_{n+1}-A_n)(t) &= \frac{2(1-\vartheta)}{M(\vartheta)(2-\vartheta)}\Big(\Big((\alpha+\beta)A_n(t)+(\gamma+\delta)B(t)+L\Big)-\Big((\alpha+\beta)A_{n-1}(t)+(\gamma+\delta)B(t)+L\Big)\Big)\\
&\quad+\frac{2\vartheta}{M(\vartheta)(2-\vartheta)}\int_0^t\Big(\Big((\alpha+\beta)A_n(x)+(\gamma+\delta)B(x)+L\Big)\\
&\quad-\Big((\alpha+\beta)A_{n-1}(x)+(\gamma+\delta)B(x)+L\Big)\Big)dx,\\
(B_{n+1}-B_n)(t) &= \frac{2(1-\vartheta)}{M(\vartheta)(2-\vartheta)}\Big(\Big((\theta+\rho)B_n(t)+(\sigma+\eta)A(t)\Big)-\Big((\theta+\rho)B_{n-1}(t)+(\sigma+\eta)A(t)\Big)\Big)\\
&\quad+\frac{2\vartheta}{M(\vartheta)(2-\vartheta)}\int_0^t\Big(\Big((\theta+\rho)B_n(x)+(\sigma+\eta)A(x)\Big)-\Big((\theta+\rho)B_{n-1}(x)+(\sigma+\eta)A(x)\Big)\Big)dx.
\end{aligned}
\tag{26}
$$

By taking the norm for both sides of Equations (26), we get

$$\begin{aligned}
\|(A_{n+1}-A_n)\|(t) &= \frac{2(1-\vartheta)}{M(\vartheta)(2-\vartheta)}\Big\|\Big(\big((\alpha+\beta)A_n(t)+(\gamma+\delta)B(t)+L\big)\\
&\quad -\big((\alpha+\beta)A_{n-1}(t)+(\gamma+\delta)B(t)+L\big)\Big)\Big\|\\
&\quad +\frac{2\vartheta}{M(\vartheta)(2-\vartheta)}\int_0^t\Big\|\Big(\big((\alpha+\beta)A_n(x)+(\gamma+\delta)B(x)\\
&\quad +L\big)-\big((\alpha+\beta)A_{n-1}(x)+(\gamma+\delta)B(x)+L\big)\Big)\Big\|dx,\\
\|(B_{n+1}-B_n)\|(t) &= \frac{2(1-\vartheta)}{M(\vartheta)(2-\vartheta)}\Big\|\Big(\big((\theta+\rho)B_n(t)+(\sigma+\eta)A(t)\big)\\
&\quad -\big((\theta+\rho)B_{n-1}(t)+(\sigma+\eta)A(t)\big)\Big)\Big\|\\
&\quad +\frac{2\vartheta}{M(\vartheta)(2-\vartheta)}\int_0^t\Big\|\Big(\big((\theta+\rho)B_n(x)+(\sigma+\eta)A(x)\big)\\
&\quad -\big((\theta+\rho)B_{n-1}(x)+(\sigma+\eta)A(x)\big)\Big)\Big\|dx.
\end{aligned} \tag{27}$$

Theorem 3.2. *The mathematical model in system (19) has a solution if the following inequality is achieved:*

$$\Delta = max\{\Xi_i\} < 1, \quad i = 1, 2. \tag{28}$$

Proof. Now we define the functions as follows

$$\mathfrak{L}_{g\mathfrak{n}}(\mathrm{t}) = (A_{n+1}-A_n)(t), \quad \mathfrak{L}_{\mathfrak{n}}(\mathrm{t}) = (B_{n+1}-B_n)(t). \tag{29}$$

Firstly, we will proof with $\mathfrak{L}_{g\mathfrak{n}}(\mathrm{t})$,

$$\begin{aligned}
\|\mathfrak{L}_{g\mathfrak{n}}(\mathrm{t})\| &= \frac{2(1-\vartheta)}{M(\vartheta)(2-\vartheta)}\Big\|\Big(\big((\alpha+\beta)A_n(t)+(\gamma+\delta)B(t)+L\big)-\big((\alpha+\beta)A_{n-1}(t)+(\gamma+\delta)B(t)+L\big)\Big)\Big\|\\
&\quad +\frac{2\vartheta}{M(\vartheta)(2-\vartheta)}\int_0^t\Big\|\Big(\big((\alpha+\beta)A_n(x)+(\gamma+\delta)B(x)\\
&\quad +L\big)-\big((\alpha+\beta)A_{n-1}(x)+(\gamma+\delta)B(x)+L\big)\Big)\Big\|dx\\
&\le \frac{2(1-\vartheta)}{M(\vartheta)(2-\vartheta)}\Xi_1\|A_n-A_{n-1}\|+\frac{2\vartheta}{M(\vartheta)(2-\vartheta)}\Xi_1\|A_n-A_{n-1}\|\\
&= \Big(\frac{2(1-\vartheta)}{M(\vartheta)(2-\vartheta)}+\frac{2\vartheta}{M(\vartheta)(2-\vartheta)}\Big)\Xi_1\|A_n-A_{n-1}\|
\end{aligned} \tag{30}$$

Secondly, we will proof with $\mathfrak{L}_{\mathfrak{n}}(\mathrm{t})$,

$$\begin{aligned}
\|\mathfrak{L}_{\mathfrak{n}}(\mathrm{t})\| &= \frac{2(1-\vartheta)}{M(\vartheta)(2-\vartheta)}\Big\|\Big(\big((\theta+\rho)B_n(t)+(\sigma+\eta)A(t)\big)-\big((\theta+\rho)B_{n-1}(t)+(\sigma+\eta)A(t)\big)\Big)\Big\|\\
&\quad +\frac{2\vartheta}{M(\vartheta)(2-\vartheta)}\int_0^t\Big\|\Big(\big((\theta+\rho)B_n(x)+(\sigma+\eta)A(x)\big)-\big((\theta+\rho)B_{n-1}(x)+(\sigma+\eta)A(x)\big)\Big)\Big\|dx\\
&\le \frac{2(1-\vartheta)}{M(\vartheta)(2-\vartheta)}\Xi_2\|B_n-B_{n-1}\|+\frac{2\vartheta}{M(\vartheta)(2-\vartheta)}\Xi_2\|B_n-B_{n-1}\|\\
&= \Big(\frac{2(1-\vartheta)}{M(\vartheta)(2-\vartheta)}+\frac{2\vartheta}{M(\vartheta)(2-\vartheta)}\Big)\Xi_2\|B_n-B_{n-1}\|
\end{aligned} \tag{31}$$

Applying this process recursively, it can be found the following,

$$\|\mathfrak{L}_{gn}(t)\| \leq \left(\frac{2(1-\vartheta)}{M(\vartheta)(2-\vartheta)} + \frac{2\vartheta}{M(\vartheta)(2-\vartheta)}\right)^n \Delta^n \mathfrak{S}_g$$

and

$$\|\mathfrak{L}_{n}(t)\| \leq \left(\frac{2(1-\vartheta)}{M(\vartheta)(2-\vartheta)} + \frac{2\vartheta}{M(\vartheta)(2-\vartheta)}\right)^n \Delta^n \mathfrak{S}.$$

So that we can find $\mathfrak{L}_{gn}(t) \to 0$ and $\mathfrak{L}_{n}(t) \to 0$ as $n \to \infty$. Thus, the proof is complete. □

3.2. Uniqueness solution

First, the following theorem can be written to show that the mathematical model has a unique solution.

Theorem 3.3. *The mathematical model (19) will have a unique solution if the inequality hold true:*

$$\left(\frac{2(1-\vartheta)}{M(\vartheta)(2-\vartheta)} + \frac{2\vartheta}{M(\vartheta)(2-\vartheta)}\right)\Xi_i \leq 1, \quad i = 1, 2. \tag{32}$$

Proof. Let us assume that the system (19) has solutions $A(t), B(t)$ as well as $\tilde{A}(t), \tilde{B}(t)$. Then,

$$\begin{aligned} A(t) - \tilde{A}(t) &= \frac{2(1-\vartheta)}{M(\vartheta)(2-\vartheta)}\Big(\big((\alpha+\beta)A(t) + (\gamma+\delta)B(t) + L\big) - \big((\alpha+\beta)\tilde{A}(t) + (\gamma+\delta)B(t) + L\big)\Big) \\ &\quad + \frac{2\vartheta}{M(\vartheta)(2-\vartheta)} \int_0^t \Big(\big((\alpha+\beta)A(x) + (\gamma+\delta)B(x) \\ &\quad + L\big) - \big((\alpha+\beta)\tilde{A}(x) + (\gamma+\delta)B(x) + L\big)\Big)dx \\ &\leq \frac{2(1-\vartheta)}{M(\vartheta)(2-\vartheta)}\Xi_1(A-\tilde{A}) + \frac{2\vartheta}{M(\vartheta)(2-\vartheta)}\Xi_1(A-\tilde{A}). \end{aligned} \tag{33}$$

Similarly,

$$\begin{aligned} B(t) - \tilde{B}(t) &= \frac{2(1-\vartheta)}{M(\vartheta)(2-\vartheta)}\Big(\big((\theta+\rho)B(t) + (\sigma+\eta)A(t)\big) - \big((\theta+\rho)\tilde{B}(t) + (\sigma+\eta)A(t)\big)\Big) \\ &\quad + \frac{2\vartheta}{M(\vartheta)(2-\vartheta)} \int_0^t \Big(\big((\theta+\rho)B(x) + (\sigma+\eta)A(x)\big) \\ &\quad - \big((\theta+\rho))\tilde{B}(x) + (\sigma+\eta)A(x)\big)\Big)dx \\ &\leq \frac{2(1-\vartheta)}{M(\vartheta)(2-\vartheta)}\Xi_2(B-\tilde{B}) + \frac{2\vartheta}{M(\vartheta)(2-\vartheta)}\Xi_2(B-\tilde{B}). \end{aligned} \tag{34}$$

When the norm is taken from both sides of the system of equations (33-34),

$$\begin{aligned} \|A(t) - \tilde{A}(t)\| &= \frac{2(1-\vartheta)}{M(\vartheta)(2-\vartheta)}\left\|\Big(\big((\alpha+\beta)A(t) + (\gamma+\delta)B(t) + L\big) - \big((\alpha+\beta)\tilde{A}(t) + (\gamma+\delta)B(t) + L\big)\Big)\right\| \\ &\quad + \frac{2\vartheta}{M(\vartheta)(2-\vartheta)} \int_0^t \left\|\Big(\big((\alpha+\beta)A(x) + (\gamma+\delta)B(x) \right. \\ &\quad \left. + L\big) - \big((\alpha+\beta)\tilde{A}(x) + (\gamma+\delta)B(x) + L\big)\Big)\right\|dx \\ &\leq \frac{2(1-\vartheta)}{M(\vartheta)(2-\vartheta)}\Xi_1\|(A-\tilde{A})\| + \frac{2\vartheta}{M(\vartheta)(2-\vartheta)}\Xi_1\|(A-\tilde{A})\|. \end{aligned} \tag{35}$$

Similarly,

$$\begin{aligned}\|B(t)-\tilde{B}(t)\| &= \frac{2(1-\vartheta)}{M(\vartheta)(2-\vartheta)}\left\|\Big(\big((\theta+\rho)B(t)+(\sigma+\eta)A(t)\big)-\big((\theta+\rho)\tilde{B}(t)+(\sigma+\eta)A(t)\big)\Big)\right\| \\ &\quad+\frac{2\vartheta}{M(\vartheta)(2-\vartheta)}\int_0^t\left\|\Big(\big((\theta+\rho)B(x)+(\sigma+\eta)A(x)\big)\right. \\ &\quad\left.-\big((\theta+\rho))\tilde{B}(x)+(\sigma+\eta)A(x)\big)\Big)\right\|dx \\ &\le \frac{2(1-\vartheta)}{M(\vartheta)(2-\vartheta)}\Xi_2\|(B-\tilde{B})\| \\ &\quad+\frac{2\vartheta}{M(\vartheta)(2-\vartheta)}\Xi_2\|(B-\tilde{B})\|. \end{aligned} \tag{36}$$

The following inequality can be written from (35-36),

$$\begin{aligned}&\Big(\frac{2(1-\vartheta)}{M(\vartheta)(2-\vartheta)}\Xi_1+\frac{2\vartheta}{M(\vartheta)(2-\vartheta)}\Xi_1-1\Big)\|A-\tilde{A}\|\ge 0,\\ &\Big(\frac{2(1-\vartheta)}{M(\vartheta)(2-\vartheta)}\Xi_2+\frac{2\vartheta}{M(\vartheta)(2-\vartheta)}\Xi_2-1\Big)\|B-\tilde{B}\|\ge 0.\end{aligned} \tag{37}$$

Thus, according to equation (32) and (37) $\|A-\tilde{A}\|=0$, $\|B-\tilde{B}\|=0$. This implies $A(t)=\tilde{A}(t), B(t)=\tilde{B}(t)$. According to these results, the model has a unique solution. □

3.3. *Equilibrium points of system and asymptotic stability*

Let consider the model with Caputo-Fabrizio derivative and ϑ satisfying $0<\vartheta\le 1$ below:

$$\begin{aligned}{}_0^{CF}\mathfrak{D}_t^{\vartheta}A(t) &= (\alpha+\beta)A(t)+(\gamma+\delta)B(t)+L,\\ {}_0^{CF}\mathfrak{D}_t^{\vartheta}B(t) &= (\theta+\rho)B(t)+(\sigma+\eta)A(t).\end{aligned} \tag{38}$$

with the inital conditions $A_0(t)=A(0), B_0(t)=B(0)$.
Let us consider the system (38),

$$\begin{aligned}{}_0^{CF}\mathfrak{D}_t^{\vartheta}A(t) &= g_1(A,B),\\ {}_0^{CF}\mathfrak{D}_t^{\vartheta}B(t) &= g_2(A,B),\end{aligned} \tag{39}$$

where $g_1(A,B)=(\alpha+\beta)A(t)+(\gamma+\delta)B(t)+L$ and $g_2(A,B)=(\theta+\rho)B(t)+(\sigma+\eta)A(t)$.

The Jacobian matrix $E_1(A^\star,B^\star)$ for the system given in (38),

$$J(E_1)=\begin{bmatrix}\alpha+\beta & \gamma+\delta\\ \sigma+\eta & \theta+\rho\end{bmatrix}$$

To evaluate the asymptotic stability of the $E_1(A^\star,B^\star)$ equilibrium of the system given by (38).

$$A^\star=\frac{-(\theta+\rho)B^\star}{(\sigma+\eta)},\quad B^\star=\frac{-(\alpha+\beta)A^\star-L}{\gamma+\delta} \tag{40}$$

The characteristic equation of system is obtained via determination of (41)

$$det(J-\lambda I)=0.$$

So that the characteristic equation of the linearized system is of the form

$$\lambda^2 - (A+B)\lambda + AB - CD = 0$$

where

$$\begin{aligned} A &= \alpha + \beta \\ B &= \theta + \rho \\ C &= \gamma + \delta \\ D &= \sigma + \eta \end{aligned} \tag{41}$$

Theorem 3.4. *Let A,B,C and D be as given in (41). If $A + B < 0$ and $AB - CD > 0$ is satisfied then the equilibrium point $E_1(A^\star, B^\star)$ of the system (38) is stable.*

Proof. If $A+B < 0$ and $AB - CD > 0$ is satisfied, from Routh-Hurwitz Criterion, all characteristics roots have negative real parts. Therefore equilibrium points is asymptotic stable. □

3.4. Numerical scheme for the mathematical Model

The numerical approach introduced by Atangana-Owolabi [3] will be used to calculate the numerical solutions of the mathematical model.With this method, the Caputo-Fabrizio fractional operator is discretized and solved.

$${}_0^{CF}\mathfrak{D}_t^{\vartheta} x(t) = (f(t, x(t))), \tag{42}$$

or

$$(f(t, x(t))) = \frac{M(\vartheta)}{1-\vartheta}\int_0^t x'(\tau) exp\Big[-\frac{\vartheta}{1-\vartheta}(t-\tau)\Big]d\tau. \tag{43}$$

When the above equation is arranged, the following equation is obtained,

$$x(t) - x(0) = \frac{1-\vartheta}{M(\vartheta)} f(t, x(t)) + \frac{\vartheta}{M(\vartheta)}\int_0^t f(\tau, x(\tau))d\tau, \tag{44}$$

consequently,

$$x(t_{n+1}) - x(0) = \frac{1-\vartheta}{M(\vartheta)} f(t_n, x(t_n)) + \frac{\vartheta}{M(\vartheta)}\int_0^{t_{n+1}} f(t, x(t))dt, \tag{45}$$

and

$$x(t_{n+1}) - x(0) = \frac{1-\vartheta}{M(\vartheta)} f(t_{n-1}, x(t_{n-1})) + \frac{\vartheta}{M(\vartheta)}\int_0^{t_n} f(t, x(t))dt. \tag{46}$$

Using equations (47) and (48), the following equation can be found.

$$\begin{aligned} x(t_{n+1}) - x(t_n) = &\frac{1-\vartheta}{M(\vartheta)}\{f(t_n, x(t_n)) - f(t_{n-1}, x_{n-1})\} \\ &+ \frac{\vartheta}{M(\vartheta)}\int_0^{t_n} f(t, x(t))dt, \end{aligned} \tag{47}$$

where

$$\begin{aligned}\int_{t_n}^{t_{n+1}} f(t,x(t))dt &= \int_{t_n}^{t_{n+1}} \left\{\frac{f(t_n,x_n)}{h}(t-t_{n-1}-\frac{f(t_{n-1},x_{n-1})}{h})(t-t_n)\right\}dt \\ &= \frac{3h}{2}f(t_n,x_n)-\frac{h}{2}f(t_{n-1},x_{n-1}).\end{aligned} \tag{48}$$

Thus,

$$x(t_{n+1})-x(t_n)=\frac{1-\vartheta}{M(\vartheta)}\Big[f(t_n,x_n)-f(t_{n-1},x_{n-1})\Big]+\frac{3\vartheta h}{2M(\vartheta)}f(t_n,x_n)-\frac{\vartheta h}{2M(\vartheta)}f(t_{n-1},x_{n-1}), \tag{49}$$

which implies that

$$x(t_{n+1})-x(t_n)=\left(\frac{1-\vartheta}{M(\vartheta)}+\frac{3\vartheta h}{2M(\vartheta)}\right)f(t_n,x_n)+\left(\frac{1-\vartheta}{M(\vartheta)}+\frac{3\vartheta h}{2M(\vartheta)}\right)f(t_{n-1},x_{n-1}). \tag{50}$$

Hence, the following equation is written,

$$x_{n+1}=x_n+\left(\frac{1-\vartheta}{M(\vartheta)}+\frac{3\vartheta h}{2M(\vartheta)}\right)f(t_n,x_n)+\left(\frac{1-\vartheta}{M(\vartheta)}+\frac{3\vartheta h}{2M(\vartheta)}\right)f(t_{n-1},x_{n-1}). \tag{51}$$

Theorem 3.5. *Let $x(t)$ be a solution of ${}_a^{CF}\mathfrak{D}_t^{\vartheta}(x(t))=f(t,x(t))$ where f is a continuous function bounded for the Caputo-Fabrizio fractional derivative [3],*

$$x_{n+1}=x_n+\left(\frac{1-\vartheta}{M(\vartheta)}+\frac{3\vartheta h}{2M(\vartheta)}\right)f(t_n,x_n)+\left(\frac{1-\vartheta}{M(\vartheta)}+\frac{\vartheta h}{2M(\vartheta)}\right)f(t_{n-1},x_{n-1})+R_{\vartheta}^n, \tag{52}$$

where $\|R_{\vartheta}^n\|\leq M$.

When the mathematical model given in (19) is arranged and written with the help of the fundamental theorem of analysis, the next system of equations is obtained for $G_i, i=1,2$ kernels as follows:

$$\begin{aligned} A(t)-A(0)&=\frac{1-\vartheta}{M(\vartheta)}G_1(t,A(t))+\frac{\vartheta}{M(\vartheta)}\int_0^t G_1(\tau,A(\tau))d\tau,\\ B(t)-B(0)&=\frac{1-\vartheta}{M(\vartheta)}G_2(t,B(t))+\frac{\vartheta}{M(\vartheta)}\int_0^t G_2(\tau,B(\tau))d\tau.\end{aligned} \tag{53}$$

Thus,

$$\begin{aligned} A_{n+1}-A(0)&=\frac{1-\vartheta}{M(\vartheta)}G_1(t_n,A(t_n))+\frac{\vartheta}{M(\vartheta)}\int_0^{t_{n+1}} G_1(t,A(t))dt,\\ B_{n+1}-B(0)&=\frac{1-\vartheta}{M(\vartheta)}G_2(t_n,B(t_n))+\frac{\vartheta}{M(\vartheta)}\int_0^{t_{n+1}} G_2(t,B(t))dt,\end{aligned} \tag{54}$$

and

$$\begin{aligned} A_n-A(0)&=\frac{1-\vartheta}{M(\vartheta)}G_1(t_{n-1},A(t_{n-1}))+\frac{\vartheta}{M(\vartheta)}\int_0^{t_n} G_1(t,A(t))dt,\\ B_n-B(0)&=\frac{1-\vartheta}{M(\vartheta)}G_2(t_{n-1},B(t_{n-1}))+\frac{\vartheta}{M(\vartheta)}\int_0^{t_n} G_2(t,B(t))dt.\end{aligned} \tag{55}$$

Using the (56) and (57) equations, the following system is obtained.

$$
\begin{aligned}
A_{n+1} - A(0) &= \frac{1-\vartheta}{M(\vartheta)}\Big\{G_1(t_n, A(t_n)) - G_1(t_{n-1}, A(t_{n-1}))\Big\} + \frac{\vartheta}{M(\vartheta)}\int_{t_n}^{t_{n+1}} G_1(t, A(t))dt,\\
B_{n+1} - B(0) &= \frac{1-\vartheta}{M(\vartheta)}\Big\{G_2(t_n, B(t_n)) - G_2(t_{n-1}, B(t_{n-1}))\Big\} + \frac{\vartheta}{M(\vartheta)}\int_{t_n}^{t_{n+1}} G_2(t, B(t))dt,
\end{aligned} \tag{56}
$$

where

$$
\begin{aligned}
\int_{t_n}^{t_{n+1}} G_1(t, A(t))dt &= \int_{t_n}^{t_{n+1}} \left\{\frac{G_1(t_n, A_n)}{h}(t - t_{n-1}) - \frac{G_1(t_{n-1}, A_{n-1})}{h}(t - t_n)\right\}\\
&= \frac{3h}{2}G_1(t_n, A_n) - \frac{h}{2}G_1(t_{n-1}, A_{n-1}),\\
\int_{t_n}^{t_{n+1}} G_2(t, B(t))dt &= \int_{t_n}^{t_{n+1}} \left\{\frac{G_2(t_n, B_n)}{h}(t - t_{n-1}) - \frac{G_2(t_{n-1}, B_{n-1})}{h}(t - t_n)\right\}\\
&= \frac{3h}{2}G_2(t_n, B_n) - \frac{h}{2}G_2(t_{n-1}, B_{n-1}).
\end{aligned} \tag{57}
$$

Therefore,

$$
\begin{aligned}
A_{n+1} - A_n &= \frac{1-\vartheta}{M(\vartheta)}\Big\{G_1(t_n, A_n) - G_1(t_{n-1}, A_{n-1})\Big\} + \frac{3\vartheta h}{2M(\vartheta)}G_1(t_n, A_n) - \frac{\vartheta h}{2M(\vartheta)}G_1(t_{n-1}, A_{n-1}),\\
B_{n+1} - B_n &= \frac{1-\vartheta}{M(\vartheta)}\Big\{G_2(t_n, B_n) - G_2(t_{n-1}, B_{n-1})\Big\} + \frac{3\vartheta h}{2M(\vartheta)}G_2(t_n, B_n) - \frac{\vartheta h}{2M(\vartheta)}G_2(t_{n-1}, B_{n-1}),
\end{aligned} \tag{58}
$$

which implies that,

$$
\begin{aligned}
A_{n+1} &= A_n + \left(\frac{1-\vartheta}{M(\vartheta)} + \frac{3\vartheta h}{2M(\vartheta)}\right)G_1(t_n, A_n) + \left(\frac{1-\vartheta}{M(\vartheta)} + \frac{\vartheta h}{2M(\vartheta)}\right)G_1(t_{n-1}, A_{n-1}),\\
B_{n+1} &= B_n + \left(\frac{1-\vartheta}{M(\vartheta)} + \frac{3\vartheta h}{2M(\vartheta)}\right)G_2(t_n, B_n) + \left(\frac{1-\vartheta}{M(\vartheta)} + \frac{\vartheta h}{2M(\vartheta)}\right)G_2(t_{n-1}, B_{n-1}).
\end{aligned} \tag{59}
$$

According to theorem (5.1), we get,

$$
\begin{aligned}
A_{n+1} &= A_n + \left(\frac{1-\vartheta}{M(\vartheta)} + \frac{3\vartheta h}{2M(\vartheta)}\right)G_1(t_n, A_n) + \left(\frac{1-\vartheta}{M(\vartheta)} + \frac{\vartheta h}{2M(\vartheta)}\right)G_1(t_{n-1}, A_{n-1}) + {}^1R_\vartheta^n,\\
B_{n+1} &= B_n + \left(\frac{1-\vartheta}{M(\vartheta)} + \frac{3\vartheta h}{2M(\vartheta)}\right)G_2(t_n, B_n) + \left(\frac{1-\vartheta}{M(\vartheta)} + \frac{\vartheta h}{2M(\vartheta)}\right)G_2(t_{n-1}, B_{n-1}) + {}^2R_\vartheta^n,
\end{aligned} \tag{60}
$$

where

$$
\|{}^iR_\vartheta^n\|_\infty < \frac{\vartheta}{M(\vartheta)}(n+1)!h^{n+1}M, \quad i = 1, 2, 3, 4.
$$

4. Discussions with simulations

In this study, the mathematical model of the poem *Yar Qasidi* written by Shahriyar has been developed. In addition, the mathematical model has been extended to the Caputo-Fabrizio fractional derivative. It was then analyzed with the Adams–Bashforth numerical approach. In this section, simulations of the model are shown and interpreted.

The mathematical model has an important temporal distinction. We considered this distinction at time $t = 40$. In other words, it is assumed that the beloved woman at $t = 40$ married another man.

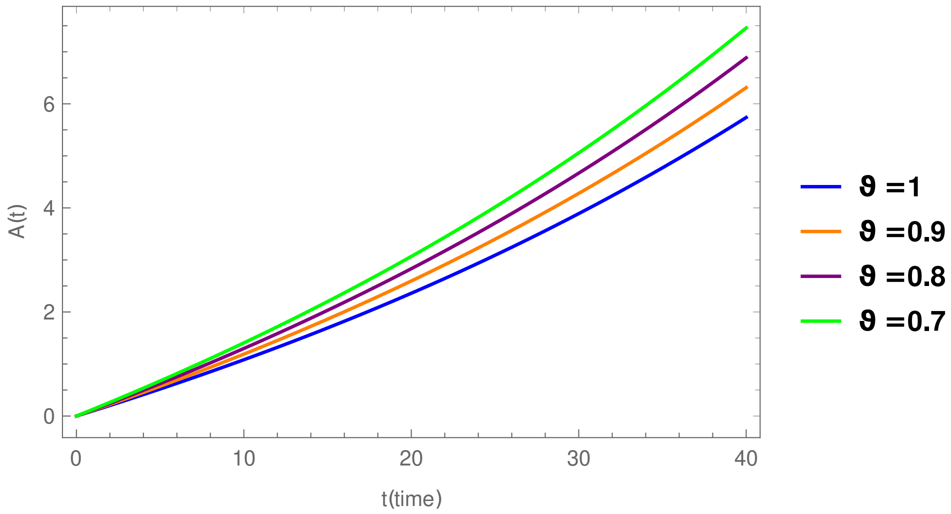

Figure 1: Positive feelings of the beloved man

Fig. 1 shows the lover man's positive feelings before the beloved woman gets married. Naturally, the feelings of someone who loves are expected to be positive. In addition, it continues to increase depending on time unless there is a negative situation. In fact, it can be easily observed that there is a significant increase with the hope of reunion. Also, this situation is calculated for different fractional derivative values. The graph showed that it gives correct results for different fractional derivative values.

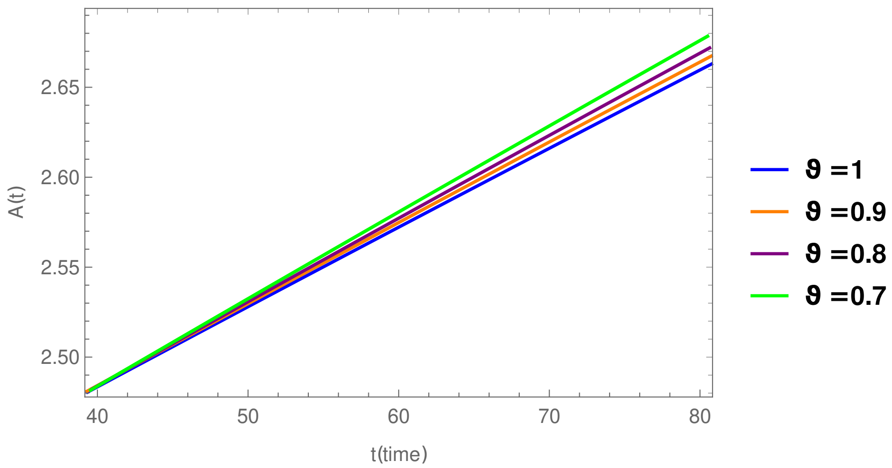

Figure 2: The feelings of a man who loves a woman after she marries someone else.

Fig. 2 shows the lover man's process after the beloved woman's marriage. Although the increase is not as rapid as the period before the woman's marriage with another man, it is seen that love continues in this process. The decrease in the acceleration of increase in Fig. 2 is due to the decrease in the hope of reunion compared to Fig. 1.

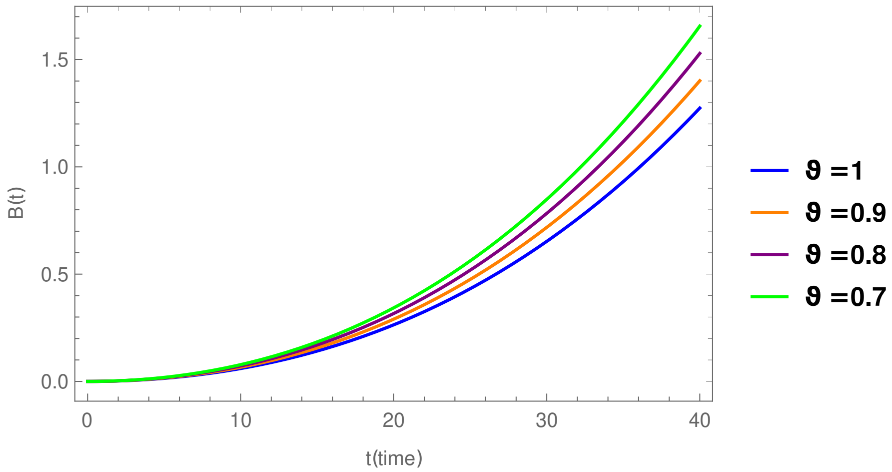

Figure 3: Positive feelings of the beloved woman

In Figure 3, it is true that the beloved woman who loves has an interest in the lover man. But it is obvious that this is not as strong as the feelings of the lover man.

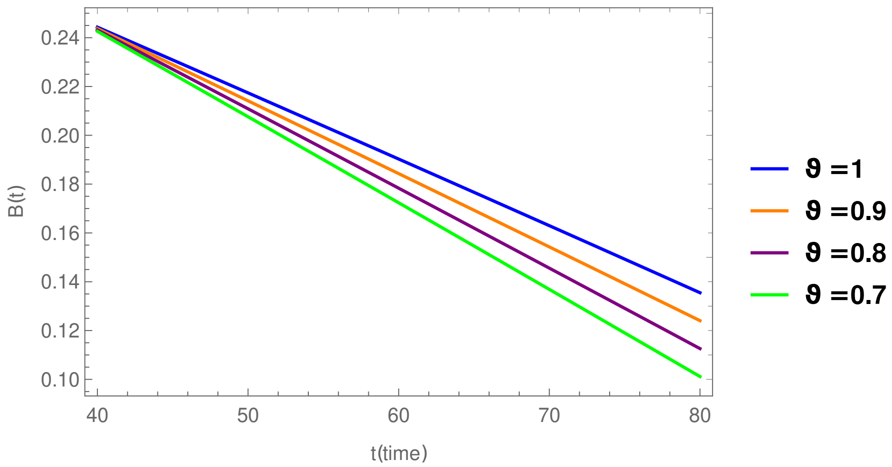

Figure 4: The feelings of the beloved woman towards the beloved man after marrying someone else

In Figure 4, the beloved woman's feelings towards the lover man after marrying another man were examined. The graph naturally shows the decline in a married beloved woman's feelings for a different man.

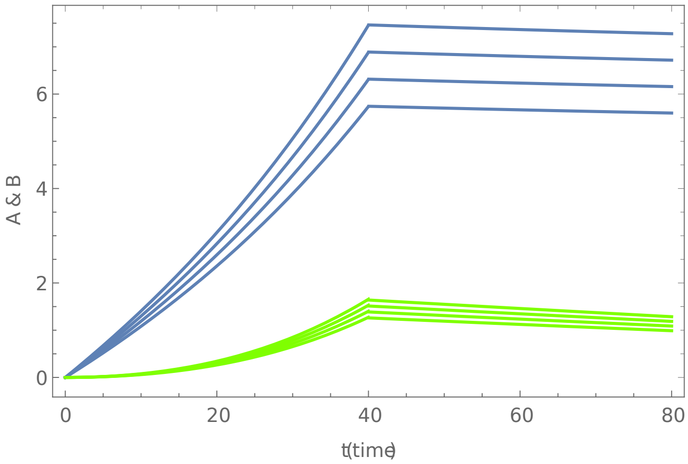

Figure 5: The feelings of the beloved man and the beloved woman

Figure 5 shows the differences in the feelings of the two protagonists against time. The situation of the beloved woman until she marries another man is positive for both parties ($0 < t < 40$). When the post-marriage process is examined, a decrease in feelings is observed. This situation describes that the process is progressing correctly ($40 < t < 80$).

In addition, this situation is calculated for different fractional derivative values. The graph showed that it gives correct results for different fractional derivative values.

5. Conclusions

In this study, the analysis of the poem *Yar Qasidi* written by Shahriar was made. Later, based on this analysis, a mathematical model was obtained to examine the variability of the feelings of the protagonists. This model is extended with the help of the Caputo-Fabrizio fractional derivative operator and the existence, uniqueness and stability of its solution are analyzed. In addition, the mathematical model is solved with the Adam-Bashford numerical approach. In the context of numerical solution, the reflections of the events in the poem on the feelings of the protagonists were taken into account, and simulations were obtained according to the values of the fractional derivative operator.

There are many parameters that can be considered in the analysis of poetry. In our study, the feelings, which are clarified by the concepts in the poem, have been analyzed from the perspective of what is described and understood. By examining different literary works and genres, researchers can deal with new contents through new protagonists and develop the mathematical model.

Authors' contributions

Ülkü Eliuz and Burak Armağan made a literary analysis of *Yar Qasidi*'s poetry. Ahmet Selçuk Akdemir, Ülkü Eliuz and Burak Armağan wrote the introduction of the article and contributed to the writing and interpretation of other chapters. Mustafa Ali Dokuyucu contributed to the establishment of the mathematical model, existence and uniqueness solutions and stability analysis. Mustafa Ali Dokuyucu also contributed to the making of numerical solutions and drawing simulations. Ahmet Selçuk Akdemir worked in the format design and editing. All authors have read and approved final version of the paper.

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