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Error bounds of multiplicative Boole's type inequalities for twice differentiable functions with applications to numerical integration

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Abstract. This paper introduces a new multiplicative integral identity for functions that are twice differentiable in the multiplicative sense. Using this identity, we establish Boole-type inequalities under the assumption of convexity within the framework of multiplicative calculus. The results provide improved absolute error bounds for integral approximations compared to those obtained through classical calculus, especially for higher-degree polynomials. To demonstrate the usefulness of these inequalities, we apply them to numerical quadrature formulas and special means. Finally, numerical examples accompanied by graphical representations are presented to validate the theoretical findings and demonstrate their practical relevance.

1. Introduction

Inequalities are the most fundamental concepts in mathematics, serving as powerful tools for analysis, estimation and problem-solving in various domains. They provide a framework for comparing quantities, establishing bounds and refining approximations, making them indispensable in both theoretical and applied research. Many optimization problems in economics, engineering and operations research rely on inequalities to define constraints and feasible regions. Inequalities play a crucial role in checking the stability of numerical schemes and in machine learning, minimization of a loss function. Throughout history, inequalities have played a big role in the advancement of mathematical sciences, leading to the development of entire fields such as optimization, functional analysis, differential equations and numerical analysis. The concept of inequalities has been a fundamental part of mathematics for a long time, with records of their use dating back to ancient civilizations. However, the systematic study of inequalities and their properties gained prominence in the 17th and 18th centuries. German mathematician Karl Hermann Amandus Schwarz is credited with introducing convex functions in the late 19th century, significantly influencing mathematical theory and its applications. These functions play a crucial role in optimization

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problems across various fields, including economics, engineering, and computer science, aiding in tasks like portfolio optimization and control theory. As a result, inequalities are now regarded as essential tools in the modern mathematical research landscape. For further insights into the history and applications of convexity, refer to [1, 2].

Definition 1.1. *A function $F : I \subset \mathbb{R} \to \mathbb{R}^+$ is considered convex, if*

$$F(\tau\omega + (1-\tau)\varpi) \leq \tau F(\omega) + (1-\tau) F(\varpi), \tag{1}$$

where $\omega, \varpi \in I$ and $\tau \in [0,1]$.

On the other hand, if the inequality (1) is inverted, the function F is deemed a concave function.

The Hermite–Hadamard inequality is a fundamental tool in various fields of applied mathematics. Its versatility extends across multiple disciplines, rendering it essential for mathematical analysis in real-world contexts. Let's review it once more:

Theorem 1.2. *The double inequalities for a convex function $F : [\omega, \varpi] \subseteq \mathbb{R} \to \mathbb{R}^+$ is stated as:*

$$F\left(\frac{\omega+\varpi}{2}\right) \leq \frac{1}{\varpi-\omega}\int_{\omega}^{\varpi} F(\varkappa)\, d\varkappa \leq \frac{F(\omega)+F(\varpi)}{2}. \tag{2}$$

If F is a concave function, both inequalities hold in the reverse direction. For more examples and insights, please refer to [3–9] and the sources cited therein.

In calculus and analysis, differentiation and integration are fundamentally based on the operations of addition and subtraction. However, alternative formulations exist in which these operations are replaced by multiplication and division, leading to the development of multiplicative calculus, also known as non-Newtonian calculus. Although this framework satisfies the essential axioms of calculus, it has not achieved the same prominence as the Newton–Leibnitz formulation. Given its relatively limited range of applications compared to classical calculus, one may question the necessity of developing a system with a narrower scope. However, this concern is analogous to questioning the use of polar coordinates in planar geometry, despite the ubiquity and generality of Cartesian coordinates. The utility of a mathematical framework is not solely determined by its breadth, but also by its suitability to specific classes of problems. Multiplicative calculus offers a natural framework for modeling systems governed by proportional change, particularly in domains such as economics, finance, and population dynamics, where exponential growth and ratio-based behavior are intrinsic. In such contexts, multiplicative calculus provides analytical tools that are often more appropriate and effective than those offered by classical methods. The concept of a multiplicatively convex function is particularly important and can be defined as follows:

Definition 1.3. *[10] Multiplicative convex function $F : I \subseteq \mathbb{R} \to [0, \infty)$ is stated as:*

$$F(\tau\omega + (1-\tau)\varpi) \leq [F(\omega)]^{\tau}[F(\varpi)]^{1-\tau}, \quad \text{for all } \omega, \varpi \in I \text{ and } \tau \in [0,1].$$

From Definition 1.3, we have

$$F(\tau\omega + (1-\tau)\varpi) \leq [F(\omega)]^{\tau}[F(\varpi)]^{1-\tau} \leq \tau F(\omega) + (1-\tau) F(\varpi).$$

This shows that every function that is convex in the multiplicative sense is also convex in the traditional sense. However, it is important to emphasize that the converse is not always true. Throughout this section, let F and Ω be positive functions from $\mathbb{R}$ to $\mathbb{R}^+$. In 2008, Bashirov [11] introduced multiplicative operators denoted by $\int_{\omega}^{\varpi} (F(\varkappa))^{d\varkappa}$, where $\int_{\omega}^{\varpi} F(\varkappa) d\varkappa$ represents the ordinary integral. It's noteworthy that F is considered multiplicatively integrable on the interval $[\omega, \varpi]$ if it is positive and Riemann integrable on that interval, and satisfies the condition:

$$\int_{\omega}^{\varpi} (F(\varkappa))^{d\varkappa} = e^{\int_{\omega}^{\varpi} \ln(F(\varkappa)) d\varkappa}.$$

Additionally, Bashirov et al. established the following properties of the multiplicative integral:

Proposition 1.4. *[11] Assuming that F and Ω are both positive functions that are Riemann integrable over the interval $[\omega,\varpi]$, then F is multiplicatively integrable on $[\omega,\varpi]$ and*

(i) $\int_{\omega}^{\varpi}((F(\varkappa))^{p})^{d\varkappa}=\int_{\omega}^{\varpi}\left((F(\varkappa))^{d\varkappa}\right)^{p}$,

(ii) $\int_{\omega}^{\varpi}(F(\varkappa)\,\Omega(\varkappa))^{d\varkappa}=\int_{\omega}^{\varpi}(F(\varkappa))^{d\varkappa}.\int_{\omega}^{\varpi}(\Omega(\varkappa))^{d\varkappa}$,

(iii) $\int_{\omega}^{\varpi}\left(\frac{F(\varkappa)}{\Omega(\varkappa)}\right)^{d\varkappa}=\frac{\int_{\omega}^{\varpi}(F(\varkappa))^{d\varkappa}}{\int_{\omega}^{\varpi}(\Omega(\varkappa))^{d\varkappa}}$,

(iv) $\int_{\omega}^{\varpi}(F(\varkappa))^{d\varkappa}=\int_{\omega}^{c}(F(\varkappa))^{d\varkappa}.\int_{c}^{\varpi}(F(\varkappa))^{d\varkappa}$, $\omega\leq c\leq\varpi$,

(v) $\int_{\omega}^{\omega}(F(\varkappa))^{d\varkappa}=1$ *and* $\int_{\omega}^{\varpi}(F(\varkappa))^{d\varkappa}=\left(\int_{\varpi}^{\omega}(F(\varkappa))^{d\varkappa}\right)^{-1}$.

Lemma 1.5. *[11] Suppose $F:[\omega,\varpi]\to\mathbb{R}$ be multiplicative differentiable and suppose $\Omega:[\omega,\varpi]\to\mathbb{R}$ to be differentiable, so the function F^{Ω} is multiplicative integrable, then*

$$\int_{\omega}^{\varpi}\left((F^{*}(\varkappa))^{\Omega(\varkappa)}\right)^{d\varkappa}=\frac{F(\varpi)^{\Omega(\varpi)}}{F(\omega)^{\Omega(\omega)}}.\frac{1}{\int_{\omega}^{\varpi}\left((F(\varkappa))^{\Omega'(\varkappa)}\right)^{d\varkappa}}.$$

Lemma 1.6. *Suppose $F:[\omega,\varpi]\to\mathbb{R}$ be multiplicative differentiable, suppose $\Omega:[\omega,\varpi]\to\mathbb{R}$ and $F:I\subset\mathbb{R}\to[\omega,\varpi]$ be both differentiable functions, then*

$$\int_{\omega}^{\varpi}\left(F^{*}(F(\varkappa))^{\Omega(\varkappa)F'(\varkappa)}\right)^{d\varkappa}=\frac{F(F(\varpi))^{\Omega(\varpi)}}{F(F(\omega))^{\Omega(\omega)}}.\frac{1}{\int_{\omega}^{\varpi}\left((F(\varkappa))^{\Omega'(\varkappa)}\right)^{d\varkappa}}.$$

Definition 1.7. *[11] The definition of the multiplicative derivative for a positive function F is given by:*

$$\frac{d^{*}F}{d\tau}(\tau)=F^{*}(\tau)=\lim_{h\to0}\left(\frac{F(\tau+h)}{F(h)}\right)^{\frac{1}{h}}.$$

When F is positive and differentiable at τ, the multiplicative derivative F^{*} can be defined. The connection between F^{*} and the ordinary derivative F' is described as follows:

$$F^{*}(\tau)=e^{[\ln F(\tau)]'}=e^{\frac{F'(\tau)}{F(\tau)}}.$$

Definition 1.8. *The second derivative of multiplicative function F is stated as:*

$$F^{**}(\tau)=e^{[\ln F^{*}(\tau)]'}=e^{[\ln F(\tau)]''}.$$

Here, $(\ln F)''(\tau)$ is occurred because $F''(\tau)$ is extant. If we continue this process to n-times for F and its derivative of nth order subsist at τ, then $F^{*n}(\tau)$ also exists:

$$F^{*(n)}(\tau)=e^{(\ln F)^{(n)}(\tau)},\quad n=1,2,3,\cdots.$$

Theorem 1.9. *[11] Considering the multiplicative differentiable functions F and Ω. If c is an arbitrary constant, then the functions cF, $F\Omega$, $F+\Omega$, $\frac{F}{\Omega}$ and F^{Ω} are also multiplicative differentiable, then*

(i) $(cF)^{*}(\tau)=cF^{*}(\tau)$,

(ii) $(F\Omega)^{*}(\tau)=F^{*}(\tau)\,\Omega^{*}(\tau)$,

(iii) $(F+\Omega)^{*}(\tau)=F^{*}(\tau)^{\frac{F(\tau)}{F(\tau)+\Omega(\tau)}}\,\Omega^{*}(\tau)^{\frac{\Omega(\tau)}{F(\tau)+\Omega(\tau)}}$,

(iv) $\left(\frac{F}{\Omega}\right)^{*}(\tau)=\frac{F^{*}(\tau)}{\Omega^{*}(\tau)}$,

(v) $\left(F^{\Omega}\right)^{*}(\tau)=F^{*}(\tau)^{\Omega(\tau)}F(\tau)^{\Omega'(\tau)}$.

This study also utilizes some special functions, whose definitions are provided below for clarity [18]:

- **Gamma function**:

$$\Gamma(z) = \int_0^\infty u^{z-1} e^{-u} du, \quad \mathrm{Re}(z) > 0.$$

- **Beta function**:

$$B(\omega, \varpi) = \frac{\Gamma(\omega)\Gamma(\varpi)}{\Gamma(\omega + \varpi)} = \int_0^1 u^{\omega-1}(1-u)^{\varpi-1} du.$$

- **Incomplete Beta function**:

$$B_z(\omega, \varpi) = \int_0^z u^{\omega-1}(1-u)^{\varpi-1} du.$$

- **Gauss hypergeometric function**:

$${}_2F_1(\omega, \varpi; c; z) = \sum_{n=0}^\infty \frac{(\omega)_n(\varpi)_n}{(c)_n} \frac{z^n}{n!},$$

where $(\omega)_n = \omega(\omega+1)\cdots(\omega+n-1)$ is the Pochhammer symbol.

These functions appear frequently in our analysis of the integral expressions presented in Sections 2. In particular, the incomplete Beta function and hypergeometric function arise naturally when evaluating integrals of the form $\int |P(\tau)|^p d\tau$ where $P(\tau)$ is a quadratic polynomial.

Ali et al. [19] formulated a Hermite–Hadamard type inequality in the context of multiplicatively convex functions, it is stated as:

Theorem 1.10. *[19] Consider multiplicative convex function F on $[\omega, \varpi]$, then the double inequalities hold:*

$$F\left(\frac{\omega + \varpi}{2}\right) \leq \left(\int_\omega^\varpi (F(\varkappa))^{d\varkappa}\right)^{\frac{1}{\varpi - \omega}} \leq \sqrt{F(\omega) F(\varpi)}.$$

See [19, 21, 22] for more information and additional references. In [7], Khan and Budak explored midpoint and trapezoidal type inequalities for multiplicative integrals, establishing fundamental bounds for multiplicatively convex functions. Mateen et al. discussed error bounds for several inequalities, including Simpson's, Hermite–Hadamard, midpoint, trapezoid, and Bullen-type, focusing on twice differentiable functions in the context of multiplicatively convex functions in [23], providing a unified framework for analyzing different quadrature rules. Additionally, Chasreechai et al. proved the multiplicative version of Newton and Simpson's 1/3 formula type inequalities in [24], extending classical numerical integration techniques to the multiplicative calculus setting. Recent work has extended fractional calculus to multiplicative settings, with applications to inequalities and numerical analysis. Du and Ai [12] established Katugampola-type multiplicative fractional integral inequalities, introducing new tools for handling non-local multiplicative operators. Meanwhile, Peng and Du [13] derived Maclaurin-type inequalities for multiplicatively convex functions, refining earlier results on exponential-type convexity. These were further improved by Peng et al. [14] for exponential-kernel integrals, where sharper error estimates were obtained. For applications, Ai and Du [15] studied Newton-type bounds, demonstrating their use in multiplicative numerical schemes, while Zhou and Du [16] developed error estimates for multiplicative quadrature, highlighting their efficiency in approximating multiplicative integrals.

Inspired by ongoing research, this research presents a novel formulation of Boole's formula-type inequalities for twice differentiable functions, framed within the innovative context of multiplicative calculus. The motivation for exploring Boole's formula within the context of multiplicative calculus stems from the desire to enhance the accuracy and applicability of integral approximations in various fields, including finance and engineering. Traditional approaches to numerical integration often rely on assumptions and properties inherent to classical calculus, which may not capture the unique characteristics of multiplicative processes. Since multiplicative calculus is modern calculus with a lot of applications in banking and finance, the study of multiplicative calculus is valuable. The main advantage of using multiplicative calculus is its ability to capture more accurate error bounds, especially for higher-order derivatives, particularly when dealing with exponential functions. Classical calculus often struggles to achieve the required precision in these scenarios, whereas multiplicative calculus provides a more robust framework. By proving integral inequalities through multiplicative calculus, the aim is to unveil novel insights that enhance our understanding of mathematical concepts and their applications [17, 20].

The remaining paper is outlined as follows: Section 2, presents the key results regarding Boole's formula-type integral inequalities for log-convex functions within the framework of multiplicative calculus. Applications to quadrature formulas and special means in the context of multiplicative calculus for real numbers are developed in Section 3. Section 4 includes numerical examples and graphical representations that illustrate and validate the findings. Lastly, Section 5 concludes with reflections on the significance of this work and proposes potential directions for future research.

2. Main Results

In this section, we derive an integral identity for multiplicatively twice differentiable functions. This identity acts as a foundational tool for exploring properties and inequalities within the realm of multiplicative calculus.

Lemma 2.1. *Consider $F : [\omega, \varpi] \subset \mathbb{R} \to \mathbb{R}^+$ to be a multiplicative differentiable function with a continuous derivative on (ω, ϖ) and $F^{**} \in L_1[\omega, \varpi]$. Then subsequent equality is satisfied:*

$$\frac{\left(\int_{\omega}^{\varpi} (F(\varkappa))^{d\varkappa}\right)^{\frac{1}{\varpi-\omega}}}{\left[(F(\omega))^7 \left(F\left(\frac{\omega+\varpi}{2}\right)\right)^{12} \left(F\left(\frac{3\omega+\varpi}{4}\right) F\left(\frac{\omega+3\varpi}{4}\right)\right)^{32} (F(\varpi))^7\right]^{\frac{1}{90}}}$$
$$= \left[\int_0^1 \left(F^{**}\left[(\tau\varpi + (1-\tau)\omega)\right]^{\Delta(\tau)}\right)^{d\tau}\right]^{\frac{(\varpi-\omega)^2}{2}}, \tag{3}$$

where

$$\Delta(\tau) := \begin{cases} \tau^2 - \dfrac{7}{45}\tau, & 0 \le \tau \le \dfrac{1}{4}; \\ \tau^2 - \dfrac{13}{15}\tau + \dfrac{8}{45}, & \dfrac{1}{4} \le \tau \le \dfrac{1}{2}; \\ \tau^2 - \dfrac{17}{15}\tau + \dfrac{14}{45}, & \dfrac{1}{2} \le \tau \le \dfrac{3}{4}; \\ \tau^2 - \dfrac{83}{45}\tau + \dfrac{38}{45}, & \dfrac{3}{4} \le \tau \le 1. \end{cases}$$

Proof. By multiplicative integration and change of variable $\varkappa = \tau\varpi + (1-\tau)\omega$, it suffices to note that:

$$I = \left[\int_0^1 \left(\left[F^{**}(\tau\varpi + (1-\tau)\omega)\right]^{\Delta(\tau)}\right)^{d\tau}\right]^{\frac{(\varpi-\omega)^2}{2}}$$
$$= \left[\int_0^{\frac{1}{4}} \left(\left[F^{**}(\tau\varpi + (1-\tau)\omega)\right]^{\left(\tau^2 - \frac{7}{45}\tau\right)}\right)^{d\tau}\right]^{\frac{(\varpi-\omega)^2}{2}}$$

$$\times\left[\int_{\frac{1}{4}}^{\frac{1}{2}}\left([F^{**}(\tau\varpi+(1-\tau)\omega)]^{\left(\tau^2-\frac{13}{15}\tau+\frac{8}{45}\right)}\right)^{d\tau}\right]^{\frac{(\varpi-\omega)^2}{2}}$$
$$\times\left[\int_{\frac{1}{2}}^{\frac{3}{4}}\left([F^{**}(\tau\varpi+(1-\tau)\omega)]^{\left(\tau^2-\frac{17}{15}\tau+\frac{14}{45}\right)}\right)^{d\tau}\right]^{\frac{(\varpi-\omega)^2}{2}}$$
$$\times\left[\int_{\frac{3}{4}}^{1}\left([F^{**}(\tau\varpi+(1-\tau)\omega)]^{\left(\tau^2-\frac{83}{45}\tau+\frac{38}{45}\right)}\right)^{d\tau}\right]^{\frac{(\varpi-\omega)^2}{2}}$$
$$= I_1\times I_2\times I_3\times I_4. \tag{4}$$

Now using multiplicative integration to solve these integrals as follows:

$$\begin{aligned}
I_1 &= \left[\int_{0}^{\frac{1}{4}}\left([F^{**}(\tau\varpi+(1-\tau)\omega)]^{\left(\tau^2-\frac{7}{45}\tau\right)}\right)^{d\tau}\right]^{\frac{(\varpi-\omega)^2}{2}}\\
&= \exp\left[\frac{(\varpi-\omega)^2}{2}\int_0^{\frac{1}{4}}\left(\tau^2-\frac{7}{45}\tau\right)(\ln\circ F)''\left(\tau\varpi+(1-\tau)\,\omega\right)d\tau\right]\\
&= \exp\left[\frac{(\varpi-\omega)^2}{2}\left(\left(\tau^2-\frac{7}{45}\tau\right)\frac{\ln F'\left(\tau\varpi+(1-\tau)\,\omega\right)}{\varpi-\omega}\Bigg|_0^{\frac{1}{4}}\right.\right.\\
&\left.\left.-\frac{1}{\varpi-\omega}\int_0^{\frac{1}{4}}\left(2\tau-\frac{7}{45}\right)(\ln\circ F)'\left(\tau\varpi+(1-\tau)\,\omega\right)\right)\right]\\
&= \exp\left[\frac{(\varpi-\omega)^2}{2}\left(\frac{17}{720\,(\varpi-\omega)}\ln F'\left(\frac{3\omega+\varpi}{4}\right)\right.\right.\\
&-\left(\frac{1}{(\varpi-\omega)^2}\left(2\tau-\frac{7}{45}\right)(\ln\circ F)\left(\tau\varpi+(1-\tau)\,\omega\right)\Bigg|_0^{\frac{1}{4}}\right.\\
&\left.\left.\left.-\frac{2}{(\varpi-\omega)^2}\int_0^{\frac{1}{4}}(\ln\circ F)\left(\tau\varpi+(1-\tau)\,\omega\right)d\tau\right)\right)\right]\\
&= \exp\left[\frac{(\varpi-\omega)^2}{2}\left(\frac{17}{720\,(\varpi-\omega)}\ln F'\left(\frac{3\omega+\varpi}{4}\right)-\frac{31}{90\,(\varpi-\omega)^2}\ln F\left(\frac{3\omega+\varpi}{4}\right)\right.\right.\\
&\left.\left.-\frac{7}{45\,(\varpi-\omega)^2}\ln F\left(\omega\right)+\frac{2}{(\varpi-\omega)^3}\int_{\omega}^{\frac{3\omega+\varpi}{4}}\ln\left(F\left(\varkappa\right)\right)d\varkappa\right)\right].
\end{aligned}\tag{5}$$

Similarly,

$$\begin{aligned}
I_2 &= \left[\int_{\frac{1}{4}}^{\frac{1}{2}}\left([F^{**}(\tau\varpi+(1-\tau)\omega)]^{\left(\tau^2-\frac{13}{15}\tau+\frac{8}{45}\right)}\right)^{d\tau}\right]^{\frac{(\varpi-\omega)^2}{2}}\\
&= \exp\left[\frac{(\varpi-\omega)^2}{2}\left(\frac{-1}{180\,(\varpi-\omega)}\ln F'\left(\frac{\omega+\varpi}{2}\right)-\frac{17}{720\,(\varpi-\omega)^2}\ln F'\left(\frac{3\omega+\varpi}{4}\right)\right.\right.\\
&-\frac{2}{15\,(\varpi-\omega)^2}\ln F\left(\frac{\omega+\varpi}{2}\right)-\frac{11}{30\,(\varpi-\omega)^2}\ln F\left(\frac{3\omega+\varpi}{4}\right)\\
&\left.\left.+\frac{2}{(\varpi-\omega)^3}\int_{\frac{3\omega+\varpi}{4}}^{\frac{\omega+\varpi}{2}}\ln\left(F\left(\varkappa\right)\right)d\varkappa\right)\right],
\end{aligned}\tag{6}$$

$$I_3 = \left[\int_{\frac{1}{2}}^{\frac{3}{4}} \left([F^{**}(\tau\varpi + (1-\tau)\omega)]^{\left(\tau^2 - \frac{17}{15}\tau + \frac{14}{45}\right)}\right)^{d\tau}\right]^{\frac{(\varpi-\omega)^2}{2}}$$
$$= \exp\left[\frac{(\varpi-\omega)^2}{2}\left(\frac{17}{720(\varpi-\omega)}\ln F'\left(\frac{\omega+3\varpi}{4}\right) + \frac{1}{180(\varpi-\omega)}\ln F'\left(\frac{\omega+\varpi}{2}\right)\right.\right.$$
$$- \frac{11}{30(\varpi-\omega)^2}\ln F\left(\frac{\omega+3\varpi}{4}\right) - \frac{2}{15(\varpi-\omega)^2}\ln F\left(\frac{\omega+\varpi}{2}\right)$$
$$\left.\left. + \frac{2}{(\varpi-\omega)^3}\int_{\frac{\omega+\varpi}{2}}^{\frac{\omega+3\varpi}{4}} \ln(F(\varkappa))\,d\varkappa\right)\right], \tag{7}$$

and

$$I_4 = \left[\int_{\frac{3}{4}}^{1} \left([F^{**}(\tau\varpi + (1-\tau)\omega)]^{\left(\tau^2 - \frac{83}{45}\tau + \frac{38}{45}\right)}\right)^{d\tau}\right]^{\frac{(\varpi-\omega)^2}{2}}$$
$$= \exp\left[\frac{(\varpi-\omega)^2}{2}\left(\frac{-17}{720(\varpi-\omega)}\ln F'\left(\frac{\omega+3\varpi}{4}\right) - \frac{7}{45(\varpi-\omega)^2}\ln F(\varpi)\right.\right.$$
$$\left.\left. - \frac{31}{90(\varpi-\omega)^2}\ln F\left(\frac{\omega+3\varpi}{4}\right) + \frac{2}{(\varpi-\omega)^3}\int_{\frac{\omega+3\varpi}{4}}^{\varpi} \ln(F(\varkappa))\,d\varkappa\right)\right]. \tag{8}$$

Substituting (5)-(8) into (4), we have

$$\left[\int_0^1 \left([F^{**}(\tau\varpi + (1-\tau)\omega)]^{\Delta(\tau)}\right)^{d\tau}\right]^{\frac{(\varpi-\omega)^2}{2}}$$
$$= \frac{\left(\int_\omega^\varpi (F(\varkappa))^{d\varkappa}\right)^{\frac{1}{\varpi-\omega}}}{\left[(F(\omega)F(\varpi))^7\left(F\left(\frac{\omega+\varpi}{2}\right)\right)^{12}\left(F\left(\frac{3\omega+\varpi}{4}\right)F\left(\frac{\omega+3\varpi}{4}\right)\right)^{32}\right]^{\frac{1}{90}}},$$

which completes the proof. □

Theorem 2.2. *Considering the conditions stated in Lemma 2.1, if F^{**} is multiplicatively convex on $[\omega, \varpi]$, then the following inequality is satisfied:*

$$\left|\frac{\left(\int_\omega^\varpi (F(\varkappa))^{d\varkappa}\right)^{\frac{1}{\varpi-\omega}}}{\left[(F(\omega)F(\varpi))^7\left(F\left(\frac{\omega+\varpi}{2}\right)\right)^{12}\left(F\left(\frac{3\omega+\varpi}{4}\right)F\left(\frac{\omega+3\varpi}{4}\right)\right)^{32}\right]^{\frac{1}{90}}}\right| \le [F^{**}(\varpi)F^{**}(\omega)]^{\frac{509(\varpi-\omega)^2}{273375}}. \tag{9}$$

Proof. From Lemma 2.1, it becomes

$$\left|\frac{\left(\int_\omega^\varpi (F(\varkappa))^{d\varkappa}\right)^{\frac{1}{\varpi-\omega}}}{\left[(F(\omega)F(\varpi))^7\left(F\left(\frac{\omega+\varpi}{2}\right)\right)^{12}\left(F\left(\frac{3\omega+\varpi}{4}\right)F\left(\frac{\omega+3\varpi}{4}\right)\right)^{32}\right]^{\frac{1}{90}}}\right|$$

$$\leq \exp\left[\frac{(\varpi-\omega)^2}{2}\int_0^1 \left|\ln\left(F^{**}(\tau\varpi+(1-\tau)\omega)\right)^{\Delta(\tau)}\right| d\tau\right]$$
$$\leq \exp\left[\frac{(\varpi-\omega)^2}{2}\int_0^1 |\Delta(\tau)|\left|\ln\left(F^{**}(\tau\varpi+(1-\tau)\omega)\right)\right| d\tau\right]$$
$$= \exp\left[\frac{(\varpi-\omega)^2}{2}\int_0^{\frac{1}{4}} \left|\tau^2-\frac{7}{45}\tau\right|\left|\ln\left(F^{**}(\tau\varpi+(1-\tau)\omega)\right)\right| d\tau\right]$$
$$\times \exp\left[\frac{(\varpi-\omega)^2}{2}\int_{\frac{1}{4}}^{\frac{1}{2}} \left|\tau^2-\frac{13}{15}\tau+\frac{8}{45}\right|\left|\ln\left(F^{**}(\tau\varpi+(1-\tau)\omega)\right)\right| d\tau\right]$$
$$\times \exp\left[\frac{(\varpi-\omega)^2}{2}\int_{\frac{1}{2}}^{\frac{3}{4}} \left|\tau^2-\frac{17}{15}\tau+\frac{14}{45}\right|\left|\ln\left(F^{**}(\tau\varpi+(1-\tau)\omega)\right)\right| d\tau\right]$$
$$\times \exp\left[\frac{(\varpi-\omega)^2}{2}\int_{\frac{3}{4}}^{1} \left|\tau^2-\frac{83}{45}\tau+\frac{38}{45}\right|\left|\ln\left(F^{**}(\tau\varpi+(1-\tau)\omega)\right)\right| d\tau\right]. \tag{10}$$

Since F^{**} is multiplicatively convex,

$$\exp\left[\frac{(\varpi-\omega)^2}{2}\int_0^{\frac{1}{4}} \left|\tau^2-\frac{7}{45}\tau\right|\left|\ln\left(F^{**}(\tau\varpi+(1-\tau)\omega)\right)\right| d\tau\right]$$
$$\leq \exp\left[\frac{(\varpi-\omega)^2}{2}\int_0^{\frac{1}{4}} \left|\tau^2-\frac{7}{45}\tau\right|\left(\tau\ln F^{**}(\varpi)+(1-\tau)\ln F^{**}(\omega)\right) d\tau\right]$$
$$= \exp\left[\frac{(\varpi-\omega)^2}{2}\left(\left(\int_0^{\frac{1}{4}} \tau\left|\tau^2-\frac{7}{45}\tau\right| d\tau\right)\ln F^{**}(\varpi)+\left(\int_0^{\frac{1}{4}} (1-\tau)\left|\tau^2-\frac{7}{45}\tau\right| d\tau\right)\ln F^{**}(\omega)\right)\right]. \tag{11}$$

Similarly,

$$\exp\left[\frac{(\varpi-\omega)^2}{2}\int_{\frac{1}{4}}^{\frac{1}{2}} \left|\tau^2-\frac{13}{15}\tau+\frac{8}{45}\right|\left|\ln\left(F^{**}(\tau\varpi+(1-\tau)\omega)\right)\right| d\tau\right]$$
$$\leq \exp\left[\frac{(\varpi-\omega)^2}{2}\int_{\frac{1}{4}}^{\frac{1}{2}} \left|\tau^2-\frac{13}{15}\tau+\frac{8}{45}\right|\left(\tau\ln\left(F^{**}(\varpi)\right)+(1-\tau)\ln\left(F^{**}(\omega)\right)\right) d\tau\right]$$
$$= \exp\left[\frac{(\varpi-\omega)^2}{2}\left(\left(\int_{\frac{1}{4}}^{\frac{1}{2}} \tau\left|\tau^2-\frac{13}{15}\tau+\frac{8}{45}\right| d\tau\right)\ln\left(F^{**}(\varpi)\right)\right.\right.$$
$$\left.\left.+\left(\int_{\frac{1}{4}}^{\frac{1}{2}} (1-\tau)\left|\tau^2-\frac{13}{15}\tau+\frac{8}{45}\right| d\tau\right)\ln\left(F^{**}(\omega)\right)\right)\right], \tag{12}$$

$$\exp\left[\frac{(\varpi-\omega)^2}{2}\int_{\frac{1}{2}}^{\frac{3}{4}} \left|\tau^2-\frac{17}{15}\tau+\frac{14}{45}\right|\left|\ln\left(F^{**}(\tau\varpi+(1-\tau)\omega)\right)\right| d\tau\right]$$
$$\leq \exp\left[\frac{(\varpi-\omega)^2}{2}\int_{\frac{1}{2}}^{\frac{3}{4}} \left|\tau^2-\frac{17}{15}\tau+\frac{14}{45}\right|\left(\tau\ln\left(F^{**}(\varpi)\right)+(1-\tau)\ln\left(F^{**}(\omega)\right)\right) d\tau\right]$$
$$= \exp\left[\frac{(\varpi-\omega)^2}{2}\left(\left(\int_{\frac{1}{2}}^{\frac{3}{4}} \tau\left|\tau^2-\frac{17}{15}\tau+\frac{14}{45}\right| d\tau\right)\ln\left(F^{**}(\varpi)\right)\right.\right.$$
$$\left.\left.+\left(\int_{\frac{1}{2}}^{\frac{3}{4}} (1-\tau)\left|\tau^2-\frac{17}{15}\tau+\frac{14}{45}\right| d\tau\right)\ln\left(F^{**}(\omega)\right)\right)\right], \tag{13}$$

and

$$\exp\left[\frac{(\varpi-\omega)^2}{2}\int_{\frac{3}{4}}^{1}\left|\tau^2-\frac{83}{45}\tau+\frac{38}{45}\right|\left|\ln\left(F^{**}(\tau\varpi+(1-\tau)\omega)\right)\right|d\tau\right]$$
$$\leq\exp\left[\frac{(\varpi-\omega)^2}{2}\int_{\frac{3}{4}}^{1}\left|\tau^2-\frac{83}{45}\tau+\frac{38}{45}\right|\left(\tau\ln\left(F^{**}(\varpi)\right)+(1-\tau)\ln\left(F^{**}(\omega)\right)\right)d\tau\right]$$
$$=\exp\left[\frac{(\varpi-\omega)^2}{2}\left(\left(\int_{\frac{3}{4}}^{1}\tau\left|\tau^2-\frac{83}{45}\tau+\frac{38}{45}\right|d\tau\right)\ln\left(F^{**}(\varpi)\right)\right.\right.$$
$$\left.\left.+\left(\int_{\frac{3}{4}}^{1}(1-\tau)\left|\tau^2-\frac{83}{45}\tau+\frac{38}{45}\right|d\tau\right)\ln\left(F^{**}(\omega)\right)\right)\right]. \tag{14}$$

Using (11)-(14) into (10), it becomes

$$\left|\frac{\left(\int_{\omega}^{\varpi}(F(\varkappa))^{d\varkappa}\right)^{\frac{1}{\varpi-\omega}}}{\left[(F(\omega)F(\varpi))^{7}\left(F\left(\frac{\omega+\varpi}{2}\right)\right)^{12}\left(F\left(\frac{3\omega+\varpi}{4}\right)F\left(\frac{\omega+3\varpi}{4}\right)\right)^{32}\right]^{\frac{1}{90}}}\right|$$
$$\leq\left[F^{**}(\varpi)F^{**}(\omega)\right]^{\frac{509(\varpi-\omega)^2}{273375}}.$$

Here, we used the following equalities:

$$\int_{0}^{\frac{1}{4}}\tau\left|\tau^2-\frac{7}{45}\tau\right|d\tau=\int_{\frac{3}{4}}^{1}(1-\tau)\left|\tau^2-\frac{83}{45}\tau+\frac{38}{45}\right|d\tau=\frac{3325187}{12597120000},$$
$$\int_{\frac{3}{4}}^{1}\tau\left|\tau^2-\frac{83}{45}\tau+\frac{38}{45}\right|d\tau=\int_{0}^{\frac{1}{4}}(1-\tau)\left|\tau^2-\frac{7}{45}\tau\right|d\tau=\frac{16854253}{12597120000},$$
$$\int_{\frac{1}{2}}^{\frac{3}{4}}\tau\left|\tau^2-\frac{17}{15}\tau+\frac{14}{45}\right|d\tau=\int_{\frac{1}{4}}^{\frac{1}{2}}(1-\tau)\left|\tau^2-\frac{13}{15}\tau+\frac{8}{45}\right|d\tau=\frac{1679}{1244160},$$
$$\int_{\frac{1}{4}}^{\frac{1}{2}}\tau\left|\tau^2-\frac{13}{15}\tau+\frac{8}{45}\right|d\tau=\int_{\frac{1}{2}}^{\frac{3}{4}}(1-\tau)\left|\tau^2-\frac{17}{15}\tau+\frac{14}{45}\right|d\tau=\frac{961}{1244160}.$$

Thus, the proof is completed. □

Theorem 2.3. *Considering the conditions stated in Lemma 2.1, if* $(\ln(F^{**}))^q$ *is convex on* $[\omega,\varpi]$ *and* $q>1$, *then the following inequality is satisfied:*

$$\left|\frac{\left(\int_{\omega}^{\varpi}(F(\varkappa))^{d\varkappa}\right)^{\frac{1}{\varpi-\omega}}}{\left[(F(\omega)F(\varpi))^{7}\left(F\left(\frac{\omega+\varpi}{2}\right)\right)^{12}\left(F\left(\frac{3\omega+\varpi}{4}\right)F\left(\frac{\omega+3\varpi}{4}\right)\right)^{32}\right]^{\frac{1}{90}}}\right|$$
$$\leq\left[F^{**}(\varpi)F^{**}(\omega)\right]^{\frac{4(\varpi-\omega)^2}{(32)^{\frac{1}{q}}}\left[\Upsilon_1(p)+\Upsilon_2(p)+\Upsilon_3(p)+\Upsilon_4(p)\right]}, \tag{15}$$

where

$$\Upsilon_1(p) := \left(\frac{\left(\frac{45}{7}\right)^{-2p-1}\left(-\sqrt{\pi}4^{-p}(\sec(\pi p)-2)\Gamma(p+1) - 4\Gamma\left(p+\frac{3}{2}\right)B_{\frac{28}{45}}(-2p-1,p+1)\right)}{4\Gamma\left(p+\frac{3}{2}\right)} \right)^{\frac{1}{p}};$$

$$\Upsilon_2(p) := \left(\frac{1}{3}2^{-2(p+1)}5^{-2p-1}\left(4\,{}_2F_1\left(\frac{1}{2},-p;\frac{3}{2};\frac{4}{9}\right) - 3B_{\frac{36}{121}}\left(-p-\frac{1}{2},p+1\right) - \frac{3\sqrt{\pi}(\sec(\pi p)-1)\Gamma(p+1)}{\Gamma\left(p+\frac{3}{2}\right)}\right)\right)^{\frac{1}{p}};$$

$$\Upsilon_3(p) := \left(\frac{1}{3}2^{-2(p+1)}5^{-2p-1}\left(4\,{}_2F_1\left(\frac{1}{2},-p;\frac{3}{2};\frac{4}{9}\right) - 3B_{\frac{36}{121}}\left(-p-\frac{1}{2},p+1\right) - \frac{3\sqrt{\pi}(\sec(\pi p)-1)\Gamma(p+1)}{\Gamma\left(p+\frac{3}{2}\right)}\right)\right)^{\frac{1}{p}};$$

$$\Upsilon_4(p) := \left(2^{-2(p+1)}7^{2p+1}45^{-2p-1}\left(-B_{\frac{196}{961}}\left(-p-\frac{1}{2},p+1\right) - \frac{\sqrt{\pi}(\sec(\pi p)-2)\Gamma(p+1)}{\Gamma\left(p+\frac{3}{2}\right)}\right)\right)^{\frac{1}{p}},$$

and $\frac{1}{p}+\frac{1}{q}=1$. *The* $\Gamma(.)$, $B_.(.,.)$ *and* ${}_2F_1(.,.;.;.)$ *are well-known special functions.*

Proof. By employing Lemma 2.1 and Hölder's inequality, it follows that

$$\begin{aligned}
&\left|\frac{\left(\int_\omega^\varpi (F(\varkappa))^{d\varkappa}\right)^{\frac{1}{\varpi-\omega}}}{\left[(F(\omega)F(\varpi))^7\left(F\left(\frac{\omega+\varpi}{2}\right)\right)^{12}\left(F\left(\frac{3\omega+\varpi}{4}\right)F\left(\frac{\omega+3\varpi}{4}\right)\right)^{32}\right]^{\frac{1}{90}}}\right| \\
&\le \exp\left[\frac{(\varpi-\omega)^2}{2}\int_0^{\frac{1}{4}}\left|\tau^2-\frac{7}{45}\tau\right|\left|\ln\left(F^{**}(\tau\varpi+(1-\tau)\omega)\right)\right|d\tau\right] \\
&\times\exp\left[\frac{(\varpi-\omega)^2}{2}\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\tau^2-\frac{13}{15}\tau+\frac{8}{45}\right|\left|\ln\left(F^{**}(\tau\varpi+(1-\tau)\omega)\right)\right|d\tau\right] \\
&\times\exp\left[\frac{(\varpi-\omega)^2}{2}\int_{\frac{3}{4}}^{1}\left|\tau^2-\frac{83}{45}\tau+\frac{38}{45}\right|\left|\ln\left(F^{**}(\tau\varpi+(1-\tau)\omega)\right)\right|d\tau\right] \\
&\le \exp\left[\frac{(\varpi-\omega)^2}{2}\left(\int_0^{\frac{1}{4}}\left|\tau^2-\frac{7}{45}\tau\right|^p d\tau\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{4}}\left(\ln\left(F^{**}(\tau\varpi+(1-\tau)\omega)\right)\right)^q d\tau\right)^{\frac{1}{q}}\right] \\
&\times\exp\left[\frac{(\varpi-\omega)^2}{2}\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left|\tau^2-\frac{13}{15}\tau+\frac{8}{45}\right|^p d\tau\right)^{\frac{1}{p}}\left(\int_{\frac{1}{4}}^{\frac{1}{2}}\left(\ln\left(F^{**}(\tau\varpi+(1-\tau)\omega)\right)\right)^q d\tau\right)^{\frac{1}{q}}\right] \\
&\times\exp\left[\frac{(\varpi-\omega)^2}{2}\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left|\tau^2-\frac{17}{15}\tau+\frac{14}{45}\right|^p d\tau\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{\frac{3}{4}}\left(\ln\left(F^{**}(\tau\varpi+(1-\tau)\omega)\right)\right)^q d\tau\right)^{\frac{1}{q}}\right] \\
&\times\exp\left[\frac{(\varpi-\omega)^2}{2}\left(\int_{\frac{3}{4}}^{1}\left|\tau^2-\frac{83}{45}\tau+\frac{38}{45}\right|^p d\tau\right)^{\frac{1}{p}}\left(\int_{\frac{3}{4}}^{1}\left(\ln\left(F^{**}(\tau\varpi+(1-\tau)\omega)\right)\right)^q d\tau\right)^{\frac{1}{q}}\right], \qquad (16)
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$. Using the convexity of $(\ln(F^{**}))^q$, one can obtains

$$\begin{aligned}
\int_0^{\frac{1}{4}} (\ln F^{**}(\tau\varpi+(1-\tau)\omega))^q\,d\tau &\le \int_0^{\frac{1}{4}} \left[\tau(\ln F^{**}(\varpi))^q+(1-\tau)(\ln F^{**}(\omega))^q\right]d\tau \\
&= \frac{(\ln F^{**}(\varpi))^q+7(\ln F^{**}(\omega))^q}{32}, \qquad (17)\\
\int_{\frac{1}{4}}^{\frac{1}{2}} (\ln F^{**}(\tau\varpi+(1-\tau)\omega))^q\,d\tau &\le \int_{\frac{1}{4}}^{\frac{1}{2}} \left[\tau(\ln F^{**}(\varpi))^q+(1-\tau)(\ln F^{**}(\omega))^q\right]d\tau \\
&= \frac{3(\ln F^{**}(\varpi))^q+5(\ln F^{**}(\omega))^q}{32}, \qquad (18)\\
\int_{\frac{1}{2}}^{\frac{3}{4}} (\ln F^{**}(\tau\varpi+(1-\tau)\omega))^q\,d\tau &\le \int_{\frac{1}{2}}^{\frac{3}{4}} \left[\tau(\ln F^{**}(\varpi))^q+(1-\tau)(\ln F^{**}(\omega))^q\right]d\tau \\
&= \frac{5(\ln F^{**}(\varpi))^q+3(\ln F^{**}(\omega))^q}{32}, \qquad (19)\\
\int_{\frac{3}{4}}^{1} (\ln F^{*}(\tau\varpi+(1-\tau)\omega))^q\,d\tau &\le \int_{\frac{3}{4}}^{1} \left[\tau(\ln F^{**}(\varpi))^q+(1-\tau)(\ln F^{**}(\omega))^q\right]d\tau \\
&= \frac{7(\ln F^{**}(\varpi))^q+(\ln F^{**}(\omega))^q}{32}. \qquad (20)
\end{aligned}$$

Here, one used the following facts:

$$\begin{aligned}
&\int_0^{\frac{1}{4}} \left|\tau^2-\frac{7}{45}\tau\right|^p d\tau \\
&= \frac{\left(\frac{45}{7}\right)^{-2p-1}\left(-\sqrt{\pi}4^{-p}(\sec(\pi p)-2)\Gamma(p+1)-4\Gamma\left(p+\frac{3}{2}\right)B_{\frac{28}{45}}(-2p-1,p+1)\right)}{4\Gamma\left(p+\frac{3}{2}\right)}, \qquad (21)\\
&\int_{\frac{1}{4}}^{\frac{1}{2}} \left|\tau^2-\frac{13}{15}\tau+\frac{8}{45}\right|^p d\tau \\
&= \frac{1}{3}2^{-2(p+1)}5^{-2p-1}\left(4\,{}_2F_1\left(\frac{1}{2},-p;\frac{3}{2};\frac{4}{9}\right)-3B_{\frac{36}{121}}\left(-p-\frac{1}{2},p+1\right)-\frac{3\sqrt{\pi}(\sec(\pi p)-1)\Gamma(p+1)}{\Gamma\left(p+\frac{3}{2}\right)}\right), \qquad (22)
\end{aligned}$$

$$\begin{aligned}
&\int_{\frac{1}{2}}^{\frac{3}{4}} \left|\tau^2-\frac{17}{15}\tau+\frac{14}{45}\right|^p d\tau \\
&= \frac{1}{3}2^{-2(p+1)}5^{-2p-1}\left(4\,{}_2F_1\left(\frac{1}{2},-p;\frac{3}{2};\frac{4}{9}\right)-3B_{\frac{36}{121}}\left(-p-\frac{1}{2},p+1\right)-\frac{3\sqrt{\pi}(\sec(\pi p)-1)\Gamma(p+1)}{\Gamma\left(p+\frac{3}{2}\right)}\right), \qquad (23)\\
&\int_{\frac{3}{4}}^{1} \left|\tau^2-\frac{83}{45}\tau+\frac{38}{45}\right|^p d\tau \\
&= 2^{-2(p+1)}7^{2p+1}45^{-2p-1}\left(-B_{\frac{196}{961}}\left(-p-\frac{1}{2},p+1\right)-\frac{\sqrt{\pi}(\sec(\pi p)-2)\Gamma(p+1)}{\Gamma\left(p+\frac{3}{2}\right)}\right), \qquad (24)
\end{aligned}$$

and

$$\int_0^{\frac{1}{4}} \tau d\tau = \int_{\frac{3}{4}}^1 (1-\tau)\, d\tau = \frac{1}{32},$$
$$\int_0^{\frac{1}{4}} (1-\tau)\, d\tau = \int_{\frac{3}{4}}^1 \tau d\tau = \frac{7}{32},$$
$$\int_{\frac{1}{4}}^{\frac{1}{2}} \tau d\tau = \int_{\frac{1}{2}}^{\frac{3}{4}} (1-\tau)\, d\tau = \frac{3}{32},$$
$$\int_{\frac{1}{4}}^{\frac{1}{2}} (1-\tau)\, d\tau = \int_{\frac{1}{2}}^{\frac{3}{4}} \tau d\tau = \frac{5}{32}.$$

If we apply the inequalities (17) to (20) and (21)-(24) in (16), then obtains the inequality (15).

Now, let $\omega_1 = 7\left(\ln F^{**}(\omega)\right)^q$, $\varpi_1 = \left(\ln F^{**}(\varpi)\right)^q$, $\omega_2 = \left(\ln F^{**}(\omega)\right)^q$ and $\varpi_2 = 7\left(\ln F^{**}(\varpi)\right)^q$. Using the facts that

$$\sum_{i=1}^{n} (\omega_i + \varpi_i)^s \leq \sum_{i=1}^{n} \omega_i^s + \sum_{i=1}^{n} \varpi_i^s, \ 0 \leq s < 1,$$

and $1 + 7^{\frac{1}{q}} \leq 8$, one obtains

$$\begin{aligned} &\left(\frac{7\left(\ln F^{**}(\omega)\right)^q + \left(\ln F^{**}(\varpi)\right)^q}{32}\right)^{\frac{1}{q}} \\ \leq\ & \left(\frac{1}{32}\right)^{\frac{1}{q}} \left(1 + 7^{\frac{1}{q}}\right) \left[\ln F^{**}(\omega) + \ln F^{**}(\varpi)\right] \\ \leq\ & 8\left(\frac{1}{32}\right)^{\frac{1}{q}} \ln\left(F^{**}(\omega) F^{**}(\varpi)\right), \\ &\left(\frac{\left(\ln F^{**}(\omega)\right)^q + 7\left(\ln F^{**}(\varpi)\right)^q}{32}\right)^{\frac{1}{q}} \\ \leq\ & 8\left(\frac{1}{32}\right)^{\frac{1}{q}} \ln\left(F^{**}(\omega) F^{**}(\varpi)\right). \end{aligned}$$

Similarly, let $\omega_3 = 3\left(\ln F^{**}(\omega)\right)^q$, $\varpi_3 = 5\left(\ln F^{**}(\varpi)\right)^q$, $\omega_4 = 5\left(\ln F^{**}(\omega)\right)^q$ and $\varpi_4 = 3\left(\ln F^{**}(\varpi)\right)^q$. Using the facts that one obtains

$$\sum_{i=1}^{n} (\omega_i + \varpi_i)^s \leq \sum_{i=1}^{n} \omega_i^s + \sum_{i=1}^{n} \varpi_i^s, \ 0 \leq s < 1,$$

and $3^{\frac{1}{q}} + 5^{\frac{1}{q}} \leq 8$, one obtains

$$\begin{aligned} &\left(\frac{3\left(\ln F^{**}(\omega)\right)^q + 5\left(\ln F^{**}(\varpi)\right)^q}{32}\right)^{\frac{1}{q}} \\ \leq\ & 8\left(\frac{1}{32}\right)^{\frac{1}{q}} \ln\left(F^{**}(\omega) F^{**}(\varpi)\right). \end{aligned}$$

This completes the proof. □

3. Applications

Boole's type inequalities play a vital role in numerical integration and the analysis of special means for real numbers. In quadrature formulas, they improve the accuracy and efficiency of integral approximations by establishing precise error bounds. For special means, these inequalities uncover fundamental relationships and bounds that are essential in various mathematical investigations. These applications underscore the significance of Boole's type inequalities in advancing both theoretical and applied mathematics.

3.1. Application to Quadrature Formula

In this part, the application of the derived Boole's type inequalities to quadrature formulas within the framework of multiplicative calculus is investigated. By incorporating these inequalities into the quadrature formula, we significantly enhance the accuracy and efficiency of numerical integration techniques. The use of Boole's inequalities refines the approximation process, especially for functions with specific characteristics, such as convexity. Additionally, these inequalities optimize the selection of sampling points and weights in the quadrature formula, thereby improving the precision of definite integral estimations. This application not only extends the theoretical understanding of Boole's inequalities but also demonstrates their practical significance in numerical analysis and computational mathematics.

Let Υ be a partition of the points $\omega = \sigma_0 < \sigma_1 < \sigma_2 < .... < \sigma_{n-1} = \varpi$ of the interval $[\omega, \varpi]$, and consider Boole's quadrature formula within the framework of multiplicative calculus:

$$\int_{\omega}^{\varpi} F(\sigma)^{d\sigma} := E_B(F, \Upsilon) \cdot S_B(F, \Upsilon), \tag{25}$$

where $S_B(F, \Upsilon)$ is the multiplicative quadrature sum, which is always positive for $F(\sigma) > 0$, and defined as follows:

$$S_B(F, \Upsilon) \quad := \quad \Pi_{i=0}^{n-1} \left((F(\sigma_i))^7 F\left(\frac{\sigma_i + \sigma_{i+1}}{2}\right)^{12} \left(F\left(\frac{3\sigma_i + \sigma_{i+1}}{4}\right)\right)^{32} \left(F\left(\frac{\sigma_i + 3\sigma_{i+1}}{4}\right)\right)^{32} (F(\sigma_{i+1}))^7 \right)^{\frac{(\sigma_{i+1}-\sigma_i)}{90}}$$

is the multiplicative Boole's version and $E_B(F, \Upsilon)$ denotes the approximation error.

Now, we derive some error estimates for Boole's formula.

Proposition 3.1. *Let $F : [\omega, \varpi] \subset \mathbb{R} \to \mathbb{R}^+$ be *differentiable function on (ω, ϖ). If F^{**} be a multiplicatively convex function on $[\omega, \varpi]$, then in (25), for every division Υ of $[\omega, \varpi]$, we have*

$$|E_B(F, \Upsilon)| \leq \Pi_{i=0}^{n-1} \left[F^{**}(\sigma_i) F^{**}(\sigma_{i+1})\right]^{\frac{509(\sigma_{i+1}-\sigma_i)^3}{273354}}. \tag{26}$$

Proof. Applying Theorem 2.2 on the subinterval $[\sigma_i, \sigma_{i+1}]$ of the division Υ, we get

$$\left| \frac{\int_{\sigma_i}^{\sigma_{i+1}} (F(\sigma))^{d\sigma}}{\left((F(\sigma_i))^7 F\left(\frac{\sigma_i+\sigma_{i+1}}{2}\right)^{12} \left(F\left(\frac{3\sigma_i+\sigma_{i+1}}{4}\right)\right)^{32} \left(F\left(\frac{\sigma_i+3\sigma_{i+1}}{4}\right)\right)^{32} (F(\sigma_{i+1}))^7 \right)^{\frac{(\sigma_{i+1}-\sigma_i)}{90}}} \right|$$
$$\leq \quad \left[F^{**}(\sigma_i) F^{**}(\sigma_{i+1})\right]^{\frac{509(\sigma_{i+1}-\sigma_i)^3}{273375}},$$

where $i = 0, 1, 2, \ldots, n-1$. Hence, we have

$$\begin{aligned}
&\left|\frac{\int_\omega^\varpi (F(\sigma))^{d\sigma}}{S_B(F,\Upsilon)}\right| \\
= &\left|\Pi_{i=0}^{n-1}\frac{\int_{\sigma_i}^{\sigma_{i+1}}(F(\sigma))^{d\sigma}}{\left((F(\sigma_i))^7 F\left(\frac{\sigma_i+\sigma_{i+1}}{2}\right)^{12}\left(F\left(\frac{3\sigma_i+\sigma_{i+1}}{4}\right)\right)^{32}\left(F\left(\frac{\sigma_i+3\sigma_{i+1}}{4}\right)\right)^{32}(F(\sigma_{i+1}))^7\right)^{\frac{(\sigma_{i+1}-\sigma_i)}{90}}}\right| \\
\leq &\ \Pi_{i=0}^{n-1}\left|\frac{\int_{\sigma_i}^{\sigma_{i+1}}(F(\sigma))^{d\sigma}}{\left((F(\sigma_i))^7 F\left(\frac{\sigma_i+\sigma_{i+1}}{2}\right)^{12}\left(F\left(\frac{3\sigma_i+\sigma_{i+1}}{4}\right)\right)^{32}\left(F\left(\frac{\sigma_i+3\sigma_{i+1}}{4}\right)\right)^{32}(F(\sigma_{i+1}))^7\right)^{\frac{(\sigma_{i+1}-\sigma_i)}{90}}}\right| \\
\leq &\ \Pi_{i=0}^{n-1}\left[F^{**}(\sigma_i)F^{**}(\sigma_{i+1})\right]^{\frac{509(\sigma_{i+1}-\sigma_i)^2}{273375}}.
\end{aligned}$$

□

Proposition 3.2. *Let $F : [\omega, \varpi] \subset \mathbb{R} \to \mathbb{R}^+$ be *differentiable function on (ω, ϖ). If $(\ln(F^{**}))^q$ be a convex function on $[\omega, \varpi]$, then in (25), for every division Υ of $[\omega, \varpi]$, we have*

$$\begin{aligned}
&|E_B(F,\Upsilon)| \\
\leq &\ \Pi_{i=0}^{n-1}\left[F^{**}(\sigma_i)F^{**}(\sigma_{i+1})\right]^{\frac{4(\sigma_{i+1}-\sigma_i)^2}{(32)^{\frac{1}{q}}}\left[\Upsilon_1(p)+\Upsilon_2(p)+\Upsilon_3(p)+\Upsilon_4(p)\right]}.
\end{aligned}$$

where $\Upsilon_1(p) - \Upsilon_4(p)$ are defined as in Theorem 2.3.

Proof. With the help of Theorem 2.3, the proof is similar to Proposition 3.1. □

3.2. Application to Special Means

Special means, like arithmetic, geometric, and harmonic means, are crucial in various mathematical applications. Boole's type inequalities provide valuable bounds and insights into the relationships between these means, offering tighter estimates and enhancing their mathematical and statistical understanding.

We shall consider the means for arbitrary real numbers ω, ϖ

(i)- The arithmetic mean: $A(\omega,\varpi) := \frac{\omega+\varpi}{2}$.

(ii)- The harmonic mean: $H(\omega,\varpi) := \frac{2\omega\varpi}{\omega+\varpi}$, $\omega, \varpi > 0$.

(iii)- The logarithmic mean: $L(\omega,\varpi) := \frac{\varpi-\omega}{\ln\varpi-\ln\omega}$, $\omega, \varpi > 0$ and $\omega \neq \varpi$.

(iv)- The p-logarithmic mean: $L_p(\omega,\varpi) := \left(\frac{\varpi^{p+1}-\omega^{p+1}}{(p+1)(\varpi-\omega)}\right)^{\frac{1}{p}}$, $\omega, \varpi > 0$, $\omega \neq \varpi$ and $p \in \mathbb{Z}\backslash\{-1, 0\}$.

Proposition 3.3. *Let $\omega, \varpi \in \mathbb{R}$ with $0 < \omega < \varpi$, then we have*

$$\begin{aligned}
&\left|\frac{\exp\left(\frac{1}{90}\left(2A^{7p}(\omega,\varpi)+A^{12p}(\omega,\varpi)+A^{32p}(\omega,\omega,\omega,\varpi)+A^{32p}(\omega,\varpi,\varpi,\varpi)\right)\right)}{e^{L_p^p(\omega,\varpi)}}\right| \\
\leq &\ e^{p(p-1)\frac{509(\varpi-\omega)}{273375}\left(\omega^{p-2}+\varpi^{p-2}\right)}.
\end{aligned}$$

Proof. The assertion follows from Theorem 2.2, applied to the function $F(u) = e^{u^p}$ with $p \geq 2$ whose $\left(\int_\omega^\varpi F(u)^{du}\right)^{\frac{1}{\varpi-\omega}} = \left(e^{\int_\omega^\varpi \ln e^{u^p} du}\right)^{\frac{1}{\varpi-\omega}} = e^{L_p^p(\omega,\varpi)}$ and $F^{**}(u) = e^{p(p-1)u^{p-2}}$. □

Proposition 3.4. *Let $\omega, \varpi \in \mathbb{R}$ with $0 < \omega < \varpi$, then we have*

$$\left|\frac{\exp\left(\frac{1}{90}\left(14H^{-1}(\omega,\varpi)+4A^{-1}(\omega,\varpi)+32A^{-1}(\omega,\omega,\omega,\varpi)+32A^{-1}(\omega,\varpi,\varpi,\varpi)\right)\right)}{e^{L^{-1}(\omega,\varpi)}}\right|$$

$$\leq \left[\frac{1}{e^{4H^{-1}(\omega^3,\varpi^3)}}\right]^{\frac{4(\sigma_{i+1}-\sigma_i)^2}{(32)^{\frac{1}{q}}}\left[\Upsilon_1(p)+\Upsilon_2(p)+\Upsilon_3(p)+\Upsilon_4(p)\right]},$$

where $\Upsilon_1(p) - \Upsilon_4(p)$ are defined as in Theorem 2.3.

Proof. This inequality can be obtained from Theorem 2.3, for the function $F(u) = e^{\frac{1}{u}}$. □

4. Numerical Examples and Graphical Analysis

This section provides some numerical examples with graphical representations of newly established Boole's formula type inequalities in multiplicative calculus. This graphical representation demonstrates that the main results are practically valid for multiplicatively convex functions.

Example 4.1. *Let $F : [\omega, \varpi] \to \mathbb{R}^+$ be a function defined by $F(\varkappa) = e^{\frac{1}{\varkappa}}$, then by applying inequality (9) to the function $F(\varkappa) = e^{\frac{1}{\varkappa}}$. Then, solved by Mathematica, the left-hand side of (9) becomes:*

$$\left|\frac{\left(\int_\omega^\varpi (F(\varkappa))^{d\varkappa}\right)^{\frac{1}{\varpi-\omega}}}{\left[(F(\omega)F(\varpi))^7\left(F\left(\frac{\omega+\varpi}{2}\right)\right)^{12}\left(F\left(\frac{3\omega+\varpi}{4}\right)F\left(\frac{\omega+3\varpi}{4}\right)\right)^{32}\right]^{\frac{1}{90}}}\right|$$

$$= \left|\left(\frac{\varpi\left(e^{1/\omega}\right)^{-\omega}\left(e^{1/\varpi}\right)^{\varpi}}{\omega}\right)^{\frac{1}{\varpi-\omega}}\right| \exp\left(-\frac{1}{90}Re\left(8\left(\frac{16}{3\omega+b}+\frac{16}{\omega+3\varpi}+\frac{3}{\omega+\varpi}\right)+\frac{7}{\omega}+\frac{7}{\varpi}\right)\right), \tag{27}$$

and the right-hand side of (9) becomes:

$$\left[F^{**}(\varpi)F^{**}(\omega)\right]^{\frac{509(\varpi-\omega)^2}{273375}}$$

$$= \left(e^{e^{\frac{2}{\omega^3}}+e^{\frac{2}{\varpi^3}}}\right)^{\frac{509(\omega-\varpi)^2}{273375}}. \tag{28}$$

From Figure 1, it is confirmed that inequality (9) of Theorem 2.2 is valid.

Example 4.2. *Let $F : [\omega, \varpi] \to \mathbb{R}^+$ be a function defined by $F(\varkappa) = e^{\varkappa^2}$, then by applying inequality (9) to the function $F(\varkappa) = e^{\varkappa^2}$. Then, solved by Mathematica, the left-hand side of (9) becomes:*

$$\left|\frac{\left(\int_\omega^\varpi (F(\varkappa))^{d\varkappa}\right)^{\frac{1}{\varpi-\omega}}}{\left[(F(\omega)F(\varpi))^7\left(F\left(\frac{\omega+\varpi}{2}\right)\right)^{12}\left(F\left(\frac{3\omega+\varpi}{4}\right)F\left(\frac{\omega+3\varpi}{4}\right)\right)^{32}\right]^{\frac{1}{90}}}\right|$$

$$= \left|\left(e^{\frac{2}{3}\left(\omega^3-\varpi^3\right)}\left(e^{\omega^2}\right)^{-\omega}\left(e^{\varpi^2}\right)^{\varpi}\right)^{\frac{1}{\varpi-\omega}}\right| e^{-\frac{1}{3}Re\left(\omega^2+\omega\varpi+\varpi^2\right)}, \tag{29}$$

and the right-hand side of (9) becomes:

$$\left[F^{**}(\varpi)F^{**}(\omega)\right]^{\frac{509(\varpi-\omega)^2}{273375}}$$

$$= e^{\frac{2036(\varpi-\omega)^2}{273375}}. \tag{30}$$

From Figure 2, it is confirmed that inequality (9) of Theorem 2.2 is valid.

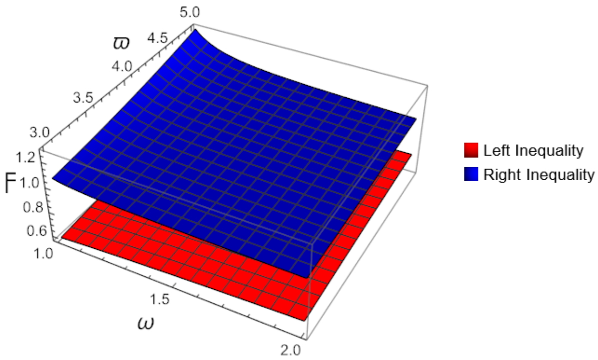

Figure 1: 3D plot for $\omega \in [3,5]$ and $\varpi \in [11,12]$ in Example 4.1, computed and plotted with Mathematica.

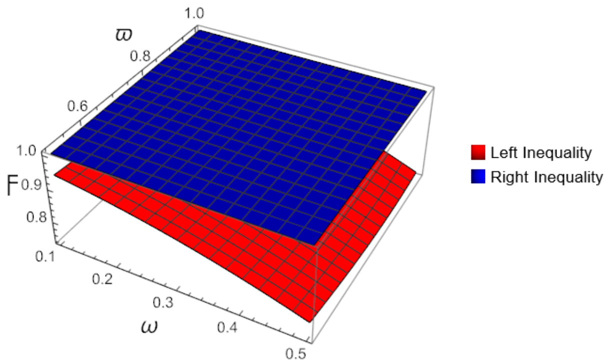

Figure 2: 3D plot for $\omega \in [0.1,0.5]$ and $\varpi \in [0.5,1]$ in Example 4.2, computed and plotted with Mathematica.

Example 4.3. *Let* $F : [\omega,\varpi] \to \mathbb{R}^{+}$ *be a function defined by* $F(\varkappa) = e^{\frac{1}{\varkappa}}$*, then by applying inequality (15) to the function* $F(\varkappa) = e^{\frac{1}{\varkappa}}$ *and* $p = q = 2$*. Then, solved by Mathematica, the right-hand side of (15) becomes:*

$$
\begin{aligned}
&\left[F^{**}(\varpi)F^{**}(\omega)\right]^{\frac{4(\varpi-\omega)^2}{(32)^{\frac{1}{q}}}\left[\Upsilon_1(p)+\Upsilon_2(p)+\Upsilon_3(p)+\Upsilon_4(p)\right]} \\
&= \left(e^{e^{\frac{2}{\omega^3}}+e^{\frac{2}{\varpi^3}}}\right)^{\frac{\left(\sqrt{327}+21\right)(\omega-\varpi)^2}{2160\sqrt{2}}}. \qquad (31)
\end{aligned}
$$

From Figure 3, it is confirmed that inequality (15) of Theorem 2.3 is valid.

Example 4.4. *Let* $F : [\omega,\varpi] \to \mathbb{R}^{+}$ *be a function defined by* $F(\varkappa) = e^{\varkappa^2}$*, then by applying inequality (15) to the function* $F(\varkappa) = e^{\varkappa^2}$*. Then, solved by Mathematica, the right-hand side of (15) becomes:*

$$
\begin{aligned}
&\left[F^{**}(\varpi)F^{**}(\omega)\right]^{\frac{4(\varpi-\omega)^2}{(32)^{\frac{1}{q}}}\left[\Upsilon_1(p)+\Upsilon_2(p)+\Upsilon_3(p)+\Upsilon_4(p)\right]} \\
&= e^{\frac{\left(\sqrt{327}+21\right)e^2(\omega-\varpi)^2}{1080\sqrt{2}}}. \qquad (32)
\end{aligned}
$$

From Figure 4, it is confirmed that inequality (15) of Theorem 2.3 is valid.

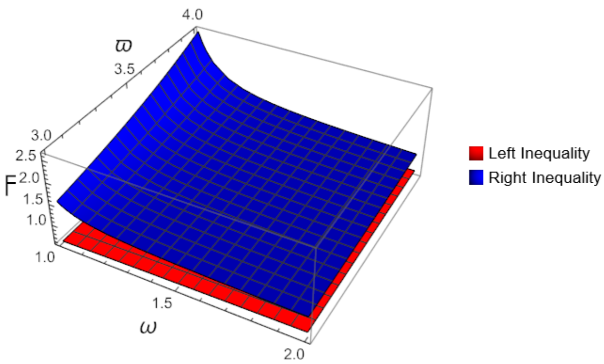

Figure 3: 3D plot for $\omega \in [2,5]$ and $\varpi \in [11,12]$ in Example 4.3, computed and plotted with Mathematica.

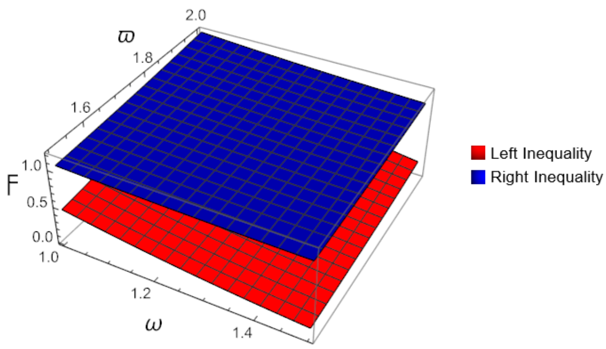

Figure 4: 3D plot for $\omega \in [1,1.5]$ and $\varpi \in [1.5,2]$ in Example 4.4, computed and plotted with Mathematica.

Table 1: Comparison of Absolute Errors in Multiplicative and Classical Calculus.

Function	Absolute Error in Multiplicative Calculus	Absolute Error in Classical Calculus
$e^{\varkappa^2}$	0.0131237	4980.83
$\varkappa^6$	0.0204848	2224.23

5. Conclusion

This article has established significant advancements in the field of multiplicative calculus by presenting refined inequalities for Boole's formula applicable to twice differentiable functions. The rigorous proofs provided demonstrate that these inequalities yield sharp error bounds for integral approximations, thereby enhancing the effectiveness of Boole's formula in numerical methods. Applications to quadrature formulas and special means for real numbers show the practical utility of derived findings. Additionally, illustrative examples and graphical representations underscore the efficacy of the proposed enhancements. As shown in Table 1, multiplicative calculus attains better absolute error bounds compared to classical calculus. This work not only contributes to the theoretical framework of multiplicative calculus but also opens new avenues for further research, particularly in optimizing techniques for integral approximation and expanding the applicability of multiplicative calculus in diverse fields.

Key Points

- Multiplicative calculus provides sharper error bounds compared to classical calculus, especially for higher-degree polynomials.
- The error in classical calculus grows exponentially for functions like $\varkappa^6$, while multiplicative calculus maintains controlled error growth.
- Multiplicative calculus offers better convergence, stability, and reliability in numerical approximation, integral estimation, and error control.
- The theoretical foundation of multiplicative calculus supports its enhanced performance, as it captures proportional changes rather than absolute changes.

This work contributes to the ongoing development of multiplicative calculus and suggests future research directions involving q-calculus, symmetrized q-calculus, and multiplicative fractional calculus in higher-dimensional contexts.

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