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On rank-one perturbations in semi-Hilbertian spaces

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Abstract. Let A be a positive bounded linear operator on a Hilbert space $(\mathcal{H}, \langle\cdot\ ,\cdot\rangle)$ and $\langle u, v\rangle_A := \langle Au, v\rangle$, $u, v \in \mathcal{H}$ the induced semi-inner product. In this paper, we study rank one perturbations of bounded linear operators in the semi-Hilbertian space $(\mathcal{H}, \langle\cdot\ ,\cdot\rangle_A)$, $R = T + u \otimes (Av)$, for $u, v \in \mathcal{H}\backslash\mathcal{N}(A)$. In addition to proving some properties of these operators related to (A, m)-isometries and (A, n)-symmetries, we characterize some conditions for R to be A-hyponormal or A-normal.

1. Introduction and preliminaries

In this section, we introduce some basic notations, definitions and known results that are needed to prove our main results.

Throughout this paper we write $\mathbb{N}$ and $\mathbb{R}$ for the sets of positive integers and real numbers, respectively. We shall denote the set of all complex numbers and the complex conjugate of a complex number λ by $\mathbb{C}$ and $\overline{\lambda}$, respectively. Let $\mathcal{H}$ be an infinite dimensional separable complex Hilbert space with inner product $\langle\cdot\ ,\cdot\rangle$, and let A be a nonzero positive operator on $\mathcal{H}$. By $\mathcal{B}(\mathcal{H})$ we denote the Banach algebra of all bounded linear operators on $\mathcal{H}$. For every $T \in \mathcal{B}(\mathcal{H})$ we denote by $\mathcal{R}(T)$ and $\mathcal{N}(T)$ the range and null space of T, respectively. We write $\sigma(T)$, $\sigma_p(T)$ and $\sigma_{ap}(T)$, respectively, for the spectrum, the point spectrum and the approximate point spectrum of the operator T. Recall that a closed subspace $\mathcal{M}$ of $\mathcal{H}$ is said to be an invariant subspace under T if $T\mathcal{M} \subseteq \mathcal{M}$, that is $Tx \in \mathcal{M}$ for $x \in \mathcal{M}$. Moreover, $\mathcal{M}$ is called a reducing subspace for T if both $\mathcal{M}$ and $\mathcal{M}^\perp$ are invariant under T. The closure of $\mathcal{R}(T)$ will be denoted by $\overline{\mathcal{R}(T)}$. The cone of positive (semi-definite) operators is given by

$$\mathcal{B}(\mathcal{H})^+ \quad := \quad \Big\{A \in \mathcal{B}(\mathcal{H}) : \langle Au, u\rangle \geq 0,\ \forall\, u \in \mathcal{H}\Big\}.$$

Definition 1.1. ([8]) *Let $A \in \mathcal{B}(\mathcal{H})^+$ and $T \in \mathcal{B}(\mathcal{H})$. We say that T is A-positive if $AT \in \mathcal{B}(\mathcal{H})^+$, which is equivalent to the condition*

$$\langle Tu, u\rangle_A \geq 0, \quad \forall\, u \in \mathcal{H}.$$

We note $T \geq_A 0$.

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The following theorem is due to Douglas (see [17] or [21] for its proof).

Theorem 1.2. (***Douglas***) *Let $A, B \in \mathcal{B}(\mathcal{H})$. The following conditions are equivalent.*

1. $R(B) \subseteq R(A)$.
2. *There exists a positive number λ such that $BB^* \leq \lambda AA^*$.*
3. *There exists $C \in \mathcal{B}(\mathcal{H})$ such that $AC = B$.*

Any $A \in \mathcal{B}(\mathcal{H})^+$ defines a positive semi-definite sesquilinear form :

$$\langle \cdot\,, \cdot \rangle_A : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{C}, \quad \langle u, v\rangle_A := \langle Au, v\rangle.$$

By $\|\cdot\|_A$ we denote the semi-norm induced by $\langle \cdot\,, \cdot \rangle_A$, i.e. $\|u\|_A = \langle u, u\rangle_A^{\frac{1}{2}}$. Observe that $\|u\|_A = 0$ if and only if $u \in \mathcal{N}(A)$. Then $\|\cdot\|_A$ is a norm if and only if A is injective. For a positive operator A, we say that u, $v \in \mathcal{H}$ are A-orthogonal, and we note $u \perp_A v$, if $\langle u, v\rangle_A = 0$. A vector $u \in \mathcal{H}$ is said to be A-unitary if $\|u\|_A = 1$. Given a subspace M of $\mathcal{H}$, its A-orthogonal subspace is the subspace

$$M^{\perp_A} := \{u \in \mathcal{H} : \langle u, v\rangle_A = 0,\ \forall\, v \in M\}.$$

Definition 1.3. *([38]) Let $T \in \mathcal{B}(\mathcal{H})$, an operator $W \in \mathcal{B}(\mathcal{H})$ is called an A-adjoint of T if*

$$\langle Tu, v\rangle_A = \langle u, Wv\rangle_A, \quad \forall\, u,\, v \in \mathcal{H},$$

i.e.,

$$AW = T^*A.$$

If T is an A-adjoint of itself, then T is called A-selfadjoint, that is $AT = T^*A$.

By Douglas theorem, T admits an A-adjoint if and only if $R(T^*A) \subseteq R(A)$. From now on, we denote by $\mathcal{B}_A(\mathcal{H})$ the set of all $T \in \mathcal{B}(\mathcal{H})$ which admit an A-adjoint, namely

$$\mathcal{B}_A(\mathcal{H}) \quad := \quad \Big\{T \in \mathcal{B}(\mathcal{H}) : \mathcal{R}(T^*A) \subseteq \mathcal{R}(A)\Big\}.$$

If $T \in \mathcal{B}_A(\mathcal{H})$, then there exists a distinguished A-adjoint operator of T, namely, the reduced solution of the equation $AX = T^*A$, i.e., $A^\dagger T^*A$ in which $A^\dagger$ is the Moore-Penrose inverse of A. This operator is denoted by $T^{\sharp_A}$. Therefore, $T^{\sharp_A} = A^\dagger T^*A$ and

$$AT^{\sharp_A} = T^*A, \ \ \mathcal{R}(T^{\sharp_A}) \subset \overline{\mathcal{R}(A)} \ \ \textit{and} \ \ \mathcal{N}(T^{\sharp_A}) = \mathcal{N}(T^*A).$$

Note that if $A = I$, then $T^{\sharp_A} = T^*$. It is well known ([25, Proposition 1.1]) that if $T \in \mathcal{B}_A(\mathcal{H})$ then $T^{\sharp_A} \in \mathcal{B}_A(\mathcal{H})$ and $(T^{\sharp_A})^{\sharp_A} = PTP$, where P is the orthogonal projection onto $\overline{R(A)}$. Recall that, for $T \in \mathcal{B}_A(\mathcal{H})$, the operators $TT^{\sharp_A}$ and $T^{\sharp_A}T$ both are A-positive operators, that is $\langle TT^{\sharp_A}u, u\rangle_A \geq 0$ and $\langle T^{\sharp_A}Tu, u\rangle_A \geq 0$ for all $u \in \mathcal{H}$. For more details see [5] and [6].
For $X, Y \in \mathcal{B}(\mathcal{H})$, we denote by $[X, Y] := XY - YX$ the commutator of X and Y. For a bounded operator T on $\mathcal{H}$, we denote its commutant by $\{T\}'$, that is,

$$\{T\}' := \{Q \in \mathcal{B}(\mathcal{H}) : [T, Q] = 0\}.$$

Definition 1.4. *An operator $T \in \mathcal{B}_A(\mathcal{H})$ is said to be:*

- *A-normal if $[T^{\sharp_A}, T] = 0$ ([43]).*
- *A-hyponormal if $[T^{\sharp_A}, T]$ is A-positive, i.e., $[T^{\sharp_A}, T] \geq_A 0$ ([37]).*

It is known that the class of A-hyponormal operators is a larger class containing A-normal operators.

Remark 1.5. *([25]) Let $T \in \mathcal{B}_A(\mathcal{H})$. The following statements hold true:*

1. *$\mathcal{N}(A)$ is an invariant subspace for T.*
2. *$\mathcal{N}(A)$ is a reducing subspace for T if and only if $\mathcal{N}(A)$ is invariant for T^*.*
3. *If $\mathcal{N}(A)$ is a reducing subspace for T, then we have*

$$TP = PT \quad and \quad AP = PA = A.$$

The theory of operators on semi-Hilbertian spaces is rich and promising as a branch of functional analysis. The interest on semi-Hilbertian analysis stems from the fact that this theory provides a more general point of view regarding the definition of operators representing physical observables in the context of quantum mechanics [12]. For some references, the reader can see [4–6, 10, 11, 19, 20, 34, 38, 40–43, 45]. In view of this, it is natural to consider the rank one perturbations of bounded linear operators in such a spaces.

If u and v are nonzero vectors in $\mathcal{H}$, we write $u \otimes v$ for the rank one operator defined by

$$(u \otimes v)x := \langle x\,,\,v\rangle u,\ x \in \mathcal{H},$$

where $\langle \cdot\,,\,\cdot\rangle$ denotes the inner product of the Hilbert space $\mathcal{H}$. Note that all rank-one operators have such representation. For $T \in \mathcal{B}(\mathcal{H})$, we say that an operator $R \in \mathcal{B}(\mathcal{H})$ is a rank one perturbation of T if there exist nonzero vectors u and v in $\mathcal{H}$ such that $R = T + u \otimes v$. The rank-one perturbations of bounded operators can be applied to several related areas in mathematical physics ([15, 27, 39]). For more details on this topic, we refer the readers for example to [18, 22–24, 26, 32].

Following by the rank one operators in $\mathcal{B}(\mathcal{H})$, for $u,\ v \in \mathcal{H}$ and for $A \in \mathcal{B}(\mathcal{H})^+$, A. Zamani ([46]) introduced the "A-rank one operator" $(u \otimes_A v)$ as follows:

Definition 1.6. *For u nonzero vector in $\mathcal{H}$ and $v \in \mathcal{H}\backslash\mathcal{N}(A)$, let $u \otimes_A v$ be the "A-rank one operator", namely the map*

$$\begin{aligned}(u \otimes_A v) : \mathcal{H} &\longrightarrow \mathcal{H}\\ x &\longmapsto (u \otimes_A v)(x) := (u \otimes (Av))(x) = \langle x, v\rangle_A u.\end{aligned}$$

Remark 1.7. *Referring to [44, Proposition 1.2.1], it is easy to verify that for $u, v, w, x \in \mathcal{H}$ and $T \in \mathcal{B}(\mathcal{H})$, we have:*

1. $\mathcal{N}(u \otimes_A v) = (\mathbb{C}Av)^{\perp}$.
2. $\mathcal{R}(u \otimes_A v) = \mathbb{C}u$.
3. $(u \otimes_A v)^* = (Av) \otimes_I u$.
4. $(u \otimes_A v)(w \otimes_A x) = \langle w, v\rangle_A (u \otimes_A x)$.
5. $u \otimes_A v = w \otimes_A z$ *if and only if there exist* $\lambda,\ \mu \in \mathbb{C}^*$ *such that* $u = \lambda w$, $Av = \mu Az$ *and* $\lambda\overline{\mu} = 1$.
6. $((Av) \otimes_I u)A = (Av) \otimes_I (Au)$.
7. $A(u \otimes_A v) = (Au) \otimes_I (Av)$.
8. $T(u \otimes_A v) = (Tu) \otimes_A v$ *and* $(u \otimes_A v)T = u \otimes (T^*Av)$.
9. $T(u \otimes_A v) = (u \otimes_A v)T$ *if and only if there exist* $\alpha,\ \beta \in \mathbb{C}^*$ *such that* $Tu = \alpha u$, $Av = \beta T^*Av$ *and* $\alpha\overline{\beta} = 1$.
10. $(u \otimes_A v)^p = \langle u, v\rangle_A^{p-1}(u \otimes_A v)$, *for* $p \geq 1$.

Recall that the space of trace class operators, denoted by τc, is predual to $\mathcal{B}(\mathcal{H})$ with the dual action $\langle T, f\rangle := tr(Tf)$ for $T \in \mathcal{B}(\mathcal{H})$ and $f \in \tau c$ ([9, 31]). For a A-rank one operator $(u \otimes_A v)$, we have

$$tr(T(u \otimes_A v)) = \langle T, u \otimes (Av)\rangle = \langle ATu, v\rangle = \langle AT, u \otimes v\rangle = tr(AT(u \otimes v)).$$

Let $\mathcal{D} \subset \mathcal{B}(\mathcal{H})$ be a closed subspace. Define by $\mathcal{D}_{\perp_A}$ the A-preannihilator of $\mathcal{D}$, i.e.,

$$\mathcal{D}_{\perp_A} := \Big\{t \in \tau c : tr(ADt) = 0,\ for\ all\ D \in \mathcal{D}\Big\}.$$

The subspace $\mathcal{D}$ is said to be A-transitive if the A-preannihilator of $\mathcal{D}$ does not contain any rank-one operators, equivalently, if the preannihilator of $\mathcal{D}$ does not contain any A-rank-one operators.

For R, $S \in \mathcal{B}(\mathcal{H})$, consider the map $\Theta_{\bullet}(R,S) : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$, $\Theta_X(R,S) := RXS - X$. An induction argument shows that

$$\Theta_X^{(m)}(R,S) = \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}R^kXS^k, \quad m \geq 1. \tag{1}$$

For $m \geq 1$, an operator $T \in \mathcal{B}(\mathcal{H})$ is said to be an (A,m)-isometry if $\Theta_A^{(m)}(T) := \Theta_A^{(m)}(T^*,T) = 0$, that is T is left (A,m)-invertible with T^* as a left inverse ([42]). Let $\mathcal{S}_{\bullet}(R,S) : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ be the generalized derivation operator defined by $\mathcal{S}_X(R,S) := RX - XS$. For $n \geq 1$, we have

$$\mathcal{S}_X^{(n)}(R,S) = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}R^kXS^{n-k}. \tag{2}$$

For $n \geq 1$, an operator $T \in \mathcal{B}(\mathcal{H})$ is said to be (A,n)-symmetric $\Big($resp. skew-(A,n)-symmetric$\Big)$ if $\mathcal{S}_A^{(n)}(T) := \mathcal{S}_A^{(n)}(T^*,T) = 0$ $\Big($resp. if (iT) is (A,n)-symmetric, that is T satisfies

$$\xi_A^{(n)}(T) = \sum_{k=0}^{n}\binom{n}{k}T^{*k}AT^{n-k} = 0\Big). \tag{3}$$

Some notable similarities between (A,m)-isometric and A-isometric operators, as well as between (A,n)-symmetries and A-symmetries, are their properties $\Big($see [29, Theorem 5.1] and [38, Proposition 4.1]$\Big)$ given by : if $0 \notin \sigma_{ap}(A)$; then

$$\Big(T \text{ is } (A,m)\text{-isometric, } m \geq 1\Big) \implies \Big(\sigma(T) = \overline{\mathbb{D}} \text{ or } \sigma(T) \subseteq \partial\mathbb{D}\Big) \tag{4}$$

and

$$\Big(T \text{ is } (A,n)\text{-symmetric, } n \geq 1\Big) \implies \Big(\sigma(T) \subseteq \mathbb{R}\Big) \tag{5}$$

where $\overline{\mathbb{D}} := \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$ and $\partial\mathbb{D} := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. For proofs and more facts about (A,m)-isometries, (A,n)-symmetries and skew-(A,n)-symmetries, we refer the reader to [14, 29, 38, 40–42] and the references therein. Some other related topics can be found in [1–3, 33].

The contents of the paper are the following. In section 2, we focus on (A,m)-isometric rank-one perturbations. Especially, in Theorem 2.1, we extend the Nakamura characterization established in [35, 36] to the semi-Hilbertian case. The study of rank one perturbations of the form $R = I_{\mathcal{H}} + u \otimes (Av)$, for u, $v \in \mathcal{H}\backslash\mathcal{N}(A)$ is examined in Theorems 2.3. Section 3 is devoted to giving some results characterizing (A,n)-symmetric rank one perturbations. In section 4, we restrict our selves to proving the two main results, Theorem 4.2 and 4.8, that characterize some conditions for $R = T + u \otimes_A v$ to be A-hyponormal or A-normal.

2. (A,m)-isometric rank-one perturbations

Since $\mathcal{N}(u \otimes v) = (\mathbb{C}v)^{\perp}$ (for u, v non zero) and since every m-isometry is necessarily injective, generally a rank-one operator can not be m-isometric. For $A \geq 0$, if T is (A,m)-isometric then $\mathcal{N}(T) \subseteq \mathcal{N}(A)$. Let $(e_n)_{n\geq 1}$ be an orthonormal basis of $\mathcal{H}$, $T := e_k \otimes_A e_k$ for a fixed $k \geq 1$ and A the positive operator defined by

$$Ae_j := \frac{1}{j}\,e_j, \quad j \geq 1.$$

It follows that

$$\mathcal{N}(T) = (\mathbb{C}Ae_k)^{\perp} = \left\{ \sum_{j\geq 1,\, j\neq k} \alpha_j e_j,\ \alpha_j \in \mathbb{C} \right\}.$$

Moreover, for $\alpha_j \in \mathbb{C}^*$, we have

$$A\left(\sum_{j\geq 1,\, j\neq k} \alpha_j e_j \right) = \sum_{j\geq 1,\, j\neq k} \frac{1}{j} \alpha_j e_j \neq 0.$$

So, $x = \sum_{j\geq 1,\, j\neq k} \alpha_j e_j \in \mathcal{N}(T)$ and $x \notin \mathcal{N}(A)$, that is $\mathcal{N}(T) \nsubseteq \mathcal{N}(A)$. Hence, the A-rank-one operator T can not be (A, m)-isometric.

In this section, we aim to give a characterization of (A, m)-isometric rank one perturbations.

Given an isometry $T \in \mathcal{B}(\mathcal{H})$ and vectors $u, v \in \mathcal{H}$, the perturbation $R = T + u \otimes v$ is an isometry if and only if there exist a vector $h \in \mathcal{H}$ ($\|h\| = 1$) and $\alpha \in \mathbb{C}$ ($|\alpha| = 1$) such that $u = (\alpha - 1)h$ and $v = T^*h$. Equivalently, we have the following result which is due to Nakamura [35, 36] : a rank-one perturbation F of the isometry T is an isometry if and only if there exists $h \in \mathcal{H}$ ($\|h\| = 1$) and $\alpha \in \mathbb{C}$ ($|\alpha| = 1$) such that

$$R = T + (\alpha - 1)h \otimes T^*h. \tag{6}$$

In the following theorem we aim to extend the Nakamura chracterization given by (6) to the semi-Hilbertian case. In other words, we give some conditions under which the operator $T + F$ is A-isometric, when F is a A-rank-one operator.

Theorem 2.1. *Let $T \in \mathcal{B}(\mathcal{H})$ be an A-isometry and F be a A-rank one operator. Then, the following properties are equivalent:*

1. *$R = T + F$ is A-isometric.*
2. *There exists a vector $h \in \mathcal{H}$ ($\|h\|_A = 1$), $\alpha \in \mathbb{C}$ with $|\alpha| = 1$, and*

$$F = (\alpha - 1)h \otimes T^*Ah.$$

Proof. ($\Rightarrow$) Assume that $R = T + u \otimes_A v$ ($u, v \in \mathcal{H}\backslash\mathcal{N}(A)$) is A-isometric, then we have

$$R^*AR = A \Longleftrightarrow 0 = (T + u \otimes_A v)^*A(T + u \otimes_A v) - A.$$

Let $a = \frac{u}{\|u\|_A}$, $b = \frac{v}{\|v\|_A}$ and $\gamma = \|u\|_A \|v\|_A$. It follows that $\|a\|_A = \|b\|_A = 1$ and

$$\begin{aligned} F = u \otimes_A v &= (\|u\|_A a) \otimes_A (\|v\|_A b) \\ &= \|u\|_A \|v\|_A (a \otimes_A b) \\ &= \gamma(a \otimes_A b). \end{aligned}$$

Since $F^* = \overline{\gamma}(Ab) \otimes a$, we have

$$\begin{aligned} 0 &= (T + F)^*A(T + F) - A \\ &= T^*AT + T^*AF + F^*AT + F^*AF - A \\ &= T^*AF + F^*AT + F^*AF \\ &= \gamma(T^*Aa) \otimes (Ab) + \overline{\gamma}(Ab) \otimes (T^*Aa) + |\gamma|^2\Big((Ab) \otimes a\Big)\Big((Aa) \otimes (Ab)\Big) \\ &= \gamma(T^*Aa) \otimes (Ab) + \overline{\gamma}(Ab) \otimes (T^*Aa) + |\gamma|^2\|a\|^2(Ab) \otimes (Ab). \end{aligned}$$

By adding the term $\Big((T^*Aa)\otimes(T^*Aa)\Big)$ to the tow sides of the above equality, we obtain

$$\begin{aligned} (T^*Aa)\otimes(T^*Aa) &= \gamma(T^*Aa)\otimes(Ab)+\overline{\gamma}(Ab)\otimes(T^*Aa)+|\gamma|^2\|a\|^2(Ab)\otimes(Ab)+(T^*Aa)\otimes(T^*Aa)\\ \Longleftrightarrow \underbrace{(T^*Aa)}_{(=X)}\otimes\underbrace{(T^*Aa)}_{(=X)} &= \underbrace{\Big(T^*Aa+\overline{\gamma}(Ab)\Big)}_{(=Y)}\otimes\underbrace{\Big(T^*Aa+\overline{\gamma}(Ab)\Big)}_{(=Y)}. \end{aligned}$$

Equivalently, we have

$$\Big(X\otimes X = Y\otimes Y\Big) \iff \Big(\exists\ \alpha\in\mathbb{C}\Big/|\alpha|=1 \text{ and } X=\alpha Y\Big).$$

It follows from that

$$\exists\ \alpha\in\mathbb{C}\Big/|\alpha|=1,\quad T^*Aa+\overline{\gamma}(Ab)=\alpha T^*Aa$$

and, so, $Ab=\frac{\alpha-1}{\overline{\gamma}}T^*Aa$. Hence, we obtain

$$\begin{aligned} F=\gamma(a\otimes_A b) &= \gamma(a\otimes(Ab))\\ &= \gamma a\otimes(\frac{\alpha-1}{\overline{\gamma}}T^*Aa)\\ &= (\overline{\alpha}-1)a\otimes T^*Aa. \end{aligned}$$

($\Leftarrow$) Assume, now, that T is A-isometric and there exists an A-unitary vector $h\in\mathcal{H}$, $\alpha\in\mathbb{C}$ with $|\alpha|=1$, and $F=(\alpha-1)h\otimes T^*Ah$. Then, we have

$$\begin{aligned} &(T+F)^*A(T+F)-A = T^*AF+F^*AT+F^*AF\\ =\ & T^*A\Big((\alpha-1)h\otimes T^*Ah\Big)+\Big(T^*Ah\otimes(\alpha-1)h\Big)AT+\Big(T^*Ah\otimes(\alpha-1)h\Big)A\Big((\alpha-1)h\otimes T^*Ah\Big)\\ =\ & (|\alpha-1|^2+\alpha-1+\overline{\alpha}-1)\|h\|_A^2(T^*Ah)\otimes(T^*Ah)\\ =\ & 0 \end{aligned}$$

which implies that $T+F$ is A-isometric.

□

Referring to [26, Lemma 2.7], if T is invertible then, for an A-unitary vector $h\in\mathcal{H}$, the operator $T+\Big((\alpha-1)h\otimes T^*Ah\Big)$ is invertible if and only if $\langle T^{-1}\Big((\alpha-1)h\Big),T^*Ah\rangle\neq -1$. Moreover, for $\alpha\in\mathbb{C}$ with $|\alpha|=1$, we have

$$\langle T^{-1}\Big((\alpha-1)h\Big),T^*Ah\rangle = \langle(\alpha-1)h,Ah\rangle=(\alpha-1)\|h\|_A^2=(\alpha-1)\neq -1.$$

As an immediate consequence of Theorem 2.1, we recover the characterization of $(A,2)$-isometric rank-one perturbations.

Corollary 2.2. *Let $T\in\mathcal{B}(\mathcal{H})$ be an invertible A-isometric operator and F be a A-rank one operator. Then, the following properties are equivalent:*

1. *$R=T+F$ is $(A,2)$-isometric.*
2. *There exists an A-unitary vector $h\in\mathcal{H}$, $\alpha\in\mathbb{C}$ with $|\alpha|=1$, and*

$$F=(\alpha-1)h\otimes T^*Ah.$$

In the case $m\geq 3$ and $T=I_{\mathcal{H}}$, the following characterization holds true.

Theorem 2.3. *Let $u, v \in \mathcal{H}\backslash\mathcal{N}(A)$ and let $m \geq 3$. Then $I + u \otimes_A v$ is (A, m)-isometric if and only if either $u \perp_A v$ or $Av = \lambda Au$ where $\lambda = \|u\|_A^{-2}(\mu - 1)$ for some $\mu \in \mathbb{C}\backslash\{1\}$ with $|\mu| = 1$.*

Proof. ($\Rightarrow$) Assume that $I + u \otimes_A v$ is (A, m)-isometric and $\langle u, v\rangle_A \neq 0$. We aim to show that u and v are linearly dependant. We have

$$\begin{aligned}
0 &= \Theta_A^{(m)}(I + u \otimes_A v)\\
&= (-1)^m A + \sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}(I + (Av) \otimes_I u)^k A(I + u \otimes_A v)^k\\
&= (-1)^m A + \sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}\left\{\sum_{j=0}^{k}\binom{k}{j}((Av) \otimes_I u)^j\right\}A\left\{\sum_{l=0}^{k}\binom{k}{l}(u \otimes_A v)^l\right\}\\
&= (-1)^m A + \sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}\left\{I + \sum_{j=1}^{k}\binom{k}{j}\langle v, u\rangle_A^{j-1}((Av) \otimes_I u)\right\}A\left\{I + \sum_{l=1}^{k}\binom{k}{l}\langle u, v\rangle_A^{l-1}(u \otimes_A v)\right\}\\
&= (-1)^m A + \sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}\left\{I + \sum_{j=1}^{k}\binom{k}{j}\langle v, u\rangle_A^{j-1}((Av) \otimes_I u)\right\}\left\{A + \sum_{l=1}^{k}\binom{k}{l}\langle u, v\rangle_A^{l-1}((Au) \otimes_A v)\right\}\\
&= (-1)^m A + \sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}A + \underbrace{\sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}\sum_{j=1}^{k}\binom{k}{j}\langle v, u\rangle_A^{j-1}}_{\alpha}((Av) \otimes_I (Au))\\
&\quad + \underbrace{\sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}\sum_{l=1}^{k}\binom{k}{l}\langle u, v\rangle_A^{l-1}}_{\beta}((Au) \otimes_A v)\\
&\quad + \underbrace{\sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}\sum_{j=1}^{k}\sum_{l=1}^{k}\binom{k}{j}\binom{k}{l}\langle v, u\rangle_A^{j-1}\langle u, v\rangle_A^{l-1}\|u\|_A^2}_{\gamma}((Av) \otimes_A v),
\end{aligned}$$

and so,

$$0 = \alpha\Big((Av) \otimes_I (Au)\Big) + \beta\Big((Au) \otimes_A v\Big) + \gamma\Big((Av) \otimes_A v\Big). \tag{7}$$

Since $\langle u, v\rangle_A \neq 0$, we have

$$0 \neq \frac{\langle v, u\rangle_A^m}{\langle v, u\rangle_A} = \sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}\frac{(\langle v, u\rangle_A + 1)^k - 1}{\langle v, u\rangle_A} = \alpha = \overline{\beta}.$$

It follows that

$$\begin{aligned}
Span\{Au\} &= \mathcal{R}\Big(-\beta(Au) \otimes_A v\Big)\\
&= \mathcal{R}\Big(\alpha((Av) \otimes_I (Au)) + \gamma((Av) \otimes_A v)\Big)\\
&= Span\{Av\},
\end{aligned}$$

that is Au and Av are linearly dependent. So, there exists $\lambda = \frac{1}{\|u\|_A^2}\langle v, u\rangle_A \in \mathbb{C}^*$ such that $Av = \lambda Au$. By using Equation (7), we get

$$\begin{aligned}
0 &= \sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}\sum_{j=1}^{k}\binom{k}{j}\lambda^{j}(\|u\|_A^2)^{j-1}((Au)\otimes_I(Au))\\
&\quad+\sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}\sum_{l=1}^{k}\binom{k}{l}\overline{\lambda}^{l}(\|u\|_A^2)^{l-1}((Au)\otimes_I(Au))\\
&\quad+\sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}\Bigg(\sum_{j=1}^{k}\binom{k}{j}\lambda^{j-1}(\|u\|_A^2)^{j-1}\Bigg)\Bigg(\sum_{l=1}^{k}\binom{k}{l}\overline{\lambda}^{l-1}(\|u\|_A^2)^{l-1}\Bigg)\|u\|_A^2\lambda\overline{\lambda}((Au)\otimes_I Au)\\
&= \sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}\sum_{j=1}^{k}\binom{k}{j}\frac{1}{\|u\|_A^2}\Big((1+\lambda\|u\|_A^2)^k-1\Big)((Au)\otimes_I(Au))\\
&\quad+\sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}\sum_{l=1}^{k}\binom{k}{l}\frac{1}{\|u\|_A^2}\Big((1+\overline{\lambda}\|u\|_A^2)^k-1\Big)((Au)\otimes_I(Au))\\
&\quad+\sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}\frac{1}{\|u\|_A^2}\Big((1+\lambda\|u\|_A^2)^k-1\Big)\Big((1+\overline{\lambda}\|u\|_A^2)^k-1\Big)((Au)\otimes_I Au)\\
&= \sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}\left\{\frac{1}{\|u\|_A^2}\Big((1+\lambda\|u\|_A^2)(1+\overline{\lambda}\|u\|_A^2)\Big)^k-\frac{1}{\|u\|_A^2}\right\}((Au)\otimes_I Au)\\
&= \frac{1}{\|u\|_A^2}\Big((1+\lambda\|u\|_A^2)(1+\overline{\lambda}\|u\|_A^2)-1\Big)^m((Au)\otimes_I Au)\\
&= \frac{1}{\|u\|_A^2}\Big((1+\langle v,u\rangle_A)(1+\langle u,v\rangle_A)-1\Big)^m((Au)\otimes_I Au)\\
&= \frac{1}{\|u\|_A^2}\Big(|1+\langle u,v\rangle_A|^2-1\Big)^m((Au)\otimes_I Au),
\end{aligned}$$

which implies that $|1+\langle u,v\rangle_A| = 1$. Hence, $\langle u,v\rangle_A = \mu - 1$ for some $\mu \in \mathbb{C}\backslash\{1\}$, $|\mu| = 1$.

($\Leftarrow$) If $u \perp_A v$, then the operator $(u \otimes_A v)$ is 2-nilpotent. Referring to [13, Theorem 2.1]), $(I + u \otimes_A v)$ is $(A,3)$-isometric, and hence it is (A,m)-isometric for $m \geq 3$. On the other hand, if $Av = \lambda Au$ where $\lambda = \|u\|_A^{-2}(-1+\mu)$ for some $\mu \in \mathbb{C}\backslash\{1\}$ with $|\mu| = 1$, then we get

$$\begin{aligned}
&(I + u\otimes_A v)^* A (I + u\otimes_A v)\\
&= (I + (Av)\otimes u)\Big(A + ((Au)\otimes(Av)\Big)\\
&= A + (Av)\otimes(Au) + (Au)\otimes(Av) + \|u\|_A^2\Big((Av)\otimes(Av)\Big)\\
&= A + \|u\|_A^{-2}(\overline{\mu}-1)\Big((Au)\otimes(Au)\Big) + \|u\|_A^{-2}(\mu-1)\Big((Au)\otimes(Au)\Big)\\
&\quad+\|u\|_A^2\|u\|_A^{-2}(\mu-1)\|u\|_A^{-2}(\overline{\mu}-1)\Big((Au)\otimes(Au)\Big)\\
&= A + \Big(\overline{\mu}+\mu-2+(\overline{\mu}-1)(\mu-1)\Big)\|u\|_A^{-2}\Big((Au)\otimes(Au)\Big)\\
&= A.
\end{aligned}$$

It follows that $(I + u \otimes_A v)$ is A-isometric and so is (A,m)-isometric.

□

3. (A, n)-symmetric rank-one perturbations

The main purpose of this section is to develop some characterizations for the class of rank-one operators in semi-Hilbertian spaces, in view of (A, n)-symmetries.

It is well-known that a rank-one operator $u \otimes v$ is symmetric if and only if $u = \lambda v$ for some $\lambda \in \mathbb{R}\backslash\{0\}$. The same property holds true for 2-symmetric rank-one operators as well, since the following equivalence holds:

$$\Big(T \text{ is 2-symmetric}\Big) \iff \Big(T \text{ is 1-symmetric}\Big);$$

as shown in [33, Theorem 3.1]. In the following theorem we show that we have the same for A-symmetric operators, as well as $(A, 2)$-symmetric operators since we have shown (see [29, Theorem 2.5]) that

$$\Big(T \text{ is } (A, 2)\text{-symmetric}\Big) \iff \Big(T \text{ is } A\text{-symmetric}\Big).$$

Theorem 3.1. *Let u, $v \in \mathcal{H}\backslash\mathcal{N}(A)$. Then, he following assertions hold:*

1. *$u \otimes_A v$ is A-symmetric if and only if $(Au) = \lambda(Av)$ for some $\lambda \in \mathbb{R}\backslash\{0\}$.*
2. *The subspace of A-symmetric operators is A-transitive.*

Proof. 1. By the statements of Remark 1.7, we have

$$\begin{aligned}\Big(u \otimes_A v \text{ is } A\text{-symmetric}\Big) &\iff \Big((Au) \otimes_I (Av) = (Av) \otimes_I (Au)\Big)\\ &\iff \Big(Au = \lambda(Av) \text{ for some } \lambda \in \mathbb{R}\backslash\{0\}\Big).\end{aligned}$$

2. Let denote by Γ_A the subspace of A-symmetric operators. Let $(e_n)_n$ be an orthonormal basis of $\mathcal{H}$ such that $e_n \notin \mathcal{N}(A)$ for all $n \in \mathbb{N}$, then $(e_n \otimes_A e_n) \in \Gamma_A$, for all $n \in \mathbb{N}$. Let $(x \otimes y) \in \Gamma_A^{\perp_A}$. It holds that

$$0 = \langle A(e_n \otimes_A e_n), x \otimes y\rangle = \langle A(e_n \otimes_A e_n)x, y\rangle = \langle x, e_n\rangle_A \langle e_n, y\rangle_A.$$

Moreover, since $e_n \notin \mathcal{N}(A)$ for all $n \in \mathbb{N}$, we have $x \perp_A e_n$ or $y \perp_A e_n$ for all $n \in \mathbb{N}$. Assume that $x \neq 0$ and $y \neq 0$. Let $k \in \mathbb{N}$ be the smallest number such that $\langle x, e_k\rangle_A \neq 0$ and $l \in \mathbb{N}$ be the smallest number such that $\langle y, e_l\rangle_A \neq 0$. It is easily seen that $k \neq l$ and $\langle x, e_l\rangle_A = \langle y, e_k\rangle_A = 0$. For $\alpha \neq 0$ and $\beta \neq 0$, let consider $u = \alpha e_l + \overline{\beta} e_k$. Then $(\alpha e_l + \overline{\beta} e_k) \otimes_A (\alpha e_l + \overline{\beta} e_k) \in \Gamma_A$ for any $\alpha, \beta \neq 0$. Thus

$$\begin{aligned}0 &= \langle(\alpha e_l + \overline{\beta} e_k) \otimes_A (\alpha e_l + \overline{\beta} e_k), x \otimes_A y\rangle\\ &= \langle x, \alpha e_l + \overline{\beta} e_k\rangle_A \langle \alpha e_l + \overline{\beta} e_k, y\rangle_A\\ &= \overline{\overline{\beta}} \langle x, e_k\rangle_A \alpha \langle e_l, y\rangle_A\end{aligned}$$

which is impossible since $\alpha, \beta \neq 0$ and $\langle x, e_k\rangle_A \neq 0$, $\langle y, e_l\rangle_A \neq 0$. Hence, $x = 0$ or $y = 0$.

□

In the case of (A, n)-symmetric rank-one operators, with $n \geq 3$, the following characterization holds true.

Theorem 3.2. *Let u, $v \in \mathcal{H}\backslash\mathcal{N}(A)$ and $n \geq 3$. Then, $(u \otimes_A v)$ is (A, n)-symmetric if and only if either $u \perp_A v$ or $Au = \lambda(Av)$ for some $\lambda \in \mathbb{R}\backslash\{0\}$.*

Proof. $(\Rightarrow)$ Assume that $(u \otimes_A v)$ is (A, n)-symmetric. By using Remark 1.7, we have

$$\begin{aligned}0 &= \mathcal{S}_A^{(m)}(u \otimes_A v)\\ &= \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} ((Av) \otimes_I u)^k A(u \otimes_A v)^{m-k}\end{aligned}$$

$$\begin{aligned}
&= (-1)^m A(u\otimes_A v)^m + \sum_{k=1}^{m-1}(-1)^{m-k}\binom{m}{k}((Av)\otimes_I u)^k A(u\otimes_A v)^{m-k} + ((Av)\otimes_I u)^m A \\
&= (-1)^m\langle u,v\rangle_A^{m-1}((Au)\otimes_I (Av)) + \langle v,u\rangle_A^{m-1}((Av)\otimes_I (Au)) \\
&\quad + \sum_{k=1}^{m-1}(-1)^{m-k}\binom{m}{k}\langle u,v\rangle_A^{m-k-1}\langle v,u\rangle_A^{k-1}\|u\|_A^2((Av)\otimes_I (Av)) \qquad (8)\\
&= \alpha((Au)\otimes_I (Av)) + \beta((Av)\otimes_I (Au)) + \gamma((Av)\otimes_I (Av)) \\
&= (\alpha(Au)+\gamma((Av))\otimes_I (Av) + \beta((Av)\otimes_I (Au))
\end{aligned}$$

for some $\alpha,\ \beta,\ \gamma \in \mathbb{C}$. If $\langle u,v\rangle_A \neq 0$ and $u,\ v\in\mathcal{H}\backslash\mathcal{N}(A)$, then $\beta((Av)\otimes_I (Au))\neq 0$ and $(\alpha(Au)+\gamma((Av))\otimes_I (Av)\neq 0$. It follows that

$$\beta((Av)\otimes_I (Au)) = -(\alpha(Au)+\gamma((Av))\otimes_I (Av).$$

A simple computation shows that $\mathcal{N}\Big(\beta((Av)\otimes_I (Au))\Big)^{\perp} = Span\{Au\}$ and $\mathcal{N}\Big((\alpha(Au)+\gamma((Av))\otimes_I (Av)\Big)^{\perp} = Span\{Av\}$. Hence, $Au=\lambda(Av)$ for some non-zero $\lambda = \frac{1}{\|v\|_A^2}\langle u,v\rangle_A$. By using Equation (8), we obtain

$$\begin{aligned}
0 &= (-1)^m\lambda^m\|v\|_A^{2(m-1)}((Av)\otimes_I (Av)) + \overline{\lambda}^m\|v\|_A^{2(m-1)}((Av)\otimes_I (Av)) \\
&\quad + \sum_{k=1}^{m-1}(-1)^{m-k}\binom{m}{k}\lambda^{m-k-1}\|v\|_A^{2(m-k-1)}\overline{\lambda}^{k-1}\|v\|_A^{2(k-1)}\|u\|_A^2((Av)\otimes_I (Av)) \\
&= (\overline{\lambda}-\lambda)^m\|v\|_A^{2(m-1)}((Av)\otimes_I (Av)) \\
&= \Big(-i\mathcal{I}m(\lambda)\Big)^m\|v\|_A^{2(m-1)}((Av)\otimes_I (Av))
\end{aligned}$$

which implies that $\mathcal{I}m(\lambda)=0$, so $\lambda\in\mathbb{R}$.

($\Leftarrow$) Referring to Theorem 3.1, if $Au=\lambda(Av)$ for some non-zero real λ, then the operator $(u\otimes_A v)$ is A-symmetric and hence is (A,n)-symmetric for any $n\geq 1$. On the other hand, if $\langle u,v\rangle_A=0$, then the operator $(u\otimes_A v)$ is 2-nilpotent. Since every k-nilpotent operator is $(A,2k-1)$-symmetric, we have $(u\otimes_A v)$ is $(A,3)$-symmetric, and hence it is (A,n)-symmetric for any $n\geq 3$.

□

Example 3.3. *Let $(e_n)_{n\geq1}$ be an orthonormal basis of $\mathcal{H}$. Define the operators A, R and S as follows*

$$\begin{aligned}
Ae_n &= \frac{1}{n}e_n \ \ (n\geq 1),\\
R &= e_i\otimes_A e_j \ \ (i\neq j),\\
S &= e_k\otimes_A e_k \ \ (k\geq 1).
\end{aligned}$$

We can easily verify that A is a positive operator and $(e_n)_{n\geq1}$ is A-orthogonal. Applying Theorem 3.2 (respc. Theorem 3.1) we can deduce that R (respc. S) is (A,n)-symmetric for $n\geq 3$ (respc. A-symmetric).

Recall that a bounded linear operator T is called $(A,2)$-symmetric if it satisfies

$$\langle T^2h,h\rangle_A - 2\|Th\|_A^2 + \langle h,T^2h\rangle_A = 0,\ \ \forall\, h\in\mathcal{H}. \qquad (9)$$

Proposition 3.4. *Let $\delta\in\mathbb{R}$, $\lambda\in\mathbb{C}$ and $T\in\mathcal{B}(\mathcal{H})$ be A-symmetric. Then the following statements hold.*

1. *The operator $R=T+u\otimes_A v$ is A-symmetric $\Big($resp. $(A,2)$-symmetric$\Big)$ if and only if $Au=\lambda(Av)$ for some $\lambda\in\mathbb{R}\backslash\{0\}$.*
2. *Suppose that $\|v\|_A=1$ and $Av\in\mathcal{N}(T^*-\lambda)$. Then, $S=T+i\delta(v\otimes_A v)$ is $(A,2)$-symmetric if and only if $\mathfrak{I}m(\lambda)=\delta$.*

3. *For $\lambda \in \sigma_p(T)$, there exists $u \in \mathcal{H}$ such that $Tu = \lambda u$ and $u \otimes_A u \in \{T\}'$.*

Proof. 1. If $R = T + u \otimes_A v$, then $R^* = T^* + (Av) \otimes_I u$ and $R^*A = T^*A + (Av) \otimes_I (Au)$. Since T is A-symmetric (i.e. $AT = T^*A$), it follows that $AR = T^*A + (Au) \otimes_I (Av)$. Hence, R is A-symmetric if and only if $(Au) \otimes_I (Av) = (Av) \otimes_I (Au)$, which in turns is equivalent to the fact that $Au = \lambda(Av)$ for some $\lambda \in \mathbb{R}\backslash\{0\}$.

2. Since T is A-symmetric and $S = T + i\delta(v \otimes_A v)$, $AS = AT + i\delta(Av) \otimes (Av)$ and $S^*Ah = ASh - 2i\delta\langle h, v\rangle_A(Av)$ for every $h \in \mathcal{H}$. It follows from that

$$\begin{aligned}
\langle S^2h, h\rangle_A &= \langle Sh, S^*Ah\rangle \\
&= \langle Sh, ASh - 2i\delta\langle h, v\rangle_A(Av)\rangle \\
&= \|Sh\|_A^2 + 2i\delta\langle v, h\rangle_A\langle h, S^*Av\rangle \\
&= \|Sh\|_A^2 + 2i\delta\langle v, h\rangle_A\langle h, (\lambda - i\delta\|v\|_A^2)Av\rangle \\
&= \|Sh\|_A^2 + 2i\delta(\overline{\lambda} + i\delta)|\langle v, h\rangle_A|^2.
\end{aligned}$$

By using similar arguments as above, we obtain

$$\langle h, S^2h\rangle_A = \|Sh\|_A^2 - 2i\delta(\lambda - i\delta)|\langle v, h\rangle_A|^2.$$

Substituting in (9) we obtain that

$$\langle S^2h, h\rangle_A - 2\|Sh\|_A^2 + \langle h, S^2h\rangle_A = 4i\delta\Big(\delta - \Im m(\lambda)\Big)|\langle v, h\rangle_A|^2.$$

Hence, S is $(A, 2)$-symmetric if and only if $\Im m(\lambda) = \delta$.

3. Since $\lambda \in \sigma_p(T)$, there exists $u \in \mathcal{H}$ such that $Tu = \lambda u$. This implies that

$$\begin{aligned}
T(u \otimes_A u) = (\lambda u) \otimes (Au) &= u \otimes A(\lambda u) \\
&= u \otimes (T^*Au) \\
&= (u \otimes_A u)T.
\end{aligned}$$

□

The next theorem shows a characterization of (A, n)-symmetric ran-one perturbation of the identity operator for some $n \geq 3$.

Theorem 3.5. *Let u, $v \in \mathcal{H}\backslash\mathcal{N}(A)$ and $n \geq 3$. Then, $I + (u \otimes_A v)$ is (A, n)-symmetric if and only if either $u \perp_A v$ or $Au = \lambda(Av)$ for some $\lambda \in F_n$, where*

$$F_n := \Big\{\lambda \in \mathbb{C}^* : 1 - \Big(1 + \overline{\lambda}\|u\|_A^2\Big)^n + \Big(-2i\Im m\lambda\|u\|_A^2\Big)^n = 0\Big\}.$$

Proof. $(\Rightarrow)$ Assume that $I + u \otimes_A v$ is (A, n)-symmetric and $\langle u, v\rangle_A \neq 0$. We have

$$\begin{aligned}
0 &= \mathcal{S}_A^{(n)}(I + u \otimes_A v) \\
&= \sum_{k=1}^{n}(-1)^{n-k}\binom{n}{k}(I + (Av) \otimes_I u)^k A(I + u \otimes_A v)^{n-k} + (-1)^n A + (-1)^n \sum_{l=1}^{n}\binom{n}{l}A(u \otimes_A v)^l\Bigg) \\
&= \sum_{k=1}^{n}(-1)^{n-k}\binom{n}{k}\left\{I + \sum_{j=1}^{k}\binom{k}{j}((Av) \otimes_I u)^j\right\}A\left\{I + \sum_{l=1}^{n-k}\binom{n-k}{l}(u \otimes_A v)^l\right\} \\
&\quad + (-1)^n A + \sum_{l=1}^{n}(-1)^n\binom{n}{l}\langle u, v\rangle_A^{l-1}(Au) \otimes_A v \\
&= \sum_{k=1}^{n}(-1)^{n-k}\binom{n}{k}\left\{I + \sum_{j=1}^{k}\binom{k}{j}\langle v, u\rangle_A^{j-1}((Av) \otimes u)\right\}A\left\{I + \sum_{l=1}^{n-k}\binom{n-k}{l}\langle u, v\rangle_A^{l-1}(u \otimes Av)\right\}
\end{aligned}$$

$$
\begin{aligned}
&+(-1)^n A+\sum_{l=1}^{n}(-1)^n\binom{n}{l}\langle u,v\rangle_A^{l-1}(Au)\otimes_A v\\
=\;& \underbrace{(-1)^n A+\sum_{k=1}^{n}(-1)^{n-k}\binom{n}{k}A}_{(=0)}+\underbrace{\sum_{k=1}^{n}(-1)^{n-k}\binom{n}{k}\sum_{l=1}^{n-k}\binom{n-k}{l}\langle u,v\rangle_A^{l-1}}_{\alpha}(Au\otimes Av)\\
&+\underbrace{\sum_{k=1}^{n}(-1)^{n-k}\binom{n}{k}\sum_{j=1}^{k}\binom{k}{j}\langle v,u\rangle_A^{j-1}}_{\beta}((Av)\otimes(Au))\\
&+\underbrace{\sum_{k=1}^{n}(-1)^{n-k}\binom{n}{k}\sum_{j=1}^{k}\sum_{l=1}^{n-k}\binom{k}{j}\binom{n-k}{l}\langle v,u\rangle_A^{j-1}\langle u,v\rangle_A^{l-1}\|u\|_A^2}_{\gamma}(Av\otimes Av)\\
=\;& \alpha(Au\otimes Av)+\beta(Av\otimes Au)+\gamma(Av\otimes Av)
\end{aligned}
\tag{10}
$$

where

$$
\begin{aligned}
\alpha &= \frac{(-1)^{n+1}}{\langle u,v\rangle_A}\Big\{\Big(\langle u,v\rangle_A+1\Big)^n-\langle u,v\rangle_A^n-1\Big\},\\
\beta &= \langle v,u\rangle_A^{n-1}\quad(\neq 0),\\
\gamma &= \frac{\|u\|_A^2}{|\langle u,v\rangle_A|^2}\Big\{\Big(2i\Im m(\langle v,u\rangle_A)\Big)^n-\langle v,u\rangle_A^n-(-1)^n\langle u,v\rangle_A^n\Big\}.
\end{aligned}
$$

It follows that

$$0 = (\alpha(Au)+\gamma(Av))\otimes(Av)+\beta(Av\otimes Au).$$

Then,

$$
\begin{aligned}
Span\{Au\} &= \mathcal{R}\Big(-\beta(Au)\otimes_A v\Big)\\
&= \mathcal{R}\Big(\alpha((Av)\otimes_I(Au))+\gamma((Av)\otimes_A v)\Big)\\
&= Span\{Av\}.
\end{aligned}
$$

Hence, $Av=\lambda(Au)$ for some non-zero $\lambda=\frac{1}{\|u\|_A^2}\langle v,u\rangle_A$. Replacing in Equation (10), we obtain

$$0=\frac{(-1)^n}{\|u\|_A^2}\Big\{1-\Big(1+\overline{\lambda}\|u\|_A^2\Big)^n+\Big(-2i\Im m\lambda\|u\|_A^2\Big)^n\Big\}(Au\otimes Au) \tag{11}$$

and, so, $\lambda\in F_n$.

($\Leftarrow$) If $u\perp_A v$, then the operator $(u\otimes_A v)$ is 2-nilpotent. According to [29, Theorem 4.2.]), $(I+u\otimes_A v)$ is $(A,3)$-symmetric, and hence it is (A,n)-symmetric for $n\geq 3$. On the other hand, if $Au=\lambda(Av)$ for some $\lambda\in F_n$, then Equation (11) allows to conclude.

□

Remark that, for $u,\, v \in \mathcal{H} \setminus \mathcal{N}(A)$, we have

$$\sigma(I + u \otimes_A v) = \sigma(I + u \otimes (Av) = \{1, 1 + \langle u, v\rangle_A\},$$

which is finite. For $A = I$ and Referring to (5), this shows that $\langle u, v\rangle \in \mathbb{R}$. The following corollary is an immediate consequence of Theorem 3.6.

Corollary 3.6. *Let $u,\, v \in \mathcal{H}\backslash\{0\}$ and $n \geq 3$. Then, $I + (u \otimes v)$ is n-symmetric if and only if either $u \perp v$ or $u = \lambda\, v$ for some $\lambda \in F_n$, where*

$$F_n \quad := \quad \left\{\lambda \in \mathbb{R}^* : \left(1 + \lambda \|u\|_A^2\right)^n = 1\right\}.$$

Example 3.7. *Let $\mathcal{H} = \mathbb{C}^2$ equipped with the norm $\|(x, y)\|^2 = |x|^2 + |y|^2$, $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, $u = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then, $A \in \mathcal{B}(\mathcal{H})^+$ and $u,\, v \in \mathcal{H} \setminus \mathcal{N}(A)$ satisfying $\langle u, v\rangle_A = -2$. According to Theorem 3.6, the operators $(I + u \otimes_A v)$ is $(A, 6)$-symmetric (and so $(A, 5)$-symmetric).*

4. A-hyponormality and A-normality of rank-one perturbations

We aim in this section to characterize the A-hyponormality (resp. the A-normality) of rank one perturbations of A-hyponormal (resp. A-normal) bounded linear operators.

Let begin our study with the following result.

Proposition 4.1. *Let $u,\, v \in \mathcal{H}$. Then the following statements hold.*

1. *$u \otimes_A v \in \mathcal{B}_{A^{\frac{1}{2}}}(\mathcal{H})$ and $(u \otimes_A v)^{\sharp_A} = v \otimes_A u$.*
2. *$u \otimes_A v$ is A-normal if and only if u and v are linearly dependant.*

Proof. 1. Let $x \in \mathcal{H}$. Since $A \in \mathcal{B}(\mathcal{H})^+$, we have

$$\begin{aligned} \|(u \otimes_A v)(x)\|_A &= |\langle x\,,\, v\rangle_A|\, \|u\|_A \\ &= |\langle A^{\frac{1}{2}} x\,,\, A^{\frac{1}{2}} v\rangle|\, \|u\|_A \\ &\leq \|A^{\frac{1}{2}} x\|\, \|A^{\frac{1}{2}} v\|\, \|u\|_A \leq \lambda\, \|u\|_A. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \langle (u \otimes_A v)^{\sharp_A}(x)\,,\, x\rangle_A &= \langle A(u \otimes_A v)^{\sharp_A}(x)\,,\, x\rangle \\ &= \langle (u \otimes_A v)^* A(x)\,,\, x\rangle \\ &= \langle ((Av) \otimes_I u) A(x)\,,\, x\rangle \\ &= \langle \langle x\,,\, u\rangle_A v\,,\, x\rangle_A = \langle (v \otimes_A u)x\,,\, x\rangle_A. \end{aligned}$$

2. Let $T = (u \otimes_A v)$. Then we have

$$\begin{aligned} T \text{ is } A\text{-normal} &\iff (u \otimes_A v)(u \otimes_A v)^{\sharp_A} = (u \otimes_A v)^{\sharp_A}(u \otimes_A v) \\ &\iff (u \otimes_A v)(v \otimes_A u) = (v \otimes_A u)(u \otimes_A v) \\ &\iff (\|v\|_A^2 u) \otimes_A u = (\|u\|_A^2 v) \otimes_A v \\ &\iff \exists\, \lambda,\, \mu \in \mathbb{C}^*,\ \|v\|_A^2 u = \lambda \|u\|_A^2 v,\ Au = \mu A v;\ \ \lambda \overline{\mu} = 1. \end{aligned}$$

□

The following theorem gives a characterization of rank-one perturbation of an A-hyponormal operator.

Theorem 4.2. *Let $T \in \mathcal{B}_A(\mathcal{H})$ be A-hyponormal, $u \in \mathcal{H} \setminus \{0_{\mathcal{H}}\}$, $v \in \mathcal{H} \setminus \mathcal{N}(A)$ and set $R = T + u \otimes_A v$. Then the following statements hold.*

1. *If u and v are linearly dependent, then R is A-hyponormal if and only if, for $\Theta_A(T) := (\lambda T^*A - \overline{\lambda}(T^*A)^*)$ and $\lambda = \frac{\langle u,v\rangle_A}{\|v\|_A^2}$, the following inequality holds true*
$$\Re e\Big(\langle x,v\rangle_A\langle\Theta_A(T)v,x\rangle\Big) \geq -\frac{1}{2}\Big(\|Tx\|_A^2 - \|T^{\sharp_A}x\|_A^2\Big), \;\; \forall\, x\in\mathcal{H}. \tag{12}$$
2. *If u and v are linearly independent, then R is A-hyponormal if and only if, for $x \in Span\{u,v\}$ and $\Psi_A(T) := \langle x, T^*Au\rangle A - \langle x,u\rangle_A AT$, we have*
$$\Re e(\langle\Psi_A(T)v,x\rangle) + \frac{1}{2}\Big(\|u\|_A^2|\langle x,v\rangle_A|^2 - \|v\|_A^2|\langle x,u\rangle_A|^2\Big) \geq -\frac{1}{2}\Big(\|Tx\|_A^2 - \|T^{\sharp_A}x\|_A^2\Big). \tag{13}$$

Proof. 1. We assume that u,v are linearly dependent. Then, there exists $\lambda\in\mathbb{C}$ such that $u = \lambda v = \frac{1}{\|v\|_A^2}\langle u,v\rangle_A\, v$. So,
$$\begin{aligned}\|u\|_A^2(v\otimes_A v) &= |\lambda|^2\|v\|_A^2(v\otimes_A v)\\ &= \|v\|_A^2(u\otimes_A u).\end{aligned}$$
Moreover, we have
$$[R^{\sharp_A},R] = [T^{\sharp_A},T] + (T^{\sharp_A}u)\otimes_A v + v\otimes_A (T^{\sharp_A}u) - (Tv)\otimes_A u - u\otimes_A (T^{\sharp_A})^{\sharp_A}v.$$
For all $x\in\mathcal{H}$, the identity (14) gives
$$\begin{aligned}
&\langle[R^{\sharp_A},R]x,x\rangle_A\\
=\;& \langle[T^{\sharp_A},T]x,x\rangle_A + \langle\Big((\lambda T^{\sharp_A}v - \overline{\lambda}Tv)\otimes_A v\Big)x,x\rangle_A + \langle v\otimes_A(\lambda T^{\sharp_A}v - (\overline{\lambda}T^{\sharp_A})^{\sharp_A}v)x,x\rangle_A\\
=\;& \langle[T^{\sharp_A},T]x,x\rangle_A + \langle\Big((\lambda AT^{\sharp_A}v - \overline{\lambda}ATv)\otimes Av\Big)x,x\rangle + \langle Av\otimes(\lambda AT^{\sharp_A}v - A(\overline{\lambda}T^{\sharp_A})^{\sharp_A}v)x,x\rangle\\
=\;& \langle[T^{\sharp_A},T]x,x\rangle_A + \langle\Big(\underbrace{(\lambda T^*Av - \overline{\lambda}(T^*A)^*v)}\otimes Av\Big)x,x\rangle + \langle Av\otimes\underbrace{(\lambda T^*Av - \overline{\lambda}(T^*A)^*v)}x,x\rangle\\
=\;& \langle[T^{\sharp_A},T]x,x\rangle_A + \langle\Big(\Theta_A(T)v\otimes Av\Big)x,x\rangle + \langle\Big(Av\otimes\Theta_A(T)v\Big)x,x\rangle\\
=\;& \langle[T^{\sharp_A},T]x,x\rangle_A + 2\Re e\Big(\langle x,v\rangle_A\langle\Theta_A(T)v,x\rangle\Big).
\end{aligned}$$
Hence R is A-hyponormal if and only if
$$\Re e\Big(\langle x,v\rangle_A\langle\Theta_A(T)v,x\rangle\Big) \geq -\frac{1}{2}\langle[T^{\sharp_A},T]x,x\rangle_A = -\frac{1}{2}\Big(\|Tx\|_A^2 - \|T^{\sharp_A}x\|_A^2\Big).$$

2. We assume now that u, v are linearly independent vectors. For $x\in\mathcal{H}$, we have
$$\begin{aligned}
&\langle[R^{\sharp_A},R]x,x\rangle_A\\
=\;& \langle[T^{\sharp_A},T]x,x\rangle_A + \langle\Big((T^{\sharp_A}u)\otimes_A v\Big)x,x\rangle_A + \langle v\otimes_A(T^{\sharp_A}u)x,x\rangle_A\\
&+\|u\|_A^2\langle(v\otimes_A v)x,x\rangle_A - \langle\Big((Tv)\otimes_A u\Big)x,x\rangle_A - \langle\Big(u\otimes_A(T^{\sharp_A})^{\sharp_A}v\Big)x,x\rangle_A - \|v\|_A^2\langle(u\otimes_A u)x,x\rangle_A\\
=\;& \langle[T^{\sharp_A},T]x,x\rangle_A + \langle x,v\rangle_A\langle T^{\sharp_A}u,x\rangle_A + \langle x,T^{\sharp_A}u\rangle_A\langle v,x\rangle_A + \|u\|_A^2|\langle x,v\rangle_A|^2\\
&-\langle x,u\rangle_A\langle Tv,x\rangle_A - \langle x,(T^{\sharp_A})^{\sharp_A}v\rangle_A\langle u,x\rangle_A - \|v\|_A^2|\langle x,u\rangle_A|^2\\
=\;& \langle[T^{\sharp_A},T]x,x\rangle_A + \langle\langle x,T^{\sharp_A}u\rangle_A v - \langle x,u\rangle_A Tv,x\rangle_A + \|u\|_A^2|\langle x,v\rangle_A|^2\\
&-\|v\|_A^2|\langle x,u\rangle_A|^2 + \langle x,\overline{\langle T^{\sharp_A}u,x\rangle_A}v - \overline{\langle u,x\rangle_A}(T^{\sharp_A})^{\sharp_A}v\rangle_A\\
=\;& \langle[T^{\sharp_A},T]x,x\rangle_A + \langle\langle x,T^{\sharp_A}u\rangle_A v - \langle x,u\rangle_A Tv,x\rangle_A + \|u\|_A^2|\langle x,v\rangle_A|^2\\
&-\|v\|_A^2|\langle x,u\rangle_A|^2 + \langle x,\langle x,T^{\sharp_A}u\rangle_A v - \langle x,u\rangle_A(T^{\sharp_A})^{\sharp_A}v\rangle_A\\
=\;& \langle[T^{\sharp_A},T]x,x\rangle_A + \langle\underbrace{\langle x,T^*Au\rangle Av - \langle x,u\rangle_A ATv},x\rangle\\
&+\langle x,\underbrace{\langle x,T^*Au\rangle Av - \langle x,u\rangle_A ATv}\rangle + \|u\|_A^2|\langle x,v\rangle_A|^2 - \|v\|_A^2|\langle x,u\rangle_A|^2\\
=\;& \langle[T^{\sharp_A},T]x,x\rangle_A + 2\Re e(\langle\Psi_A(T)v,x\rangle) + \Big(\|u\|_A^2|\langle x,v\rangle_A|^2 - \|v\|_A^2|\langle x,u\rangle_A|^2\Big).
\end{aligned}$$

($\Rightarrow$) If $x \in Span\{u, v\}$ and

$$\Re e(\langle \Psi_A(T)v, x\rangle + \frac{1}{2}\Big(\|u\|_A^2 |\langle x, v\rangle_A|^2 - \|v\|_A^2 |\langle x, u\rangle_A|^2\Big) \geq -\frac{1}{2}\langle [T^{\sharp_A}, T]x, x\rangle_A$$

then $\langle [R^{\sharp_A}, R]x, x\rangle_A \geq 0$ which in turn implies that $R^{\sharp_A}$ is A-hyponormal. On the other hand, if $x \in \big(Span\{u, v\}\big)^{\perp_A}$, then $\langle [R^{\sharp_A}, R]x, x\rangle_A = \langle [T^{\sharp_A}, T]x, x\rangle_A \geq 0$, and thus $R^{\sharp_A}$ is A-hyponormal.

($\Leftarrow$) If $R^{\sharp_A}$ is A-hyponormal, then arguing as above we prove that Inequality (13) holds true.

□

The following example is an application of Theorem 4.2.

Example 4.3. *Let* $T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ *and* $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ *be operators acting on two-dimensional Hilbert space* $\mathbb{C}^2$. *Set* $R = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + (u \otimes_A v)$. *Then,* $T^{\sharp_A} = 0$ *([25]) and* $\Theta_A(T)v = \Psi_A(T)v = 0$. *Referring to Theorem 4.2, we deduce that:*

1. *If u and v are linearly dependent, then R is A-hyponormal if and only if* $(u \otimes_A v)$ *is A-hyponormal.*
2. *If u and v are linearly independent, then R is A-hyponormal if and only if*

$$\|u\|_A^2 |\langle x, v\rangle_A|^2 \geq \|v\|_A^2 |\langle x, u\rangle_A|^2, \quad x \in Span\{u, v\}.$$

Corollary 4.4. *Let* $A \in \mathcal{B}(\mathcal{H})^+$ *satisfying* $\langle e_i, e_j\rangle_A = 0$ *for all* $i \neq j$, $T \in \mathcal{B}_A(\mathcal{H})$ *be a unilateral shift,* $u \in \mathcal{H} \setminus \{0_{\mathcal{H}}\}$ *and* $v \in \mathcal{H} \setminus \mathcal{N}(A)$. *Then* $R = T + u \otimes_A v$ *is A-hyponormal if and only if either*

$$\Re e\Big(\langle x, v\rangle_A \langle \Theta_A(T)v, x\rangle\Big) \geq -\frac{1}{2}\Lambda_A, \quad \forall\, x \in \mathcal{H} \tag{14}$$

or

$$\Re e(\langle \Psi_A(T)v, x\rangle) + \frac{1}{2}\Big(\|u\|_A^2 |\langle x, v\rangle_A|^2 - \|v\|_A^2 |\langle x, u\rangle_A|^2\Big) \geq -\frac{1}{2}\Lambda_A, \quad \forall\, x \in \mathcal{H} \tag{15}$$

where

$$\Lambda_A = \sum_{n \neq m} x_n \overline{x}_m \langle T^{\sharp_A} e_n, T^{\sharp_A} e_m\rangle_A + \sum_{n} |x_n|^2 \Big(\|e_{n+1}\|_A^2 - \|T^{\sharp_A} e_n\|_A^2\Big).$$

Proof. Let $T \in \mathcal{B}(\mathcal{H})$ be the unilateral shift defined by $Te_n = e_{n+1}$, $n \geq 1$. Then, for $x = \sum_{n\geq 1} x_n e_n \in \mathcal{H}$, we have

$$\begin{aligned} \|Tx\|_A^2 &= \underbrace{\sum_{n \neq m} x_n \overline{x}_m \langle e_{n+1}, e_{m+1}\rangle_A}_{(=0)} + \sum_{n \geq 1} |x_n|^2 \|e_{n+1}\|_A^2, \\ \|T^{\sharp_A} x\|_A^2 &= \sum_{n \neq m} x_n \overline{x}_m \langle T^{\sharp_A} e_n, T^{\sharp_A} e_m\rangle_A + \sum_{n \geq 1} |x_n|^2 \|T^{\sharp_A} e_n\|_A^2. \end{aligned}$$

By applying Theorem 4.2 we finish the proof. □

Example 4.5. *Let $Ae_n = \frac{1}{n}e_n$, $n \geq 1$ and $Te_n = e_{n+1}$, $n \geq 1$. Then, $\langle e_i, e_j\rangle_A = \frac{1}{i}\langle e_i, e_j\rangle = 0$. Moreover, for $u = \sum_{n\geq 1}\rho_n e_n \in \mathcal{H}$, we have*

$$\begin{aligned} \|Tu\|_A^2 &= \langle Tu, Tu\rangle_A \\ &= \underbrace{\sum_{i\neq j\geq 1}\rho_i\overline{\rho_j}\langle Te_i, Te_j\rangle_A}_{(=0)} + \sum_{i\geq 1}|\rho_i|^2\langle Te_i, Te_i\rangle_A \\ &= \sum_{i\geq 1}|\rho_i|^2\frac{1}{i+1}\langle e_{i+1}, e_{i+1}\rangle \\ &= \sum_{i\geq 1}|\rho_i|^2\frac{i}{i+1}\langle \frac{1}{i}e_i, e_i\rangle \leq \|u\|_A^2, \end{aligned}$$

which implies that $T \in \mathcal{B}_A(\mathcal{H})$. By a simple calculation, we obtain

$$A^{\dagger}(x_1, x_2, x_3, \cdots) = (x_1, 2x_2, 3x_3, \cdots),\ i.e.\ A^{\dagger}e_n = ne_n,\ n \geq 1,$$

and it follows from that

$$T^{\sharp_A}e_n = A^{\dagger}T^*Ae_n = T^*e_n = \begin{cases} 0, & \text{if } n = 1; \\ e_{n-1}, & \text{if } n \geq 2. \end{cases}$$

Moreover, we have T is A-hyponormal since $\langle [T^{\sharp_A}, T]u, u\rangle_A = \langle [T^*, T]u, u\rangle_A = 0$, for all $u \in \mathcal{H}$.

Corollary 4.6. *Let $T \in \mathcal{B}_A(\mathcal{H})$ be a unilateral shift, $u \in \mathcal{H} \setminus \{0_{\mathcal{H}}\}$, $v \in \mathcal{H} \setminus \mathcal{N}(A)$ and set $R = T + u \otimes_A v$. Then the following statements hold.*

1. *Assume that u and v are linearly dependent vectors. If $\Theta_A(T)v = \eta Av$, then*
$$\eta = \frac{2i}{\|v\|_A^4}\Im m\Big(\langle u, v\rangle_A\langle v, Tv\rangle_A\Big)$$
and R is A-hyponormal.
2. *Assume that u and v are linearly independent with $\|u\|_A = \|v\|_A = 1$. If $\Psi_A(T)v = 0$ or $\Psi_A(T)v = x$ for $x = y + z$ where $y = \langle y, u\rangle_A u + \langle y, v\rangle_A v$ with $|\langle y, v\rangle_A| \geq |\langle y, u\rangle_A|$ and $z \in \{u, v\}^{\perp_A}$, then R is A-hyponormal.*

Proof. 1. Assume that $\Theta_A(T)v = \eta Av$. Then,

$$\begin{aligned} \eta\|v\|_A^4 &= \|v\|_A^2\langle\Theta_A(T)v, v\rangle \\ &= \langle u, v\rangle_A\langle T^*Av, v\rangle - \overline{\langle u, v\rangle_A\langle v, (T^*A)^*v\rangle} \\ &= 2i\Im m\Big(\langle u, v\rangle_A\langle v, Tv\rangle_A\Big). \end{aligned}$$

It follows that $\eta = \frac{2i}{\|v\|_A^4}\Im m\Big(\langle u, v\rangle_A\langle v, Tv\rangle_A\Big)$. Moreover, for any $x \in \mathcal{H}$, we have

$$\begin{aligned} \Re e\Big(\langle x, v\rangle_A\langle\Theta_A(T)v, x\rangle\Big) &= \Re e\Big(\eta\langle x, v\rangle_A\langle v, x\rangle_A\Big) \\ &= \Re e\Big(\eta|\langle x, v\rangle_A|^2\Big) = 0. \end{aligned}$$

By applying (1)-Theorem 4.2 we obtain the desired claim.

2. Under the assumptions, we obtain

$$\begin{aligned} &\Re e(\langle\Psi_A(T)v, x\rangle) + \frac{1}{2}\Big(\|u\|_A^2|\langle x, v\rangle_A|^2 - \|v\|_A^2|\langle x, u\rangle_A|^2\Big) \\ &= \begin{cases} \frac{1}{2}\Big(|\langle y, v\rangle_A|^2 - |\langle y, u\rangle_A|^2\Big), & \text{if } \Psi_A(T)v = 0; \\ \|x\|^2 + \frac{1}{2}\Big(|\langle y, v\rangle_A|^2 - |\langle y, u\rangle_A|^2\Big), & \text{if } \Psi_A(T)v = x. \end{cases} \end{aligned}$$

By applying (2)-Theorem 4.2 we finish the proof.

□

Example 4.7. *Let A and T as in Example 4.5 and let $u = e_1 + e_2,\ v = i(e_1 + e_2)$. Then*

$$\eta = \frac{2i}{\|v\|_A^4}\Im m\Big(\langle u, v\rangle_A \langle v, Tv\rangle_A\Big) = -\frac{4}{5}i$$

and the operator $R = T + \Big((e_1 + e_2) \otimes_A i(e_1 + e_2)\Big)$ is A-hyponormal.

The following theorem gives a necessary and sufficient conditions that a A-rank one perturbation of an A-normal operator to be A-normal.

Theorem 4.8. *Let $u, v \in \mathcal{H}\backslash\mathcal{N}(A)$ and $N \in \mathcal{B}_A(\mathcal{H})$ be A-normal such that $\mathcal{N}(A)$ is a reducing subspace for N. Then $T = N + u \otimes_A v \in \mathcal{B}_A(\mathcal{H})$ is A-normal if and only if one of the following conditions holds true:*

(*i*) *u, v are linearly dependent and $(v \otimes \Theta_A(N)v)$ is a skew A-symmetric operator.*

(*ii*) *u, v are linearly independent and there exist $\sigma, \eta \in \mathbb{C}$ such that $\Re e(\eta) = -\frac{1}{2}$ and*

$$(N - \sigma)^{\sharp_A} u = \|u\|_A^2 \eta P v \ \ and \ \ (N - \sigma)v = \|v\|_A^2 \overline{\eta} u. \tag{16}$$

Proof. Remark that, for all $x \in \mathcal{H}$, we have

$$\begin{aligned} u \otimes_A ((N^{\sharp_A})^{\sharp_A})v(x) &= \langle x, ((N^{\sharp_A})^{\sharp_A})v\rangle_A u \\ &= \langle x, Nv\rangle_A u \\ &= u \otimes_A (Nv)(x). \end{aligned}$$

Since N is A-normal, the operator $T = N + u \otimes_A v$ is A-normal if and only if

$$(N^{\sharp_A} u) \otimes_A v + v \otimes_A (N^{\sharp_A} u) + \|u\|_A^2 (v \otimes_A v) \quad = \quad (Nv) \otimes_A u + u \otimes_A Nv + \|v\|_A^2 (u \otimes_A u). \tag{17}$$

Obviously, it follows from (17) that

$$ran([T^{\sharp_A}, T]) \subset Span\{u, v, Nv, N^{\sharp_A} u\}.$$

Moreover, we have

$$\langle x, v\rangle_A N^{\sharp_A} u + \langle x, N^{\sharp_A} u + \|u\|_A^2 v\rangle_A v \quad = \quad \langle x, Nv + \|v\|_A^2 u\rangle_A u + \langle x, u\rangle_A (Nv). \tag{18}$$

If u, v are linearly dependent, then there exists $\lambda \in \mathbb{C}$ such that $u = \lambda v = \frac{1}{\|v\|_A^2}\langle u, v\rangle_A\, v$. So,

$$\begin{aligned} \|u\|_A^2 (v \otimes_A v) &= |\lambda|^2 \|v\|_A^2 (v \otimes_A v) \\ &= \|v\|_A^2 (u \otimes_A u). \end{aligned}$$

Then, for $x \in \mathcal{H}$, the identity (18) gives

$$\langle [T^{\sharp_A}, T]x, x\rangle_A \quad = \quad \langle\Big(\Theta_A(N)v \otimes Av\Big)x, x\rangle + \langle\Big(Av \otimes \Theta_A(N)v\Big)x, x\rangle. \tag{19}$$

Then T is A-normal if and only if

$$\langle\Big(\Theta_A(N)v \otimes Av\Big)x, x\rangle = -\langle\Big(Av \otimes \Theta_A(N)v\Big)x, x\rangle,\ \forall\, x \in \mathcal{H}.$$

Equivalently, we have

$$A(v \otimes \Theta_A(N)v) = -(v \otimes \Theta_A(N)v)^* A,$$

and this identity holds if and only if $(v \otimes \Theta_A(N)v)$ is skew A-symmetric.
Now, assume that u, v are linearly independent and (16) holds true. According to Remark 1.5, note first that since $\mathcal{N}(A)$ is a reducing subspace for N, we have $NP = PN$ and $AP = PA = A$. Then for all $x \in \mathcal{H}$ we have

$$\begin{aligned}
&\langle [T^{\sharp_A}, T]x, x\rangle_A \\
=\ & \left\langle \Big((\overline{\sigma}Pu + \|u\|_A^2 \eta Pv) \otimes_A v + v \otimes_A (\overline{\sigma}Pu + \|u\|_A^2 \eta Pv) + \|u\|_A^2 v \otimes_A v \Big) x, x \right\rangle_A \\
& - \left\langle \Big((\sigma v + \|v\|_A^2 \overline{\eta} u) \otimes_A u + u \otimes_A (\sigma v + \|v\|_A^2 \overline{\eta} u) + \|v\|_A^2 u \otimes_A u \Big) x, x \right\rangle_A \\
=\ & \overline{\sigma}\langle x, v\rangle_A \langle u, x\rangle_A + \sigma \langle x, u\rangle_A \langle v, x\rangle_A + (2\mathfrak{Re}(\eta) + 1)\|u\|_A^2 \langle x, v\rangle_A \langle v, x\rangle_A \\
& - \sigma \langle x, u\rangle_A \langle v, x\rangle_A - \overline{\sigma}\langle x, v\rangle_A \langle u, x\rangle_A - (2\mathfrak{Re}(\eta) + 1)\|v\|_A^2 \langle x, u\rangle_A \langle u, x\rangle_A \\
=\ & 0.
\end{aligned}$$

Conversely, suppose that T is A-normal. If u and v are dependant, then obviously (19) implies that $(v \otimes \Theta_A(N)v)$ is skew A-symmetric. So, (i) holds true.
We assume now that u and v are linearly independent vectors. The identity (18) gives

$$\langle x, N^{\sharp_A}u\rangle_A \, v - \langle x, Nv\rangle_A \, u = 0, \quad x \in \left(Span\{u, v\}\right)^{\perp_A} = \left(Span\{Au, Av\}\right)^{\perp}$$

which implies that

$$\langle x, N^{\sharp_A}u\rangle_A = \langle x, Nv\rangle_A = 0, \quad \forall\, x \in \left(Span\{u, v\}\right)^{\perp_A}.$$

It follow from that

$$N^{\sharp_A}u = \alpha Pu + \beta Pv, \quad Nv = \gamma u + \sigma v. \tag{20}$$

On the other hand, we have

$$Pv \otimes_A P^*Pu(z) = \langle z, P^*APu\rangle = Pv \otimes_A P^*u(z), \quad \forall z \in \mathcal{H}.$$

By multiplying both sides of (17) by P and P^* and using (20), we obtain

$$\begin{aligned}
& \alpha Pu \otimes_A P^*v + \overline{\alpha} Pv \otimes_A P^*u + \left(\beta + \overline{\beta} + \|u\|_A^2\right) Pv \otimes_A P^*v \\
=\ & \overline{\sigma} Pu \otimes_A P^*v + \sigma Pv \otimes_A P^*u + \left(\gamma + \overline{\gamma} + \|v\|_A^2\right) Pu \otimes_A P^*u.
\end{aligned}$$

This last equality holds if

$$\alpha = \overline{\sigma}, \quad 2\mathfrak{Re}(\beta) + \|u\|_A^2 = 2\mathfrak{Re}(\gamma) + \|v\|_A^2 = 0.$$

Combining this with (20), there exists $\nu_1, \nu_2 \in \mathbb{R}$ such that

$$(N - \sigma)^{\sharp_A} u = (N^{\sharp_A} - \overline{\sigma}P)u = \left(-\frac{1}{2}\|u\|_A^2 + i\nu_1 \right) Pv \tag{21}$$

and

$$(N - \sigma)v = \left(-\frac{1}{2}\|v\|_A^2 + i\nu_2 \right) u. \tag{22}$$

Since N is A-normal and $NP = PN$ (see Remark 1.5), it follows that

$$(N - \sigma)^{\sharp_A}(N - \sigma)u = \left(-\frac{1}{2}\|u\|_A^2 + i\nu_1 \right)\left(-\frac{1}{2}\|v\|_A^2 + i\nu_2 \right) Pu.$$

Thus, since $(N - \sigma)$ is A-normal, the operator $(N - \sigma)^{\sharp_A}(N - \sigma)$ is A-positive, that is

$$\langle (N - \sigma)^{\sharp_A}(N - \sigma)u, u\rangle_A \;=\; \left(\frac{\|u\|_A^2 \|v\|_A^2}{4} - i(\nu_2 \|u\|_A^2 + \nu_1 \|v\|_A^2) - \nu_1\nu_2 \right) \|u\|_A^2 \;\geq 0$$

which implies that $\nu_2\|u\|_A^2 = -\nu_1\|v\|_A^2$. By using (21) and (22) we see that

$$(N-\sigma)^{\sharp_A}u = \|u\|_A^2\Big(-\frac{1}{2}+i\frac{\nu_1}{\|u\|_A^2}\Big)Pv$$

and

$$(N-\sigma)v = \|v\|_A^2\Big(-\frac{1}{2}-i\frac{\nu_1}{\|u\|_A^2}\Big)u.$$

Hence, u and v satisfies (16). $\square$

Corollary 4.9. *Let $N \in \mathcal{B}_A(\mathcal{H})$ be A-normal such that $\mathcal{N}(A)$ is a reducing subspace for N and set $T = N + u \otimes_A v \in \mathcal{B}_A(\mathcal{H})$. If u and v are dependant, then T is A-normal if and only if $v \in \mathcal{N}\Big(2\Im m(\lambda N^* A) + itA\Big)$ for some $t \in \mathbb{R}$.*

Proof. By using Theorem 4.8, we have

$$\begin{aligned}
[T^{\sharp_A}, T] = 0 \quad &\Longleftrightarrow \quad A\Big(v\otimes\Theta_A(N)v\Big) = -\Big(v\otimes\Theta_A(N)v\Big)^* A\\
&\Longleftrightarrow \quad (Av)\otimes\Theta_A(N)v = -\Theta_A(N)v\otimes(Av)\\
&\Longleftrightarrow \quad \Theta_A(N)v = -itAv \quad (\text{for some } t\in\mathbb{R})\\
&\Longleftrightarrow \quad v \in \mathcal{N}\Big(2\Im m(\lambda N^* A) + itA\Big) \quad (\text{for some } t\in\mathbb{R}).
\end{aligned}$$

$\square$

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