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Study on topological properties of Pythagorean neutrosophic normed space

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Abstract. We introduce the concept of Pythagorean neutrosophic normed spaces. We study that every Pythagorean neutrosophic normed space need not to be neutrosophic normed space. We define convergence of a sequence in Pythagorean neutrosophic normed spaces and Cauchy sequence. We define open ball, boundedness and compact with respect to PNNS. We induce a topology generated by these open balls. We prove that topology generated by these balls is Hausdorff.

1. Introduction

The introduction and development of the Pythagorean neutrosophic set (PNS) constitute a fascinating journey deeply rooted in the evolution of mathematical theories. To comprehend the emergence of PNS, it is essential to traverse through the milestones of fuzzy set theory, neutrosophic set theory, and the timeless influence of Pythagorean triples.

Firstly, the journey begins with the advent of fuzzy set theory in the 1960s. Lotfi Zadeh's [19] groundbreaking work transformed the landscape of mathematical modeling by introducing the concept of fuzzy sets, enabling the representation of uncertainty through degrees of membership. This foundational framework laid the groundwork for subsequent theories seeking to enhance our understanding of imprecise information.

Fuzzy norm spaces, an extension of traditional normed vector spaces, have emerged as a valuable mathematical framework to address uncertainty and imprecision. Rooted in the theory of fuzzy sets, these spaces replace crisp norms with fuzzy norms, allowing for a nuanced representation of vague information. The pioneering work of Katsaras [7], [8] and Liu [9] laid the foundation for fuzzy norm spaces, introducing the concept of a fuzzy norm and exploring its applications. Subsequent research by Kaleva and Seikkala [6] and Gähler[4] further expanded the theoretical underpinnings, providing insights into the convergence of sequences and continuity of operators in this novel setting. The interdisciplinary nature of fuzzy norm

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spaces finds applications in diverse fields, as exemplified by studies in optimization, control theory, and decision-making (Yager and Rybalov, 1996) [17]. This flexible mathematical framework serves as a powerful tool to model and analyze systems in which uncertainties play a pivotal role.

Intuitionistic fuzzy normed spaces extend classical fuzzy norm spaces, capturing uncertainty more comprehensively. Pioneered by Atanassov's intuitionistic fuzzy sets(1986)[2], this concept has evolved with contributions from Saadati et al[14]., refining mathematical foundations and finding applications in decision-making and pattern recognition.

In the 1990s, Florentin Smarandache extended the fuzzy set paradigm by introducing neutrosophic set theory [15]. Neutrosophic sets introduced a third component, indeterminacy, alongside truth and falsity, providing a more comprehensive tool for handling uncertain information. This evolution addressed the limitations of fuzzy sets, opening new avenues for researchers to explore the intricacies of ambiguity in various contexts. Kirişci, M., & Şimşek, N. studied neutrosophic norm and metric [12], [13]. Khan and Faisal studied more about neutrosophic norm [10], [11].

Parallelly, the Pythagorean triples, deeply rooted in ancient mathematical traditions, fascinated scholars for centuries due to their relationship with the Pythagorean theorem. The synthesis of these Pythagorean principles with fuzzy sets marked the genesis of the Pythagorean fuzzy set [16] in the early 21st century. This innovative integration sought to harness the mathematical elegance of Pythagorean relationships to enhance the descriptive power of neutrosophic sets.

The Pythagorean neutrosophic set [5] thus stands at the nexus of these mathematical progressions, combining the fuzzy set's foundations, the nuanced representation of neutrosophic sets, and the timeless principles of Pythagorean triples. The integration of Pythagorean concepts into neutrosophic sets not only expanded their theoretical framework but also offered a versatile tool for addressing uncertainty in decision-making, pattern recognition, and artificial intelligence.

Pythagorean neutrosophic set represents a culmination of mathematical thought, where ideas from fuzzy sets, neutrosophic sets, and Pythagorean triples converge to create a powerful framework capable of handling complex and uncertain information in a structured and elegant manner. This evolutionary journey showcases the collaborative efforts of mathematicians across different disciplines, pushing the boundaries of mathematical theory to address real-world challenges [1], [3], [18]. We will define Pythagorean neutrosophic norm and study its topological properties.

Definition 1.1. *[15] Let $\mathcal{V}$ be a non fuzzy set (fixed). A neutrosophic set A in $\mathcal{V}$ is of the form*

$$A = \left\{ \langle \nu, \mathcal{F}_A(\nu), \mathcal{G}_A(\nu), \mathcal{H}_A(\nu) \rangle \middle| \nu \in \mathcal{V} \right\} \tag{1}$$

where

$$\mathcal{F}_A : \mathcal{V} \to [0,1], \ \ \mathcal{G}_A : \mathcal{V} \to [0,1] \ \ \text{and} \ \ \mathcal{H}_A : \mathcal{V} \to [0,1]$$

be membership, non-membership and hesitancy functions respectively. If no confusion arises throughout our paper if, we say that A is a neutrosophic set in $\mathcal{V}$, then this is as in equation 1.

Definition 1.2. *[5] Let A be a neutrosophic set in $\mathcal{V}$ then A is said to be Pythagorean neutrosophic set in $\mathcal{V}$, if for all $\nu \in \mathcal{V}$,*

(a) $0 \le \mathcal{F}_A^2(\nu) + \mathcal{G}_A^2(\nu) + \mathcal{H}_A^2(\nu) \le 2$;
(b) $0 \le \mathcal{F}_A^2(\nu) + \mathcal{H}_A^2(\nu) \le 1$;
(c) $0 \le \mathcal{G}_A(\nu) \le 1$.

Definition 1.3. *[12] 6-tuple $(\mathcal{V}, \mathcal{F}, \mathcal{G}, \mathcal{H}, \star, \diamond)$ is said to be neutrosophic normed space if $\mathcal{V}$ is a vector space over a field $\mathbb{F}$ (real or complex), $\star$ is continuous $t-$ norm, $\diamond$ is continuous $t-$ conorm and $\mathcal{F}, \mathcal{G}, \mathcal{H}$ are fuzzy sets in $\mathcal{V} \times (0,\infty)$ satisfying the following conditions; for any $\nu \in \mathcal{V}$, $ŧ > 0$, $\alpha \in \mathbb{F}$*

(a) $\mathcal{F}(\nu, ŧ) > 0$,

(b) $\mathcal{F}(\nu, ł) = 1$ *if and only if* $\nu = 0$,
(c) $\mathcal{F}(\alpha\nu, ł) = \mathcal{F}(\nu, \frac{ł}{|\alpha|})$ *for* $\alpha \neq 0$
(d) $\mathcal{F}(\nu, s) \star \mathcal{F}(\mu, ł) \leq \mathcal{F}(\nu + \mu, s + ł)$
(e) $\mathcal{F}(\nu, .)$ *is continuous and non-decreasing*
(f) $\lim_{ł\to\infty} \mathcal{F}(\nu, ł) = 1$,
(g) $\mathcal{G}(\nu, ł) > 0$
(h) $\mathcal{G}(\nu, ł) = 0$ *if and only if* $\nu = 0$,
(i) $\mathcal{G}(\alpha\nu, ł) = \mathcal{G}(\nu, \frac{ł}{|\alpha|})$ *for* $\alpha \neq 0$,
(j) $\mathcal{G}(\nu, s) \diamond \mathcal{G}(\mu, ł) \geq \mathcal{G}(\nu + \mu, s + ł)$
(k) $\mathcal{G}(\nu, .)$ *is continuous and non-increasing,*
(l) $\lim_{ł\to\infty} \mathcal{G}(\nu, ł) = 0$,
(m) $\mathcal{H}(\nu, ł) > 0$
(n) $\mathcal{H}(\nu, ł) = 0$ *if and only if* $\nu = 0$,
(o) $\mathcal{H}(\alpha\nu, ł) = \mathcal{H}(\nu, \frac{ł}{|\alpha|})$ *if* $\alpha \neq 0$,
(p) $\mathcal{H}(\nu, s) \diamond \mathcal{H}(\mu, ł) \geq \mathcal{H}(\nu + \mu, s + ł)$,
(q) $\mathcal{H}(\nu, .)$ *is continuous and non-increasing,*
(r) $\lim_{ł\to\infty} \mathcal{H}(\nu, ł) = 0$,

3-tuple $(\mathcal{F}, \mathcal{G}, \mathcal{H})$ *is called neutrosophic norm on* $\mathcal{V}$. *We will denote neutrosophic norm by* $[[.]] = (\mathcal{F}, \mathcal{G}, \mathcal{H})$.

2. Main Results

In this section we will define the notion of Pythagorean neutrosophic norm(Pnn) on Pythagorean neutrosophic set(Pns).

Definition 2.1. *Let* $\mathcal{V}$ *denote a vector space over a field F, and let A represent a Pythagorean neutrosophic set in* $\mathcal{V} \times (0, \infty)$. *Suppose* $\star$ *and* $\diamond$ *are continuous t-norm and continuous t-conorm respectively. A triplet* $(\mathcal{F}, \mathcal{G}, \mathcal{H})$ *is termed a Pythagorean neutrosophic norm on* $\mathcal{V}$ *if the membership function* $\mathcal{F}$, *non-membership function* $\mathcal{G}$, *and indeterminacy function* $\mathcal{H}$ *satisfy the following conditions for* $\nu \in \mathcal{V}$ *and* $ł > 0$

(a) $0 \leq \mathcal{F}^2(\nu, ł) + \mathcal{G}^2(\nu, ł) + \mathcal{H}^2(\nu, ł) \leq 2$
(b) $0 \leq \mathcal{F}^2(\nu, ł) + \mathcal{H}^2(\nu, ł) \leq 1$ *for all* $ł \in \mathbb{R}^+$,
(c) $\mathcal{F}(\nu, ł) > 0$,
(d) $\mathcal{F}(\nu, ł) = 1$ *if and only if* $\nu = 0$,
(e) $\mathcal{F}(\alpha\nu, ł) = \mathcal{F}(\nu, \frac{ł}{|\alpha|})$ *for* $\alpha \neq 0$
(f) $\mathcal{F}(\nu, s) \star \mathcal{F}(\mu, ł) \leq \mathcal{F}(\nu + \mu, s + ł)$
(g) $\mathcal{F}(\nu, .)$ *is continuous and non-decreasing*
(h) $\lim_{ł\to\infty} \mathcal{F}(\nu, ł) = 1$,
(i) $\mathcal{G}(\nu, ł) > 0$,
(j) $\mathcal{G}(\nu, ł) = 0$ *for* $(ł > 0)$ *if and only if* $\nu = 0$,
(k) $\mathcal{G}(\alpha\nu, ł) = \mathcal{G}(\nu, \frac{ł}{|\alpha|})$ *for* $\alpha \neq 0$,
(l) $\mathcal{G}(\nu, s) \diamond \mathcal{G}(\mu, ł) \geq \mathcal{G}(\nu + \mu, s + ł)$
(m) $\mathcal{G}(\nu, .)$ *is continuous and non-increasing,*
(n) $\lim_{ł\to\infty} \mathcal{G}(\nu, ł) = 0$,
(o) $\mathcal{H}(\nu, ł) > 0$
(p) $\mathcal{H}(\nu, ł) = 0$ *(for* $ł > 0$*) if and only if* $\nu = 0$,
(q) $\mathcal{H}(\alpha\nu, ł) = \mathcal{H}(\nu, \frac{ł}{|\alpha|})$ *if* $\alpha \neq 0$,
(r) $\mathcal{H}(\nu, s) \diamond \mathcal{H}(\mu, ł) \geq \mathcal{H}(\nu + \mu, s + ł)$,
(s) $\mathcal{H}(\nu, .)$ *is continuous and non-increasing,*

(t) $\lim_{ŧ\to\infty} \mathcal{H}(\nu, ŧ) = 0$,

From the definition 2.1 of Pythagorean neutrosophic norm, we can say that every Pythagorean neutrosophic normed space is neutrosophic normed space.

Example 2.2. *Consider $(\mathcal{V}, \|.\|)$ as a normed space over the real numbers. Assume that $\star$ and $\diamond$ are continuous t-norm and continuous t-conorm operations, where $\star$ is defined as the product $a \star b = ab$, and $\diamond$ is defined as $a \diamond b = a + b - ab$ for all $a, b \in [0,1]$. Let $\mathcal{F}, \mathcal{G}$, and $\mathcal{H}$ denote fuzzy sets on $\mathcal{V} \times (0,\infty)$, defined as follows: For $\nu \in \mathcal{V}$, consider $ŧ > \|\nu\|$ and $ŧ > 0$*

$$\mathcal{F}(\nu, ŧ) = \frac{ŧ}{ŧ + \|\nu\|}, \quad \mathcal{G}(\nu, ŧ) = \frac{\|\nu\|}{ŧ + \|\nu\|}, \quad \mathcal{H}(\nu, ŧ) = \frac{\|\nu\|}{ŧ + \|\nu\|}. \tag{2}$$

If $\nu = 0$,

$$\mathcal{F}(\nu, ŧ) = 1 \quad \mathcal{G}(\nu, ŧ) = 0 \quad \mathcal{H}(\nu, ŧ) = 0. \tag{3}$$

Then $[[.]] = (\mathcal{F}, \mathcal{G}, \mathcal{H})$ is Pythagorean neutrosophic norm on $\mathcal{V}$. For $\nu \in \mathcal{V}$ such that $ŧ \geq \|\nu\|$, since $\mathcal{F}(\nu, ŧ) \geq 0$ and $\mathcal{H}(\nu, ŧ) \geq 0$, we have

$$\mathcal{F}^2(\nu, ŧ) + \mathcal{H}^2(\nu, ŧ) \geq 0$$

and

$$\begin{aligned}
\mathcal{F}^2(\nu, ŧ) + \mathcal{H}^2(\nu, ŧ) &= \frac{ŧ^2}{(ŧ + \|\nu\|)^2} + \frac{\|\nu\|^2}{(ŧ + \|\nu\|)^2} \\
&= \frac{ŧ^2 + \|\nu\|^2}{(ŧ + \|\nu\|)^2} \\
&= \frac{ŧ^2 + \|\nu\|^2 + 2ŧ\|\nu\| - 2ŧ\|\nu\|}{(ŧ + \|\nu\|)^2} \\
&= \frac{(ŧ + \|\nu\|)^2 - 2ŧ\|\nu\|}{(ŧ + \|\nu\|)^2} \\
&= 1 - \frac{2ŧ\|\nu\|}{(ŧ + \|\nu\|)^2}
\end{aligned}$$

$$\mathcal{F}^2(\nu, ŧ) + \mathcal{H}^2(\nu, ŧ) \leq 1.$$

$$\begin{aligned}
\mathcal{F}(\nu, ŧ)^2 + \mathcal{G}(\nu, ŧ)^2 + \mathcal{H}(\nu, ŧ)^2 &= \frac{ŧ^2}{(ŧ + \|\nu\|)^2} + \frac{\|\nu\|^2}{(ŧ + \|\nu\|)^2} + \frac{\|\nu\|^2}{(ŧ + \|\nu\|)^2} \\
&= \frac{ŧ^2}{(ŧ + \|\nu\|)^2} + 2\frac{\|\nu\|^2}{(ŧ + \|\nu\|)^2} \\
&= \frac{ŧ^2 + 2\|\nu\|^2}{(ŧ + \|\nu\|)^2} \\
&= \frac{ŧ^2 + \|\nu\|^2 + 2ŧ\|\nu\| - 2ŧ\|\nu\| + \|\nu\|^2}{(ŧ + \|\nu\|)^2} \\
&= \frac{(ŧ + \|\nu\|)^2 - 2ŧ\|\nu\| + \|\nu\|^2}{(ŧ + \|\nu\|)^2} \\
&= 1 + \frac{\|\nu\|^2 - 2ŧ\|\nu\|}{(ŧ + \|\nu\|)^2} \\
&= 1 + \frac{\|\nu\|(\|\nu\| - 2ŧ)}{(ŧ + \|\nu\|)^2}
\end{aligned}$$

Since, $\text{ł} \geq \|\nu\|$*, we have* $2\text{ł} \geq \|\nu\| \implies \|\nu\| - 2\text{ł} \leq 0 \implies \dfrac{\|\nu\|(\|\nu\| - 2\text{ł})}{(\text{ł} + \|\nu\|)^2} \leq 0$*. Hence, we get*

$$\mathcal{F}(\nu, \text{ł})^2 + \mathcal{G}(\nu, \text{ł})^2 + \mathcal{H}(\nu, \text{ł})^2 \leq 1 < 2.$$

So, $[[.]] = (\mathcal{F}, \mathcal{G}, \mathcal{H})$ *is Pythagorean neutrosophic norm on* $\mathcal{V}$.

Now, we will give an example of neutrosophic normed space which is not Pythagorean neutrosophic normed space.

Example 2.3. *Let* $(\mathcal{V}, \|.\|)$ *be a normed over the reals. Let* $\star$ *and* $\diamond$ *be continuous* $t-$*norm and continuous* $t-$*conorm respectively. Let* $\mathcal{F}$, $\mathcal{G}$ *and* $\mathcal{H}$ *be fuzzy sets on* $\mathcal{V} \times (0, 1)$ *and defined as follows, For* $\nu \in \mathcal{V}$, $\text{ł} > 0$ *and* $\|\nu\| < \text{ł}$

$$\mathcal{F}(\nu, \text{ł}) = \frac{\text{ł}}{\text{ł} + \|\nu\|}, \ \mathcal{G}(\nu, \text{ł}) = \frac{\|\nu\|}{\text{ł}}, \ \mathcal{H}(\nu, \text{ł}) = \frac{\|\nu\|}{\text{ł}}.$$

Let $\star(a, b) = \max\{a, b\}$ *and* $\diamond(a, b) = \min\{a, b\}$*, for all* $a, b \in [0, 1]$*. Then* $(\mathcal{F}, \mathcal{G}, \mathcal{H})$ *is neutrosophic norm on* $\mathcal{V}$*. Since,*

$$\mathcal{F}(\nu, \text{ł})^2 + \mathcal{G}(\nu, \text{ł})^2 + \mathcal{H}(\nu, \text{ł})^2 \leq \frac{\text{ł}^2}{(\text{ł} + \|\nu\|)^2} + 1 + 1 \leq 3.$$

Therefore $(\mathcal{F}, \mathcal{G}, \mathcal{H})$ *is not a Pythagorean neutrosophic norm on* $\mathcal{V}$.

3. Topology induced by Pythagorean neutrosophic norm

In this section, we will define open ball with the help of Pythagorean neutrosophic norm.

Definition 3.1. *Let* $(\mathcal{V}, \mathcal{F}, \mathcal{G}, \mathcal{H}, \star, \diamond)$ *be a Pythagorean neutrosophic normed space. Then for* $\nu \in \mathcal{V}$ *and* $\delta \in (0, 1)$*, the set*

$$B_{[[.]]}(\nu, \delta, \text{ł}) = \left\{ \mu \in \mathcal{V} : \mathcal{F}(\nu - \mu, \text{ł}) > 1 - \delta, \ \mathcal{G}(\nu - \mu, \text{ł}) < \delta, \ \mathcal{H}(\nu - \mu, \text{ł}) < \delta \right\} \tag{4}$$

is called open ball with center ν*, radius* δ.

Definition 3.2. *Let* $(\mathcal{V}, \mathcal{F}, \mathcal{G}, \mathcal{H}, \star, \diamond)$ *be a Pythagorean neutrosophic normed space. Let* $\mathcal{T}$ *be the collection of open balls with center* ν *and radius* δ

$$\mathcal{T} = \left\{ B_{[[.]]}(\nu, \delta, \text{ł}) : \nu \in \mathcal{V}, \text{ł and } \delta \in (0, 1) \right\} \tag{5}$$

Definition 3.3. *Let* M *be a subset of a neutrosophic normed space* $\mathcal{V}$*,* M *is said to be open set if it contains an open ball around each of its points.*

Definition 3.4. *Let* $(\mathcal{V}, \mathcal{F}, \mathcal{G}, \mathcal{H}, \star, \diamond)$ *be a PNNS and* A *be a subset of* $(\mathcal{V}, \mathcal{F}, \mathcal{G}, \mathcal{H}, \star, \diamond)$*. Then* A *is called Pythagorean neutrosophic-bounded (PNB), if there exist* $\text{ł} > 0$ *and* $\delta \in (0, 1)$ *such that* $\mathcal{F}(a, \text{ł}) > 1 - \delta$, $\mathcal{G}(a, \text{ł}) < \delta$ *and* $\mathcal{H}(a, \text{ł}) < \delta$ $(\forall a \in A)$.

Definition 3.5. *Let* $(\mathcal{V}, \mathcal{F}, \mathcal{G}, \mathcal{H}, \star, \diamond)$ *be a PNNS and* $\nu = (\nu_n)$ *a sequence in* $(\mathcal{V}, \mathcal{F}, \mathcal{G}, \mathcal{H}, \star, \diamond)$*. Then* $\nu = (\nu_n)$ *is said to be convergent to* ℓ*, if for each* $\delta \in (0, 1)$ *and* $\text{ł} > 0$ *there exist positive integer* n_0 *such that* $\nu_n \in B_{[[.]]}(\ell, \delta, \text{ł})$ *for all* $n \geq n_0$.

Definition 3.6. *Let* $\nu = (\nu_n)$ *be a sequence in PNNS* $(\mathcal{V}, \mathcal{F}, \mathcal{G}, \mathcal{H}, \star, \diamond)$*. Then* $\nu = (\nu_n)$ *is said to be Cauchy, if for each* $\delta \in (0, 1)$ *and* $\text{ł} > 0$ *there exist positive integer* N *such that* $\nu_n \in B_{[[.]]}(\nu_m, \delta, \text{ł})$ *for all* $n, m \geq N$*. PNNS* $(\mathcal{V}, \mathcal{F}, \mathcal{G}, \mathcal{H}, \star, \diamond)$ *is complete if each Cauchy sequence is convergent.*

Definition 3.7. *The collection $A = \{B : B$ is an open ball, $\nu \in \mathcal{V}, \delta \in (0,1), ŧ0\}$ in a Pythagorean neutrosophic normed space $(\mathcal{V}, \mathcal{F}, \mathcal{G}, \mathcal{H}, \star, \diamond)$ is said to be open cover of a subspace M of $(\mathcal{V}, \mathcal{F}, \mathcal{G}, \mathcal{H}, \star, \diamond)$ if*

$$M \subseteq \bigcup_{B \in A} B.$$

Definition 3.8. *A subspace A of a Pythagorean neutrosophic normed space $(\mathcal{V}, \mathcal{F}, \mathcal{G}, \mathcal{H}, \star, \diamond)$ is said to be compact if every open cover of A has a finite subcover.*

Theorem 3.9. *Every open ball $B_{[[.]]}(\mathcal{V}, \delta, ŧ)$ is an open set.*

Proof. Let $B_{[[.]]}(\mathcal{V}, \delta, ŧ)$ be an open ball with center ν and radius δ, for $ŧ > 0, 0 < \delta < 1$. Choose $\mu \in B_{[[.]]}(\nu, \delta, ŧ)$, by the definition of open ball, we have

$$\mathcal{F}(\nu - \mu, ŧ) > 1 - \delta, \quad \mathcal{G}(\nu - \mu, ŧ) < \delta \quad \text{and} \quad \mathcal{H}(\nu - \mu, ŧ) < \delta.$$

Since $\mathcal{F}(\nu - \mu, ŧ) > 1 - \delta$ there exists $ŧ_0 \in (0, ŧ)$ such that

$$\mathcal{F}(\nu - \mu, ŧ_0) > 1 - \delta, \quad \mathcal{G}(\nu - \mu, ŧ_0) < \delta \quad \text{and} \quad \mathcal{H}(\nu - \mu, ŧ_0) < \delta.$$

Since μ is fixed and $ŧ_0$ is in $(0, ŧ)$, let us suppose that $\mathcal{F}(\nu - \mu, ŧ_0) = \delta_0$. Then there exists $\xi \in (0,1)$ such that $\delta_0 > 1 - \xi > 1 - \delta$. Then we have δ_1, δ_2 and δ_3 such that

$$\delta_0 \star \delta_1 > 1 - \xi$$

$$(1 - \delta_0) \diamond (1 - \delta_2) \leq \xi$$

$$(1 - \delta_0) \diamond (1 - \delta_3) \leq \xi.$$

Let $\delta_4 = \max\{\delta_1, \delta_2, \delta_3\}$. Now our aim is to show that

$$B_{[[.]]}(\mu, 1 - \delta_4, ŧ - ŧ_0) \subseteq B_{[[.]]}(\nu, \delta, ŧ) \tag{6}$$

Suppose $z \in B_{[[.]]}(\mu, 1 - \delta_4, ŧ - ŧ_0)$ then we have

$$\mathcal{F}(z - \mu, ŧ - ŧ_0) > \delta_4, \quad \mathcal{G}(z - \mu, ŧ) < 1 - \delta_4 \quad \text{and} \quad \mathcal{H}(z - \mu, ŧ) < 1 - \delta_4$$

Now we have

$$\mathcal{F}(\nu - z, ŧ) \geq \mathcal{F}(\nu - \mu, ŧ_0) \star \mathcal{F}(\mu - z, ŧ - ŧ_0) \geq \delta_0 \star \delta_4 \geq \delta_0 \star \delta_1 \geq 1 - \xi > 1 - \delta.$$

$$\mathcal{G}(\nu - z, ŧ) \leq \mathcal{G}(\nu - \mu, ŧ_0) \diamond \mathcal{G}(\mu - z, ŧ - ŧ_0) \leq (1 - \delta_0) \diamond (1 - \delta_4) \leq (1 - \delta_0) \diamond (1 - \delta_2) \leq \xi < \delta$$

$$\mathcal{H}(\nu - z, ŧ) \leq \mathcal{H}(\nu - \mu, ŧ_0) \diamond \mathcal{H}(\mu - z, ŧ - ŧ_0) \leq (1 - \delta_0) \diamond (1 - \delta_4) \leq (1 - \delta_0) \diamond (1 - \delta_3) \leq \xi < \delta$$

From the above, we get $z \in B_{[[.]]}(\nu, \delta, ŧ)$. Hence $B_{[[.]]}(\mu, 1 - \delta_4, ŧ - ŧ_0) \subseteq B_{[[.]]}(\nu, \delta, ŧ)$. This proves that every open ball is an open set in Pythagorean neutrosophic normed space. □

Theorem 3.10. *Let $(\mathcal{V}, \mathcal{F}, \mathcal{G}, \mathcal{H}, \star, \diamond)$ be a Pythagorean neutrosophic normed space. Then topology $\mathcal{T}$ generated as 5 is Hausdorff.*

Proof. Let ν and μ be two distinct points in Pythagorean neutrosophic normed space $(\mathcal{V}, \mathcal{F}, \mathcal{G}, \mathcal{H}, \star, \diamond)$. Then by the definition of Pythagorean neutrosophic norm for $ŧ > 0$, we have

$$0 \leq \mathcal{F}^2(\nu - \mu, ŧ) + \mathcal{G}^2(\nu - \mu, ŧ) + \mathcal{H}^2(\nu - \mu, ŧ) \leq 2;$$

$$0 \leq \mathcal{F}^2(\nu - \mu, ŧ) + \mathcal{H}^2(\nu - \mu, ŧ) \leq 1;$$

$0 \leq \mathcal{G}(\nu - \mu, \mathfrak{t}) \leq 1.$

Let $\delta_1 = \mathcal{F}(\nu - \mu, \mathfrak{t})$, $\delta_2 = \mathcal{G}(\nu - \mu, \mathfrak{t})$, $\delta_3 = \mathcal{H}(\nu - \mu, \mathfrak{t})$ and $\delta = \max\{\delta_1, 1 - \delta_2, 1 - \delta_3\}$

$$\delta_1^2 + \delta_2^2 + \delta_3^2 \leq 2;$$

$$\delta_1^2 + \delta_3^2 \leq 1;$$

$$0 \leq \delta_2 \leq 1.$$

Choose $\delta_0 \in (\delta, 1)$. There exists δ_4, δ_5 and δ_6 such that

$$\delta_4 \star \delta_4 \geq \delta_0$$

$$(1 - \delta_5) \diamond (1 - \delta_5) \leq 1 - \delta_0$$

$$(1 - \delta_6) \diamond (1 - \delta_6) \leq 1 - \delta_0.$$

If we take $\delta_7 = \max\{\delta_4, \delta_5, \delta_6\}$ and construct two open balls $B_{[[.]]}(\nu, 1 - \delta_7, \mathfrak{t})$ and $B_{[[.]]}(\mu, 1 - \delta_7, \mathfrak{t})$. Then these two balls are disjoint. Let us suppose that $c \in B_{[[.]]}(\nu, 1 - \delta_7, \mathfrak{t}) \cap B_{[[.]]}(\mu, 1 - \delta_7, \mathfrak{t})$. Then

$$\begin{aligned} \delta_1 = \mathcal{F}(\nu - \mu, \mathfrak{t}) \geq \mathcal{F}(\nu - c, \frac{\mathfrak{t}}{2}) \star \mathcal{F}(c - \mu, \frac{\mathfrak{t}}{2}) &\geq \delta_7 \star \delta_7 \\ &\geq \delta_4 \star \delta_4 \\ &\geq \delta_0 > \delta_1 \end{aligned}$$

$$\begin{aligned} \delta_2 = \mathcal{G}(\nu - \mu, \mathfrak{t}) \leq \mathcal{G}(\nu - c, \frac{\mathfrak{t}}{2}) \diamond \mathcal{G}(c - \mu, \frac{\mathfrak{t}}{2}) &\leq (1 - \delta_7) \diamond (1 - \delta_7) \\ &\leq (1 - \delta_5) \diamond (1 - \delta_5) \\ &\leq (1 - \delta_0) < \delta_2 \end{aligned}$$

$$\begin{aligned} \delta_3 = \mathcal{H}(\nu - \mu, \mathfrak{t}) \leq \mathcal{H}(\nu - c, \frac{\mathfrak{t}}{2}) \diamond \mathcal{G}(c - \mu, \frac{\mathfrak{t}}{2}) &\leq (1 - \delta_7) \diamond (1 - \delta_7) \\ &\leq (1 - \delta_6) \diamond (1 - \delta_6) \\ &\leq (1 - \delta_0) < \delta_3 \end{aligned}$$

which is a contradiction. Hence balls $B_{[[.]]}(\nu, 1 - \delta_7, \mathfrak{t})$ and $B_{[[.]]}(\mu, 1 - \delta_7, \mathfrak{t})$. are disjoint. This Proves that topology generated by Pythagorean neutrosophic norm is Hausdorff. □

Theorem 3.11. *Every compact subset of a Pythagorean neutrosophic normed space is PNB.*

Proof. Let $(\mathcal{V}, \mathcal{F}, \mathcal{G}, \mathcal{H}, \star, \diamond)$ be a PNNS and M be a compact subset of $(\mathcal{V}, \mathcal{F}, \mathcal{G}, \mathcal{H}, \star, \diamond)$. Let $C = \{B_{[[.]]}(\nu, \delta, \mathfrak{t}) : \nu \in M\}$ be an open cover of M.

$$M \subseteq \bigcup_{B_{[[.]]}(\nu, \mathfrak{t}) \in C} B_{[[.]]}(\nu, \delta, \mathfrak{t}). \tag{7}$$

Since M is compact, open cover C has a finite sub-cover $\mathcal{SC} = \{B_{[[.]]}(\nu_k, \delta, \mathfrak{t}) : \nu_k \in M\}$ i.e., there exists $\nu_1, \nu_2, ..., \nu_k$ in M such that

$$M \subseteq \bigcup_{i=1}^{k} B_{[[.]]}(\nu_i, \delta, \mathfrak{t}).$$

For some m and n and $\mu, z \in M$, we have $\mu \in B_{[[.]]}(\nu_m, \delta, \mathfrak{t})$ and $z \in B_{[[.]]}(\nu_n, \delta, \mathfrak{t})$. This implies that

$$\mathcal{F}(\mu - \nu_m, \mathfrak{t}) > 1 - \delta$$

$$\mathcal{G}(\mu - \nu_m, \mathfrak{t}) < \delta$$
$$\mathcal{H}(\mu - \nu_m, \mathfrak{t}) < \delta.$$

Similarly,

$$\mathcal{F}(z - \nu_n, \mathfrak{t}) > 1 - \delta$$
$$\mathcal{F}(z - \nu_n, \mathfrak{t}) < \delta$$
$$\mathcal{F}(z - \nu_n, \mathfrak{t}) < \delta.$$

Now, let $\delta_1 = \min\{\mathcal{F}(\nu_m - \nu_n, \mathfrak{t}) : 1 \le m, n \le k\}$, $\delta_2 = \max\{\mathcal{G}(\nu_m - \nu_n, \mathfrak{t}) : 1 \le m, n \le k\}$ and $\delta_3 = \max\{\mathcal{H}(\nu_m - \nu_n, \mathfrak{t}) : 1 \le m, n \le k\}$.

$$\begin{aligned}
\mathcal{F}(\nu - \mu, \mathfrak{t}) &= \mathcal{F}(\nu - \nu_m + \nu_m + \mu_n - \mu_n - \mu, 3\mathfrak{t}) \\
&= \mathcal{F}(\nu - \nu_m + \nu_m - \mu_n + \mu_n - \mu, \mathfrak{t} + \mathfrak{t} + \mathfrak{t}) \\
&\ge \mathcal{F}(\nu - \nu_m, \mathfrak{t}) \star \mathcal{F}(\nu_m - \mu_n, \mathfrak{t}) \star \mathcal{F}(\mu - \mu_n, \mathfrak{t}) \\
&\ge (1 - \delta) \star \delta_1 \star (1 - \delta) \\
&\ge (1 - \delta) \star (1 - \delta) \star \delta_1 \\
&> 1 - \xi_1.
\end{aligned}$$

$$\begin{aligned}
\mathcal{G}(\nu - \mu, \mathfrak{t}) &= \mathcal{G}(\nu - \nu_m + \nu_m + \mu_n - \mu_n - \mu, 3\mathfrak{t}) \\
&= \mathcal{G}(\nu - \nu_m + \nu_m - \mu_n + \mu_n - \mu, \mathfrak{t} + \mathfrak{t} + \mathfrak{t}) \\
&\le \mathcal{G}(\nu - \nu_m, \mathfrak{t}) \diamond \mathcal{G}(\nu_m - \mu_n, \mathfrak{t}) \diamond \mathcal{G}(\mu - \mu_n, \mathfrak{t}) \\
&\le \delta \diamond \delta_2 \diamond \delta \\
&\le \delta \diamond \delta \diamond \delta_2 \\
&< \xi_2.
\end{aligned}$$

$$\begin{aligned}
\mathcal{H}(\nu - \mu, \mathfrak{t}) &= \mathcal{H}(\nu - \nu_m + \nu_m + \mu_n - \mu_n - \mu, 3\mathfrak{t}) \\
&= \mathcal{H}(\nu - \nu_m + \nu_m - \mu_n + \mu_n - \mu, \mathfrak{t} + \mathfrak{t} + \mathfrak{t}) \\
&\le \mathcal{H}(\nu - \nu_m, \mathfrak{t}) \diamond \mathcal{H}(\nu_m - \mu_n, \mathfrak{t}) \diamond \mathcal{H}(\mu - \mu_n, \mathfrak{t}) \\
&\le \delta \diamond \delta_3 \diamond \delta \\
&\le \delta \diamond \delta \diamond \delta_3 \\
&< \xi_3.
\end{aligned}$$

Let $\xi = \max\{\xi_1, \xi_2, \xi_3\}$ and $\mathfrak{t}_0 = 3\mathfrak{t}$, we have

$$\mathcal{F}(\nu - \mu, \mathfrak{t}) > 1 - \xi$$
$$\mathcal{G}(\nu - \mu, \mathfrak{t}) < \xi$$
$$\mathcal{F}(\nu - \mu, \mathfrak{t}) < \xi.$$

This completes our proof that every compact subset in a PNNS is PNB. □

Theorem 3.12. *Let $(\mathcal{V}, \mathcal{F}, \mathcal{G}, \mathcal{H}, \star, \diamond)$ be a complete PNNS and $(\mathcal{U}, \mathcal{F}, \mathcal{G}, \mathcal{H}, \star, \diamond)$ be a subspace of $(\mathcal{V}, \mathcal{F}, \mathcal{G}, \mathcal{H}, \star, \diamond)$. Then $(\mathcal{U}, \mathcal{F}, \mathcal{G}, \mathcal{H}, \star, \diamond)$ is complete if and only if $(\mathcal{U}, \mathcal{F}, \mathcal{G}, \mathcal{H}, \star, \diamond)$ is closed subspace of $(\mathcal{V}, \mathcal{F}, \mathcal{G}, \mathcal{H}, \star, \diamond)$.*

Proof. Let $(\mathcal{U}, \mathcal{F}, \mathcal{G}, \mathcal{H}, \star, \diamond)$ be a closed subspace of $(\mathcal{V}, \mathcal{F}, \mathcal{G}, \mathcal{H}, \star, \diamond)$. Let $\nu = (\nu_n)$ be a Cauchy sequence in $(\mathcal{U}, \mathcal{F}, \mathcal{G}, \mathcal{H}, \star, \diamond)$. Then $\nu = (\nu_n)$ be Cauchy in $(\mathcal{V}, \mathcal{F}, \mathcal{G}, \mathcal{H}, \star, \diamond)$. Since $(\mathcal{V}, \mathcal{F}, \mathcal{G}, \mathcal{H}, \star, \diamond)$ is complete there exists $\ell \in (\mathcal{V}, \mathcal{F}, \mathcal{G}, \mathcal{H}, \star, \diamond)$ such that $\nu = (\nu_n)$ will converge to ℓ. Since $(\mathcal{U}, \mathcal{F}, \mathcal{G}, \mathcal{H}, \star, \diamond)$ is closed, $\ell \in (\mathcal{U}, \mathcal{F}, \mathcal{G}, \mathcal{H}, \star, \diamond)$. Hence every Cauchy sequence in $(\mathcal{U}, \mathcal{F}, \mathcal{G}, \mathcal{H}, \star, \diamond)$ is convergent in $(\mathcal{U}, \mathcal{F}, \mathcal{G}, \mathcal{H}, \star, \diamond)$. Hence $(\mathcal{U}, \mathcal{F}, \mathcal{G}, \mathcal{H}, \star, \diamond)$ is complete.

Suppose on the contrary that, $(\mathcal{U},\mathcal{F},\mathcal{G},\mathcal{H},\star,\diamond)$ is complete but not closed. Since Y is not closed i.e. all limit points of Y are not in Y. This implies there exists a $\ell_1 \in \overline{Y}/Y$. Let $\nu = (\nu_n)$ be a sequence in $(\mathcal{U},\mathcal{F},\mathcal{G},\mathcal{H},\star,\diamond)$ such that sequence (ν_n) converges to ℓ_1 with respect to PNN. Hence, for all $ƚ > 0$ and $\delta \in (0,1)$ there exists a positive integer N_0 such that

$$\nu_n \in B_{[[.]]}(\ell_1, \delta, ƚ) \text{ for all } n \geq N_0.$$

Hence $\nu = (\nu_n)$ is a Cauchy sequence in Y. Since, Y is complete there exists $\ell_2 \in Y$ such that sequence (ν_n) will converge to ℓ_2 with respect to PNN. Hence, we have that sequence converges to ℓ_1 and ℓ_2. Since $\ell_1 \notin Y$ and $\ell_2 \in Y$, we have $\ell_1 \neq \ell_2$. This is a contradiction. Hence Y is closed. □

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