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Inner estimates of the solution set of interval linear systems with precise coefficient matrices

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Abstract. Finding "the best" interval solution for an interval system of linear equations is a problem known to be NP-hard. This paper provides a full characterization of inner interval estimates of the united solution set of formally consistent interval linear systems, where the coefficient matrix of arbitrary size is precise. This characterization is based on the generalized inverses of the real coefficient matrices. A straightforward approach for obtaining the united inner solution set of systems of linear interval equations with real coefficient matrices of arbitrary size is presented, accompanied by illustrative examples. Additionally, a necessary and sufficient condition for the equality of inner estimates and the maximal inner estimates of such systems is provided.

1. Introduction

The concept of generalized inverses plays a central role in solving systems of linear equations, particularly when the system is consistent but the coefficient matrix is not square or invertible. A consistent system's general solution can be represented through any arbitrary {1}-inverse of its coefficient matrix, which is why these {1}-inverses are often referred to as "equation-solving generalized inverses" in the literature [3, 4, 11]. Among the most well-known of these is the Moore-Penrose inverse, originally introduced by Moore (1920) and later independently by Penrose (1955), which remains a pivotal tool in matrix theory and linear algebra [3]. Recent studies have reinforced the continued relevance of generalized inverses, as seen in [8, 17, 23].

In the domain of fuzzy linear systems (FLS), Lodwick and Dubois [10] emphasized the anomalies that can arise from insufficient understanding of interval linear equations. They applied an interval analysis approach, where both the coefficient matrix A and the vector b have fuzzy entries, and the unknown vector—i.e., the solution—depends on the specific interpretation of fuzzy numbers. When the solution set is non-empty, it forms a real-valued multidimensional fuzzy set, and the interval hulls of this set are proper intervals. Mihailović et al. [12] further clarified these issues in connection with Friedman et al.'s

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(1990) approach, showing that to solve a fuzzy linear system, one must first solve a corresponding family of interval linear equations, with the intervals being α-cuts of fuzzy numbers. However, when the coefficient matrix in the classical linear system is neither square nor invertible, the use of generalized inverses of matrices becomes necessary. Recently, methods for solving fuzzy linear systems and dual fuzzy linear systems that involve {1}-inverses were introduced in [5, 13, 15], generalizing the work of Alahviranloo and Gambari, who had considered the classical inverses of square coefficient matrices [2].

The field of interval linear systems (ILS) has been shaped by significant early contributions, beginning with Moore's (1966) foundational work on interval analysis [16], followed by Neumaier's methods for solving interval systems of equations [18], and Alefeld et al.'s exploration of specific classes of linear interval systems [1]. An exhaustive overview of granular computation based on interval analysis, fuzzy sets and rough set was presented by Pedritz et. al in [19]. Jaulin et al. also contributed to the practical application of interval arithmetic in solving these systems [7]. More recent contributions include the comprehensive and detailed study by Shary [20], the work of Thipwiwatpotjana et al. on various solution types for two-sided interval systems, particularly in the context of interval linear programming problems [21], and Hladík et al., who investigated relationships between different solution methods for interval systems [6]. Thipwiwatpotjana et al. also introduced methods for transforming mixed solution types into binary linear inequalities in [22].

In this paper, we focus on *formally consistent interval linear systems with a precise coefficient matrix of arbitrary size*. By applying the general solution of the classical linear system in the form of generalized {1}-inverses, developed in the context of fuzzy linear systems, we aim to extend this methodology to the realm of interval linear systems. Our main objectives are to:

1. Formulate a necessary and sufficient condition for the existence of solutions of the associated interval linear systems;
2. Obtain the exact algebraic form of any solution of the consistent associated interval linear systems;
3. Present an efficient algorithm for determining the inner estimates and the maximal inner estimates of a formally consistent interval linear system with a precise coefficient matrix of arbitrary size; and
4. Provide a necessary and sufficient condition for the equality of inner estimates and the maximal inner estimates of such systems.

The remainder of the paper is organized as follows. Section 2 revisits the foundational concepts necessary for the development of the main results, which are presented in Section 3. Section 4 provides a series of numerical examples that are specifically designed to illustrate the core findings of the paper. Finally, Section 5 offers concluding remarks and delineates potential directions for future research.

2. Preliminaries

In this section, we recall the fundamentals of interval linear systems and other related concepts necessary for understanding the subsequent results.

Definition 2.1. *For arbitrary closed classical proper intervals of the real line* $\underline{\overline{x}} = [\,\underline{x}, \overline{x}\,]$ *and* $\underline{\overline{y}} = [\,\underline{y}, \overline{y}\,]$, $\underline{\overline{x}}, \underline{\overline{y}} \in \mathbb{IR}$ *and* $c \in \mathbb{R}$ *we define:*

1. *Addition:* $\underline{\overline{x}} + \underline{\overline{y}} = [\,\underline{x}, \overline{x}\,] + [\,\underline{y}, \overline{y}] = [\underline{x} + \underline{y}, \overline{x} + \overline{y}\,]$,
2. *Scalar multiplication:* $c\underline{\overline{x}} = c \cdot [\underline{x}, \overline{x}\,] = \begin{cases} [\,c\underline{x}, c\overline{x}\,], & c \geq 0, \\ [\,c\overline{x}, c\underline{x}\,], & c < 0. \end{cases}$

Equality of two intervals is defined as standard equality of two sets, i.e. $\underline{\overline{x}} = \underline{\overline{y}}$ iff $\underline{x} = \underline{y}$ and $\overline{x} = \overline{y}$.

Definition 2.2. *An interval linear system (ILS) is an $m \times n$ linear system defined as follows:*

$$
\begin{aligned}
\overline{\underline{a_{11}}}x_1 + \overline{\underline{a_{12}}}x_2 + \ldots + \overline{\underline{a_{1n}}}x_n &\approx \overline{\underline{b_1}},\\
\overline{\underline{a_{21}}}x_1 + \overline{\underline{a_{22}}}x_2 + \ldots + \overline{\underline{a_{2n}}}x_n &\approx \overline{\underline{b_2}},\\
&\vdots\\
\overline{\underline{a_{m1}}}x_1 + \overline{\underline{a_{m2}}}x_2 + \ldots + \overline{\underline{a_{mn}}}x_n &\approx \overline{\underline{b_m}},
\end{aligned}
\tag{1}
$$

where the coefficient matrix $[A] = [[a_{ij}, \overline{\underline{a_{ij}}}]]$, with $1 \le i \le m$ and $1 \le j \le n$, is an interval $m \times n$ matrix, and $(x_1, \ldots, x_n) \in \mathbb{R}^n$ are the unknowns.

Research on interval linear systems has explored various solution sets for the system $[A]X \approx [B]$, where both the coefficient matrix and the right-hand side vector are interval-valued. Specifically, $[A] = [[\underline{a}_{ij}, \overline{a}_{ij}]]$ and $[B] = [[\underline{b}_i, \overline{b}_i]]$, with $\underline{a}_{ij} \le \overline{a}_{ij}$ and $\underline{b}_i \le \overline{b}_i$ for all i and j (see e.g. [10]). The solution sets for such systems are classified into the following cases:

- **Case 1:** $[A]X \cap [B] \neq \emptyset$. The solution set is referred to as the *united set of solutions*, denoted by $\Sigma_{\exists\exists}$, and is defined as:

$$\Sigma_{\exists\exists} = \{X' \in \mathbb{R}^n \mid (\exists A \in [A])(\exists B \in [B])(AX' = B)\}.$$

- **Case 2:** $[A]X \subseteq [B]$. The solution set is referred to as the *tolerable set of solutions*, denoted by $\Sigma_{\forall\exists}$, and is defined as:

$$\Sigma_{\forall\exists} = \{X' \in \mathbb{R}^n \mid (\forall A \in [A])(\exists B \in [B])(AX' = B)\}.$$

- **Case 3:** $[A]X \supseteq [B]$. The solution set is referred to as the *controllable set of solutions*, denoted by $\Sigma_{\exists\forall}$, and is defined as:

$$\Sigma_{\exists\forall} = \{X' \in \mathbb{R}^n \mid (\forall B \in [B])(\exists A \in [A])(AX' = B)\}.$$

- **Case 4:** $[A]X = [B]$. This (usually empty) set is denoted by $\Sigma_{\forall\forall}$, and is defined as:

$$\Sigma_{\forall\forall} = \{X' \in \mathbb{R}^n \mid (\forall A \in [A])(\forall B \in [B])(AX' = B)\}.$$

Note that the notation $B \in [B]$ here indicates that the n-dimensional vector B belongs to the Cartesian product of the interval components of $[B]$. For further details, refer to [2, 9, 10, 20].

In this paper, we consider interval linear systems with a *precise coefficient matrix*. Recall that a matrix $[A]$ is precise if $\underline{a}_{ij} = \overline{a}_{ij}$ for all i and j, which means the interval linear system $AX \approx [B]$ effectively reduces to one with a real matrix A. The solution sets for such systems satisfy the following:

$$\Sigma_{\exists\exists} = \Sigma_{\forall\exists} = \left\{(x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n \mid \textstyle\sum_{j=1}^{n} a_{ij}x_j \in \overline{\underline{b_i}}, i = 1, \ldots, m\right\},$$

$$\Sigma_{\exists\forall} = \Sigma_{\forall\forall} = \emptyset \quad (\mathit{if}\ \ \underline{b}_k \neq \overline{b}_k,\ \ \mathit{for\ some}\ k).$$

Definition 2.3. *The following $m \times n$ linear system:*

$$
\begin{aligned}
a_{11}\overline{\underline{x_1}} + a_{12}\overline{\underline{x_2}} + \ldots + a_{1n}\overline{\underline{x_n}} &= \overline{\underline{b_1}},\\
a_{21}\overline{\underline{x_1}} + a_{22}\overline{\underline{x_2}} + \ldots + a_{2n}\overline{\underline{x_n}} &= \overline{\underline{b_2}},\\
&\vdots\\
a_{m1}\overline{\underline{x_1}} + a_{m2}\overline{\underline{x_2}} + \ldots + a_{mn}\overline{\underline{x_n}} &= \overline{\underline{b_m}},
\end{aligned}
\tag{2}
$$

with unknown interval vector $(\overline{\underline{x_1}}, \ldots, \overline{\underline{x_n}}) \in \mathbb{IR}^n$, is called an associated interval linear system (AILS) with respect to ILS (1), where the coefficient matrix $A = [a_{ij}]$, $1 \le i \le m$, $1 \le j \le n$ is a real $m \times n$ matrix.

An interval vector $[U] = (\underline{\overline{u_1}}, \ldots, \underline{\overline{u_n}})^T$ is a solution (*an algebraic solution* or *a formal solution*) of (2) if it satisfies $A[U] = [B]$. This implies that, after applying interval arithmetic, the interval vectors on the left-hand side and the right-hand side are equal. An interval linear system (ILS) is considered *formally consistent* if its associated interval linear system (AILS) has a solution.

Let $\mathcal{SS}$ denote the solution set of (2). As shown in the proof of Theorem 3.7 in [2], for any $[U] \in \mathcal{SS}$, it holds that:

$$\bigtimes_{i=1}^{n} \underline{\overline{u_i}} \subseteq \Sigma_{\exists\exists},$$

where $\bigtimes_{i=1}^{n} \underline{\overline{u_i}} = [\underline{u_1}, \overline{u_1}] \times [\underline{u_2}, \overline{u_2}] \times \cdots \times [\underline{u_n}, \overline{u_n}]$ is the Cartesian product of the intervals $\underline{\overline{u_i}}$, $i = 1, \ldots, n$, and $\Sigma_{\exists\exists}$ is the united set of solutions of the ILS (1). Finally,

$$\bigcup_{U \in \mathcal{SS}} \bigtimes_{i=1}^{n} \underline{\overline{u_i}} \subseteq \Sigma_{\exists\exists}.$$

Since A is precise, this result also follows from Theorem 11.2.1 in [20]. According to this theorem, each interval vector $[U] = (\underline{\overline{u_1}}, \ldots, \underline{\overline{u_n}})^T \in \mathcal{SS}$ that is a solution of AILS (2) is referred to as an *inner interval estimate of the united set of solutions* $\Sigma_{\exists\exists}$ of the ILS (1). Generally, an inner interval estimate of the united set of solutions is not unique. Therefore, we will refer to $\bigcup_{U \in \mathcal{SS}} \times_{i=1}^{n} \underline{\overline{u_i}}$ as the *united inner solution set* of the ILS (1). The definition of the *maximal inner estimate* is provided as follows.

Definition 2.4. *The maximal inner estimate for the united set of solutions $\Sigma_{\exists\exists}$ of the ILS (1) with the precise coefficient matrix $A \in \mathbb{R}^{m \times n}$ is an interval vector $[U^m] = (\underline{\overline{u^m}}_1, \ldots, \underline{\overline{u^m}}_n)^T \in \mathcal{SS}$ from the solution set $\mathcal{SS}$ of (2) such that for every interval vector $[U] = (\underline{\overline{u_1}}, \ldots, \underline{\overline{u_n}})^T \in \mathcal{SS}$, it holds that if $\underline{\overline{u^m}}_i \subseteq \underline{\overline{u_i}}$ for all $i = 1, \ldots, n$, then it follows that $[U^m] = [U]$.*

As stated in [3], the principal application of {1}-inverses is in solving linear systems, where they function similarly to ordinary inverses in the nonsingular case. Recall that for every finite matrix A (whether square or rectangular) with real or complex entries, a matrix X that satisfies the condition

$$AXA = A \tag{3}$$

is called a {1}-inverse of matrix A, denoted by $A^{(1)}$.

For $A \in \mathbb{R}^{n \times n}$, the *index of A*, denoted by *ind*(A), is defined as the smallest non-negative integer k such that $rank(A^{k+1}) = rank(A^k)$. O denotes the null matrix of size $m \times n$.

The most prominent {1}-inverses discussed in the literature include the Moore-Penrose inverse, which is uniquely determined for each matrix A and will be denoted by $A^\dagger$. Additionally, there is a group inverse (denoted by $A^\sharp$), which exists only for singular square matrices A with index 1 and is uniquely determined in such cases. Finally, the classical inverse of a nonsingular square matrix A, denoted by A^{-1}, is a uniquely determined {1}-inverse and the sole solution to the equation (3) for such matrices. For further details, see [3].

Based on Corollary 3, Chapter 2 of [3], and as a consequence of Penrose's result (Theorem 1, Chapter 2 in [3]), the necessary and sufficient condition for the consistency of a linear system with a real coefficient matrix can be formulated as follows. Note that this result is also valid for linear systems with complex parameters. The set of $m \times n$ matrices with elements in $\mathbb{R}$ is denoted by $\mathbb{R}^{m \times n}$.

Theorem 2.5. *Let $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{m \times 1}$. The linear system given in matrix form by*

$$AX = B \tag{4}$$

is consistent if and only if there exists an $A^{(1)}$ such that

$$AA^{(1)}B = B. \tag{5}$$

Moreover, the general solution of (4) is

$$X = A^{(1)}B + (I_n - A^{(1)}A)V, \tag{6}$$

where V is an arbitrary vector in $\mathbb{R}^{n\times 1}$.

For a comprehensive study of generalized inverses, we recommend [3]. For techniques related to the calculation of generalized inverses based on block representations, see [14].

3. Main results

In order to determine the united inner solution set of interval linear system (1), where $A \in \mathbb{R}^{m\times n}$, we need to characterize a general solution of associated interval linear system (2).

Let $A = [a_{ij}]$, $A \in \mathbb{R}^{m\times n}$ and $|A| = [|a_{ij}|]$, $|A| \in \mathbb{R}^{m\times n}$. Denote $A^+ = [a^+_{ij}] \in \mathbb{R}^{m\times n}$ and $A^- = [a^-_{ij}] \in \mathbb{R}^{m\times n}$, where $a^+_{ij} = \frac{a_{ij}+|a_{ij}|}{2}$ and $a^-_{ij} = \frac{|a_{ij}|-a_{ij}}{2}$, for all $i = 1,\dots,m$, $j = 1,\dots,n$. For $[X] = (\underline{\overline{x_1}},\dots,\underline{\overline{x_n}})^T \in \mathbb{IR}^{n\times 1}$, let us denote $\underline{X} = (\underline{x_1},\dots,\underline{x_n}) \in \mathbb{R}^{n\times 1}$ and $\overline{X} = (\overline{x_1},\dots,\overline{x_n})^T \in \mathbb{R}^{n\times 1}$. Now, $[X^0]$ is a solution of $A[X] = [B]$ if and only if

$$A^+\underline{X}^0 - A^-\overline{X}^0 = \underline{B}, \tag{7}$$

$$A^+\overline{X}^0 - A^-\underline{X}^0 = \overline{B}. \tag{8}$$

where $\underline{X}^0 = (\underline{x}^0_1,\dots,\underline{x}^0_n,)^T$ and $\overline{X}^0 = (\overline{x}^0_1,\dots,\overline{x}^0_n)^T$. By adding and subtracting (7) and (8), it is evident that the sum $\overline{X}^0 + \underline{X}^0$ is a solution to the classical $m \times n$ linear system

$$A(\overline{X} + \underline{X}) = \overline{B} + \underline{B}, \tag{9}$$

while the difference $\overline{X}^0 - \underline{X}^0$ is a solution to the system

$$|A|(\overline{X} - \underline{X}) = \overline{B} - \underline{B}. \tag{10}$$

Thus, (9) and (10) are consistent linear systems. If a solution exists, we say that the AILS is consistent.

On the other hand, let (9) and (10) be consistent linear systems, with solutions denoted by $\overline{X}^0 + \underline{X}^0$ and $\overline{X}^0 - \underline{X}^0$. In this case, we will obtain a solution of AILS (2) only if it holds that $\underline{x}^0_i \le \overline{x}^0_i$, for all $i = 1,\dots,n$, i.e. if $\overline{X}^0 - \underline{X}^0 \ge O$.

Our main result presents a necessary and sufficient condition for the consistency of AILS (2), for given $[B]$ and given coefficient matrix $A \in \mathbb{R}^{m\times n}$. Moreover, the following theorem establishes the general solution form for consistent AILS (2). Let $A^{(1)} \in \mathbb{R}^{n\times m}$ be an arbitrary {1}-inverse matrix of A. Denote $A^{(1)} = H = [h_{ij}]$, $|H| = [|h_{ij}|]$, $H^+ = [h^+_{ij}]$, and $H^- = [h^-_{ij}]$, for all $i = 1,\dots,n$, $j = 1,\dots,m$.

Theorem 3.1. *Let $A \in \mathbb{R}^{m\times n}$ be the coefficient matrix of the AILS (2), for given $[B] = (\underline{\overline{b_1}},\dots,\underline{\overline{b_m}})^T$. The AILS (2) is consistent if and only if for some $M = A^{(1)}$, $AM(\overline{B}+\underline{B}) = \overline{B}+\underline{B}$, and for some $\Lambda, \Theta \in \mathbb{R}^{n\times 1}$, such that $\Theta \le \frac{1}{2}|M|(\overline{B}-\underline{B})$, it holds*

$$A\Lambda = O \ \ and \ \ |A|\Theta = \frac{1}{2}\left(|A||M|(\overline{B} - \underline{B}) - \overline{B} + \underline{B}\right). \tag{11}$$

Moreower, the general solution of the AILS (2) is

$$[X] = (\underline{\overline{x_1}},\dots,\underline{\overline{x_n}})^T,\ \underline{\overline{x_i}} = [\underline{x_i},\overline{x_i}],\ i = 1,\dots,n,\ \ where\ \underline{X} = M^+\underline{B} - M^-\overline{B} + \frac{1}{2}\Lambda + \Theta\ and\ \overline{X} = M^+\overline{B} - M^-\underline{B} + \frac{1}{2}\Lambda - \Theta,$$

$$\Lambda = (I_n - MA)V_1, \tag{12}$$

$$\Theta = \frac{1}{2}N(I_m - |A||M|)(\underline{B} - \overline{B}) + (I_n - N|A|)V_2, \tag{13}$$

for arbitrary $M = A^{(1)}$, $N = |A|^{(1)}$ *and* $V_1, V_2 \in \mathbb{R}^{n\times 1}$ *such that* $\Theta \le \frac{1}{2}|M|(\overline{B} - \underline{B})$.

Proof. ($\Rightarrow$) Suppose that AILS (2) is consistent and $[X^0] = (\underline{\overline{x_1}^0}, \dots, \underline{\overline{x_n}^0})^T$ is one of its solutions. Then $\overline{X}^0 + \underline{X}^0$ is one of the solutions of linear system (9), whereas $\overline{X}^0 - \underline{X}^0 \ge O$ is one of the solutions of linear system (10). Hence, according to Theorem 2.5, due to the consistency of (9) and (10) there exist $M = A^{(1)}$, $N = |A|^{(1)}$, and $V_1^0, V_2^0 \in \mathbb{R}^{n\times 1}$ such that

$$\begin{aligned}\overline{X}^0 + \underline{X}^0 &= M(\overline{B} + \underline{B}) + V_1^0,\\ \overline{X}^0 - \underline{X}^0 &= N(\overline{B} - \underline{B}) + V_2^0.\end{aligned}$$

From the previous equations, we get:

$$\begin{aligned}\overline{X}^0 &= \frac{1}{2}\left(M(\overline{B} + \underline{B}) + N(\overline{B} - \underline{B}) - |M|(\overline{B} - \underline{B}) + |M|(\overline{B} - \underline{B}) + V_1^0 + V_2^0\right)\\ &= M^+\overline{B} - M^-\underline{B} + \frac{1}{2}(N - |M|)(\overline{B} - \underline{B}) + \frac{1}{2}(V_1^0 + V_2^0),\end{aligned}$$

and similarly,

$$\begin{aligned}\underline{X}^0 &= \frac{1}{2}\left(M(\overline{B} + \underline{B}) - N(\overline{B} - \underline{B}) + |M|(\overline{B} - \underline{B}) - |M|(\overline{B} - \underline{B}) + V_1^0 - V_2^0\right)\\ &= M^+\underline{B} - M^-\overline{B} + \frac{1}{2}(N - |M|)(\underline{B} - \overline{B}) + \frac{1}{2}(V_1^0 - V_2^0).\end{aligned}$$

Now, denote $\Lambda^0 = V_1^0$ and $\Theta^0 = \frac{1}{2}(N - |M|)(\underline{B} - \overline{B}) - \frac{1}{2}V_2^0$.

Clearly, it holds that:

$$\overline{X}^0 + \underline{X}^0 = M(\overline{B} + \underline{B}) + \Lambda^0, \qquad \overline{X}^0 - \underline{X}^0 = |M|(\overline{B} - \underline{B}) - 2\Theta^0.$$

Further, since $\overline{X}^0 - \underline{X}^0 \ge O$, we get $\Theta^0 \le \frac{1}{2}|M|(\overline{B} - \underline{B})$. We have

$$\overline{B} + \underline{B} = A(\overline{X}^0 + \underline{X}^0) = AM(\overline{B} + \underline{B}) + A\Lambda^0,$$

$$\overline{B} - \underline{B} = |A|(\overline{X}^0 - \underline{X}^0) = |A||M|(\overline{B} - \underline{B}) - 2|A|\Theta^0.$$

Since $M(\overline{B} + \underline{B})$ is a solution of the consistent linear system (9), we conclude that $A\Lambda^0 = O$, and if we denote $2W = |A||M|(\overline{B} - \underline{B}) - \overline{B} + \underline{B}$, obviously $|A|\Theta^0 = W$.

($\Leftarrow$) Suppose that there exist $\Lambda^0, \Theta^0 \in \mathbb{R}^{n\times 1}$, such that (11) holds and $2\Theta^0 \le |M|(\overline{B} - \underline{B})$, for some $M = A^{(1)}$. Let $[X^0] = (\underline{\overline{x_1}^0}, \dots, \underline{\overline{x_n}^0})^T$, $\underline{\overline{x_i}^0} = [\underline{x}_i^0, \overline{x}_i^0]$, where $\underline{X}^0 = M^+\underline{B} - M^-\overline{B} + \frac{1}{2}\Lambda^0 + \Theta^0$ and $\overline{X}^0 = M^+\overline{B} - M^-\underline{B} + \frac{1}{2}\Lambda^0 - \Theta^0$. Then $\overline{X}^0 + \underline{X}^0 = M(\overline{B} + \underline{B}) + \Lambda^0$ and $\overline{X}^0 - \underline{X}^0 = |M|(\overline{B} - \underline{B}) - 2\Theta^0 \ge O$. Using (11), we obtain that linear systems (9) and (10) are consistent, therefore $A[X^0] = [B]$, hence we get the claim.

By Theorem 2.5, we obtain the general solution form of consistent AILS. Namely, $M = A^{(1)}$ is arbitrary {1}-inverse of A, whereas Λ and Θ are solutions of the consistent linear systems $A\Lambda = O$ and $|A|\Theta = \frac{1}{2}\left(|A||M|(\overline{B} - \underline{B}) - \overline{B} + \underline{B}\right)$, respectively, therefore,

$$\Lambda = (I_n - MA)V_1, \qquad \Theta = \frac{1}{2}N\left(|A||M|(\overline{B} - \underline{B}) - \overline{B} + \underline{B}\right) + (I_n - N|A|)V_2,$$

for arbitrary $N = |A|^{(1)}$, and $V_1, V_2 \in \mathbb{R}^{n\times 1}$ such that $2\Theta \le |M|(\overline{B} - \underline{B})$ holds. From the previous equations, we get (12) and (13). □

Notice that the previous result generalizes Theorem 2.5. Specifically, if $[B]$ is precise, i.e., $\underline{B} = \overline{B} \in \mathbb{R}^{m\times 1}$, then the claim made above regarding systems of linear interval equations reduces to the classical case discussed in Theorem 2.5. In the following, we present some consequences of Theorem 3.1.

Corollary 3.2. *Any associated interval linear system (2) with the coefficient matrix $A \in \mathbb{R}^{m\times n}$ has one of the following exclusive conclusions:*

(i) no solution,

(ii) a unique solution,

(iii) infinitely many solutions.

Proof. Let $[X^0]$ be one of the solutions of consistent $A[X] = [B]$, for given $[B]$. Then, $\overline{X}^0 + \underline{X}^0$ is a solution of (9). Hence, (9) is consistent, from Theorem 2.5 it follows that for some $M = A^{(1)}$, $M\left(\overline{B} + \underline{B}\right)$ is a solution of (9), i.e. $AM\left(\overline{B} + \underline{B}\right) = \overline{B} + \underline{B}$. By Theorem 3.1, we conclude:
(i) The system of linear interval equations (2) has no solution (it is inconsistent) if for each $M = A^{(1)}$, $AM\left(\overline{B} + \underline{B}\right) \neq \overline{B} + \underline{B}$ or there is no solution of $|A|\Theta = W$, where $W = \frac{1}{2}\left(|A||M|(\overline{B} - \underline{B}) - \overline{B} + \underline{B}\right)$, such that $[X]$, defined with $\underline{X} = M^{+}\underline{B} - M^{-}\overline{B} + \frac{1}{2}\Lambda + \Theta$, $\overline{X} = M^{+}\overline{B} - M^{-}\underline{B} + \frac{1}{2}\Lambda - \Theta$, is an interval vector. Here, Λ is any solution of $A\Lambda = O$.
(ii) The system of linear interval equations (2) has a unique solution (it is consistent) if for some $M = A^{(1)}$, $AM\left(\overline{B} + \underline{B}\right) = \overline{B} + \underline{B}$, and $|A|\Theta = W$ has the unique solution Θ^0 such that $2\Theta^0 \leq |M|(\overline{B} - \underline{B})$. In this case, $A\Lambda = O$ has only the trivial solution, i.e. $\Lambda^0 = O$.
(iii) The system (2) has infinitely many solutions (it is consistent) if for some $M = A^{(1)}$, $AM\left(\overline{B} + \underline{B}\right) = \overline{B} + \underline{B}$, $|A|\Theta = W$ or $A\Lambda = O$ has infinitely many solutions such that $2\Theta \leq |M|(\overline{B} - \underline{B})$. □

Corollary 3.3. *Let $A \in \mathbb{R}^{n\times n}$ be the coefficient matrix of the consistent AILS (2), for given $[B] = (\underline{\overline{b_1}}, \dots, \underline{\overline{b_n}})^T$. If A and $|A|$ are invertible matrices then the AILS has a unique solution $[X] = (\underline{\overline{x_1}}, \dots, \underline{\overline{x_n}})^T$, $\underline{\overline{x_i}} = [\underline{x_i}, \overline{x_i}]$, $i = 1, \dots, n$, given by*

$$\underline{X} = \left(A^{-1}\right)^{+}\underline{B} - \left(A^{-1}\right)^{-}\overline{B} + \frac{1}{2}\left(|A|^{-1} - |A^{-1}|\right)\left(\underline{B} - \overline{B}\right), \tag{14}$$

$$\overline{X} = \left(A^{-1}\right)^{+}\overline{B} - \left(A^{-1}\right)^{-}\underline{B} - \frac{1}{2}\left(|A|^{-1} - |A^{-1}|\right)\left(\underline{B} - \overline{B}\right) \tag{15}$$

Moreover, if $|A|^{-1} = |A^{-1}|$, then a unique solution $[X]$ is determined by

$$\underline{X} = \left(A^{-1}\right)^{+}\underline{B} - \left(A^{-1}\right)^{-}\overline{B},$$
$$\overline{X} = \left(A^{-1}\right)^{+}\overline{B} - \left(A^{-1}\right)^{-}\underline{B}.$$

Proof. Let $A \in \mathbb{R}^{n\times n}$ be the coefficient matrix of the consistent square AILS (2), such that A is completely nonsingular matrix, i.e. such that both A and $|A|$ are nonsingular. The unique {1}-inverse for A invertible is its classical inverse A^{-1}, and it holds that $A^{-1}A = I_n$. Similarly, for $|A|$, we have that $|A|^{-1}|A| = I_n$. Using these facts, from (12) and (13), we obtain the unique solution of (2). Obviously, if $|A|^{-1} = |A^{-1}|$, by (14) and (15), we obtain the last solution form. □

Notice that for a matrix $A \in \mathbb{R}^{m\times n}$ of full column rank, it holds that $A^{(1)}A = I_n$ for each $A^{(1)}$. Hence, we obtain an analogue of the first statement in Corollary 3.3, with the condition that A and $|A|$ are full column rank matrices. More generally, the analogue statement to the second one in Corollary 3.3 is as follows.

Corollary 3.4. *Let $A \in \mathbb{R}^{m\times n}$ be the coefficient matrix of the consistent AILS (2), for given $[B] = (\underline{\overline{b_1}}, \dots, \underline{\overline{b_m}})^T$. If there exists {1}-inverse of $|A|$ such that $N = |A|^{(1)} = |A^{(1)}| = |M|$, for some $M = A^{(1)}$, then $[X] = (\underline{\overline{x_1}}, \dots, \underline{\overline{x_n}})^T$, $\underline{\overline{x_i}} = [\underline{x_i}, \overline{x_i}]$, $i = 1, \dots, n$, given by*

$$\begin{aligned}\underline{X} &= M^+\underline{B} - M^-\overline{B},\\ \overline{X} &= M^+\overline{B} - M^-\underline{B}.\end{aligned}$$

is a solution of (2).

Proof. Due to $N = |M|$ and

$$\Theta = \frac{1}{2}N(I_n - |A||M|)(\underline{B} - \overline{B}) + (I_n - N|A|)V_2 = \frac{1}{2}(N - |M|)(\underline{B} - \overline{B}) + (I_n - N|A|)\left(V_2 + \frac{1}{2}|M|(\underline{B} - \overline{B})\right),$$

for $V_2 = \frac{1}{2}|M|(\overline{B} - \underline{B})$, and $V_1 = O$ u (12), we get the claim. □

Finally, based on the above results, we introduce the straightforward way for obtaining the inner solution set of an $m\times n$ ILS (1) using its associated interval linear system (2), with the real coefficient matrix $A \in \mathbb{R}^{m\times n}$. For arbitrary $M = A^{(1)}$, denote

$$\underline{X}^* = M^+\underline{B} - M^-\overline{B}, \tag{16}$$

$$\overline{X}^* = M^+\overline{B} - M^-\underline{B}. \tag{17}$$

For M we can choose any of {1}-inverses of A, e.g, the Moore-Penrose inverse $A^\dagger$, the group inverse $A^\sharp$ (if it exists), etc. For more details about these {1}-inverses and algorithms for their computation, see, e.g. [3, 12, 13]. In order to reduce the computational cost, it should be noticed that $W = (w_1, \dots, w_m)^T$, defined by $W = \underline{B} - A^+\underline{X}^* + A^-\overline{X}^*$, satisfies $W = \underline{B} - A^+\underline{X}^* + A^-\overline{X}^* = A^+\overline{X}^* - A^-\underline{X}^* - \overline{B}$. The last claim follows from the fact that $\overline{X}^* + \underline{X}^*$ is one of the solutions of (9), if it is consistent. Therefore, $2W = |A||M|(\overline{B} - \underline{B}) - \overline{B} + \underline{B}$. Using Corollary 3.3, if the AILS is consistent, for each solution $\Lambda = (\lambda_1, \dots, \lambda_n)^T$ of the homogenous system $A\Lambda = O$, and each solution $\Theta = (\theta_1, \dots, \theta_n)^T$ of $|A|\Theta = W$, such that $\theta_i \le \frac{\overline{x}_i^* - \underline{x}_i^*}{2}$, for all $i = 1, \dots, n$, we compute:

$$\underline{x}_i = \underline{x}_i^* + \frac{\lambda_i}{2} + \theta_i, \quad \text{and} \quad \overline{x}_i = \overline{x}_i^* + \frac{\lambda_i}{2} - \theta_i, \quad , \quad i = 1, \dots, n.$$

As the final step, we obtain the united inner solution set as follows:

$$\bigcup_{\Lambda,\Theta \text{ feasible}} \bigtimes_{i=1}^{n} \underline{\overline{x_i}} \subseteq \Sigma_{\exists\exists}.$$

The following theorem, concerning maximal inner estimates from Definition 2.4 is our result that corresponds to Kuprianova´s Fundamental theorem on inner estimation (Theorem 2.2 in [9]). It provides a necessary and sufficient condition for the equality of inner estimates and the maximal inner estimates of ILS (1).

Theorem 3.5. *Let $A \in \mathbb{R}^{m\times n}$ be the coefficient matrix of formally consistent ILS* (1)*, and let* $[V]$ *be a solution of AILS* (2)*. The matrix A does not contain a zero column if and only if* $[V]$ *is the maximal inner estimate of the united solution set* $\Sigma_{\exists\exists}$ *of ILS* (1)*.*

Proof. Let $[V]$ be a solution of *AILS* (2). Assume that there is a solution $[U]$ of *AILS* (2) such that for some $j \in \{1, \dots, n\}$ it holds that $\underline{\overline{u_i}} = \underline{\overline{v_i}}$ for all $i \neq j$ and that $\underline{u_j} = \underline{v_j}$ and $\overline{u_j} = \overline{v_j} + \epsilon$ for some $\epsilon > 0$ holds, too (the case $\underline{u_j} = \underline{v_j} - \epsilon$ and $\overline{u_j} = \overline{v_j}$ is considered analogously).

Based on Theorem 3.1, there exist $\Lambda^1, \Theta^1 \in \mathbb{R}^{n\times 1}$ and $\Lambda^2, \Theta^2 \in \mathbb{R}^{n\times 1}$, which are solutions of the system of equations

$$A\Lambda = O \ \text{and} \ |A|\Theta = \frac{1}{2}\left(|A||M|(\overline{B} - \underline{B}) - \overline{B} + \underline{B}\right), \tag{18}$$

for some $M = A^{(1)}$, and the following holds:

$$\underline{U} = M^{+}\underline{B} - M^{-}\overline{B} + \frac{1}{2}\Lambda^{1} + \Theta^{1}, \quad \overline{U} = M^{+}\overline{B} - M^{-}\underline{B} + \frac{1}{2}\Lambda^{1} - \Theta^{1},$$

and

$$\underline{V} = M^{+}\underline{B} - M^{-}\overline{B} + \frac{1}{2}\Lambda^{2} + \Theta^{2}, \quad \overline{V} = M^{+}\overline{B} - M^{-}\underline{B} + \frac{1}{2}\Lambda^{2} - \Theta^{2}.$$

Considering the previous equalities component-wise, we obtain:

$$\begin{aligned}
&\frac{1}{2}\lambda_i^1 + \theta_i^1 = \frac{1}{2}\lambda_i^2 + \theta_i^2, \text{ for } i \neq j,\\
&\frac{1}{2}\lambda_i^1 - \theta_i^1 = \frac{1}{2}\lambda_i^2 - \theta_i^2, \text{ for } i \neq j,\\
&\qquad \text{and}\\
&\frac{1}{2}\lambda_j^1 + \theta_j^1 = \frac{1}{2}\lambda_j^2 + \theta_j^2,\\
&\frac{1}{2}\lambda_j^1 - \theta_j^1 = \frac{1}{2}\lambda_j^2 - \theta_j^2 + \epsilon.
\end{aligned} \tag{19}$$

From this, for $i \neq j$, we get that $\lambda_i^1 = \lambda_i^2$ and $\theta_i^1 = \theta_i^2$, while $\lambda_j^1 + \epsilon = \lambda_j^2$ and $\theta_j^1 - \frac{\epsilon}{2} = \theta_j^2$.

To show the "if" part of the statement, assume that [V] is not the maximal inner estimate of the united set of solutions $\Sigma_{\exists\exists}$ of *ILS* (1). Since, by the assumption, none of the columns of matrix A is a zero column, owing to $\Lambda_i^1 = \Lambda_i^2$ for all $i \neq j$, from (18) we get $(A\Lambda^1)_k = (A\Lambda^2)_k$, where k is such that $a_{k,j} \neq 0$. Finally, it follows that $\epsilon = 0$, which is an obvious contradiction.

The opposite direction will be obtained by contraposition. If the j-th column of matrix A consists entirely of zeros, then the same holds for matrix $|A|$. Thus, if Λ^2 and Θ^2 as solutions of the system (18) determine the solution $[V]$ of *AILS* (2), then Λ^1 and Θ^1 are also solutions of the system (18), where for $i \neq j$ it holds that $\lambda_i^1 = \lambda_i^2$, $\theta_i^1 = \theta_i^2$, and $\lambda_j^1 = \lambda_j^2 - \epsilon$, $\theta_j^1 = \theta_j^2 + \frac{\epsilon}{2}$ for each $\epsilon > 0$. Then, if Λ^1 and Θ^1 determine the solution $[U]$ of *AILS* (2), it will hold that $[V] \subsetneq [U]$. □

Corollary 3.6. *Regarding solutions of a given AILS* (2), *one of the following statements holds:*

1. *Every solution is the maximal inner estimate of the united set of solutions $\Sigma_{\exists\exists}$ of formally consistent ILS* (1).

2. *No solution is the maximal inner estimate of the united set of solutions $\Sigma_{\exists\exists}$ of formally consistent ILS* (1).

4. Numerical examples

In order to demonstrate the efficiency and applicability of the proposed results, this section presents several numerical examples. To determine the inner solution set and the maximal inner estimate of the interval linear system $AX \approx [B]$, where the coefficient matrix is precise, we will solve the associated interval linear system $A[X] = [B]$.

Before presenting the detailed numerical results, we first illustrate a real-life scenario where interval linear systems naturally arise due to measurement uncertainties. This example serves as a practical motivation for the theory discussed in the paper.

Consider the following situation: we aim to design a two-day nutritional supplement plan for a pregnant woman. In order to ensure safety, the total intake from two products, denoted A and B, should not exceed 10 mg per day. The required daily intake of folic acid is estimated to be between 0.74 mg and 0.84 mg on the first day, and between 0.65 mg and 0.75 mg on the second day. According to the planned regimen, she takes four times more of product B than A on the first day, and equal amounts of both products on the second day.

This leads to an interval linear model, where x_1 and x_2 represent the concentrations (in percent of the 10 mg daily intake) of folic acid in products A and B, respectively:

$$\begin{array}{rcl} 0.2 \cdot x_1 + 0.8 \cdot x_2 & \approx & [7.4\%, 8.4\%] \\ 0.5 \cdot x_1 + 0.5 \cdot x_2 & \approx & [6.5\%, 7.5\%] \end{array}.$$

Using the proposed inner estimation approach, we conclude that product A should contain between 5% and 6% folic acid, while product B should contain between 8% and 9%, relative to the total daily intake (i.e., up to 10 mg). This ensures that, for any combination of these concentrations within the estimated ranges, the daily intake of folic acid remains within medically recommended bounds on both days.

A detailed explanation of the computational procedure for determining such inner estimates is provided in the following examples. In Example 4.1, we consider a nonsingular associated interval linear system (AILS) with a completely nonsingular coefficient matrix, resulting in a unique solution. In the subsequent examples, we examine singular AILS with an infinite number of solutions. Specifically, in Example 4.2, A is nonsingular and $|A|$ is singular, whereas in Example 4.3 and Example 4.4, both matrices A and $|A|$ are singular.

Example 4.1. *Let us find the united inner solution set and the maximal inner estimate for the following* 2×2 *interval linear system:*

$$\begin{array}{rcrcl} x_1 & + & 3x_2 & \approx & [3.5, 15.5] \\ -2x_1 & + & x_2 & \approx & [-15.5, -1.5] \end{array}.$$

Observe that:

$$\Sigma_{\exists\exists} = \Sigma_{\forall\exists} = \left\{ (x_1, x_2)^T \mid 3.5 \le x_1 + 3x_2 \le 15.5, -15.5 \le -2x_1 + x_2 \le -1.5 \right\}, \; \Sigma_{\exists\forall} = \Sigma_{\forall\forall} = \emptyset.$$

We first solve the associated interval linear system:

$$\begin{array}{rcrcl} \overline{\underline{x_1}} & + & 3\overline{\underline{x_2}} & = & [3.5, 15.5] \\ -2\overline{\underline{x_1}} & + & \overline{\underline{x_2}} & = & [-15.5, -1.5] \end{array}.$$

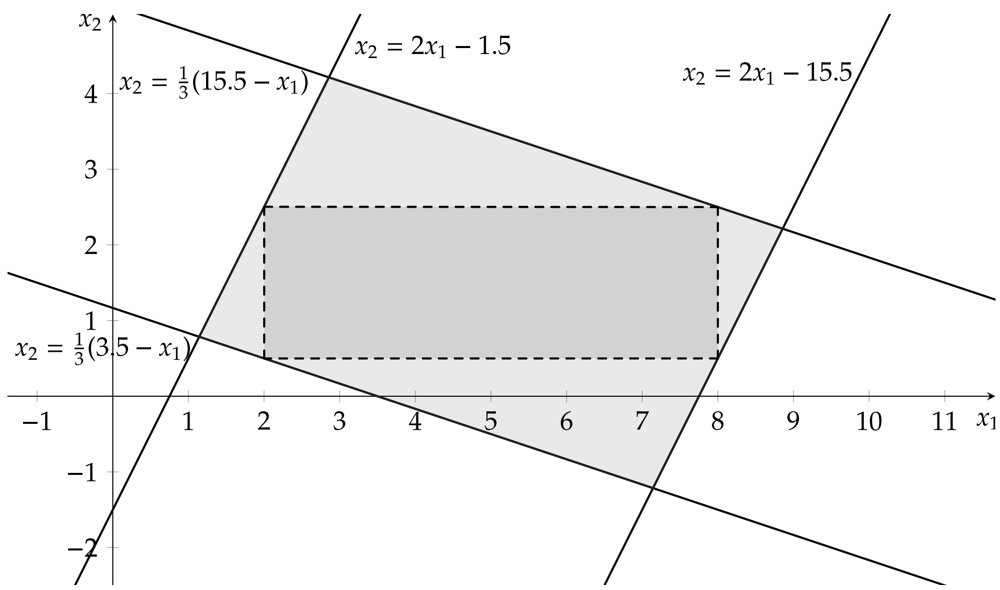

Figure 1: Graphical representation of Example 1

Matrices A, A^+ and A^- of this AILS are:

$$A = \begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix},\ A^+ = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix},\ A^- = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}.$$

Since both matrices A and $|A|$ are invertible, the matrix A is completely nonsingular, hence $ind(A) = 0$ and $ind(|A|) = 0$. The classical inverse of A is:

$$M = A^{-1} = A^{\dagger} = \frac{1}{7}\begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix}.$$

We obtain $[X^*] = (\overline{\underline{x^*}_1}, \overline{\underline{x^*}_2})^T$, where

$$\begin{aligned} \overline{\underline{x^*_1}} &= [\frac{8}{7}, \frac{62}{7}], \\ \overline{\underline{x^*_2}} &= [-\frac{17}{14}, \frac{59}{14}]. \end{aligned}$$

The homogenous system $A\Lambda = O$ has only the trivial solution $\Lambda_0 = (0,0)^T$. Now, since, $|A|$ is nonsingular, the classical system $|A|\Theta = W$, $W = (w_1, w_2)^T = (6, \frac{24}{7})^T$, has the unique solution $\Theta = (\frac{6}{7}, \frac{12}{7})^T$. In order to obtain an interval vector, i.e. in order to $\overline{x}_i \geq \underline{x}_i$ be fulfilled for $i = 1, 2$, Θ need to satisfy the following necessary condition $\theta_i \leq \frac{\overline{x}^*_i - \underline{x}^*_i}{2}$, for $i = 1, 2$. The obtained Θ is feasible, therefore, the unique solution of the AILS is

$$[X] = (\overline{\underline{x_1}}, \overline{\underline{x_2}})^T = ([2, 8], [0.5, 2.5])^T.$$

The obtained interval vector represents the inner estimate of ILS, and by Theorem 3.5, it is also the maximal inner estimate. The inner solution set of ILS is $[2, 8] \times [0.5, 2.5] \subset \Sigma_{\exists\exists}$, and since the solution of AILS is unique, it is also the united inner solution set of the ILS. The united set of solutions and the maximal inner estimate of this ILS are graphically represented in Figure 1.

Example 4.2. *Let us find the united inner solution set and the maximal inner estimate for the following 2×2 interval linear system:*

$$\begin{array}{rcrcl} 2x_1 & - & x_2 & \approx & [0,7] \\ -2x_1 & - & x_2 & \approx & [0,7] \end{array}.$$

Observe that:

$$\Sigma_{\exists\exists} = \Sigma_{\forall\exists} = \left\{(x_1, x_2)^T \mid 0 \leq 2x_1 - x_2 \leq 7, 0 \leq -2x_1 - x_2 \leq 7\right\},\ \Sigma_{\exists\forall} = \Sigma_{\forall\forall} = \emptyset.$$

The associated interval linear system is:

$$\begin{array}{rcrcl} 2\overline{\underline{x_1}} & - & \overline{\underline{x_2}} & = & [0,7] \\ -2\overline{\overline{\underline{\underline{x_1}}}} & - & \overline{\overline{\underline{\underline{x_2}}}} & = & [0,7] \end{array}.$$

The coefficient matrix A is nonsingular, $ind(A) = 0$, while $|A|$ is singular with $ind(|A|) = 1$. We compute the classical inverse of A:

$$M = A^{-1} = A^{\dagger} = \begin{bmatrix} 0.25 & -0.25 \\ -0.5 & -0.5 \end{bmatrix}.$$

We compute $[X^*] = ([-1.75, 1.75], [-7, 0])^T$.
Since A is invertible, $\Lambda_0 = (0,0)^T$ is the unique solution of $A\Lambda = O$, further, $W = (3.5, 3.5)^T$, and the next system has infinitely many solutions $\Theta = (\theta_1, \theta_2)^T$:

$$\begin{array}{ll} 2\theta_1 + \theta_2 & = 3.5 \\ 2\theta_1 + \theta_2 & = 3.5 \end{array}.$$

The solution class of the previous system is $\Theta = (\theta_1, \theta_2)^T = (C, 3.5 - 2C)^T$, $C \in \mathbb{R}$. Θ *is feasible if* $\underline{x}_i$ *and* $\overline{x}_i$ *satisfy necessary constrains* $\theta_i \leq \frac{\overline{x}_i^* - \underline{x}_i^*}{2}$, $i = 1, 2$, *hence* $0 \leq C \leq 1.75$. *Therefore, by computing* $\underline{x}_i = \underline{x}_i^* + \theta_i$, *and* $\overline{x}_i = \overline{x}_i^* - \theta_i$, *for* $i = 1, 2$, *we obtain the solution class with an infinite number of inner estimates*

$$[X] = (\overline{\underline{x_1}}, \overline{\underline{x_2}})^T = ([-1.75 + C, 1.75 - C], [-3.5 - 2C, -3.5 + 2C])^T, \quad 0 \leq C \leq 1.75.$$

All of them are, by Theorem 3.5, maximal inner estimates. Therefore, the united inner solution set of ILS is

$$\bigcup_{0 \leq C \leq 1.75} [-1.75 + C, 1.75 - C] \times [-3.5 - 2C, -3.5 + 2C] = \Sigma_{\exists\exists}.$$

The united set of solutions and two maximal inner estimates obtained for $C = 0.6$ *and* $C = \frac{7}{8}$ *are graphically represented in Figure 2. Note that despite the interval vectors' Lebesgue measure (in this case area of a rectangle) reaches its maximum only for the estimate obtained for* $C = \frac{7}{8}$, *by Definition 2.4 and Theorem 3.5 all inner estimates within the range* $0 \leq C \leq 1.75$ *are the maximal ones.*

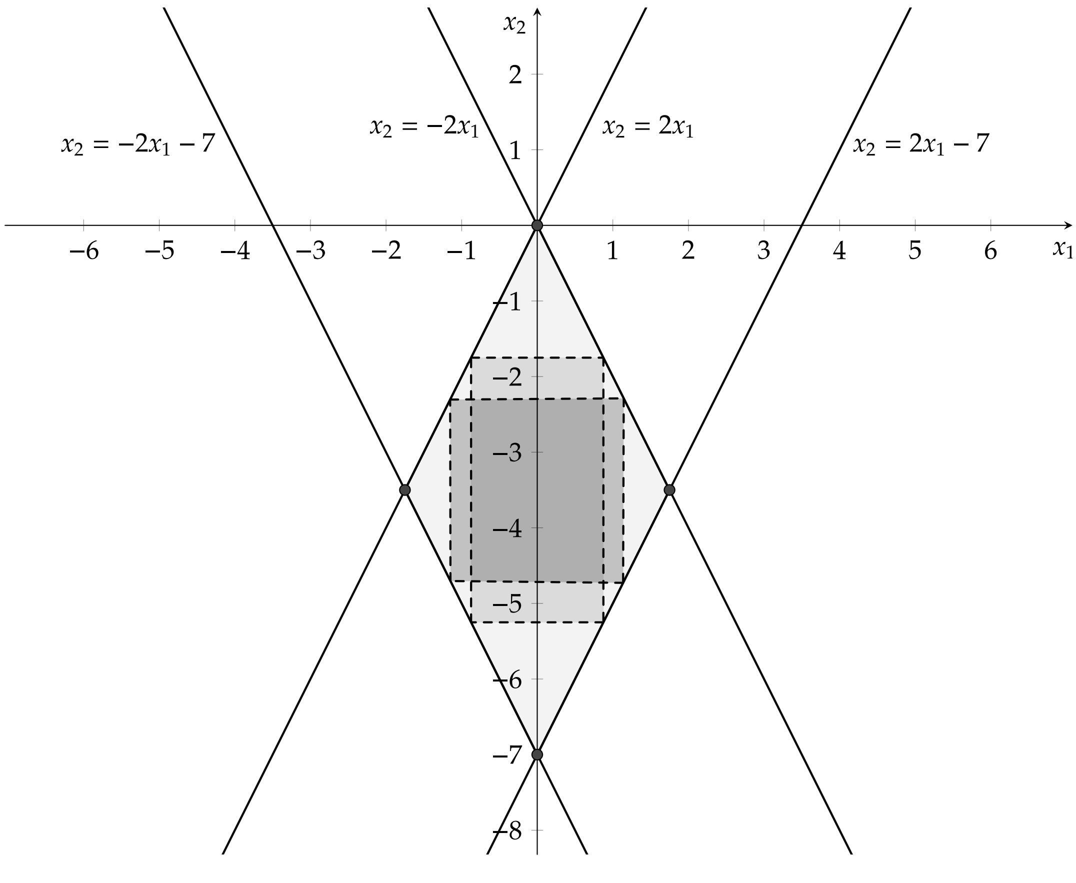

Figure 2: Graphical representation of Example 2

Example 4.3. *Let us find the united inner solution set and the maximal inner estimate for the following* 2×2 *interval linear system:*

$$\begin{array}{rcl} -2x_1 + x_2 & \approx & [-1, 3] \\ 4x_1 - 2x_2 & \approx & [-6, 2] \end{array}.$$

Observe that:

$$\Sigma_{\exists\exists} = \Sigma_{\forall\exists} = \left\{ (x_1, x_2)^T \mid -1 \leq -2x_1 + x_2 \leq 3 \right\}, \Sigma_{\exists\forall} = \Sigma_{\forall\forall} = \emptyset.$$

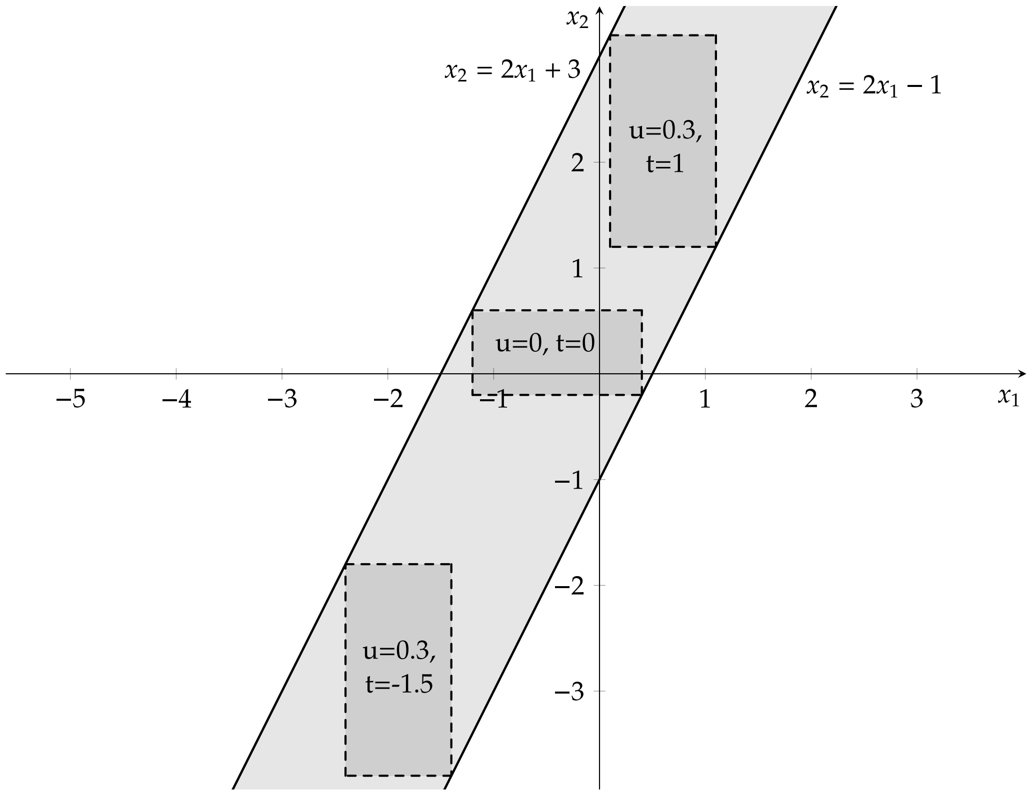

Figure 3: Graphical representation of Example 3

Now, consider the corresponding AILS where both A and $|A|$ are singular, with $ind(A) = ind(|A|) = 1$.

$$\begin{array}{rcl} -2\underline{\overline{x_1}} + \underline{\overline{x_2}} & = & [-1,3] \\ 4\underline{\overline{x_1}} - 2\underline{\overline{x_2}} & = & [-6,2] \end{array}.$$

Again, we compute the Moore-Penrose inverse of the singular matrix A:

$$M = A^{\dagger} = \begin{bmatrix} -0.08 & 0.16 \\ 0.04 & -0.08 \end{bmatrix}.$$

Hence, we obtain: $[X^*] = ([-1.2,-0.4],[-0.2,-0.6])^T$, $\Lambda = (2t,4t)^T$, $t \in \mathbb{R}$, $W = (0,0)^T$ *, which implies that the obtained* $[X^*]$ *is a solution of the given AILS. All solutions of* $|A|\Theta = W$ *are* $\Theta = (u,-2u)^T$*, where* $u \in \mathbb{R}$*. We require additional necessary constrains for* Θ*:*

$$\theta_1 \leq \frac{\overline{x}_1^{*\Lambda} - \underline{x}_1^{*\Lambda}}{2} = \frac{\overline{x}_1^{*} - \underline{x}_1^{*}}{2} = 0.8,$$
$$\theta_2 \leq \frac{\overline{x}_2^{*\Lambda} - \underline{x}_2^{*\Lambda}}{2} = \frac{\overline{x}_2^{*} - \underline{x}_2^{*}}{2} = 0.4.$$

Since $\theta_2 = -2\theta_1$*, we get* $-0.2 \leq \theta_1 \leq 0.8$*. Finally, we obtain* $[X] = (\underline{\overline{x_1}},\underline{\overline{x_2}})^T \in SS$*, where* $t \in \mathbb{R}$*,* $\theta_1 = u$*, and* $-0.2 \leq u \leq 0.8$*:*

$$\begin{array}{rcl} \underline{\overline{x_1}} & = & [-1.2 + t + u, 0.4 + t - u], \\ \underline{\overline{x_2}} & = & [-0.2 + 2t - 2u, 0.6 + 2t + 2u]. \end{array}$$

For example, for $t = 0$ *and* $u = 0$, *we obtain* $[X] = ([-1.2, 0.4], [-0.2, 0.6])^T$, etc. *Therefore, the united inner solution set of this ILS is*

$$\bigcup_{t\in\mathbb{R}, -0.2\leq u\leq 0.8} [-1.2 + t + u, 0.4 + t - u] \times [-0.2 + 2t - 2u, 0.6 + 2t + 2u] = \Sigma_{\exists\exists}.$$

Since the matrix A contains no zero-columns, by Theorem 3.5 all inner estimates are maximal inner estimates. The united set of solutions of this ILS, one maximal inner estimate obtained for $t = 0, u = 0$, *and two instances of maximal inner estimates with the maximal Lebesgue measure (for* $t = -1.5$, $u = 0.3$, *and* $t = 1$, $u = 0.3$*) are graphically represented in Figure 3.*

Example 4.4. *Let us find the united inner solution set and maximal inner estimate for the following* 2×2 *interval linear system:*

$$\begin{array}{rcl} 2x_1 + 0x_2 & \approx & [2, 4] \\ -3x_1 + 0x_2 & \approx & [-6, -3] \end{array}.$$

Observe that:

$$\Sigma_{\exists\exists} = \Sigma_{\forall\exists} = \left\{(x_1, x_2)^T \mid 1 \leq x_1 \leq 2, x_2 \in \mathrm{R}\right\}, \Sigma_{\exists\forall} = \Sigma_{\forall\forall} = \emptyset.$$

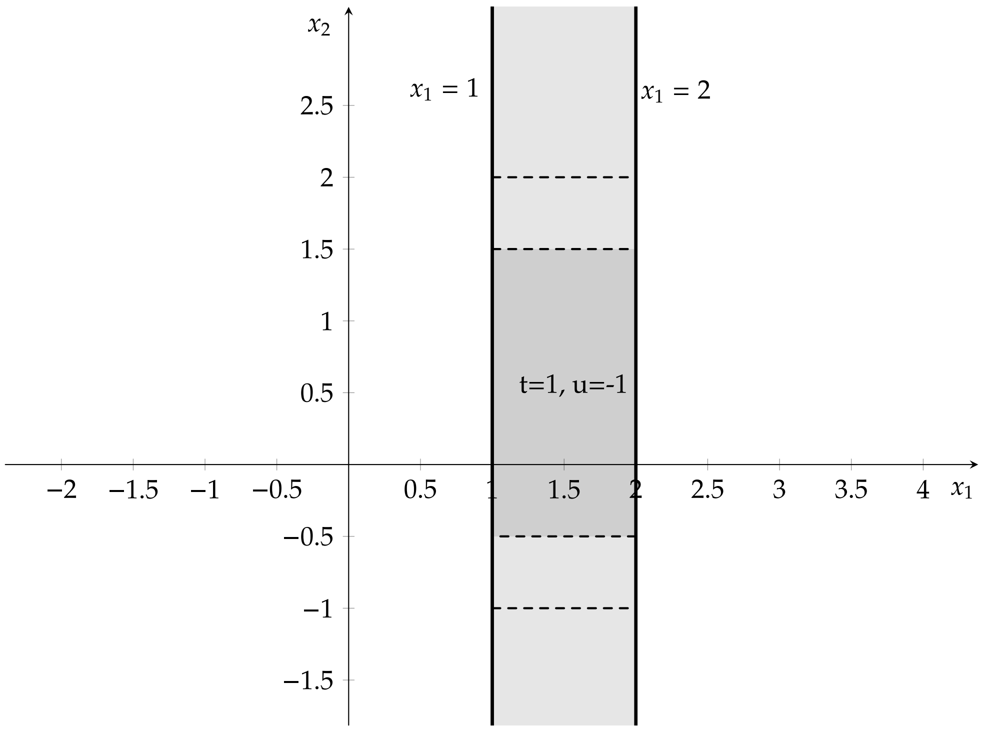

Figure 4: Graphical representation of Example 4

Now, consider the corresponding AILS where both A and $|A|$ *are singular with* $ind(A) = ind(|A|) = 1$.

$$\begin{array}{rcl} 2\overline{\underline{x_1}} + 0\overline{\underline{x_2}} & = & [2, 4] \\ -3\overline{\underline{x_1}} + 0\overline{\underline{x_2}} & = & [-6, -3] \end{array}.$$

One $\{1\}$*-inverse of the singular matrix A is:*

$$M = A^{(1)} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence, we obtain: $[X^*] = ([1, 2], [0, 0])^T$, $\Lambda = (0, t)^T$, $t \in \mathbb{R}$, $W = (0, 0)^T$, *which implies that the obtained* $[X^*]$ *is a solution of the given AILS. All solutions of* $|A|\Theta = W$ *are* $\Theta = (0, u)^T$, *where* $u \leq 0$. *We obtain* $[X] = \left([1, 2], [\frac{t}{2} + u, \frac{t}{2} - u]\right)^T$, $t \in \mathbb{R}$, $u \leq 0$. *Therefore, the united inner solution set of this ILS is*

$$\bigcup_{t\in\mathbb{R}, u\leq 0} [1, 2] \times \left[\frac{t}{2} + u, \frac{t}{2} - u\right] = \Sigma_{\exists\exists}.$$

Since matrix A contains a zero column, by Theorem 3.5 it yields that there are no maximal inner estimates. The united set of solutions of this ILS, and its inner interval estimates obtained for $t = 1$, $u = -1$ *and* $t = 1$, $u = -1.5$ *are graphically represented in Figure 4.*

5. Concluding remarks

Based on recent results related to the general solution of consistent fuzzy linear systems obtained in [12, 13, 15], the inner solution set of interval linear systems with precise coefficient matrices of arbitrary size has been characterized using their generalized {1}-inverses, as well as the maximal inner solution set of a given interval linear system (ILS). Our main result is a theorem that establishes a necessary and sufficient condition for the consistency of the associated interval linear system. Remarkably, this result generalizes the well-known result involving {1}-inverses for the consistency of linear systems. Building upon these findings, a necessary and sufficient condition for the equality of inner estimates and the maximal inner estimates of such ILS has been obtained, which is in accordance with [9].

Further research could focus on extending the methods to handle cases where interval linear systems are not formally consistent, where intervals on the right-hand side of the system are parametric, where the coefficient matrices are not precisely defined, or by incorporating uncertainty into the models to better reflect real-world scenarios.

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