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Further characterizations and representations of the Bott-Duffin core inverse

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Abstract. The purpose of this paper is to discuss some new characterizations and representations of the Bott-Duffin core inverse. We characterize the Bott-Duffin core inverse by using the orthogonal projectors, the matrix equation, the Bott-Duffin group inverse and the solution of the block-rank equation. Some representations of the Bott-Duffin core inverse are presented by using the full-rank decomposition. Finally, we provide the least-squares solution of the constrained system of linear equations.

1. Introduction

Bott and Duffin, in their famous paper [1], introduced the "constrained inverse" of a square matrix as an important tool in the electrical network theory. This inverse is called, in their honor the Bott-Duffin inverse [2]. In [31], we extend it to the Bott-Duffin core inverse and give some properties and characterizations of the Bott-Duffin core inverse. Recently, numerous scholars have generalized the Bott-Duffin inverse and investigated its properties(see [5, 6, 9, 11, 26, 28]).In this paper, $\mathbb{C}^{m\times n}$ is the set of $m \times n$ complex matrices and $\mathbb{C}^n$ is the vector space of $n-$tuples of complex number over $\mathbb{C}$. If L is a subspace of $\mathbb{C}^n$, we use the notation $L \leqslant \mathbb{C}^n$.Let $\mathbb{C}_n^{CM}$ be the set of $n \times n$ matrices of index one, that is,

$$\mathbb{C}_n^{CM} = \left\{ A \in \mathbb{C}^{n\times n} \middle| \operatorname{rank}(A^2) = \operatorname{rank}(A) \right\}.$$

For convenience, we provide the definition of the Bott-Duffin core inverse.

Definition 1.1. *[31] Let $A \in \mathbb{C}^{n\times n}$, $L \leqslant \mathbb{C}^n$. If $AP_L + P_{L^\perp} \in \mathbb{C}_n^{CM}$, then*

$$A_{(L)}^{(\#)} = P_L(AP_L + P_{L^\perp})^{\#},$$

is called the Bott-Duffin core inverse of A with respect to L.

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In this paper, we will continue to discuss some new characterizations and representations of it. Our main contributions can be summarized as below:

(1) We present some new characterizations of the Bott-Duffin core inverse and consider the relationships between it and other generalized inverses.

(2) We characterize the Bott-Duffin core inverse by using the solution of the block-rank equation: $\operatorname{rank}\left(\begin{bmatrix} A & B \\ C & X \end{bmatrix}\right) = \operatorname{rank}(A)$. Further, for the solution $X = A^{(\#\!\!\!\bigcirc)}_{(L)}$ of the block-rank equation: $\operatorname{rank}\left(\begin{bmatrix} P_L A^* P_L & B \\ C & X \end{bmatrix}\right) = \operatorname{rank}(P_L A^* P_L)$, all matrices B and C are described.

(3) The full-rank and integral representations of the Bott-Duffin core inverse are given. Moreover, using [19], a novel representation of it is also presented.

(4) We give a Cramer's rule of the least-squares solutions of the constraint system of linear equations and a condensed determinantal expression of the Bott-Duffin core inverse.

This paper is organized as follows. In Section 2, we introduce some necessary notations, definitions and lemmas. In Section 3, we present some new characterizations and properties of the Bott-Duffin core inverse. In Section 4, some new representations of the Bott-Duffin core inverse are provided. A Cramer's rule for the unique solution of a constrained matrix equation is given in Section 5.

2. Preliminaries

The symbols $\mathcal{R}(A)$, $\mathcal{N}(A)$, A^*, A^T and $\operatorname{rank}(A)$ represent the range space, null space, conjugate transpose, transpose and rank of $A \in \mathbb{C}^{m\times n}$, respectively. We denote the identity matrix in $\mathbb{C}^{n\times n}$ by I_n. The symbol O means the null matrix of appropriate size. $L^\perp$ means the orthogonal complement subspace of $L \leqslant \mathbb{C}^n$. The dimension of L is denoted by $\dim(L)$. $P_{L,M}$ stands for the oblique projector onto L along M, where $L, M \leqslant \mathbb{C}^n$ and $L \oplus M = \mathbb{C}^n$. P_L, the orthogonal projector onto L, is equivalent to $P_{L,L^\perp}$.

Additionally, the Moore–Penrose inverse $A^\dagger \in \mathbb{C}^{n\times m}$ of $A \in \mathbb{C}^{m\times n}$ is the unique matrix verifying the following matrix equations (see [2, 8, 18, 24])

$$AA^\dagger A = A,\ A^\dagger AA^\dagger = A^\dagger,\ (AA^\dagger)^* = AA^\dagger,\ (A^\dagger A)^* = A^\dagger A.$$

If $A \in \mathbb{C}^{m\times n}$, a matrix $X \in \mathbb{C}^{n\times m}$ satisfies the following three equations:

$$XAX = X, \quad \mathcal{R}(X) = T \quad \text{and} \quad \mathcal{N}(X) = S,$$

where T is a subspace of $\mathbb{C}^n$ and S is a subspace of $\mathbb{C}^m$, then X is unique and is denoted by $A^{(2)}_{T,S}$ (see [2, 24]).

The group inverse of $A \in \mathbb{C}^{CM}_n$ is the unique matrix $A^\# \in \mathbb{C}^{n\times n}$ verifying the following matrix equations (see [2, 8, 15, 24])

$$AA^\# A = A,\ A^\# AA^\# = A^\#,\ AA^\# = A^\# A.$$

For a given matrix $A \in \mathbb{C}^{CM}_n$, the core inverse of A is defined to be the unique matrix $A^{\#\!\!\!\bigcirc} \in \mathbb{C}^{n\times n}$ satisfying (see [3])

$$AA^{\#\!\!\!\bigcirc} = AA^\dagger,\ \mathcal{R}(A^{\#\!\!\!\bigcirc}) \subset \mathcal{R}(A).$$

Moreover, Wang and Liu [22] prove that the core inverse of $A \in \mathbb{C}^{CM}_n$ is the unique matrix satisfying

$$AA^{\#\!\!\!\bigcirc}A = A,\ A(A^{\#\!\!\!\bigcirc})^2 = A^{\#\!\!\!\bigcirc},\ (AA^{\#\!\!\!\bigcirc})^* = AA^{\#\!\!\!\bigcirc}. \tag{1}$$

Additional symbols used in this paper are $\mathbb{C}_n^P$, $\mathbb{C}_n^{OP}$, $\mathbb{C}_n^{TM}$ and $\mathbb{C}_n^{EP}$, and denote the sets of projectors (idempotent matrices), orthogonal projectors (Hermitian idempotent matrices), tripotent and EP matrices, respectively, i.e.

$$\begin{aligned}
\mathbb{C}_n^P &= \{A \mid A \in \mathbb{C}^{n\times n}, A^2 = A\},\\
\mathbb{C}_n^{OP} &= \{A \mid A \in \mathbb{C}^{n\times n}, A^2 = A = A^*\},\\
\mathbb{C}_n^{TM} &= \{A \mid A \in \mathbb{C}^{n\times n}, A^3 = A\},\\
\mathbb{C}_n^{EP} &= \{A \mid A \in \mathbb{C}^{n\times n}, AA^\dagger = A^\dagger A\} = \{A \mid A \in \mathbb{C}^{n\times n}, \mathcal{R}(A) = \mathcal{R}(A^*)\}.
\end{aligned}$$

Lemma 2.1. [31] *Let $A \in \mathbb{C}^{n\times n}$, $L \leqslant \mathbb{C}^n$, $T = \mathcal{R}(P_LAP_L)$ and $S = \mathcal{N}(P_LAP_L)$. If $(AP_L + P_{L^\perp}) \in \mathbb{C}_n^{CM}$, then the following statements hold:*

(i) $P_LAA_{(L)}^{(\#)} = P_T$ *and* $A_{(L)}^{(\#)}AP_L = P_{T,S}$;

(ii) $A_{(L)}^{(\#)} = A_{T,T^\perp}^{(2)} = (AP_L)_{T,T^\perp}^{(2)} = (P_LA)_{T,T^\perp}^{(2)} = (P_LAP_L)_{T,T^\perp}^{(1,2)}$;

(iii) $A_{(L)}^{(\#)} = (P_LAP_L)^{\#}$;

(iv) $AA_{(L)}^{(\#)} = P_{AT,T^\perp}$ *and* $A_{(L)}^{(\#)}A = P_{T,(A^*T)^\perp}$.

Lemma 2.2. [20] *Let $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$, $C \in \mathbb{C}^{m\times n}$. Then there exists a unique solution $X \in \mathbb{C}^{m\times m}$ to the equation:*

$$\operatorname{rank}\left(\begin{bmatrix} A & B \\ C & X \end{bmatrix}\right) = \operatorname{rank}(A)$$

if and only if $\mathcal{N}(C) \supset \mathcal{N}(A)$ and $\mathcal{R}(B) \subset \mathcal{R}(A)$, in which case $X = CA^\dagger B$.

Let $A \in \mathbb{C}^{n\times n}$ and $L \leqslant \mathbb{C}^n$. In order to discuss some properties of the Bott-Duffin core inverse, we will consider an appropriate matrix decomposition of A with respect to L. Since P_L is an orthogonal projector, then there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ such that

$$P_L = U\begin{bmatrix} I_l & O \\ O & O \end{bmatrix}U^*, \tag{2}$$

where $l = \dim(L)$, the matrix A can be written as

$$A = U\begin{bmatrix} A_L & B_L \\ C_L & D_L \end{bmatrix}U^*, \tag{3}$$

where $A_L \in \mathbb{C}^{l\times l}$, $B_L \in \mathbb{C}^{l\times(n-l)}$, $C_L \in \mathbb{C}^{(n-l)\times l}$, $D_L \in \mathbb{C}^{(n-l)\times(n-l)}$.

Lemma 2.3. [31] *Let P_L and A be given by* (2) *and* (3)*, respectively. $A_{(L)}^{(\#)}$ exists if and only if $A_L \in \mathbb{C}_l^{CM}$. In this case,*

$$A_{(L)}^{(\#)} = U\begin{bmatrix} A_L^{\#} & O \\ O & O \end{bmatrix}U^*. \tag{4}$$

3. Some new characterizations and properties of the Bott-Duffin core inverse

Motivated by (1), we provide some new characterizations of the Bott-Duffin core inverse of $A \in \mathbb{C}^{n\times n}$ (in the case when it exists).

Theorem 3.1. *Let $A \in \mathbb{C}^{n\times n}$, $L \leqslant \mathbb{C}^n$ and $\dim(L) = l$ be such that $A^{(\#)}_{(L)}$ exists and let $X \in \mathbb{C}^{n\times n}$. The following statements are equivalent:*

(a) $X = A^{(\#)}_{(L)}$;

(b) $P_LAXAP_L = P_LAP_L$, $P_LAX^2 = X$ *and* $(P_LAX)^* = P_LAX$;

(c) $XP_LAP_LX = X$, $X(AP_L)^2 = P_LAP_L$ *and* $(P_LAX)^* = P_LAX$;

(d) $X(AP_L)^2 = P_LAP_L$, $P_LAX^2 = X$ *and* $(P_LAX)^* = P_LAX$.

Proof. Let P_L, A and $A^{(\#)}_{(L)}$ be given by (2), (3) and (4), respectively. By simple calculation, $(b) - (d)$ can be derived from (a).

$(b) \Rightarrow (a)$. Suppose that X is given by the following equation:

$$X = U\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}U^*, \tag{5}$$

where $X_1 \in \mathbb{C}^{l\times l}$, $X_2 \in \mathbb{C}^{l\times(n-l)}$, $X_3 \in \mathbb{C}^{(n-l)\times l}$, $X_4 \in \mathbb{C}^{(n-l)\times(n-l)}$ and U is given in (2). Let P_L and A be given by (2) and (3), respectively. From $P_LAX^2 = X$, we have $A_LX_1{}^2 = X_1$, $A_LX_1X_2 = X_2$, $X_3 = O$ and $X_4 = O$. Since $(P_LAX)^* = P_LAX$, it follows that $(A_LX_1)^* = A_LX_1$ and $AX_2 = O$. In terms of $P_LAXAP_L = P_LAP_L$, we get $A_LX_1A_L = A_L$. By (1), we can verify $X_1 = A_L{}^{\#}$. From $A_LX_1X_2 = X_2$ and $A_LX_2 = O$, we have $\mathcal{R}(X_2) \subset \mathcal{R}(A_L)$ and $\mathcal{R}(X_2) \subset \mathcal{N}(A_L)$. It follows from $A_L \in \mathbb{C}_n^{CM}$ that $\mathcal{R}(A_L) \cap \mathcal{N}(A_L) = \{0\}$, which means $X_2 = O$. Thus, using (4), $X = A^{(\#)}_{(L)}$.

The rest of the proof follows similarly. □

Theorem 3.2. *Let $A \in \mathbb{C}^{n\times n}$ and $L \leqslant \mathbb{C}^n$ be such that $A^{(\#)}_{(L)}$ exists and let $X \in \mathbb{C}^{n\times n}$. The following statements are equivalent:*

(a) $X = A^{(\#)}_{(L)}$;

(b) $X \in \mathbb{C}_n^{EP}$, $P_LAX^2 = X$ *and* $P_LAXAP_L = P_LAP_L$;

(c) $X \in \mathbb{C}_n^{EP}$, $P_LAX^2 = X$ *and* $X(AP_L)^2 = P_LAP_L$;

(d) $X \in \mathbb{C}_n^{EP}$, $P_LAX^2 = X$ *and* $X(AP_L)^{m+2} = (P_LAP_L)^m$, *where* $m \in \mathbb{Z}^+$;

(e) $X \in \mathbb{C}_n^{EP}$, $P_LAX^2 = X$ *and* $\operatorname{rank}(P_LAP_L) = \operatorname{rank}(X)$.

Proof. Let P_L, A and $A^{(\#)}_{(L)}$ be given by (2), (3) and (4), respectively. Since $A^{(\#)}_{(L)} \in \mathbb{C}_n^{EP}$ and by simple calculation, $(b) - (e)$ can be derived from (a).

$(b) \Rightarrow (a)$. Let X be given by (5). From $X \in \mathbb{C}_n^{EP}$ and $P_LAX^2 = X$, we have $X_2 = O$, $X_3 = O$, $X_4 = O$ and $A_LX_1{}^2 = X_1$. Since $P_LAXAP_L = P_LAP_L$, it follows that $A_LX_1A_L = A_L$. By [12], the matrix A_L can be written as

$$A_L = V\begin{bmatrix} \Sigma K & \Sigma L \\ O & O \end{bmatrix}V^*, \tag{6}$$

where $V \in \mathbb{C}^{l\times l}$ is unitary, $r = \operatorname{rank}(A_L)$, $\Sigma = \operatorname{diag}(\sigma_1 I_{r_1}, \ldots, \sigma_t I_{r_t})$ is the diagonal matrix of singular values of A_L, $\sigma_1 > \sigma_2 > \cdots > \sigma_t > 0$, $r_1 + r_2 + \cdots + r_t = r$, and $K \in \mathbb{C}^{r\times r}$, $L \in \mathbb{C}^{r\times(n-r)}$ satisfy $KK^* + LL^* = I_r$. Suppose that X_1 is given by the following equation:

$$X_1 = V\begin{bmatrix} X_1{}' & X_2{}' \\ X_3{}' & X_4{}' \end{bmatrix}V^*.$$

where $X_1{}' \in \mathbb{C}^{r\times r}$, $X_2{}' \in \mathbb{C}^{r\times(l-r)}$, $X_3{}' \in \mathbb{C}^{(l-r)\times r}$ and $X_4{}' \in \mathbb{C}^{(l-r)\times(l-r)}$. From $X_1 \in \mathbb{C}_l^{EP}$ and $A_L X_1{}^2 = X_1$, we have $X_2{}' = O$, $X_3{}' = O$ and $X_4{}' = O$. It follows from $A_L \in \mathbb{C}_l^{CM}$ and $A_L X_1 A_L = A_L$ that we can obtain $X_1{}' = (\Sigma K)^{-1}$. In [3], it follows from (6) that

$$A_L{}^{\textcircled{\#}} = V\begin{bmatrix} (\Sigma K)^{-1} & O \\ O & O \end{bmatrix} V^*. \tag{7}$$

Hence $X_1 = A_L{}^{\textcircled{\#}}$ that means $X = A_{(L)}^{(\textcircled{\#})}$.

The rest of the proof follows similarly. □

In the following theorem, we present some characterizations of the Bott-Duffin core inverse in terms of just two matrix equations.

Theorem 3.3. *Let $A \in \mathbb{C}^{n\times n}$, $L \leqslant \mathbb{C}^n$, $T = \mathcal{R}(P_L A P_L)$ and $S = \mathcal{N}(P_L A P_L)$ be such that $A_{(L)}^{(\textcircled{\#})}$ exists and let $X \in \mathbb{C}^{n\times n}$. The following statements are equivalent:*

- *(a)* $X = A_{(L)}^{(\textcircled{\#})}$;
- *(b)* $P_L A X = P_T$ *and* $P_T X = X$;
- *(c)* $X A P_L = P_{T,S}$ *and* $X P_T = X$;
- *(d)* $X A P_L = P_{T,S}$ *and* $P_T X P_T = X$;
- *(e)* $P_L A X = P_T$ *and* $P_T X P_T = X$.

Proof. Item (a) implies any of the assertions $(b) - (e)$ which follows directly by Lemma 2.1 (i), (ii), (iv) and $P_L P_T = P_T P_L = P_T$. For the converse implications, we will only give the proof that (b) implies (a).

$(b) \Rightarrow (a)$. From $P_L A X = P_T$ and $P_T X = X$, we have $\mathcal{N}(X) \subset T^\perp$ and $\mathcal{R}(X) \subset T$, which imply $\mathcal{R}(X) = T$ and $\mathcal{N}(X) = T^\perp \supset L^\perp$. It is clear that $XAX = (XP_L)AX = XP_T = X$. Thus, from Lemma 2.1 (ii), we get $X = A_{(L)}^{(\textcircled{\#})}$. □

In [28], Zhang et al. introduce a new generalized inverse: Bott-Duffin group inverse, $A_{(L)}^{(\#)} = P_L(AP_L + P_{T^\perp})^\#$. In the following theorem, we use the Bott-Duffin group inverse to characterize the Bott-Duffin core inverse.

Theorem 3.4. *Let $A \in \mathbb{C}^{n\times n}$, $L \leqslant \mathbb{C}^n$ and $T = \mathcal{R}(P_L A P_L)$, be such that $A_{(L)}^{(\textcircled{\#})}$ exists and let $X \in \mathbb{C}^{n\times n}$. The following statements are equivalent:*

- *(a)* $X = A_{(L)}^{(\textcircled{\#})}$;
- *(b)* $P_L A X A_{(L)}^{(\#)} = A_{(L)}^{(\#)}$, $(P_L A X)^* = P_L A X$ *and* $\mathcal{R}(X) \subset T$;
- *(c)* $X P_L A A_{(L)}^{(\#)} = A_{(L)}^{(\#)}$, $(P_L A X)^* = P_L A X$ *and* $\mathcal{R}(X) \subset T$;
- *(d)* $A_{(L)}^{(\#)} X P_L A = A_{(L)}^{(\#)}$, $(P_L A X)^* = P_L A X$ *and* $\mathcal{R}(X) \subset T$;
- *(e)* $X A_{(L)}^{(\#)} P_L A = A_{(L)}^{(\#)}$, $(P_L A X)^* = P_L A X$ *and* $\mathcal{R}(X) \subset T$;
- *(f)* $A_{(L)}^{(\#)} P_L A X = X$, $(P_L A X)^* = P_L A X$ *and* $\mathcal{R}(X) \subset T$;
- *(g)* $P_L A A_{(L)}^{(\#)} X = X$, $(P_L A X)^* = P_L A X$ *and* $\mathcal{R}(X) \subset T$.

Proof. Item (a) implies any of the assertions $(b)-(g)$ which follows directly by (2), (3) and (4). For the converse implications, we will only give the proof that (b) implies (a).

$(b) \Rightarrow (a)$. Let P_L, A and X be given by (2), (3) and (5), respectively. From $\mathcal{R}(X) \subset \mathcal{R}(P_LAP_L)$, we have

$$X = U\begin{bmatrix} X_1 & X_2 \\ O & O \end{bmatrix}U^*.$$

Since $P_LAXA_{(L)}^{(\#)} = A_{(L)}^{(\#)}$, it follow that $A_LX_1A_L{}^{\#} = A_L{}^{\#}$. From $(P_LAX)^* = P_LAX$, we can obtain $(A_LX_1)^* = A_LX_1$ and $A_LX_2 = O$. Since $\mathcal{R}(X) \subset T$, it follows that $\mathcal{R}(X_1)+\mathcal{R}(X_2) = \mathcal{R}(A_L)$, which implies that $\mathcal{R}(X_1) \subset \mathcal{R}(A_L)$ and $\mathcal{R}(X_2) \subset \mathcal{R}(A_L)$. By $A_LX_2 = O$, we have $\mathcal{R}(X_2) \subset \mathcal{N}(A_L)$. In terms of $A_L \in \mathbb{C}_n^{CM}$, we have $\mathcal{R}(A_L) \cap \mathcal{N}(A_L) = \{0\}$, which means $X_2 = O$. Since [13, Theorem 5(ii)], it follows from $A_LX_1A_L{}^{\#} = A_L{}^{\#}$, $(A_LX_1)^* = A_LX_1$ and $\mathcal{R}(X_1) \subset \mathcal{R}(A_L)$ that $X_1 = A_L{}^{\textcircled{\#}}$. Thus $X = A_{(L)}^{(\textcircled{\#})}$. □

In [22], we know that $X = A^{\textcircled{\#}}$ is equivalent to $AX^2 = X$, $AXA = A$ and $AX = (AX)^*$. In [14], we have $X = A^{\textcircled{\#}}$ is equivalent to $XA^2 = A$, $XAX = X$ and $AX = (AX)^*$. We try to use the characterizations of $A^{\textcircled{\#}}$ to study the characterization of $A_{(L)}^{(\textcircled{\#})}$. In the following theorem, we investigate the characterization of Bott-Duffin core inverse of A on the basis of $X(P_LA)^2$ or P_LAX^2.

Theorem 3.5. *Let $A \in \mathbb{C}^{n\times n}$, $L \leqslant \mathbb{C}^n$ and $T = \mathcal{R}(P_LAP_L)$, be such that $A_{(L)}^{(\textcircled{\#})}$ exists and let $X \in \mathbb{C}^{n\times n}$. The following statements are equivalent:*

(a) $X = A_{(L)}^{(\textcircled{\#})}$;

(b) $X(P_LA)^2 = P_LA$ *and* $P_LXP_T = X$;

(c) $X(P_LA)^2 = P_LA$, $P_LAX = (P_LAX)^*$ *and* $\mathcal{R}(X) \subset \mathcal{R}(A)$;

(d) $P_LAX^2 = X$, $XP_T = X$ *and* $\operatorname{rank}(X) = \operatorname{rank}(P_LAP_L)$;

(e) $P_LAX^2 = X$, $P_LAX = (P_LAX)^*$ *and* $\operatorname{rank}(X) = \operatorname{rank}(P_LAP_L)$.

Proof. Let P_L, A and $A_{(L)}^{(\textcircled{\#})}$ be given by (2), (3) and (4), respectively. Clearly, items $(b)-(d)$ can be derived from (a).

$(b) \Rightarrow (a)$. From (2), (3), (4) and Lemma 2.1, we have

$$P_T = U\begin{bmatrix} A_LA_L{}^{\textcircled{\#}} & O \\ O & O \end{bmatrix}U^*.$$

Let X be given by (5). Since $P_LXP_T = X$, it follows that $X_1 = X_1P_{A_L}$, $X_2 = O$, $X_3 = O$ and $X_4 = O$. In terms of $X(P_LA)^2 = P_LA$, we can obtain $X_1A_L{}^2 = A_L$. Since [13, Theorem 2 (iv)], it follows that $X_1 = A_L{}^{\textcircled{\#}}$. Thus, $X = A_{(L)}^{(\textcircled{\#})}$.

The rest of the proof follows similarly. □

To discuss the relationships between the Bott-Duffin core inverse and other generalized inverses, we first review the concepts and properties of several generalized inverses (see [10, 21, 28]).

The unique matrix $X \in \mathbb{C}^{n\times n}$ satisfying

$$XAX = X, \quad AX = XA \quad \text{and} \quad A^{k+1}X = A^k,$$

where $A \in \mathbb{C}^{n\times n}$ with $k = \operatorname{Ind}(A)$, is called the Drazin inverse of $A \in \mathbb{C}^{n\times n}$ and is denoted by A^D.

The unique matrix $X \in \mathbb{C}^{n\times n}$ satisfying

$$XAX = X \quad \text{and} \quad \mathcal{R}(X) = \mathcal{R}(X^*) = \mathcal{R}(A^k),$$

where $A \in \mathbb{C}^{n\times n}$ with $k = \mathrm{Ind}(A)$, is called the core-EP inverse of A and is denoted by $A^{\textcircled{\dagger}}$.

Let $A \in \mathbb{C}^{n\times n}$ with $k = \mathrm{Ind}(A)$, the unique matrix $X \in \mathbb{C}^{n\times n}$ satisfying

$$XAX = X, \quad XA = A^{\dagger}A \quad \text{and} \quad A^kX = A^kA^{\dagger}$$

is called the DMP inverse of A and is denoted by $A^{D,\dagger}$.

Let $A \in \mathbb{C}^{n\times n}$ with $k = \mathrm{Ind}(A)$, the unique matrix $X \in \mathbb{C}^{n\times n}$ satisfying

$$AX^2 = X \quad \text{and} \quad AX = A^{\textcircled{\dagger}}A$$

is called the weak group inverse of A and is denoted by $A^{\textcircled{w}}$.

If $A \in \mathbb{C}^{n\times n}$ and $L \leq \mathbb{C}^n$, then $P_L(AP_L + P_L^{\perp})^{\#}$ is called the Bott-Duffin group inverse (BDG-inverse) and is defined by $A^{(\#)}_{(L)}$, i.e.

$$A^{(\#)}_{(L)} = P_L(AP_L + P_L^{\perp})^{\#}.$$

Lemma 3.6. [10, 21, 27, 28] *Let $A \in \mathbb{C}^{n\times n}$ with $Ind(A) = k$. Then*

(a) $A^{\dagger} = A^{(2)}_{\mathcal{R}(A^*),\mathcal{N}(A^*)}$;

(b) $A^{D} = A^{(2)}_{\mathcal{R}(A^k),\mathcal{N}(A^k)}$;

(c) $A^{\textcircled{\dagger}} = A^{(2)}_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*)}$;

(d) $A^{D,\dagger} = A^{(2)}_{\mathcal{R}(A^k),\mathcal{N}(A^kA^{\dagger})}$;

(e) $A^{\textcircled{w}} = A^{(2)}_{\mathcal{R}(A^k),\mathcal{N}(A^*A^k)^{\perp}}$;

(f) $A^{(\#)}_{(L)} = A^{(2)}_{\mathcal{R}(P_LAP_L),\mathcal{N}(P_LAP_L)}$.

Theorem 3.7. *Let $A \in \mathbb{C}^{n\times n}$, $\mathrm{ind}(A) = k$, $L \leqslant \mathbb{C}^n$ and $T = \mathcal{R}(P_LAP_L)$ be such that $A^{(\textcircled{\#})}_{(L)}$ exists. The following statements hold:*

(a) $A^{(\textcircled{\#})}_{(L)} = A^{\dagger} \Leftrightarrow T = \mathcal{R}(A)$ *and* $A \in \mathbb{C}_n^{EP}$;

(b) $A^{(\textcircled{\#})}_{(L)} = A^{D} \Leftrightarrow T = \mathcal{R}(A^k)$ *and* $A^k \in \mathbb{C}_n^{EP}$;

(c) $A^{(\textcircled{\#})}_{(L)} = A^{\textcircled{\dagger}} \Leftrightarrow T = \mathcal{R}(A^k)$;

(d) $A^{(\textcircled{\#})}_{(L)} = A^{D,\dagger} \Leftrightarrow T = \mathcal{R}(A^k)$ *and* $A^kA^{\dagger} \in \mathbb{C}_n^{EP}$;

(e) $A^{(\textcircled{\#})}_{(L)} = A^{\textcircled{w}} \Leftrightarrow T = \mathcal{R}(A^k)$ *and* $A^k(A^k)^*A \in \mathbb{C}_n^{EP}$;

(f) $A^{(\textcircled{\#})}_{(L)} = A^{(\#)}_{(L)} \Leftrightarrow P_LAP_L \in \mathbb{C}_n^{EP}$.

Proof. (a). Since $A^{\dagger} = A^{(2)}_{\mathcal{R}(A^*),\mathcal{N}(A^*)}$ and $A^{(\textcircled{\#})}_{(L)} = A^{(2)}_{T,T^{\perp}}$, we have that $A^{(\textcircled{\#})}_{(L)} = A^{\dagger}$ if and only if $T = \mathcal{R}(A^*)$ and $T^{\perp} = \mathcal{N}(A^*) = \mathcal{R}(A)^{\perp}$ that are equivalent to $T = \mathcal{R}(A)$ and $A \in \mathbb{C}_n^{EP}$.

(b)−(c). Since $A^{D} = A^{(2)}_{\mathcal{R}(A^k),\mathcal{N}(A^k)}$ and $A^{\textcircled{\dagger}} = A^{(2)}_{\mathcal{R}(A^k),\mathcal{N}((A^k)^*)}$, the proof can be obtained directly by $A^{(\textcircled{\#})}_{(L)} = A^{(2)}_{T,T^{\perp}}$.

(d). Since $A^{D,\dagger} = A^{(2)}_{\mathcal{R}(A^k),\mathcal{N}(A^kA^{\dagger})}$, it follows that $A^{(\textcircled{\#})}_{(L)} = A^{D,\dagger}$ if and only if $T = \mathcal{R}(A^k)$ and $T^{\perp} = \mathcal{N}(A^kA^{\dagger}) = \mathcal{R}((A^kA^{\dagger})^*)^{\perp}$. However $\mathcal{R}(A^kA^{\dagger}) = \mathcal{R}(A^k)$, so $A^{(\textcircled{\#})}_{(L)} = A^{D,\dagger}$ if and only if $T = \mathcal{R}(A^k)$ and $A^kA^{\dagger} \in \mathbb{C}_n^{EP}$.

(e). Note the fact that $A^{\textcircled{W}} = A^{(2)}_{\mathcal{R}(A^k),\mathcal{R}(A^*A^k)^{\perp}}$, by $\mathcal{R}(A^k) = \mathcal{R}(A^k(A^k)^*A)$ and $\mathcal{R}(A^*A^k)^{\perp} = \mathcal{N}((A^k)^*A) = \mathcal{N}(A^k(A^k)^*A)$, we have $A^{(\#\!\!\!\#)}_{(L)} = A^{\textcircled{W}}$ if and only if $T = \mathcal{R}(A^k)$ and $A^k(A^k)^*A \in \mathbb{C}_n^{EP}$.

(f). Since $A^{(\#)}_{(L)} = A^{(2)}_{T,\mathcal{N}(P_LAP_L)}$, it follows that $A^{(\#\!\!\!\#)}_{(L)} = A^{(\#)}_{(L)}$ if and only if $T^{\perp} = \mathcal{N}(P_LAP_L)$ which means $P_LAP_L \in \mathbb{C}_n^{EP}$. $\square$

Theorem 3.8. *Let $A \in \mathbb{C}^{n\times n}$, $L \leqslant \mathbb{C}^n$, $T = \mathcal{R}(P_LAP_L)$ and $S = \mathcal{N}(P_LAP_L)$ be such that $A^{(\#\!\!\!\#)}_{(L)}$ exists and* $\dim(T) = t$. *Then there is a unique matrix $P \in \mathbb{C}^{n\times n}$ such that*

$$P^2 = P,\ PAP_T = O,\ AP_TP = O \text{ and } \operatorname{rank}(P) = n - t, \tag{8}$$

a unique matrix $Q \in \mathbb{C}^{n\times n}$ such that

$$Q^2 = Q,\ QP_{T,S}A = O,\ P_TAQ = O \text{ and } \operatorname{rank}(Q) = n - t, \tag{9}$$

and a unique matrix $X \in \mathbb{C}^{n\times n}$ such that

$$\operatorname{rank}\left(\begin{bmatrix} A & I_n - Q \\ I_n - P & X \end{bmatrix}\right) = \operatorname{rank}(A). \tag{10}$$

In the case the matrix X is the Bott-Duffin core inverse of A, and

$$P = P_{T^{\perp},AT},\ Q = P_{(A^*T)^{\perp},T}.$$

Proof. Since Lemma 2.1 (iv), it follows from $\mathcal{N}(P_T) \subset \mathcal{N}(AP_T) \subset \mathcal{N}(A^{(\#\!\!\!\#)}_{(L)}AP_T) = \mathcal{N}(P_T)$ that $\mathcal{N}(P_T) = \mathcal{N}(AP_T)$ and $\operatorname{rank}(AP_T) = t$. Notice the fact that

$$\begin{aligned} \text{the condition (8) holds} \quad &\Leftrightarrow \quad (I_n - P)^2 = (I_n - P), (I_n - P)AP_T = AP_T \\ &\qquad\ AP_T(I_n - P) = AP_T, \operatorname{rank}(I_n - P) = t \\ &\Leftrightarrow \quad (I_n - P)^2 = (I_n - P), \mathcal{R}(I_n - P) = \mathcal{R}(AP_T) \\ &\qquad\ = AT, \mathcal{N}(I_n - P) = \mathcal{N}(AP_T) = T^{\perp} \\ &\Leftrightarrow \quad I_n - P = P_{AT,T^{\perp}} \\ &\Leftrightarrow \quad P = P_{T^{\perp},AT}. \end{aligned}$$

Similarly, the condition (9) has the unique solution $Q = P_{(A^*T)^{\perp},T}$. From Lemma 2.1 (ii) and (iv), we have

$$\begin{aligned} \operatorname{rank}\left(\begin{bmatrix} A & I_n - P \\ I_n - Q & X \end{bmatrix}\right) &= \operatorname{rank}\left(\begin{bmatrix} A & AA^{(\#\!\!\!\#)}_{(L)} \\ A^{(\#\!\!\!\#)}_{(L)}A & X \end{bmatrix}\right) \\ &= \operatorname{rank}\left(\begin{bmatrix} I_n & O \\ -A^{(\#\!\!\!\#)}_{(L)} & I_n \end{bmatrix}\begin{bmatrix} A & AA^{(\#\!\!\!\#)}_{(L)} \\ A^{(\#\!\!\!\#)}_{(L)}A & X \end{bmatrix}\right) \\ &= \operatorname{rank}\left(\begin{bmatrix} A & AA^{(\#\!\!\!\#)}_{(L)} \\ O & X - A^{(\#\!\!\!\#)}_{(L)} \end{bmatrix}\right) \\ &= \operatorname{rank}(A) + \operatorname{rank}(X - A^{(\#\!\!\!\#)}_{(L)}). \end{aligned}$$

Thus, from (10), $X = A^{(\#\!\!\!\#)}_{(L)}$. $\square$

In the following theorem, we characterize all matrices B and C such that $X = A^{(\#\!\!\!\#)}_{(L)}$ is the solution of $\operatorname{rank}\left(\begin{bmatrix} P_LA^*P_L & B \\ C & X \end{bmatrix}\right) = \operatorname{rank}(P_LA^*P_L)$.

Theorem 3.9. *Let $L \leqslant \mathbb{C}^n$, $A \in \mathbb{C}^{n\times n}$, $P_LAP_L \in \mathbb{C}_n^{CM}$, $\operatorname{rank}(P_LA^*P_L) = r$ and $\dim(L) = l$. Let P_L and A be given by (2) and (3). By* [12], *the matrix A_L can be written as (6). Then $X = A_{(L)}^{(\#)}$ is the solution of equation*

$$\operatorname{rank}\left(\begin{bmatrix} P_LA^*P_L & B \\ C & X \end{bmatrix}\right) = \operatorname{rank}(P_LA^*P_L) \tag{11}$$

if and only if

$$B = P\begin{bmatrix} (\Sigma K)^*D^{-1}(\Sigma K)^{-1} & O \\ (\Sigma L)^*D^{-1}(\Sigma K)^{-1} & O \\ O & O \end{bmatrix}P^* \text{ and } C = P\begin{bmatrix} D & O \\ O & O \end{bmatrix}P^*, \tag{12}$$

where $D = M_1'(\Sigma K)^* + M_2'(\Sigma L)^*$ is nonsingular, $M_1' \in \mathbb{C}^{r\times r}$, $M_2' \in \mathbb{C}^{r\times(l-r)}$ and $P = U\begin{bmatrix} V & O \\ O & V' \end{bmatrix}$ is an unitary matrix, for some unitary matrix $V' \in \mathbb{C}^{(n-l)\times(n-l)}$.

Proof. Consider first that $X = A_{(L)}^{(\#)}$ is a solution of (11). From Lemma 2.2, there exist two matrices M and Q such that $B = P_LA^*P_LQ$ and $C = MP_LA^*P_L$. Then using the unitary matrix U given in (2), the product U^*QU can be divided as follows

$$U^*QU = \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix},$$

where $Q_1 \in \mathbb{C}^{l\times l}$ and

$$B = P_LA^*P_LQ = U\begin{bmatrix} A_L{}^* & O \\ O & O \end{bmatrix}\begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix}U^* = U\begin{bmatrix} A_L{}^*Q_1 & A_L{}^*Q_2 \\ O & O \end{bmatrix}U^*. \tag{13}$$

In the same way, supposing

$$U^*MU = \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix},$$

where $M_1 \in \mathbb{C}^{l\times l}$ and we have

$$C = MP_LA^*P_L = U\begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix}\begin{bmatrix} A_L{}^* & O \\ O & O \end{bmatrix}U^* = U\begin{bmatrix} M_1A_L{}^* & O \\ M_3A_L{}^* & O \end{bmatrix}U^*. \tag{14}$$

Using Lemma 2.2, (4), (6), (7), (13) and (14), we have

$$\begin{aligned} & A_{(L)}^{(\#)} = C(P_LA^*P_L)^\dagger B \\ \Leftrightarrow\ & U\begin{bmatrix} A_L{}^{\#} & O \\ O & O \end{bmatrix}U^* = U\begin{bmatrix} M_1A_L{}^* & O \\ M_3A_L{}^* & O \end{bmatrix}\begin{bmatrix} (A_L{}^*)^\dagger & O \\ O & O \end{bmatrix}\begin{bmatrix} A_L{}^*Q_1 & A_L{}^*Q_2 \\ O & O \end{bmatrix}U^* \\ \Leftrightarrow\ & U\begin{bmatrix} A_L{}^{\#} & O \\ O & O \end{bmatrix}U^* = U\begin{bmatrix} M_1A_L{}^*Q_1 & M_1A_L{}^*Q_2 \\ M_3A_L{}^*Q_1 & M_3A_L{}^*Q_2 \end{bmatrix}U^*, \end{aligned}$$

that is, $M_1A_L{}^*Q_1 = A_L{}^{\#}$, $M_1A_L{}^*Q_2 = O$, $M_3A_L{}^*Q_1 = O$ and $M_3A_L{}^*Q_2 = O$. Using the unitary matrix V given in (6), the product V^*M_1V, V^*M_3V, V^*Q_1V and V^*Q_2V can be divided as follows

$$V^*M_1V = \begin{bmatrix} D & M_2' \\ M_3' & M_4' \end{bmatrix},\ V^*M_3V = \begin{bmatrix} M_1'' & M_2'' \\ M_3'' & M_4'' \end{bmatrix},$$
$$V^*Q_1V = \begin{bmatrix} Q_1' & Q_2' \\ Q_3' & Q_4' \end{bmatrix},\ V^*Q_2V = \begin{bmatrix} Q_1'' & Q_2'' \\ Q_3'' & Q_4'' \end{bmatrix},$$

where $D \in \mathbb{C}^{r\times r}$, $M_3' \in \mathbb{C}^{(l-r)\times r}$, $Q_1' \in \mathbb{C}^{r\times r}$ and $Q_2' \in \mathbb{C}^{r\times(l-r)}$. Therefore,

$$\begin{aligned} & A_L^{\textcircled{\#}} = M_1 A_L^* Q_1 \\ \Leftrightarrow \quad & V\begin{bmatrix} (\Sigma K)^{-1} & O \\ O & O \end{bmatrix}V^* = V\begin{bmatrix} (M_1'(\Sigma K)^* + M_2'(\Sigma L)^*)\,Q_1' & (M_1'(\Sigma K)^* + M_2'(\Sigma L)^*)\,Q_2' \\ (M_3'(\Sigma K)^* + M_4'(\Sigma L)^*)\,Q_1' & (M_3'(\Sigma K)^* + M_4'(\Sigma L)^*)\,Q_2' \end{bmatrix}V^*, \end{aligned}$$

that is,

$$\begin{aligned} &(M_1'(\Sigma K)^* + M_2'(\Sigma L)^*)\,Q_1' = (\Sigma K)^{-1} \\ &(M_3'(\Sigma K)^* + M_4'(\Sigma L)^*)\,Q_1' = O \\ &(M_1'(\Sigma K)^* + M_2'(\Sigma L)^*)\,Q_2' = O \\ &(M_3'(\Sigma K)^* + M_4'(\Sigma L)^*)\,Q_2' = O. \end{aligned}$$

It is clear that $D = M_1'(\Sigma K)^* + M_2'(\Sigma L)^*$ and Q_1' are invertible and $Q_2' = O$, $M_3'(\Sigma K)^* + M_4'(\Sigma L)^* = O$. Similarly, from $M_1 A_L^* Q_2 = O$, $M_3 A_L^* Q_1 = O$ and $M_3 A_L^* Q_2 = O$, we have $M_1''(\Sigma K)^* + M_2''(\Sigma L)^* = O$, $M_3''(\Sigma K)^* + M_4''(\Sigma L)^* = O$, $Q_1'' = O$ and $Q_2'' = O$. From (13) and (14),

$$\begin{aligned} B &= U\begin{bmatrix} V\begin{bmatrix} (\Sigma K)^* D^{-1}(\Sigma K)^{-1} & O \\ (\Sigma L)^* D^{-1}(\Sigma K)^{-1} & O \end{bmatrix}V^* & O \\ O & O \end{bmatrix}U^* \\ C &= U\begin{bmatrix} V\begin{bmatrix} D & O \\ O & O \end{bmatrix}V^* & O \\ O & O \end{bmatrix}U^*. \end{aligned}$$

Let $P = U\begin{bmatrix} V & O \\ O & V' \end{bmatrix}$, where $V' \in \mathbb{C}^{(n-l)\times(n-l)}$ is an unitary matrix. It is clear that P is unitary. Hence, (12) holds. Conversely, if B and C can be represented in the form (12), we can check $\mathcal{N}(C) \supset \mathcal{N}(P_L A^* P_L)$ and $\mathcal{R}(B) \subset \mathcal{R}(P_L A^* P_L)$. Thus, the theorem holds by using Lemma 2.2. $\square$

4. Some new representations of the Bott-Duffin core inverse

In [22, Theorem 3.5], Wang gives a representation of the core inverse. Motivated by it, we give a representation of the Bott-Duffin core inverse.

Theorem 4.1. *Let $A \in \mathbb{C}^{n\times n}$, $L \leqslant \mathbb{C}^n$, $b \in \mathbb{C}^n$ and $T = \mathcal{R}(P_L A P_L)$ with $\operatorname{rank}(T) = r$ be such that $A_{(L)}^{(\textcircled{\#})}$ exists, and let $F \in \mathbb{C}^{n\times(n-r)}$ with $\operatorname{rank}(F) = n - r$ and $\mathcal{R}(F) = T^{\perp}$. Then,*

$$A_{(L)}^{(\textcircled{\#})} = (P_L A P_L + FF^*)^{-1} P_T. \tag{15}$$

Proof. Since $\mathcal{R}(F) = T^{\perp}$, $P_L A P_L \in \mathbb{C}_n^{CM}$, $\mathcal{R}(FF^*) = \mathcal{R}(F) = T^{\perp}$ and $\mathcal{R}((P_L A P_L)^{\textcircled{\#}}) = \mathcal{R}(P_L A P_L) = T$, it follows that

$$\begin{aligned} (FF^*)(FF^*)^{\dagger} &= P_{T^{\perp},T}, \\ P_L A P_L (P_L A P_L)^{\textcircled{\#}} &= P_{\mathcal{R}(P_L A P_L)} = P_{T,T^{\perp}}, \\ FF^*(P_L A P_L)^{\textcircled{\#}} &= O. \end{aligned}$$

Thus

$$\begin{aligned} & (P_L A P_L + FF^*)((P_L A P_L)^{\textcircled{\#}} + (FF^*)^{\dagger} - (P_L A P_L)^{\textcircled{\#}} P_L A P_L (FF^*)^{\dagger}) \\ = \ & P_L A P_L (P_L A P_L)^{\textcircled{\#}} + (FF^*)(FF^*)^{\dagger} \\ = \ & P_{T,T^{\perp}} + P_{T^{\perp},T} \\ = \ & I_n. \end{aligned}$$

Therefore, $P_LAP_L + FF^*$ is invertible and $(P_LAP_L + FF^*)^{-1} = (P_LAP_L)^{\#\!\!\!\!\bigcirc} + (FF^*)^\dagger - (P_LAP_L)^{\#\!\!\!\!\bigcirc}P_LAP_L(FF^*)^\dagger$. From $\mathcal{N}((FF^*)^\dagger) = \mathcal{N}(F^*) = T$, it is clear that $(FF^*)^\dagger P_T = O$. Then, $(P_LAP_L + FF^*)^{-1}P_T = (P_LAP_L)^{\#\!\!\!\!\bigcirc}P_T = A^{(\#\!\!\!\!\bigcirc)}_{(L)}$. □

In the following theorem, we present the full-rank representation of the Bott-Duffin core inverse.

Theorem 4.2. *Let $A \in \mathbb{C}^{n\times n}$ and $L \leqslant \mathbb{C}^n$ be such that $A^{(\#\!\!\!\!\bigcirc)}_{(L)}$ exists. If $P_LAP_L = FG$ is the full-rank decomposition of P_LAP_L, where* $\mathrm{rank}(P_LAP_L) = r$*, $F \in \mathbb{C}^{n\times r}$ with* $\mathrm{rank}(F) = r$ *and $G \in \mathbb{C}^{r\times n}$ with* $\mathrm{rank}(G) = r$*, then*

$$A^{(\#\!\!\!\!\bigcirc)}_{(L)} = F(GF)^{-1}(F^*F)^{-1}F^*.$$

Proof. Due to the full-rank decompositions of the core inverse [22], it follows immediately that $A^{(\#\!\!\!\!\bigcirc)}_{(L)} = (P_LAP_L)^{\#\!\!\!\!\bigcirc} = F(GF)^{-1}(F^*F)^{-1}F^*$. □

In the following theorems, we will establish two integral representations of the Bott-Duffin core inverse.

Theorem 4.3. *Let $A \in \mathbb{C}^{n\times n}$ and $L \leqslant \mathbb{C}^n$ be such that $A^{(\#\!\!\!\!\bigcirc)}_{(L)}$ exists. If $P_LAP_L = FG$ is the full-rank decomposition of P_LAP_L, then*

$$A^{(\#\!\!\!\!\bigcirc)}_{(L)} = \int_0^\infty F(P_LAP_LF)^* \exp(-P_LAP_LF(P_LAP_LF)^*t)dt.$$

Proof. From [32, Corollary 3.3] and Lemma 2.1 (*iii*), it can be directly obtained. □

Theorem 4.4. *Let $A \in \mathbb{C}^{n\times n}$ and $L \leqslant \mathbb{C}^n$ be such that $A^{(\#\!\!\!\!\bigcirc)}_{(L)}$ exists. Then*

$$A^{(\#\!\!\!\!\bigcirc)}_{(L)} = \int_0^\infty P_LAP_L \exp(-(P_LA^*)^2(P_LA)^2P_Lt)(P_LA^*)^2P_L dt.$$

Proof. By simple calculation, it follows from [32, Corollary 4.3] and Lemma 2.1 (*iii*). □

In the following theorem, we provide a new representation of the Bott-Duffin core inverse in terms of the Bott-Duffin inverse.

Theorem 4.5. *Let $A \in \mathbb{C}^{n\times n}$, $L \leqslant \mathbb{C}^n$ and $T = \mathcal{R}(P_LAP_L)$ be such that $A^{(\#\!\!\!\!\bigcirc)}_{(L)}$ exists. Then*

$$A^{(\#\!\!\!\!\bigcirc)}_{(L)} = (P_LAP_L)^{(-1)}_{(T)}.$$

.

Proof. It is well known that $(P_LAP_L)^{(-1)}_{(T)} = P_T(P_LAP_LP_T + P_{T^\perp})^{-1}$. Let P_L and A be given by (2) and (3), respectively. We have

$$P_T = U\begin{bmatrix} A_LA_L^{\#\!\!\!\!\bigcirc} & O \\ O & O \end{bmatrix}U^*. \tag{16}$$

Then,

$$\begin{aligned} & P_T(P_LAP_LP_T + P_{T^\perp})^{-1} \\ = \;& U\begin{bmatrix} A_LA_L^{\#\!\!\!\!\bigcirc} & O \\ O & O \end{bmatrix}\left(\begin{bmatrix} A_L{}^2A_L^{\#\!\!\!\!\bigcirc} & O \\ O & O \end{bmatrix} + \begin{bmatrix} I_l - A_LA_L^{\#\!\!\!\!\bigcirc} & O \\ O & I_{n-l} \end{bmatrix}\right)^{-1}U^* \\ = \;& U\begin{bmatrix} A_LA_L^{\#\!\!\!\!\bigcirc} & O \\ O & O \end{bmatrix}\begin{bmatrix} A_L{}^2A_L^{\#\!\!\!\!\bigcirc} + I_l - A_LA_L^{\#\!\!\!\!\bigcirc} & O \\ O & I_{n-l} \end{bmatrix}^{-1}U^*. \end{aligned}$$

Below we will discuss the nonsingularity of ${A_L}^2 {A_L}^{\text{\textcircled{\#}}} + I_l - A_L {A_L}^{\text{\textcircled{\#}}}$. From (6), we can obtain

$$ {A_L}^2 {A_L}^{\text{\textcircled{\#}}} + I_l - A_L {A_L}^{\text{\textcircled{\#}}} = V \begin{bmatrix} \Sigma K & O \\ O & I_{l-r} \end{bmatrix} V^*, \tag{17} $$

where $r = \operatorname{rank}(A_L)$. It can be clearly seen from (17) that ${A_L}^2 {A_L}^{\text{\textcircled{\#}}} + I_l - A_L {A_L}^{\text{\textcircled{\#}}}$ is nonsingular and

$$ ({A_L}^2 {A_L}^{\text{\textcircled{\#}}} + I_l - A_L {A_L}^{\text{\textcircled{\#}}})^{-1} = V \begin{bmatrix} (\Sigma K)^{-1} & O \\ O & I_{l-r} \end{bmatrix} V^*. $$

Thus, from (7) and (4), $(P_L A P_L)^{(-1)}_{(T)} = P_T (P_L A P_L P_T + P_{T^\perp})^{-1} = U \begin{bmatrix} {A_L}^{\text{\textcircled{\#}}} & O \\ O & O \end{bmatrix} U^* = A^{\text{\textcircled{\#}}}_{(L)}$. □

5. Application

Consider the following equation:

$$ P_L A x = b, \tag{18} $$

where $P_L A P_L \in \mathbb{C}^{CM}_n$, $L \leqslant \mathbb{C}^n$ and $T = \mathcal{R}(P_L A P_L)$. When $b \notin \mathcal{R}(P_L A)$, (18) is unsolvalble, it has least-squares solutions. Therefore, we consider the least-squares solutions of (18) under the certain condition $x \in T$, i.e.,

$$ \|P_L A x - b\|_F = \min \quad \text{subject to} \quad x \in T, \tag{19} $$

where Frobenius norm $\|\cdot\|_F$ is defined as the square root of the sum of the squared absolute values of all matrix elements.

In [31], we have discussed $x = A^{(\text{\textcircled{\#}})}_{(L)} b$ is the unique solution of (19).

When $M \in \mathbb{C}^{n\times n}$ is nonsingular, it is well known that the Cramer's rule for the solution of x of a nonsingular equation $Mx = b$ ($M \in \mathbb{C}^{n\times n}, b \in \mathbb{C}^n, x = (x_1, x_2, \dots, x_n)^T$) is

$$ x_i = \frac{\det(M(i \to b))}{\det(M)},\ i = 1, 2, \dots, n, \tag{20} $$

where $M(i \to b)$ denotes the matrix obtained by replacing the ith column of M with b.

Using Theorem 4.1, we provide a Cramer's rule for the unique solution of (19).

Theorem 5.1. *Let $A \in \mathbb{C}^{n\times n}$, $L \leqslant \mathbb{C}^n$, $b \in \mathbb{C}^n$, $T = \mathcal{R}(P_L A P_L)$ and $\operatorname{rank}(P_L A P_L) = r$ be such that $A^{(\text{\textcircled{\#}})}_{(L)}$ exists, and let $F \in \mathbb{C}^{n\times(n-r)}$ with $\operatorname{rank}(F) = n - r$ and $\mathcal{R}(F) = T^\perp$. Then, (19) has the unique solution $x = (x_1, x_2, \dots, x_n)^T$ satisfying*

$$ x_j = \frac{\det(P_L A P_L + FF^*)(j \to P_T b)}{\det(P_L A P_L + FF^*)}, \tag{21} $$

where $j = 1, 2, \dots, n$.

Proof. Applying Theorem 4.1 to $x = A^{(\text{\textcircled{\#}})}_{(L)} b$, we have

$$ x = (P_L A P_L + FF^*)^{-1} P_T b, $$

that is,

$$ (P_L A P_L + FF^*) x = P_T b. $$

It follows from (20) that we get (21). □

Example 5.2. *Let*

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & 2 & 1 \\ 0 & 2 & 0 & 0 \\ 2 & 0 & 1 & 0 \end{bmatrix},\ L = \mathcal{R}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right),\ b = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix},\ F = \begin{bmatrix} 2 & 0 \\ -2 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

It is clear that $b \notin \mathcal{R}(P_L A)$, then (18) is unsolvable. Therefore, by using Theorem 5.1, we consider the least-squares solutions of (18). We can check $\operatorname{rank}(F) = 2$ and $\mathcal{R}(F) = T^{\perp}$. Let $x = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T$ be the unique solution of (19). Applying (21), we can derive the components of x directly, i.e.

$$x_1 = \frac{4}{3},\ x_2 = \frac{11}{9},\ x_3 = -\frac{2}{9},\ x_4 = 0.$$

In the following theorem, using Theorem 4.1, we get a condensed determinantal expression of $A_{(L)}^{(\#)}$.

Theorem 5.3. *Let A, L, F and T be defined as in Theorem 5.1. Then the Bott-Duffin core inverse $A_{(L)}^{(\#)}$ is given by:*

$$A_{(L)_{i,j}}^{(\#)} = \frac{\det(P_L A P_L + FF^*)(i \to P_T e_j)}{\det(P_L A P_L + FF^*)}, \tag{22}$$

where $i, j = 1, 2, \ldots, n$, $e_j = (\underbrace{0, \ldots, 0, 1}_{j}, 0, \ldots, 0)^T$, where $j = 1, 2, \ldots, n$.

Proof. From Theorem 4.1, it follows that $P_L A P_L + FF^*$ is invertible. Using (20), we consider

$$(P_L A P_L + FF^*)x = P_T e_j$$

and get the solution

$$e_i^T x = \frac{\det(P_L A P_L + FF^*)(i \to P_T e_j)}{\det(P_L A P_L + FF^*)},$$

where $i, j = 1, \ldots, n$. It follows from (15) and $A_{(L)_{i,j}}^{(\#)} = e_i^T A_{(L)}^{(\#)} e_j$ that we get (22). □

Example 5.4. *Let A, L and F be as in Example 5.2. Using (22) and $P_T = I - (FF^*)(FF^*)^{\dagger}$, we get*

$$A_{(L)}^{(\#)} = \begin{bmatrix} -\frac{1}{3} & 0 & \frac{2}{3} & 0 \\ -\frac{1}{9} & \frac{1}{9} & \frac{4}{9} & 0 \\ \frac{4}{9} & \frac{2}{9} & -\frac{4}{9} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

6. Conclusion

This paper presents some new characterizations and representations of the Bott-Duffin core inverse and considers the relationships between the Bott-Duffin core inverse and other generalized inverses. We also characterize the Bott-Duffin core inverse by using the solution of the block-rank equation. We provide the Cramer's rule for the least-squares solution of the constraint system of linear equations. On the basis of the current research background, there are many topics on the Bott-Duffin core inverse which can be discussed. Some ideas are given as follows:

(1) It is possible to discuss the algebraic perturbation theory of Bott-Duffin core inverse and the expression of the algebraic perturbation of Bott-Duffin core inverse.

(2) The continuity and iterative methods of the Bott-Duffin core inverse can be discussed.

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