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Topological properties on warped product submanifolds of space forms

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Abstract. The current article aspires to formulate the sufficient condition for the Laplacian and the gradient of the warping function of a compact warped product submanifold $\mathbf{F}^{\alpha_1+\alpha_2}$ in a space form $\mathbb{F}_c^{\alpha_1+\alpha_2+k_1}(c)$ that provides the trivial homology and fundamental groups. Also, we validate the instability of current flows in $\pi_1(\mathbf{F}^{\alpha_1+\alpha_2})$. The constraints also apply to the warped function eigenvalues, integral Ricci curvature, and Hessian tensor.

1. Introduction and Main Results

Any non-trivial integral homology class in $H_{\alpha_1}(\mathbf{F}^n, \mathbb{Z})$ is associated with the topological properties of submanifolds in different ambient spaces. According to [7], this was the first time this had been demonstrated. Lawson-Simons provide a variational calculus representation of geometric measure [8] by using the Federer and Fleming method [7]. As a result, the second fundamental form is escalated, which enforces the vanishing of homology in an intermediate dimension region and the nonexistence of stable currents in the connected space form submanifold. This work has achieved the following main objective, which is the following result.

Theorem 1.1. *[8, 20] If the following optimization inequality hold for a compact n-dimensional submanifold in a space form $\mathbb{F}_c(c)$ such that the curvature $c \geq 0$ and $\alpha_1 + \alpha_2 = n$,*

$$\sum_{b_1=1}^{\alpha_1} \sum_{b_2=\alpha_1+1}^{m} \Big(2\|\mathbf{A}(v_{b_1}, v_{b_2})\|^2 - g\Big(\mathbf{A}(v_{b_1}, v_{b_1}), \mathbf{A}(v_{b_2}, v_{b_2})\Big)\Big) < \alpha_1\alpha_2 c, \tag{1.1}$$

then no any stable α_1-currents flow in $\mathbf{F}^n$ and

$$H_{\alpha_1}(\mathbf{F}^n, \mathbb{Z}) = H_{\alpha_2}(\mathbf{F}^n, \mathbb{Z}) = 0,$$

for any $\alpha_1 \in \mathbf{F}^n$ and $H_i(\mathbf{F}^n, \mathbb{Z})$ is the i-th homology group of $\mathbf{F}^n$ with integer coefficients.

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Submanifold geometry and topological properties have been studied extensively in various ambient spaces in recent years. It has been demonstrated in numerous papers such as the Euclidean spaces [9], complex projective spaces [11], CR-warped product submanifolds in Sasakian space forms [14], CR-warped product submanifold in the Euclidean spaces [15], CR-warped product in nearly Kaeher manifold [2, 15], CR-warped product in hyperbolic spaces [3] and many others (see [1, 10, 20, 21]). Vanishing homology groups between submanifolds and stable currents have a closed relationship in various ambient manifold classes. A pinching condition was applied to the second fundamental form to reach these conclusions. There have been many studies on topological features in response to the nonexistence of stable currents and stable submanifolds [1, 12, 17–19], motivated by the Theorem [8] in references. In the present paper, we study some topological properties of warped product submanifolds in space form and generalize the previously proven results.

2. Preliminaries

A complete and simply connected space with a constant sectional curvature is represented by $\mathbb{F}_c^{m+k_1}(c)$, with $c = 1 \geq 0$ and $(m + k_1)$ dimensions. Given that $\mathbb{F}_c^{m+k_1}(c)$ accepts a canonical isometric immersion in $\mathbb{F}_c^{m+k_1}$ $\mathbb{H}^{m+k_1}$and $\mathbb{R}^{m+k_1+1}$ with $c = 1,\ -1$ and $c = 0$ respectively, we use this as our main argument.

The Riemannian curvature tensor is therefore $\widetilde{R}$ of space form $\mathbb{F}_c^{m+k_1}(c)$ satisfies the following

$$\widetilde{R}(V_2, V_3, V_4, V_5) = c\Big\{g(V_2, V_5)g(V_3, V_4) - g(V_3, V_5)g(V_2, V_4)\Big\}, \tag{2.1}$$

for any $V_3, V_2, V_4, V_5 \in \Gamma(T\mathbb{F})$, where $T\mathbb{F}$ is the tangent bundle of $\mathbb{F}_c^{m+k_1}(c)$ and g is the Riemannian metric. In other words, the space form $\mathbb{F}_c^{m+k_1}(c)$ is a manifold of constant sectional curvature one with co-dimension k.

Remark 2.1. The space form $\mathbb{F}_c^{m+k_1}(c)$ is equivalent to sphere $\mathbb{S}_c^{m+k_1}(1)$ for $c = 1$, hyperbolic space $\mathbb{H}_c^{m+k_1}(-1)$ for $c = -1$ and an Eculidean space $\mathbb{E}_c^{m+k_1}$ for $c = 0$, respectively.

If $\widetilde{R}$ and R are represented as the curvature tensors on $\mathbb{F}$ and $\mathbf{F}^n$, respectively, then the Gauss equation is given by:

$$\begin{aligned}\widetilde{R}(V_2, V_3, V_4, V_5) =& R(V_2, V_3, V_4, V_5) + g\Big(\mathbf{A}(V_2, V_4), \mathbf{A}(V_3, V_5)\Big)\\ &- g\Big(\mathbf{A}(V_2, V_5), \mathbf{A}(V_3, V_4)\Big),\end{aligned} \tag{2.2}$$

for any $V_2, V_3, V_4, V_5 \in \Gamma(T\mathbb{F})$. A local orthonormal frame's $\{v_1, v_2, \cdots, v_n\}$ mean curvature vector $\mathcal{H}$ on $\mathbf{F}$ is defined by

$$\|\mathcal{H}\|^2 = \frac{1}{n^2}\sum_{r=m+1}^{m+k_1}\left(\sum_{b_1=1}^{m}\mathbf{A}_{b_1b_1}\right)^2 \tag{2.3}$$

where $\mathbf{A}$ is the second fundamental form of $\mathbf{F}$. The scalar curvature of submanifold $\mathbf{F}^m$, and denoted by $\tau(T_x\mathbf{F}^m)$, is accustomed as follows:

$$\tau(T_xF^m) = \sum_{1\leq b_1<b_2\leq m} K_{b_1b_2} \tag{2.4}$$

where $K_{b_1b_2} = K\Big(v_{b_1} \wedge v_{b_2}\Big)$. The first equality (2.4) is proportionate to the following equation, it will be used often in later proofs:

$$2\tau(T_xF^m) = \sum_{1\leq b_1<b_2\leq n} K_{b_1b_2},\ \ 1 \leq b_1, b_2 \leq n. \tag{2.5}$$

Similarly, the scalar curvature $\tau(L_x)$ of an $L-$plane is given by

$$\tau(L_x) = \sum_{1\le b_1<b_2\le n} K_{b_1b_2} \tag{2.6}$$

If the plane sections are spanned by v_{b_1} and v_{b_2} at x, then the sectional curvature of the submanifold $\mathbf{F}^n$ and Riemannian manifold $\mathbb{F}^{m+k_1}$, are denoted by $K_{b_1b_2}$ and $\widetilde{K}_{b_1b_2}$, respectively. Thus $K_{b_1b_2}$ and $\widetilde{K}_{b_1b_2}$ are the intrinsic and the extrinsic sectional curvature of the span $\{v_{b_1}, v_{b_2}\}$ at x, respectively. Using Gauss equations (2.2), and (2.4), we have

$$\sum_{1\le b_1<b_2\le n+k} K_{b_1b_2} = \sum_{1\le b_1<b_2\le n+k} \widetilde{K}_{b_1b_2} + \sum_{r=m+1}^{m+k_1}\Big(\mathbf{A}^r_{b_1b_1}\mathbf{A}^r_{b_2b_2} - (\mathbf{A}^r_{b_1b_2})^2\Big). \tag{2.7}$$

A warped product manifold is product $\mathbf{F}^{\alpha_1+\alpha_2} = \mathbf{F}_1^{\alpha_1}\times_\mu \mathbf{F}_2^{\alpha_2}$ with the Riemannian metric $g = g_1 + \mu^2 g_2$ in [4]. Assuming that $\mathbf{F}^{\alpha_1+\alpha_2} = \mathbf{F}_1^{\alpha_1}\times_\mu \mathbf{F}_2^{\alpha_2}$ is a warped product manifold, then for any $V_2 \in \alpha(T\mathbf{F}_1)$ and $V_4 \in \alpha(T\mathbf{F}_2)$, we find that

$$\nabla_{V_4}V_2 = \nabla_{V_2}V_4 = (V_2 \ln\mu)V_4. \tag{2.8}$$

Therefore, the warping function is represented by the function μ on $\mathbf{F}^{\alpha_1+\alpha_2}$. Further, $\nabla(\ln\mu)$ is the gradient of $\ln\mu$, given by

$$g(\nabla\ln\mu, V_2) = V_2(\ln\mu). \tag{2.9}$$

Thus, from [4], we have

$$R(V_2, V_4)V_3 = \frac{H^\mu(V_2, V_3)}{\mu}V_4, \tag{2.10}$$

where H^μ is a Hessian tensor of μ. Then the $\|\nabla\mu\|^2$ the gradient of the positive differential function μ for an orthonormal frame $\{v_1, \ldots, v_n\}$ is defined as:

$$\|\nabla\mu\|^2 = \sum_{i=1}^{m}\big(v_i(\mu)\big)^2. \tag{2.11}$$

The gradient $\vec{\nabla}\mu$ is given by:

$$g(\vec{\nabla}\mu, V_2) = V_2\mu, \quad and \quad \vec{\nabla}\mu = \sum_{i=1}^{m} v_i(\mu)v_i. \tag{2.12}$$

The Laplacian $\Delta\mu$ of μ is expressed as:

$$\Delta\mu = -\sum_{i=1}^{n}\{(\nabla_{v_i}v_i)\mu - v_i(v_i(\mu))\} = \sum_{i=1}^{n} g(\nabla_{v_i}grad\mu, v_i) = trHess(\mu). \tag{2.13}$$

Remark 2.2. It should be emphasized that we take into account Chen's opposite sign of [6] of the function's Laplacian μ, that is $\Delta\mu = div(\nabla\mu)$. The sign convention for the Laplacian Δ adapted by the authors is that $\Delta = \frac{d^2}{dt^2}$ on the real line.

In addition, as the vector fields V_2 and V_4 are tangent to $\mathbf{F}_1^{\alpha_1}$ and $\mathbf{F}_2^{\alpha_2}$, respectively, we get:

$$\begin{aligned} K(V_2 \wedge V_4) &= g(R(V_2, V_4)V_2, V_4) = (\nabla_{V_2} V_2) \ln hg(V_4, V_4) - g\Big(\nabla_{V_2}((V_2 \ln \mu)V_5), V_4\Big) \\ &= (\nabla_{V_2} V_2) \ln hg(V_4, V_4) - g\Big(\nabla_{V_2}(V_2 \ln \mu)V_4 + (V_2 \ln \mu)\nabla_{V_2} V_4, V_4\Big) \\ &= (\nabla_{V_2} V_2) \ln \mu g(V_4, V_4) - (V_2 \ln \mu)^2 - V_2(V_2 \ln \mu). \end{aligned} \tag{2.14}$$

Taking summing up over the vector fields concerning orthonormal frame's $\{v_1, \cdots v_n\}$, one obtains:

$$\sum_{i=1}^{\alpha_1} \sum_{j=1}^{\alpha_2} K(v_i \wedge v_j) = \sum_{i=1}^{\alpha_1} \sum_{j=1}^{\alpha_2} \Big((\nabla_{v_i} v_i) \ln \mu - v_i(v_i \ln \mu) - (v_i \ln \mu)^2 \Big),$$

which implies that:

$$\sum_{i=1}^{\alpha_1} \sum_{j=1}^{\alpha_2} K(v_i \wedge v_j) = -\frac{\alpha_2 \Delta \mu}{\mu}. \tag{2.15}$$

Main results

We are also essential to use the technique which is an invaluable tool for verifying our results. In the first case, assuming that the warped product submanifold is embedding in $\mathbb{F}_c^{m+k_1}(c)$ with co-dimension k, and utilizing Theorem 1.1, we intend to get some identical conclusions of warped product submanifold hypothesis where pinching criteria on the second fundamental form shall be replaced by the warping function.

Using Theorem 1.1, the first major outcome of this memo is now provided.

Theorem 2.1. *Let $\Psi : \mathbf{F}^{\alpha_1+\alpha_2} = \mathbf{F}_1^{\alpha_1} \times_\mu \mathbf{F}_2^{\alpha_2} \longrightarrow \mathbb{F}_c^{\alpha_1+\alpha_2+k_1}$ be an $\mathbf{F}_1^{\alpha_1}$-minimal isometric embedding from a compact warped product submanifold $\mathbf{F}^{\alpha_1+\alpha_2}$ into an $(\alpha_1 + \alpha_2 + k_1)$-dimensional space form $\mathbb{F}_c^{\alpha_1+\alpha_2+k_1}(c)$ such that the following inequality satisfies*

$$3\mu\Delta\mu < 2\Big(\alpha_2 \|\nabla\mu\|^2 + c\alpha_1\mu^2\Big) \tag{2.16}$$

where $\Delta\mu$ and $\nabla\mu$ stand for the Laplacian and gradient of a function μ, respectively. Then there are the subsequent

(a) There is not existent any stable integral α_1-currents flow in a warped product submanifold $\mathbf{F}^{\alpha_1+\alpha_2}$.

(b) The i-th integral homology groups of $\mathbf{F}^{\alpha_1+\alpha_2}$, are vanished, which means:

$$\mathbb{H}_{\alpha_1}(\mathbf{F}^{\alpha_1+\alpha_2}, \mathbb{Z}) = \mathbb{H}_{\alpha_2}(\mathbf{F}^{\alpha_1+\alpha_2}, \mathbb{Z}) = 0,$$

(c) If $\alpha_1 = 1$ then the fundamental group $\pi_1(\mathbf{F})$ is null, i.e., $\pi_1(\mathbf{F}) = 0$. Moreover, $\mathbf{F}^{\alpha_1+\alpha_2}$ is a simply connected warped product manifold.

Proof. Let $\dim(\mathbf{F}_1) = \alpha_1$ and $\dim(\mathbf{F}_2) = \alpha_2$ and let $\{v_1, v_2, \cdots, v_{\alpha_1}\}$, and $\{v^*_{\alpha_1+1}, \cdots, v^*_m\}$ to be orthonormal frames of $T\mathbf{F}_1$ and $T\mathbf{F}_2$, respectively. Then, from Gauss equation (2.2) for the standard unit sphere $\mathbb{F}_c^{m+k_1}(c)$, we have

$$\begin{aligned} \sum_{b_1=1}^{\alpha_1} \sum_{b_2=1}^{\alpha_2} \Big\{ 2\|\mathbf{A}(v_{b_1}, v_{b_2})\|^2 - g\Big(\mathbf{A}(v_{b_2}, v_{b_2}), \mathbf{A}(v_{b_1}, v_{b_1})\Big) \Big\} \\ = \sum_{b_1=1}^{\alpha_1} \sum_{b_2=1}^{\alpha_2} g\Big(R(v_{b_1}, v_{b_2})v_{b_1}, v_{b_2}\Big) \\ - \sum_{b_1=1}^{\alpha_1} \sum_{b_2=1}^{\alpha_2} g\Big(\tilde{R}(v_{b_1}, v_{b_2})v_{b_1}, v_{b_2}\Big) + \sum_{r=1}^{m+k_1} \sum_{b_1=1}^{\alpha_1} \sum_{b_2=1}^{\alpha_2} (\mathbf{A}^r_{b_1 b_2})^2. \end{aligned} \tag{2.17}$$

From $R(v_{b_1}, v_{b_2})v_{b_1} = \frac{\mathcal{H}^{\mu}(v_{b_1}, v_{b_1})}{\mu} v_{b_2}$ in (2.10), we derive

$$\sum_{b_1=1}^{\alpha_1}\sum_{b_2=1}^{\alpha_2} g\Big(R(v_{b_1}, v_{b_2})v_{b_1}, v_{b_2}\Big) = \frac{\alpha_2}{\mu}\sum_{b_1=1}^{\alpha_1} g\Big(\nabla_{v_{b_1}}\nabla\mu, v_{b_1}\Big). \tag{2.18}$$

Thus, from (2.17) and (2.18), we derive

$$\begin{aligned}\sum_{b_1=1}^{\alpha_1}\sum_{b_2=1}^{\alpha_2}&\Big\{2\|\mathbf{A}(v_{b_1}, v_{b_2})\|^2 - g\Big(\mathbf{A}(v_{b_2}, v_{b_2}), \mathbf{A}(v_{b_1}, v_{b_1})\Big)\Big\}\\ &= \frac{\alpha_2}{\mu}\sum_{b_1=1}^{\alpha_1} g\Big(\nabla_{v_{b_1}}\nabla\mu, v_{b_1}\Big) + \sum_{r=1}^{m+k_1}\sum_{b_1=1}^{\alpha_1}\sum_{b_2=1}^{\alpha_2}(\mathbf{A}^r_{b_1b_2})^2\\ &\quad - \sum_{b_1=1}^{\alpha_1}\sum_{b_2=1}^{\alpha_2} g\Big(\tilde{R}(v_{b_1}, v_{b_2})v_{b_1}, v_{b_2}\Big).\end{aligned} \tag{2.19}$$

Calculation of the Laplacian $\Delta\mu$ on $\mathbf{F}^{\alpha_1+\alpha_2}$, one obtains:

$$\Delta\mu = \sum_{i=1}^{m} g\Big(\nabla_{v_i} grad\mu, v_i\Big) = \sum_{b_1=1}^{\alpha_1} g\Big(\nabla_{v_{b_1}} grad\mu, v_{b_1}\Big) + \sum_{b_2=1}^{\alpha_2} g\Big(\nabla_{v_{b_2}} grad\mu, v_{b_2}\Big).$$

We know this from the warped product submanifold, hypothesizing that $\mathbf{F}_1^{\alpha_1}$ is a totally geodesic in $\mathbf{F}^m$. It implies that $grad\mu \in \mathfrak{X}(T\mathbf{F}_1)$, and from the description of the warped product, we obtain

$$\Delta\mu = \frac{1}{\mu}\sum_{b_2=1}^{\alpha_2} g(v_{b_2}, v_{b_2})\|\nabla\mu\|^2 + \sum_{b_1=1}^{\alpha_1} g\Big(\nabla_{v_{b_1}} grad\mu, v_{b_1}\Big).$$

Multiply the above equation by $\frac{1}{\mu}$, we get:

$$\frac{\Delta\mu}{\mu} = \frac{1}{\mu}\sum_{b_1=1}^{\alpha_1} g\Big(\nabla_{v_{b_1}} grad\mu, v_{b_1}\Big) + \alpha_2\|\nabla(\ln\mu)\|^2.$$

Following some calculations, we discover that

$$\frac{\alpha_2}{\mu}\sum_{b_1=1}^{\alpha_1} g\Big(\nabla_{v_{b_1}} grad\mu, v_{b_1}\Big) = \frac{\alpha_2\Delta\mu}{\mu} - \alpha_2^2\|\nabla\ln\mu\|^2. \tag{2.20}$$

Thus, from (2.19) and (2.20), one obtains;

$$\begin{aligned}\sum_{b_1=1}^{\alpha_1}\sum_{b_2=1}^{\alpha_2}&\Big\{2\|\mathbf{A}(v_{b_1}, v_{b_2})\|^2 - g\Big(\mathbf{A}(v_{b_2}, v_{b_2}), \mathbf{A}(v_{b_1}, v_{b_1})\Big)\Big\}\\ &= \sum_{r=m+1}^{m+k_1}\sum_{b_1=1}^{\alpha_1}\sum_{b_2=1}^{\alpha_2}(\mathbf{A}^r_{b_1b_2})^2 + \frac{\alpha_2\Delta\mu}{\mu} - \alpha_2^2\|\nabla\ln\mu\|^2 - \sum_{b_1=1}^{\alpha_1}\sum_{b_2=1}^{\alpha_2} g\Big(\tilde{R}(v_{b_1}, v_{b_2})v_{b_1}, v_{b_2}\Big).\end{aligned}$$

Next, using the Gauss equation (2.2) for the unit sphere $\mathbb{F}_c^{m+k_1}(c)$, we find that

$$m^2\|\mathcal{H}\|^2 + cm(m-1) = \|\mathbf{A}\|^2 + \sum_{1\le A<B\le n} K(v_A \wedge v_B). \tag{2.21}$$

The warped product manifold $\mathbf{F}^{\alpha_1+\alpha_2}$ can be expressed using the aforementioned equation and using (2.3) and respectively (2.10), we get:

$$\begin{aligned}\sum_{r=m+1}^{m+k_1}\left(\sum_{A=1}^{m}\mathbf{A}^r_{AA}\right)^2 &= \sum_{r=m+1}^{m+k_1}\sum_{i,j=1}^{\alpha_1}(\mathbf{A}^r_{ij})^2+\sum_{r=m+1}^{m+k_1}\sum_{a,b=1}^{\alpha_2}(\mathbf{A}^r_{ab})^2\\&+2\sum_{r=m+1}^{m+k_1}\sum_{b_1=1}^{\alpha_1}\sum_{b_2=1}^{\alpha_2}(\mathbf{A}^r_{b_1b_2})^2+\sum_{b_1=1}^{\alpha_1}\sum_{b_2=1}^{\alpha_2}K(v_{b_1}\wedge v_{b_2})\\&+\sum_{1\le i<j\le\alpha_1}K(v_i\wedge v_j)+\sum_{1\le a<b\le\alpha_2}K(v_a\wedge v_b).\end{aligned}\tag{2.22}$$

Using (2.7) and (2.15) in the above equation, we derive

$$\begin{aligned}\sum_{r=m+1}^{m+k_1}\left(\sum_{A=1}^{m}\mathbf{A}^r_{AA}\right)^2 &= \sum_{r=m+1}^{m+k_1}\sum_{i,j=1}^{\alpha_1}(\mathbf{A}^r_{ij})^2+\sum_{r=m+1}^{m+k_1}\sum_{a,b=1}^{\alpha_2}(\mathbf{A}^r_{ab})^2\\&+2\sum_{r=m+1}^{m+k_1}\sum_{b_1=1}^{\alpha_1}\sum_{b_2=1}^{\alpha_2}(\mathbf{A}^r_{b_1b_2})^2-\frac{\alpha_2\Delta\mu}{\mu}-cm(m-1)\\&+\sum_{1\le i<j\le\alpha_1}\widetilde{K}(v_i\wedge v_j)+\sum_{1\le a<b\le\alpha_2}\widetilde{K}(v_a\wedge v_b)\\&+\sum_{r=m+1}^{m+k_1}\sum_{1\le i<j\le\alpha_1}\left(\mathbf{A}^r_{ii}\mathbf{A}^r_{jj}-(\mathbf{A}^r_{ij})^2\right)\\&+\sum_{r=m+1}^{m+k_1}\sum_{1\le a<b\le\alpha_2}\left(\mathbf{A}^r_{aa}\mathbf{A}^r_{bb}-(\mathbf{A}^r_{ab})^2\right).\end{aligned}\tag{2.23}$$

Thus, by modifying the previous equation and applying the sphere's curvature equation $\mathbb{F}^{m+k_1}_c(c)$, one can obtain.

$$\begin{aligned}\sum_{r=m+1}^{m+k_1}\left(\sum_{A=1}^{m}\mathbf{A}^r_{AA}\right)^2 &= \sum_{r=m+1}^{m+k_1}\sum_{i,j=1}^{\alpha_1}(\mathbf{A}^r_{ij})^2+\sum_{r=m+1}^{m+k_1}\sum_{a,b=1}^{\alpha_2}(\mathbf{A}^r_{ab})^2+2\sum_{r=m+1}^{m+k_1}\sum_{b_1=1}^{\alpha_1}\sum_{b_2=1}^{\alpha_2}(\mathbf{A}^r_{b_1b_2})^2\\&-\frac{\alpha_2\Delta\mu}{\mu}-\sum_{r=m+1}^{m+k_1}\sum_{1\le i<j\le\alpha_1}(\mathbf{A}^r_{ij})^2+\sum_{r=m+1}^{m+k_1}\sum_{1\le i<j\le\alpha_1}\mathbf{A}^r_{ii}\mathbf{A}^r_{jj}\\&+\sum_{r=m+1}^{m+k_1}\left((\mathbf{A}^r_{11})^2+\cdots+(\mathbf{A}_{\alpha_1\alpha_2})^2\right)-\sum_{r=m+1}^{m+k_1}\left((\mathbf{A}^r_{11})^2+\cdots+(\mathbf{A}_{\alpha_1\alpha_2})^2\right)\\&+\sum_{r=m+1}^{m+k_1}\sum_{1\le a<b\le\alpha_2}\mathbf{A}^r_{aa}\mathbf{A}^r_{bb}-\sum_{r=m+1}^{m+k_1}\sum_{1\le a<b\le\alpha_2}(\mathbf{A}^r_{ab})^2\\&+\sum_{r=m+1}^{m+k_1}\left((\mathbf{A}^r_{\alpha_1+1\alpha_2+1})^2+\cdots+(\mathbf{A}_{nn})^2\right)\\&-\sum_{r=m+1}^{m+k_1}\left((\mathbf{A}^r_{\alpha_1+1\alpha_2+1})^2+\cdots+(\mathbf{A}_{nn})^2\right)\\&+c\alpha_1(\alpha_1-1)+c\alpha_2(\alpha_2-1)-cm(m-1).\end{aligned}\tag{2.24}$$

The result of rearranging the equation as mentioned earlier is

$$\begin{aligned}\sum_{r=m+1}^{m+k_1}\left(\sum_{A=1}^{m}\mathbf{A}^r_{AA}\right)^2 =& \sum_{r=m+1}^{m+k_1}\sum_{i,j=1}^{\alpha_1}(\mathbf{A}^r_{ij})^2+\sum_{r=m+1}^{m+k_1}\sum_{a,b=1}^{\alpha_2}(\mathbf{A}^r_{ab})^2+2\sum_{r=m+1}^{m+k_1}\sum_{b_1=1}^{\alpha_1}\sum_{b_2=1}^{\alpha_2}(\mathbf{A}^r_{b_1b_2})^2\\
&+\sum_{r=m+1}^{m+k_1}\left\{\sum_{1\le i<j\le\alpha_1}\mathbf{A}^r_{ii}\mathbf{A}^r_{jj}+(\mathbf{A}^r_{11})^2+\cdots+(\mathbf{A}_{\alpha_1\alpha_1})^2\right\}\\
&-\sum_{r=m+1}^{m+k_1}\left\{\sum_{1\le i<j\le\alpha_1}(\mathbf{A}^r_{ij})^2+(\mathbf{A}^r_{11})^2+\cdots+(\mathbf{A}_{\alpha_1\alpha_1})^2\right\}\\
&+\sum_{r=m+1}^{m+k_1}\left\{\sum_{1\le a<b\le\alpha_2}\mathbf{A}^r_{aa}\mathbf{A}^r_{bb}+(\mathbf{A}^r_{\alpha_1+1\alpha_1+1})^2+\cdots+(\mathbf{A}_{nn})^2\right\}\\
&-\sum_{r=m+1}^{m+k_1}\left\{\sum_{1\le a<b\le\alpha_2}(\mathbf{A}^r_{ab})^2+(\mathbf{A}^r_{\alpha_1+1\alpha_1+1})^2+\cdots+(\mathbf{A}_{nn})^2\right\}\\
&-\frac{\alpha_2\Delta\mu}{\mu}+2c\alpha_1\alpha_2.\end{aligned}$$

It is straightforward to verify this using the binomial theorem and the knowledge that the base manifold $\mathbf{F}_1^{\alpha_1}$ of a warped product manifold $\mathbf{F}_1^{\alpha_1}\times_\mu\mathbf{F}_2^{\alpha_2}$ is totally geodesic. Then we have

$$\begin{aligned}\sum_{r=m+1}^{m+k_1}\left(\sum_{A=p+1}^{m}\mathbf{A}^r_{AA}\right)^2 =& 2c\alpha_1\alpha_2+\sum_{r=m+1}^{m+k_1}\sum_{i,j=1}^{\alpha_1}(\mathbf{A}^r_{ij})^2+\sum_{r=m+1}^{m+k_1}\sum_{a,b=1}^{\alpha_2}(\mathbf{A}^r_{ab})^2\\
&+2\sum_{r=m+1}^{m+k_1}\sum_{b_1=1}^{\alpha_1}\sum_{b_2=1}^{\alpha_2}(\mathbf{A}^r_{b_1b_2})^2+\sum_{r=m+1}^{m+k_1}\Big((\mathbf{A}^r_{11})^2+\cdots+(\mathbf{A}_{\alpha_1\alpha_1})^2\Big)\\
&-\sum_{r=m+1}^{m+k_1}\sum_{i,j=1}^{\alpha_1}(\mathbf{A}^r_{ij})^2-\sum_{r=m+1}^{m+k_1}\sum_{a,b=1}^{\alpha_2}(\mathbf{A}^r_{ab})^2\\
&+\sum_{r=m+1}^{m+k_1}\Big((\mathbf{A}^r_{\alpha_1+1\alpha_1+1})^2+\cdots+(\mathbf{A}_{nn})^2\Big)-\frac{\alpha_2\Delta\mu}{\mu}.\end{aligned}\tag{2.25}$$

Since the base manifold $\mathbf{F}_1^{\alpha_1}$ of the warped product submanifold $\mathbf{F}_1^{\alpha_1}\times_\mu\mathbf{F}_2^{\alpha_2}$ is known to be minimal according to the theorem's hypothesis, we can use this knowledge to determine that V^{th} term on the right side of Equation (2.25) is equal to zero and VII^{th} term is equal to the first term on the left side. So, here we are:

$$2\sum_{r=m+1}^{m+k_1}\sum_{b_1=1}^{\alpha_1}\sum_{b_2=1}^{\alpha_2}(\mathbf{A}^r_{b_1b_2})^2=\frac{\alpha_2\Delta\mu}{\mu}-2c\alpha_1\alpha_2.\tag{2.26}$$

From (2.21) and (2.26), we get:

$$\begin{aligned}\sum_{b_1=1}^{\alpha_1}\sum_{b_2=1}^{\alpha_2}&\Big\{2\|\mathbf{A}(v_{b_1},v_{b_2})\|^2-g\Big(\mathbf{A}(v_{b_2},v_{b_2}),\mathbf{A}(v_{b_1},v_{b_1})\Big)\Big\}\\
&=\frac{\alpha_2\Delta\mu}{\mu}-\alpha_2^2\|\nabla\ln\mu\|^2+\frac{\alpha_2\Delta\mu}{2\mu}-c\alpha_1\alpha_2\\
&-\sum_{b_1=1}^{\alpha_1}\sum_{b_2=1}^{\alpha_2}g\Big(\tilde{R}(v_{b_1},v_{b_2})v_{b_1},v_{b_2}\Big).\end{aligned}$$

Then, from Eq. (2.1) one obtains

$$\sum_{b_1=1}^{\alpha_1}\sum_{b_2=1}^{\alpha_2} g\Big(\tilde{R}(v_{b_1}, v_{b_2})v_{b_1}, v_{b_2}\Big) = -c\alpha_1\alpha_2. \tag{2.27}$$

From this, we get:

$$\sum_{b_1=1}^{\alpha_1}\sum_{b_2=1}^{\alpha_2}\Big\{2\|\mathbf{A}(v_{b_1}, v_{b_2})\|^2 - g\Big(\mathbf{A}(v_{b_2}, v_{b_2}), \mathbf{A}(v_{b_1}, v_{b_1})\Big)\Big\} = \frac{3\alpha_2\Delta\mu}{2\mu} - \frac{\alpha_2^2}{\mu^2}\|\nabla\mu\|^2. \tag{2.28}$$

From our assumption (2.16) and (2.28), we obtain

$$\sum_{b_1=1}^{\alpha_1}\sum_{b_2=1}^{\alpha_2}\Big\{2\|\mathbf{A}(v_{b_1}, v_{b_2})\|^2 - g\Big(\mathbf{A}(v_{b_2}, v_{b_2}), \mathbf{A}(v_{b_1}, v_{b_1})\Big)\Big\} < c\alpha_1\alpha_2. \tag{2.29}$$

Applying Theorem 1.1 for constant curvature c, we achieve that there are not existent stable α_1-currents in $\mathbf{F}^{\alpha_1+\alpha_2}$ and $H_{\alpha_1}(\mathbf{F}^{\alpha_1+\alpha_2}, \mathbb{Z}) = H_{\alpha_2}(\mathbf{F}^{\alpha_1+\alpha_2}, \mathbb{Z}) = 0$, which reach the proof (a) and (b) of the theorem. On the other hand, if it (2.28) we make the substitution $\alpha_1 = 1$, then we get:

$$\sum_{b_2=2}^{m}\Big\{2\|\mathbf{A}(v_1, v_{b_2})\|^2 - g\Big(\mathbf{A}(v_{b_2}, v_{b_2}), \mathbf{A}(v_1, v_1)\Big)\Big\} = \frac{3\alpha_2\Delta\mu}{2\mu} - \frac{\alpha_2}{\mu^2}\|\nabla\mu\|^2. \tag{2.30}$$

If the pinching condition (2.16) for $\alpha_1 = 1$ and $\alpha_2 = m-1$ holds, then we get

$$\sum_{b_2=2}^{m}\Big\{2\|\mathbf{A}(v_1, v_{b_2})\|^2 - g\Big(\mathbf{A}(v_{b_2}, v_{b_2}), \mathbf{A}(v_1, v_1)\Big)\Big\} < c(m-1). \tag{2.31}$$

Then there are no stable 1-currents in $\mathbf{F}^{1+\alpha_2}$ and $H_1(\mathbf{F}^{1+\alpha_2}, \mathbb{Z} = H_{n-1}(\mathbf{F}^{1+\alpha_2}, \mathbb{Z}) = 0$. Let's assume that $\pi_1(\mathbf{F})$ does not equal 0. The traditional theorem, which uses the findings of Cartan and Hadamard, claims that there is a minimal closed geodesic in any non-trivial homotopy class in $\pi_1(\mathbf{F})$, contradicts itself when applied to the compactness of $\mathbf{F}^{1+\alpha_2}$. Consequently, $\pi_1(\mathbf{F}) = 0$. The theorem's third component is this. This Riemannian manifold is simply connected if the finite basic group for any Riemannian manifold is null. Therefore, $\mathbf{F}^{\alpha_1+\alpha_2}$ is simply connected. □

Inspired by geometric rigidity, the second mission of this study is to show a novel vanishing result for compact warped product submanifolds utilizing the Ricci curvature and the eigenvalue of the warping function's Laplacian. The following theorem can be provided in detail.

Theorem 2.2. *if the warping function μ is an eigenfunction of the Laplacian of $\mathbf{F}^{\alpha_1+\alpha_2}$ associated with the first positive eigenvalue λ_1 under the same statement of Theorem 2.1, then the subsequent stringent inequality holds.*

$$\|\nabla^2\mu\|^2 + Ric(\nabla\mu, \nabla\mu) + \frac{\lambda_1(3\lambda_1 + 2c\alpha_1)\mu^2}{2\alpha_2} > 0. \tag{2.32}$$

Then there are the subsequent

(a) There is not existent any stable integral α_1-currents flow in a warped product submanifold $\mathbf{F}^{\alpha_1+\alpha_2}$.

(b) The i-th integral homology groups of $\mathbf{F}^{\alpha_1+\alpha_2}$ with integer coefficients, are vanished, that is

$$\mathbb{H}_{\alpha_1}(\mathbf{F}^{\alpha_1+\alpha_2}, \mathbb{Z}) = \mathbb{H}_{\alpha_2}(\mathbf{F}^{\alpha_1+\alpha_2}, \mathbb{Z}) = 0,$$

(c) The fundamental group $\pi_1(\mathbf{F})$ is null, i.e., $\pi_1(\mathbf{F}) = 0$. Further, $\mathbf{F}^{\alpha_1+\alpha_2}$ is a simply connected warped product submanifold.

Proof. If μ is the first eigenfunction of the Laplacian $\Delta\mu = div(\nabla\mu)$ of $\mathbf{F}^{\alpha_1+\alpha_2}$ associated with the first non-zero eigenvalue λ_1, that is, $\Delta\mu = -\lambda_1\mu$, then we recall now the Bochner formula (see e.g. [5]) which declares that the next connection is true for a differentiable function μ that is defined on a Riemannian manifold:

$$\frac{1}{2}\Delta\|\nabla\mu\|^2 = \|\nabla^2\mu\|^2 + Ric(\nabla\mu, \nabla\mu) + g\Big(\nabla\mu, \nabla(\Delta\mu)\Big).$$

Using the Stokes theorem to integrate the aforementioned equation, we arrive at

$$\begin{aligned}\int_{\mathbf{F}^{\alpha_1+\alpha_2}} \|\nabla^2\mu\|^2 dV + &\int_{\mathbf{F}^{\alpha_1+\alpha_2}} Ric(\nabla\mu, \nabla\mu) dV \\ &= -\int_{\mathbf{F}^{\alpha_1+\alpha_2}} g\Big(\nabla\mu, \nabla(\Delta\mu)\Big) dV \end{aligned} \tag{2.33}$$

Now, using $\Delta\mu = -\lambda_1\mu$ and change something in Eq. (2.33), we derive

$$\int_{\mathbf{F}^{\alpha_1+\alpha_2}} \|\nabla\mu\|^2 dV = \frac{1}{\lambda_1}\Big(\int_{\mathbf{F}^{\alpha_1+\alpha_2}} \|\nabla^2\mu\|^2 dV + \int_{\mathbf{F}^{\alpha_1+\alpha_2}} Ric(\nabla\mu, \nabla\mu) dV\Big). \tag{2.34}$$

If (2.32) holds, then one obtains:

$$\int_{\mathbf{F}^{\alpha_1+\alpha_2}} \Big\{\|\nabla^2\mu\|^2 + Ric(\nabla\mu, \nabla\mu)\Big\} dV + \frac{\lambda_1(3\lambda_1 + 2c\alpha_1)}{2\alpha_2}\int_{\mathbf{F}^{\alpha_1+\alpha_2}} \mu^2 dV > 0. \tag{2.35}$$

Substituting Eq. (2.35) in (2.34), we get:

$$-\frac{\lambda_1(3\lambda_1 + 2c\alpha_1)}{2\alpha_2}\int_{\mathbf{F}^{\alpha_1+\alpha_2}} \mu^2 dV < \lambda_1 \int_{\mathbf{F}^{\alpha_1+\alpha_2}} \|\nabla\mu\|^2 dV,$$

which implies that

$$-3\lambda_1 \int_{\mathbf{F}^{\alpha_1+\alpha_2}} \mu^2 dV < 2c\alpha_1 \int_{\mathbf{F}^{\alpha_1+\alpha_2}} \mu^2 dV + 2\alpha_2 \int_{\mathbf{F}^{\alpha_1+\alpha_2}} \|\nabla\mu\|^2 dV. \tag{2.36}$$

Now using $\Delta = -\lambda_1\mu$ in the left hand side of Eq. (2.36), we arrive at

$$\int_{\mathbf{F}^{\alpha_1+\alpha_2}} \Big\{3h\Delta\mu - 2\alpha_2\|\nabla\mu\|^2 - 2c\alpha_1\mu^2\Big\} dV < 0. \tag{2.37}$$

Then, one obtains:

$$3\mu\Delta\mu < 2\alpha_2\|\nabla\mu\|^2 + 2c\alpha_1\mu^2. \tag{2.38}$$

Finally, we arrive after our theorem using the aforementioned equation as well as Theorem 2.1. The theorem's proof is now complete. □

As a quick implementation of Theorem 2.2, we can give now the subsequent:

Theorem 2.3. *Assuming that $\Psi : \mathbf{F}^{\alpha_1+\alpha_2} = \mathbf{F}_1^{\alpha_1} \times_\mu \mathbf{F}_2^{\alpha_2} \longrightarrow \mathbb{F}_c^{\alpha_1+\alpha_2+k_1}$ is an $\mathbf{F}_1^{\alpha_1}$-minimal isometric embedding from a compact warped product submanifold $\mathbf{F}^{\alpha_1+\alpha_2}$ into an $(\alpha_1+\alpha_2+k)$-dimensional space form $\mathbb{F}_c^{\alpha_1+\alpha_2+k_1}(c)$ satisfying the following inequality*

$$\begin{aligned}\int_{\mathbf{F}^{\alpha_1+\alpha_2}} \|\nabla^2\mu\|^2 dV < &\int_{\mathbf{F}^{\alpha_1+\alpha_2}} \sum_{i=1}^{\alpha_1} \|\mathbf{A}\Big(\nabla\mu, v_i\Big)\|^2 dV \\ &+ \frac{(\alpha_1 - 1 - \lambda_1)(3\lambda_1 + 2c\alpha_1)}{2\alpha_2}\int_{\mathbf{F}^{\alpha_1+\alpha_2}} \mu^2 dV. \end{aligned} \tag{2.39}$$

Then the conditions (a), (b), and (c) in Theorem 2.1 are satisfied. Moreover, $\{v_i\}$ are orthonormal frame for the base $\mathbf{F}_1^{\alpha_1}$.

Proof. As we are aware $\mathbf{F}^{\alpha_1+\alpha_2}$ is $\mathbf{F}_1^{\alpha_1}$-minimal compact warped product submanifold, then from Gauss equation, one obtains

$$R^i_{jkl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} + \sum_{r=1}^{k}\Big(\mathbf{A}^r_{ik}\mathbf{A}^r_{jl} - \mathbf{A}^r_{il}\mathbf{A}^r_{jk}\Big).$$

which implies the following

$$R^i_{jij} = \delta_{ii}\delta_{jj} - \delta_{ij}\delta_{ji} + \sum_{r=1}^{k}\Big(\mathbf{A}^r_{ii}\mathbf{A}^r_{jj} - \mathbf{A}^r_{ij}\mathbf{A}^r_{ji}\Big). \tag{2.40}$$

Taking into account that $\mathbf{F}_1^{\alpha_1}$ is a minimal submanifold and using the argument of the Ricci curvature for a unit sphere, we get

$$Ric(v_i, v_j) = (\alpha_1 - 1)\delta_{ij} - \sum_{r=1}^{k}\sum_{l=1}^{\alpha_1}\mathbf{A}^r_{il}\mathbf{A}^r_{jl}$$

The above equation yields that

$$Ric(h_i v_i, v_j h_j) = (\alpha_1 - 1)\delta_{ij}h_i h_j - \sum_{r=1}^{k}\sum_{l=1}^{\alpha_1}\mathbf{A}^r_{il}\mathbf{A}^r_{jl}h_i h_j. \tag{2.41}$$

Using Eq. (2.41), we get

$$Ric(\nabla\mu, \nabla\mu) = (\alpha_1 - 1)\|\nabla\mu\|^2 - \sum_{i=1}^{\alpha_1}\|\mathbf{A}\Big(\nabla\mu, v_i\Big)\|^2.$$

Putting the aforementioned equation into practice in (2.34), we obtain

$$\begin{aligned}\int_{\mathbf{F}^{\alpha_1+\alpha_2}}\sum_{i=1}^{\alpha_1}\|\mathbf{A}\Big(\nabla\mu, v_i\Big)\|^2 dV &= \int_{\mathbf{F}^{\alpha_1+\alpha_2}}\|\nabla^2\mu\|^2 dV \\ &\quad + (\alpha_1 - 1 - \lambda_1)\int_{\mathbf{F}^{\alpha_1+\alpha_2}}\|\nabla\mu\|^2 dV.\end{aligned} \tag{2.42}$$

If our assumption (2.39) is satisfied, then

$$\begin{aligned}\int_{\mathbf{F}^{\alpha_1+\alpha_2}}\|\nabla^2\mu\|^2 dV &< \int_{\mathbf{F}^{\alpha_1+\alpha_2}}\sum_{i=1}^{\alpha_1}\|\mathbf{A}\Big(\nabla\mu, v_i\Big)\|^2 dV \\ &\quad + \frac{(\alpha_1 - 1 - \lambda_1)(3\lambda_1 + 2c\alpha_1)}{2\alpha_2}\int_{\mathbf{F}^{\alpha_1+\alpha_2}}\mu^2 dV.\end{aligned}$$

The following form can be used to express the previously mentioned equation by using $\Delta\mu = -\lambda_1\mu$

$$\begin{aligned}\frac{3(\alpha_1 - 1 - \lambda_1)}{2\alpha_2}\int_{\mathbf{F}^{\alpha_1+\alpha_2}}\mu\Delta\mu dV + \int_{\mathbf{F}^{\alpha_1+\alpha_2}}\|\nabla^2\mu\|^2 dV \\ < \int_{\mathbf{F}^{\alpha_1+\alpha_2}}\sum_{i=1}^{\alpha_1}\|\mathbf{A}\Big(\nabla\mu, v_i\Big)\|^2 dV \\ + \frac{\alpha_1(c\alpha_1 - 1 - \lambda_1)}{\alpha_2}\int_{\mathbf{F}^{\alpha_1+\alpha_2}}\mu^2 dV.\end{aligned}$$

Including the previous equation in (2.42), we derive that

$$\frac{3(\alpha_1 - 1 - \lambda_1)}{2\alpha_2} \int_{\mathbf{F}^{\alpha_1+\alpha_2}} \mu \Delta \mu dV < (\alpha_1 - 1 - \lambda_1) \int_{\mathbf{F}^{\alpha_1+\alpha_2}} \|\nabla \mu\|^2 dV + \frac{\alpha_1(c\alpha_1 - 1 - \lambda_1)}{\alpha_2} \int_{\mathbf{F}^{\alpha_1+\alpha_2}} \mu^2 dV,$$

which implies the following from the above equation

$$3\mu\Delta\mu < 2\alpha_2\|\nabla\mu\|^2 + 2c\alpha_1\mu^2. \tag{2.43}$$

Consequently, we achieve the desired outcome (2.39). This completes the proof of the result. □

Another intriguing outcome that can be attained as a consequence of Theorem 2.3 is the following one:

Corollary 2.1. *Under the same assumption as Theorem 2.3 and if $\nabla\mu \in Ker\mathbf{A}$ with the following holds*

$$\int_{\mathbf{F}^{\alpha_1+\alpha_2}} \|\nabla^2\mu\|^2 dV < \frac{(\alpha_1 - 1 - \lambda_1)(3\lambda_1 + 2c\alpha_1)}{2\alpha_2} \int_{\mathbf{F}^{\alpha_1+\alpha_2}} \mu^2 dV, \tag{2.44}$$

Then the conditions (a), (b), and (c) in Theorem 2.1 are satisfied.

Proof. Using the hypothesis of corollary,$\nabla\mu \in Ker\mathbf{A}$, we get $\mathbf{A}(\nabla\mu, v_i) = 0$. Using this condition in (2.39), we can easily obtain the desired outcome. □

Remark 2.3. If we used the Remark 2.1 for non-negative constant section curvature $c = 1$ and $c = 0$ only, then all the results proved above are generalized results to [13] and [12] for $c = 1$ and $c = 0$.

Availability of data and material

There is no data used for this manuscript.

Competing interests

The authors declare no competing interests.

Authors' contributions

All authors have equal contributions and finalized.

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References

[1] A. Ali, F. Mofarreh, C. Ozel and W. A. M. Othman, *Homology of warped product submanifolds in the unit sphere and its applications*, Int. J. Geom. Methods Mod. Phys. **17** (2020), 2050121.

[2] A. Ali, F. Mofarreh, N. Alluhaibi and P. Laurian-Ioan, *Null homology in warped product Lagrangian submanifolds of the nearly Kaehler* $\mathbb{S}^6$ *and its applications*, J. Geom. Phys. **158** (2020), 103859.

[3] A. Ali, A. H. Alkhaldi and P. Laurian-Ioan, *Stable currents and homology groups in a compact CR-warped product submanifold with negative constant sectional curvature*, J. Geom. Phys. **148** (2020), 103566.

[4] Bishop, R. L., O'Neil, B., *Manifolds of negative curvature*. Trans. Amer. Math. Soc. **145**, (1969); 1-9.

[5] Berger, M., Gauduchon, P., Mazet, E., *Le spectre d'une variété Riemannienne, Lectures Notes in Mathematics*, Vol. **194**, Springer, Berlin, (1971).

[6] Chen, B. Y., *On isometric minimal immersions from warped products into real space forms.* Proc. Edinb. Math. Soc **45** (2002); 579-587.

[7] Federer, H., Fleming, W., *Normal and integral currents*, Ann. of Math. **72** (1960) 458-520.

[8] Lawson, H. B., Simons, J., *On stable currents and their application to global problems in real and complex geometry*, Ann. Math. **98** (1973), 427-450.

[9] Lui, L., Zhang, Q., *Non-existence of stable currents in submanifolds of the Euclidean spaces,* J. Geom. **96** (2009), 125-133.

[10] Leung, P., *On a relation between the topology and the intrinsic and extrinsic geometries of a compact submanifold*, Proc. Edinburgh Math. Soc. 28 (1985) 305-311.

[11] Li Y, Alkhaldi AH, Ali A, Laurian-Ioan P., *On the topology of warped product pointwise semi-slant submanifolds with positive curvature.* Mathematics. 2021; 9**24**:3156.

[12] Li Y, Laurian-Ioan P., Ali A, Alkhaldi AH., *Null homology groups and stable currents in warped product submanifolds of Euclidean spaces.* Symmetry. 2021; 9**13**:1587.

[13] Alqahtani, L.S.; Ali, A.; Laurian-Ioan, P.; Alkhaldi, A.H. *The homology of warped product submanifolds of spheres and their Applications.* Mathematics (2023), 11, 3405. https://doi.org/10.3390/math11153405

[14] F. Sahin, B.Sahin., *Homology of contact CR-warped product submanifolds of an odd-dimensional unit sphere.* Bull. Korean Math. Soc. **52** (2015), no. 1, 215–222.

[15] F. Sahin., *On the topology of CR-warped product submanifolds,* Intern. J. Geom. Meth. Mod. Phy., **15**, (2018); 1850032.

[16] F. Sahin., *Homology of submanifolds of six dimensional sphere.* J. Geom. Phys. **145** (2019), 103471.

[17] F.Sahin, B.Sahin., *Homology of contact CR-submanifolds,* Filomat, Vol 35, No. 11 (2021), 3811-3819.

[18] F. Şahin, B Şahin, *Homology of contact 3-CR-submanifolds of an almost 3-contact hypersurface,* Chaos, Solitons & Fractals 151, (2021); 111267.

[19] Vlachos, T., *Homology vanishing theorems for submanifolds*, Proc. Amer. Math. Soc., **135**, (2007)., 2607-2617.

[20] Xin, Y. L., *An application of integral currents to the vanishing theorems*, Sci. Sinica Ser. A **27** (1984) 233-241.

[21] Zhang, Z. X., *Non-existence of stable currents in submanifolds of a product of two spheres,* Bull. Austral. Math. Soc. **44**, (1991) 325-336.