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Algebraic interpretations of hyperbolic functions

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Abstract. In this paper, we obtain an algebraic interpretation of hyperbolic functions. By using four conditions, we define a hyperbolic algebra over reals, and we prove that linear groupoids or commutative cubic algebras cannot be a hyperbolic algebra, but the quadratic algebras can be hyperbolic algebras under some conditions.

1. Introduction

The theory of groupoids [2, 3] has been introduced by some researchers. It has been combined with the theory of general algebraic structures. One of the methods for the generalization of axioms is to employ special functions, i.e., by using of proper mappings, we may generalize axioms in mathematical structures. The notion of *BCK*-algebras was formulated by Iséki [7]. The motivation of this notion is based on both set theory and propositional calculus. Recently, as the generalization of the groupoid theory, general algebraic structures have been developed combined with *BCK*-algebras (see [1, 5, 6, 8, 11–14]).

The notion of a linear groupoid has been applied to the Fibonacci sequences in groupoids. Using the notion of a flexibility, the linear groupoid was used to the study of the Fibonacci sequence in groupoids [4].

Kim *et al.* [11] introduced the notion of a generalized commutative law in algebras, and showed that every pre-commutative *BCK*-algebra is bounded. In the study of the pre-commutativity in groupoids, they proved that if a linear groupoid is left-(right-) pre-commutative, then it is also abelian. Hwang *et al.* [5] discussed on some implicativities for groupoids and *BCK*-algebras. They characterized linear groupoids for implicative groupoids.

Kim and Lee [10] introduced the notion of a quadratic *BG*-algebra, and obtained that quadratic *BG*-algebras, quadratic *Q*-algebras and quadratic *B*-algebras are logically equivalent on a field X with $|X| \geq 3$, and they showed that every quadratic *BG*-algebra is a *BCI*-algebra. Kim and Kye [9] applied the quadratic algebras to *BF*-algebras, and showed that every quadratic *BF*-algebra is also a *BCI*-algebra.

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In th In this paper, we define a hyperbolic algebra which can describe the hyperbolic functions over reals. We emphasize that we construct an algebraic interpretation of hyperbolic functions by using four axioms, and we find that there are no non-trivial proper hyperbolic algebras in linear groupoids or commutative cubic algebras. It is not yet known that there is no non-trivial proper hyperbolic algebras in non-commutative cubic algebras. It will be next research topics for finding general algebraic structures for well-known functions, e.g., $z = x^2 + y^2$, $z = ax^2 + by^2$ where $a, b > 0$ and $a \neq b$. Moreover, several quadratic surfaces, e.g., ellipsoid, elliptic paraboloid, cone and hyperboloid of one(two) sheet will be added to further research topics for their algebraic interpretations.

2. Preliminaries

Let $\mathbf{R}$ be the set of all real numbers. We define a binary operation "$*$" on $\mathbf{R}$ by

$$x * y := Ax^2 + Bxy + Cy^2 + Dx + Ey + F \tag{1}$$

for all $x, y \in \mathbf{R}$, where $A, B, C, D, E, F \in \mathbf{R}$. We call such a groupoid $(\mathbf{R}, *)$ a *quadratic algebra* [5, 9–12] over real numbers.

We may define several binary operations on $\mathbf{R}$ by, e.g., $x *_1 y := x^2 + xy - y^2 - 3x + 2y + 1$, $x *_2 y := 2x^2 + 3xy - 5y^2$, $x *_3 y := 2x + 3y - 4$, $x *_4 y := x + y, \cdots$. All $(\mathbf{R}, *_i)$ are examples of quadratic algebras.

Especially, if $A = B = C = 0$ in (1), then

$$x * y = Dx + Ey + F. \tag{2}$$

We call such an algebra $(\mathbf{R}, *)$ a *linear groupoid* [4, 5, 11, 12] over reals. If $D = E = 1$ and $F = 0$ in (2), then $x * y = x + y$ for all $x, y \in \mathbf{R}$, which is an example of an additive group of real numbers. Thus there are linear groupoids which are groups, and even abelian groups.

3. Hyperbolic algebras

We have discussed on the geometric structure of hyperbolic functions, but we did not yet mention on its algebraic interpretations. We give the following four axioms to construct an algebraic structure corresponding the hyperbolic functions as below:

Let X be a non-empty set and let e be a fixed element (or constant) of X. Let "$*$" be a binary operation on X. A groupoid $(X, *, e)$ is said to be *hyperbolic* if it satisfies the following conditions:

(H1) $x * x = e$,
(H2) $(x * y) * z = (y * x) * z$,
(H3) $x * (y * z) = x * (z * y)$,
(H4) if $x * y = y * x$, then $x * y = y * x = e$,

for all $x, y, z \in X$.

A hyperbolic groupoid $(X, *, e)$ is said to be *trivial* if $x * y = e$ for all $x, y \in X$. A hyperbolic algebra $(X, *, e)$ is said to be *non-trivial* if it satisfies the following condition:

(H5) there exist elements $x, y \in X$ such that $x * y \neq e$.

Example 3.1. *Let $\mathbf{R}$ be the set of all real numbers. We define a binary operation "$*$" on $\mathbf{R}$ by $x * y := x^2 - y^2$ for all $x, y \in \mathbf{R}$. Then $(\mathbf{R}, *, 0)$ is a hyperbolic algebra. In fact, given $x, y, z \in \mathbf{R}$, we have $x * x = x^2 - x^2 = 0$, $(x * y) * z = (x^2 - y^2)^2 - z^2 = (y^2 - x^2)^2 - z^2 = (y * x) * z$ and $x * (y * z) = x^2 - (y^2 - z^2)^2 = x^2 - (z^2 - y^2)^2 = x * (z * y)$. Moreover, if $x * y = y * x$, then $2(x^2 - y^2) = 0$, and hence $x^2 = y^2$. It shows $x * y = y * x = 0$. Hence $(\mathbf{R}, *, 0)$ is a hyperbolic algebra. If we assume that $x := 1$ and $y := 0$, then $1 * 0 = 1 - 0 = 1 \neq 0$. This proves that $(\mathbf{R}, *, 0)$ is a non-trivial hyperbolic algebra.*

Let **R** be the set of all real numbers and let $A, B, C \in \mathbf{R}$. We define a binary operation "$*$" on **R** by

$$x * y := Ax + By + C$$

for all $x, y \in \mathbf{R}$. Such an algebra $(\mathbf{R}, *)$ is said to be a *linear groupoid*.

Theorem 3.2. *There exists no non-trivial hyperbolic algebras over reals which are linear groupoids.*

Proof. Assume that $(\mathbf{R}, *, e)$ is both a hyperbolic algebra and a linear groupoid. Then there exist $A, B, C \in \mathbf{R}$ such that $x * y = Ax + By + C$ for all $x, y \in \mathbf{R}$. It follows that

$$e = x * x = (A + B)x + C$$

for all $x \in \mathbf{R}$. It implies that $A + B = 0$ and $C = e$, i.e., $x * y = A(x - y) + e$. If we apply (H2), then we obtain $A\{A(x - y) + e - z\} + e = A\{A(y - x) + e - z\} + e$ for all $x, y, z \in \mathbf{R}$. It follows that $2A^2(x - y) = 0$ for all $x, y \in \mathbf{R}$. Hence we obtain $A = 0$, i.e., $x * y = e$. It shows that there is no non-trivial hyperbolic algebras over reals which are linear groupoids. □

A non-trivial hyperbolic algebra over reals is said to be *proper* if it is not a linear groupoid. Theorem 3.2 shows that linear groupoids can not be a hyperbolic algebra.

Remark 3.3. *Let $(\mathbf{R}, -)$ be a linear groudpoid where "$-$" is the usual subtraction on **R**. Then $(\mathbf{R}, -, 0)$ is a non-trivial linear groupoid, but not a hyperbolic algebra, since $(x - y) - z \neq (y - x) - z$ if $x \neq y$.*

Proposition 3.4. *Let $(\mathbf{R}, *, e_1)$ and $(\mathbf{R}, \circledast, e_2)$ be linear groupoids where $x * y := \alpha x + \beta y - e_1$ and $x \circledast y := px + qy - e_2$ where $\alpha, \beta, p, q, e_1, e_2 \in \mathbf{R}$, $e_1 \neq e_2$. If $\varphi : (\mathbf{R}, *, e_1) \to (\mathbf{R}, \circledast, e_2)$ is a homomorphism of linear groupoids defined by $\varphi(x) := (x - e_1) + e_2$, then*

(i) $\alpha = p,\ \beta = q,\ \alpha + \beta = 2$,
(ii) $(X, *) \cong (X, \circledast)$.

Proof. Given $x, y \in \mathbf{R}$, since φ is a homomorphism of groupoids, we obtain

$$\begin{aligned} \varphi(x * y) &= (x * y - e_1) + e_2 \\ &= (\alpha x + \beta y - e_1 - e_1) + e_2 \\ &= \alpha x + \beta y - 2e_1 + e_2 \end{aligned}$$

and

$$\begin{aligned} \varphi(x) \circledast \varphi(y) &= p\varphi(x) + q\varphi(y) - e_2 \\ &= p[(x - e_1) + e_2] + q[(y - e_1) + e_2] - e_2 \\ &= px + qy + p(-e_1 + e_2) + q(-e_1 + e_2) - e_2. \end{aligned}$$

It follows that $\alpha = p, \beta = q$ and $\alpha + \beta = 2$. Since $\varphi^{-1}(x) = (x - e_2) + e_1$ for all $x \in \mathbf{R}$, we obtain

$$\begin{aligned} (\varphi \circ \varphi^{-1})(x) &= \varphi(\varphi^{-1}(x)) \\ &= (\varphi^{-1}(x) - e_1) + e_2 \\ &= \{(x - e_2 + e_1) - e_1\} + e_2 \\ &= x, \end{aligned}$$

which proves the proposition. □

Proposition 3.4 shows that if $\varphi : (\mathbf{R}, *, e_1) \to (\mathbf{R}, \circledast, e_2)$ is a homomorphism of linear groupoids defined by $\varphi(x) := (x - e_1) + e_2$, then the binary operation "$*$" on **R** is of the form $x * y = \alpha x + (2 - \alpha)y - e$ where $\alpha, e \in \mathbf{R}$.

4. Quadratic hyperbolic algebras

In this section, we may expect to find non-trivial hyperbolic algebras among the quadratic algebras. Let **R** be the set of all real numbers. We define a binary operation "$*$" on **R** by

$$x * y := Ax^2 + Bxy + Cy^2 + Dx + Ey + F$$

for all $x, y \in \mathbf{R}$, where $A, B, C, D, E, F \in \mathbf{R}$. First, we need to arrange the coefficients $A, B, \cdots, F$ by applying the conditions $(H1)$ and $(H2)$ to the quadratic algebra $(\mathbf{R}, *, e)$ as follows:

Lemma 4.1. *Let* $(\mathbf{R}, *, 0)$ *be a quadratic algebra over reals. If it satisfies the conditions* $(H1)$ *and* $(H2)$*, then either* $A = 0$ *or* $A = \pm C$.

Proof. Assume that the conditions $(H1)$ and $(H2)$ hold for the quadratic algebra $(\mathbf{R}, *)$ over reals. If we let $y := x$ in $(H1)$, then $(A + B + C)x^2 + (D + E)x + F = e$ and thus

$$A + B + C = D + E = 0 \quad \text{and} \quad F = e. \tag{3}$$

If we let $y := 1, z := 0$ in $(H2)$, then

$$\begin{aligned} x * 1 &= Ax^2 + Bx + C + Dx + E + e \\ &= Ax^2 + (B + D)x + (C + E + e), \end{aligned} \tag{4}$$

and

$$\begin{aligned} 1 * x &= A + Bx + Cx^2 + D + Ex + e \\ &= Cx^2 + (B + E)x + (A + D + e) \end{aligned} \tag{5}$$

for all $x \in \mathbf{R}$. Since $x * 0 = Ax^2 + Dx + e$ for all $x \in \mathbf{R}$, by applying (4) and (5), we obtain

$$\begin{aligned} (x * 1) * 0 &= A(x * 1)^2 + D(x * 1) + e \\ &= A[Ax^2 + (B + D)x + (C + E + e)]^2 \\ &+ D[Ax^2 + (B + D)x + (C + E + e)] + e \\ &= A^3x^4 + 2A^2(B + D)x^3 + [A(B + D)^2 + 2A^2(C + E + e) \\ &+ AD]x^2 + [2A(B + D)(C + E + e) + D(B + D)]x \\ &+ [A(C + E + e)^2 + D(C + E + e) + e], \end{aligned} \tag{6}$$

and

$$\begin{aligned} (1 * x) * 0 &= A(1 * x)^2 + D(1 * x) + e \\ &= A[Cx^2 + (B + E)x + (A + D + e)]^2 \\ &+ D[Cx^2 + (B + E)x + (A + D + e)] + e \\ &= AC^2x^4 + 2AC(B + E)x^3 + [A(B + E)^2 \\ &+ 2AC(A + D + e)CD]x^2 + [2A(B + E) + D(B + E)]x \\ &+ [A(A + D + e)^2 + D(A + D + e) + e] \end{aligned} \tag{7}$$

for all $x \in \mathbf{R}$. By comparing (6) and (7), we obtain $A^3 = AC^2$, which shows that either $A = 0$ or $A = \pm C$, which proves the lemma. □

Lemma 4.2. *Let* $(\mathbf{R}, *, 0)$ *be a quadratic algebra over reals satisfying the conditions* $(H1) \sim (H3)$*. If* $A = 0$*, then* $x * y = D(x - y) + e$ *for all* $x, y \in \mathbf{R}$.

Proof. If we let $A := 0$ in (3), then $B + C = 0, D + E = 0$ and $F = e$. It follows that

$$\begin{aligned} x * y &= Bxy - By^2 + Dx - Dy + e \\ &= B(xy - y^2) + D(x - y) + e \\ &= By(x - y) + D(x - y) + e \end{aligned} \tag{8}$$

for all $x, y \in \mathbf{R}$. If we let $y := 1$ in (8), then

$$x * 1 = (B + D)x - (B + D) + e$$

for all $x \in \mathbf{R}$. It follows from (8) that

$$\begin{aligned} (x * 1) * 0 &= D(x * 1) + e \\ &= D[(B + D)x - (B + D)] + e \\ &= D(B + D)x - D(B + D) + e. \end{aligned} \tag{9}$$

Similarly, if we let $x := 1, y := x$ in (8), then

$$1 * x = Bx(1 - x) + D(1 - x) + e$$

for all $x \in \mathbf{R}$. It follows that

$$(1 * x) * 0 = -BDx^2 + (BD - D^2)x + (D^2 + De + e) \tag{10}$$

for all $x \in \mathbf{R}$. Comparing (9) with (10), we obtain $BD = 0, D(B+D) = D(B-D)$ and $-D(B+D)+e = D^2+De+e$. We have two cases: (i) $B = 0$; (ii) $D = 0$.
Case (i). $B = 0$. Since $A = 0$, by (3), we obtain $C = 0$ and $E = -D$. Hence $x * y = D(x - y) + e$.
Case (ii). $D = 0$. Since $A + B + C = 0$ and $D + E = 0$, we have $C = -B$ and $E = 0$. It follows that

$$x * y = Bxy + Cy^2 + e = By(x - y) + e$$

for all $x, y \in \mathbf{R}$. From this, we obtain $x * 1 = B(x - 1) + e$ and $1 * x = -Bx^2 + Bx + e$. It follows that

$$\begin{aligned} 0 * (x * 1) &= B(x * 1)[0 - x * 1] + e \\ &= -B[B(x - 1)^2 + e] + e \\ &= -B^3x^2 + 2B^2(B - 1)x + [-B^3 + 2B^2 - Be^2 + e] \end{aligned} \tag{11}$$

and

$$\begin{aligned} 0 * (1 * x) &= B(1 * x)[0 - (1 * x)] + e \\ &= -B(1 * x)^2 + e \\ &= -B^3x^4 + 2B^3x^3 - B(B^2 - 2Be)x^2 - 2B^2ex - Be^2 + e. \end{aligned} \tag{12}$$

Comparing (11) with (12), by $(H3)$, we obtain $B = 0$. Since $A + B + C = 0$ and $A = 0$, we obtain $C = 0$. Hence $x * y = Dx + Ey + e = D(x - y) + e$. We prove the lemma. □

Note that, by Theorem 3.2 and Lemma 4.2, we see that if $A = 0$, then there is no non-trivial proper hyperbolic algebras over reals.

Lemma 4.3. *Let* $(\mathbf{R}, *, 0)$ *be a quadratic algebra over reals satisfying the conditions* $(H1) \sim (H3)$*. If* $A = C \neq 0$*, then the condition* $(H4)$ *does not hold.*

Proof. If we let $A = C$ in (3), then $B = -2A$, $E = -D$ and $F = e$. It follows that

$$\begin{aligned} x * y &= Ax^2 + Bxy + Ay^2 + D(x-y) + e \\ &= A(x-y)^2 + D(x-y) + e \end{aligned}$$

for all $x, y \in \mathbf{R}$. From this, we obtain $x * 1 = A(x-1)^2 + D(x-1) + e$ and $1 * x = A(x-1)^2 - D(x-1) + e$. It follows that

$$\begin{aligned} (x*1)*0 &= A(x*1)^2 + D(x*1) + e \\ &= A[A(x-1)^2 + D(x-1) + e]^2 \\ &+ D[A(x-1)^2 + D(x-1) + e] + e \\ &= A^3(x-1)^4 + 2A^2D(x-1)^3 \\ &+ [AD^2 + 2A^2e + AD](x-1)^2 \\ &+ [2ADe + D^2](x-1) + [Ae^2 + De + e], \end{aligned} \tag{13}$$

and

$$\begin{aligned} (1*x)*0 &= A(1*x)^2 + D(1*x) + e \\ &= A[A(x-1)^2 - D(x-1) + e]^2 \\ &+ D[A(x-1)^2 - D(x-1) + e] + e \\ &= A^3(x-1)^4 - 2A^2D(x-1)^3 \\ &+ [AD^2 + 2A^2e + AD](x-1)^2 \\ &- [D^2 + 2ADe](x-1) + [Ae^2 + De + e]. \end{aligned} \tag{14}$$

Comparing (13) with (14), we obtain $2A^2D = -2A^2D$ and $2ADe + D^2 = -(D^2 + 2AD)$, which shows that $4A^2D = 0$. Since $A \neq 0$, we obtain $D = 0$. Hence $x * y = A(x-y)^2 + e = A(y-x)^2 + e = y * x$ for all $x, y \in \mathbf{R}$. If we assume $(H4)$ holds, then $x * y = y * x = e$ for all $x, y \in \mathbf{R}$, i.e., $A(x-y)^2 + e = e$ for all $x, y \in \mathbf{R}$. This shows that $A = 0$, which is a contradiction. □

Note that, by Theorem 3.2 and Lemma 4.3, we see that if $A = C \neq 0$, then there is no non-trivial proper hyperbolic algebras over reals.

Theorem 4.4. *Let $(\mathbf{R}, *)$ be a hyperbolic algebra over reals. If $A = -C \neq 0$, then $x * y$ is one of the following form:*

$$x * y = \begin{cases} A(x^2 - y^2) + e, & \text{if } D = 0, \\ A\{(x-e)^2 - (y-e)^2\} + e, & \text{if } D = -2Ae,\ e \neq 0. \end{cases}$$

Proof. Letting $A = -C$ in (3), we get $B = 0$, $E = -D$ and $F = e$. By (3), we obtain

$$x * y = A(x^2 - y^2) + D(x-y) + e. \tag{15}$$

Letting $y := 1$ in (15), we get

$$x * 1 = A(x^2 - 1) + D(x-1) + e.$$

If we let $x := 1$ and $y := x$ in (15), we obtain

$$1 * x = A(1 - x^2) + D(1-x) + e.$$

We compute $(x * 1) * 0$ and $(1 * x) * 0$ as follows:

$$\begin{aligned}(x * 1) * 0 &= A(x * 1)^2 + D(x * 1) + e \\ &= A[A(x^2 - 1) + D(x - 1) + e]^2 \\ &+ D[A(x^2 - 1) + D(x - 1) + e] + e \\ &= A^3x^4 + 2A^2Dx^3 + [AD^2 - 2A^3 - 2A^2D + 2A^2e + AD]x^2 \\ &+ [-2A^2D - 2AD^2 + 2ADe + D^2]x \\ &+ [A(A + D - e)^2 - AD - D^2 + De + e], \end{aligned} \tag{16}$$

and

$$\begin{aligned}(1 * x) * 0 &= A(1 * x)^2 + D(1 * x) + e \\ &= A[A(1 - x^2) + D(1 - x) + e]^2 \\ &+ D[A(1 - x^2) + D(1 - x) + e] + e \\ &= A^3x^4 + 2A^2Dx^3 + [-2A^3 + AD^2 - 2A^2D - 2A^2e - AD]x^2 \\ &+ [-2AD^2 - 2A^2D - 2ADe - D^2]x \\ &+ [A^3 + AD^2 + Ae^2 + 2A^2D + 2ADe \\ &+ 2A^2e + AD + D^2 + De + e]. \end{aligned} \tag{17}$$

Since (16) and (17) are equal by $(H2)$, the corresponding coefficients are equal. By routine calculations we obtain the following results:

$$\begin{aligned} A(2Ae + D) &= 0, \\ D(2Ae + D) &= 0, \\ (A + D)(2Ae + D) &= 0. \end{aligned} \tag{18}$$

By (18), we obtain $A = 0$ or $D = 0$ or $D = -2Ae, e \neq 0$ or $D = -A$. We claim that $A \neq 0$. In fact, if $A = 0$, then $x * y = D(x - y) + e$ by Lemma 4.2. It shows that $(\mathbf{R}, *)$ should not be a non-trivial hyperbolic algebra over reals by Theorem 3.2. If we let $D := 0$ in (19), then $x * y = A(x^2 - y^2) + e$. If $D := -2Ae$ and $e \neq 0$, then

$$\begin{aligned} x * y &= A(x^2 - y^2) + D(x - y) + e \\ &= A(x^2 - y^2) - 2Ae(x - y) + e \\ &= A[(x^2 - 2ex) - (y^2 - 2ey)] + e \\ &= A[(x - e)^2 - e^2 - (y - e)^2 + e^2] + e \\ &= A[(x - e)^2 - (y - e)^2] + e. \end{aligned} \tag{19}$$

The case $D = -A$ is a special case of $D = -2Ae$ with $e = \frac{1}{2}$, and we omit it. We prove the theorem. □

By Theorem 4.4, we conclude that all quadratic hyperbolic algebras have of the form $x * y = A(\alpha^2 - \beta^2) + e$, where $\alpha, \beta, e \in \mathbf{R}$.

By Lemmas 4.1 and 4.3 and Theorem 4.4, we summarize our assertion as follows:

Theorem 4.5. *Let $(\mathbf{R}, *)$ be a quadraric algebra, where $x * y = Ax^2 + Bxy + Cy^2 + Dx + Ey + F$ for all $x, y \in \mathbf{R}$. Then all quadratic non-trivial proper hyperbolic algebras have of the form $x * y = A(\alpha^2 - \beta^2) + e$ where $\alpha, \beta, e \in \mathbf{R}$.*

Example 4.6. *Let $(\mathbf{R}, *)$ be a non-trivial proper hyperbolic quadratic algebra, where $x * y = Ax^2 + Bxy + Cy^2 + Dx + Ey + F$ for all $x, y \in \mathbf{R}$. If we let $A := 1$ and $C := -1$, then by (4) we obtain $B = 0, E = -D$ and $F = e$. Let $e := 10$. If we let $D := 0$, then $E = 0$ and $x * y = x^2 - y^2 + 10$. If we let $D := -2Ae = -20$, then $E = 20$, and hence $x * y = x^2 - y^2 - 20x + 20y + 10 = [(x - 10)^2 - (y - 10)^2] + 10$. We see that all of them are non-trivial proper hyperbolic quadratic algebras over reals.*

5. Cubic hyper algebras

In this section, we may expect to find non-trivial hyperbolic algebras among the cubic algebras. Let **R** be the set of all real numbers. We define a binary operation "$*$" on **R** by

$$\begin{aligned} x * y \quad := \quad & Ax^3 + Bx^2y + Cxy^2 + Dy^3 + Ex^2 \\ & + Fxy + Gy^2 + Ix + Jy + H \end{aligned} \tag{20}$$

for all $x, y \in \mathbf{R}$, where $A, B, C, D, E, F, G, H, I, J \in \mathbf{R}$. We call such an algebra $(\mathbf{R}, *)$ a *cubic algebra*.

Theorem 5.1. *Let* $(\mathbf{R}, *)$ *be a cubic algebra. If* $(\mathbf{R}, *)$ *is commutative, i.e.,* $x * y = y * x$ *for all* $x, y \in \mathbf{R}$*, and if it satisfies (H1), then*

$$x * y = A(x - y)^2(x + y) + E(x - y)^2 + e.$$

Proof. If we let $y := x$ in (20), then $e = x * x = (A + B + C + D)x^3 + (E + F + G)x^2 + (I + J)x + H = e$. It follows that

$$\begin{aligned} A + B + C + D &= 0, \\ E + F + G &= 0, \\ I + J &= 0, \\ H &= e. \end{aligned} \tag{21}$$

Assume $x * y = y * x$ for all $x, y \in \mathbf{R}$. Letting $y := 1$ in (20), we get

$$\begin{aligned} x * 1 \quad = \quad & Ax^3 + (B + E)x^2 + (C + F + I)x \\ & + (D + G + J + e). \end{aligned}$$

Letting $x := 1$ and $y := x$ in (20), we get

$$\begin{aligned} 1 * x \quad = \quad & Dx^3 + (C + G)x^2 + (B + F + J)x \\ & + (A + E + I + e). \end{aligned}$$

Since $x * 1 = 1 * x$, we obtain $A = D, B + E = C + G, C + F + I = B + F + J$ and $D + G + J + e = A + E + I + e$. Hence

$$\begin{aligned} A &= D, \\ B + E &= C + G, \\ C + I &= B + J, \\ D + G + J &= A + E + I. \end{aligned} \tag{22}$$

By comparing (21) with (22), we obtain $A = D, I = -J, J = -A - B, C = -2A - B, G = E + 2A + 2B$ and $F = -2E - 2A - 2B$. Hence, by (20), we obtain

$$\begin{aligned} x * y \quad = \quad & Ax^3 + Bx^2y - (2A + B)xy^2 + Ay^3 \\ + \quad & Ex^2 - 2(E + A + B)xy + (E + 2A + 2B)y^2 \\ + \quad & (A + B)x - (A + B)y + e, \end{aligned} \tag{23}$$

and

$$\begin{aligned} y * x \quad = \quad & Ax^3 - (2A + B)x^2y + Bxy^2 + Ay^3 \\ + \quad & (E + 2A + 2B)x^2 - 2(E + A + B)xy + Ey^2 \\ - \quad & (A + B)x + (A + B)y + e. \end{aligned} \tag{24}$$

Since (23) and (24) are equal by the assumption, we obtain $B = -2A - B$, $E = E + 2A + 2B$ and $A + B = -A - B$. Hence we get $B = -A$. By (23),

$$\begin{aligned} x * y &= Ax^3 - Ax^2y - Axy^2 + Ay^3 + Ex^2 - 2Exy + Ey^2 + e \\ &= A\{x^2(x-y) - y^2(x-y)\} + E(x-y)^2 + e \\ &= A(x-y)^2(x+y) + E(x-y)^2 + e, \end{aligned}$$

which proves the theorem. □

Theorem 5.2. *Let $(\mathbf{R}, *)$ be a cubic algebra. Then there exist no commutative cubic hyperbolic algebras over reals.*

Proof. Assume that $(\mathbf{R}, *)$ is a commutative cubic hyperbolic algebra. Then it satisfies two conditions of Theorem 5.1, and hence there exist $A, E \in \mathbf{R}$ and $A \neq 0$ such that

$$x * y = A(x-y)^2(x+y) + E(x-y)^2 + e.$$

for all $x, y \in \mathbf{R}$. Since $(\mathbf{R}, *)$ is commutative, if we apply the condition $(H4)$, we obtain $x * y = e$ for all $x, y \in \mathbf{R}$. We claim that $A = 0$. In fact, since $2 * 1 = A(2-1)^2(2+1) + E(2-1)^2 + e = 3A + E + e = e$ and $3 * 1 = A(3-1)^2(3+1) + E(3-1)^2 + e = 16A + 4E + e = e$, we obtain $4A + E = 0 = 3A + E$, which shows that $A = 0$, a contradiction. Hence there exist no commutative cubic hyperbolic algebras over reals. □

We did not discuss the case of non-commutative cubic algebras yet. We give the following conjecture.

Conjecture. There is no non-commutative cubic hyperbolic algebra over reals which is not a proper quadratic hyperbolic algebra.

6. Conclusion and future work

We studied algebraic interpretations of hyperbolic functions, and obtained four conditions to describe the hyperbolic functions. Conversely, we applied the conditions to linear groupoids, quadratic algebras, and cubic algebras. We found that there is no non-trivial proper hyperbolic algebras in linear groupoids and commutative cubic algebras, but there are non-trivial proper hyperbolic algebras in quadratic algebras over reals. We give a conjecture that there is no such a hyperbolic algebras in non-commutative cubic algebras, and we left it for reader's interest. A quadratic surface can be represented by $Ax^2 + By^2 + Cz^2 + J$ or $Ax^2 + By^2 + Iz = 0$ by translation and rotation. We will focus on latter case for finding their general algebraic structures, e.g., $z = x^2 + y^2$, $z = ax^2 + by^2$ where $a, b > 0$ and $a \neq b$. Moreover, the former cases, i.e., several quadratic surfaces, e.g., ellipsoid, elliptic paraboloid, cone and hyperboloid of one(two) sheet will be added to further research topics for their algebraic interpretations.

Declarations

Availability of data and materials

Not applicable.

Human and animal rights

We would like to mention that this article does not contain any studies with animals and does not involve any studies over human being.

Conflict of interest

The authors declare that they have no competing interests.

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