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Generalizations Euler-Maclaurin-type inequalities for conformable fractional integrals

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Abstract. In this study, we obtain a unique insight into differentiable convex functions by employing newly defined conformable fractional integrals. With this innovative approach, we unveil fresh Euler-Maclaurin-type inequalities designed specifically for these integrals. Our proofs draw on fundamental mathematical principles, including convexity, Hölder's inequality, and power mean inequality. Furthermore, we delve into new inequalities applicable to bounded functions, Lipschitzian functions, and functions of bounded variation. Notably, our findings align with established results under particular circumstances.

1. Introduction and Preliminaries

Convexity is a key term in mathematical analysis, with broad consequences in various domains. Its utility in optimisation, mathematical modelling, and algorithm design underlines its importance. Convex functions have several essential characteristics that make them invaluable in academic study and practical applications [16, 29, 35]. If the subsequent inequality is valid for a function $\mathcal{F} : [\sigma, \rho] \to \mathbb{R} \subset \mathbb{R}$ then $\mathcal{F}$ is convex:

$$\eta\mathcal{F}(\sigma) + (1-\eta)\mathcal{F}(\rho) \geq \mathcal{F}(\eta\sigma + (1-\eta)\rho), \tag{1}$$

for all $\sigma, \rho \in I,\ \eta \in [0,1]$. In contrast if the inequality (1) is reversed, $\mathcal{F}$ is classified as concave function.
Integral inequalities are widely utilized in statistical analysis, approximation theory, spectral analysis, and distribution theory. They established bounds on distributions, errors in approximation methods, spectral measures, and probability inequalities. It contributes to advances in research and technology through applications in fields like physics, biology, signal processing, and control theory, playing an important role in mathematical analysis and addressing complicated problems spanning scientific disciplines [1, 5, 10, 22]. Many scholars have studied fractional calculus, focusing on its application in inequality theory. In [7, 34],

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authors analysed fractional forms of trapezoid-type inequalities. In the context of fractional calculus, Aamir et al. have examined some new parameterized Newton-type inequalities for differentiable convex functions [3]. Using Riemann-Liouville (R-L) fractional integrals, Hezenci et al. [18] have demonstrated Newton's inequality for differentiable convex functions. They provide a graphical analysis that elucidates the validity of the recently established inequalities. Budak and Karagozoglu [14] have unveiled the fractional variant of Milne-type inequality. Ertugral and Sarikaya [8] have investigated some extended versions of Simpson-type integral inequalities by leveraging generalised fractional integral. For functions with second derivatives that exhibit convexity in absolute value, You et al. [40] have obtained numerous Simpson-type inequalities by taking advantage of generalised fractional integrals. Over the past few years, many publications have focused on forming significant inequalities by employing fractional integrals [2, 38].

Definition 1.1. *[27] Let us consider $\mathcal{F} \in L_1[\sigma, \rho]$. The R-L integrals $\Im^{\alpha}_{\sigma+}\mathcal{F}$ and $\Im^{\alpha}_{\rho-}\mathcal{F}$, with order $\alpha > 0$, defined as follows:*

$$\Im^{\alpha}_{\sigma+}\mathcal{F}(\upsilon) = \frac{1}{\Gamma(\alpha)}\int_{\sigma}^{\upsilon} (\upsilon - \eta)^{\alpha-1}\mathcal{F}(\eta)d\eta, \ \ \upsilon > \sigma \tag{2}$$

and

$$\Im^{\alpha}_{\rho-}\mathcal{F}(\upsilon) = \frac{1}{\Gamma(\alpha)}\int_{\upsilon}^{\rho} (\eta - \upsilon)^{\alpha-1}\mathcal{F}(\eta)d\eta, \ \ \upsilon < \rho, \tag{3}$$

respectively. In this context, $\mathcal{F}$ is a function belonging to the space $L_1[\sigma, \rho]$ and Γ denotes the Gamma function defined by

$$\Gamma(\alpha) = \int_0^{\infty} e^{-\eta}\eta^{\alpha-1}d\eta.$$

Numerous pioneering conformable fractional operators have arisen to overcome the limitations of classical fractional operators in formulating specific phenomena [17, 41]. Especially, the operator proposed by Jarad et al. as reported in [26], stands out for its versatility to extend various operators, including the Hadamard operators, offering a more comprehensive framework for obtaining complex systems and phenomena.

Definition 1.2. *[26] Let $\mathcal{F} \in L[\sigma, \rho]$. The left and right-sided conformable fractional integral operator of order $\beta \in \mathbb{C}$ and $\mathbb{R}(\beta) > 0$ and $\alpha \in (0, 1]$ are outlined as:*

$${}^{\beta}J^{\alpha}_{\sigma^+}\mathcal{F}(\upsilon) = \frac{1}{\Gamma(\beta)}\int_{\sigma}^{\upsilon} \left(\frac{(\upsilon-\sigma)^{\alpha} - (\eta-\sigma)^{\alpha}}{\alpha}\right)^{\beta-1} \frac{\mathcal{F}(\eta)}{(\eta-\sigma)^{1-\alpha}}d\eta, \quad \upsilon > \sigma \tag{4}$$

and

$${}^{\beta}J^{\alpha}_{\rho^-}\mathcal{F}(\upsilon) = \frac{1}{\Gamma(\beta)}\int_{\upsilon}^{\rho} \left(\frac{(\rho-\upsilon)^{\alpha} - (\rho-\eta)^{\alpha}}{\alpha}\right)^{\beta-1} \frac{\mathcal{F}(\eta)}{(\rho-\eta)^{1-\alpha}}d\eta, \quad \upsilon < \rho. \tag{5}$$

Remark 1.3. *If we set $\sigma = 0, \rho = 0$, and $\alpha = 1$ in (4) and (5), we find the R-L fractional integrals (2) and (3) accordingly.*

On the other hand, Set et al. [36] proposed the one-sided conformable fractional integral operator as follows:

$${}^{\beta}\mathcal{J}^{\alpha}\mathcal{F}(\upsilon) = \frac{1}{\Gamma(\beta)}\int_{0}^{\upsilon} \left(\frac{\upsilon^{\alpha} - \eta^{\alpha}}{\alpha}\right)^{\beta-1} \frac{\mathcal{F}(\eta)}{\eta^{1-\alpha}}d\eta. \tag{6}$$

In [37], authors have proposed a novel identity in the term of conformable fractional integral. By evolving this identity, they examined the new Ostrowski type inequalities for conformable fractional integral operators. Celik et al. [9] have extended Milne-type inequalities with the help of conformable fractional integrals.

Also, they discussed several function classes, such as bounded functions, Lipschitzian Functions, and functions of bounded variation. Hyder et al. [19] have contributed extensively by presenting midpoint-type inequalities. Based on previous work, Kara et al. [28] derived midpoint-type and trapezoid-type inequalities for twice-differentiable convex functions. Recently, considerable research devoted for inequalities in the framework of conformable fractional integrals such as Grüss inequality [30], Chebyshev inequality [31], Minkowski inequality [32], Hermite-Jensen-Mercer inequality [6], Simpson-type inequalities [20] and so on [4, 21, 39].
The subsequent Simpson's rules are applicable to Simpson's inequalities.

I. The subsequent formula represents Simpson's quadrature, alternatively referred to as Simpson's 1/3 rule:

$$\int_{\sigma}^{\rho} \mathcal{F}(\eta)\, d\eta \approx \frac{\rho-\sigma}{6}\left[\mathcal{F}(\sigma)+4\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)+\mathcal{F}(\rho)\right]. \tag{7}$$

II. The Newton-Cotes quadrature formula, commonly known as Simpson's second formula (also referred to as Simpson's 3/8 rule; see [11]), can be stated as:

$$\int_{\sigma}^{\rho} \mathcal{F}(\eta)\, d\eta \approx \frac{\rho-\sigma}{8}\left[\mathcal{F}(\sigma)+3\mathcal{F}\left(\frac{2\sigma+\rho}{3}\right)+3\mathcal{F}\left(\frac{\sigma+2\rho}{3}\right)+\mathcal{F}(\rho)\right]. \tag{8}$$

III. The Maclaurin rule, derived from the Maclaurin formula (as seen in [11]), is identical to the corresponding dual Simpson's 3/8 formula:

$$\int_{\sigma}^{\rho} \mathcal{F}(\eta)\, d\eta \approx \frac{\rho-\sigma}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\rho}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\rho}{6}\right)\right]. \tag{9}$$

Formulas (7), (8), and (9) hold true for any function $\mathcal{F}$ with a continuous fourth derivative over the interval $[\sigma, \rho]$.

The subsequent Newton-Cotes quadrature, frequently utilized, includes a three-point Simpson's- type inequality:

Theorem 1.4. *Let $\mathcal{F} : [\sigma, \rho] \to \mathbb{R}$ be a four times differentiable and continuous function on (σ, ρ), and let $\left\|\mathcal{F}^{(4)}\right\|_{\infty} = \sup_{\eta\in(\sigma,\rho)} \left|\mathcal{F}^{(4)}(\eta)\right| < \infty$. Then, the subsequent inequality is valid:*

$$\left|\frac{1}{6}\left[\mathcal{F}(\sigma)+4\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)+\mathcal{F}(\rho)\right]-\frac{1}{\rho-\sigma}\int_{\sigma}^{\rho}\mathcal{F}(\eta)d\eta\right| \le \frac{1}{2880}\left\|\mathcal{F}^{(4)}\right\|_{\infty}(\rho-\sigma)^4.$$

In accordance with the Simpson 3/8 inequality, the Simpson 3/8 rule is a well-known closed-type quadrature rule, and it is expressed as follows:

Theorem 1.5. *Assume that $\mathcal{F} : [\sigma, \rho] \to \mathbb{R}$ is a function that is differentiable four times and continuous on (σ, ρ), and $\left\|\mathcal{F}^{(4)}\right\|_{\infty} = \sup_{\eta\in(\sigma,\rho)} \left|\mathcal{F}^{(4)}(\eta)\right| < \infty$. Then, one observes the subsequent inequality:*

$$\left|\frac{1}{8}\left[\mathcal{F}(\sigma)+3\mathcal{F}\left(\frac{2\sigma+\rho}{3}\right)+3\mathcal{F}\left(\frac{\sigma+2\rho}{3}\right)+\mathcal{F}(\rho)\right]-\frac{1}{\rho-\sigma}\int_{\sigma}^{\rho}\mathcal{F}(\eta)d\eta\right| \le \frac{1}{6480}\left\|\mathcal{F}^{(4)}\right\|_{\infty}(\rho-\sigma)^4.$$

The Maclaurin rule, originating from the Maclaurin inequality, is equivalent to the corresponding dual Simpson's 3/8 formula:

Theorem 1.6. *Assume $\mathcal{F} : [\sigma, \rho] \to \mathbb{R}$ is a function that is differentiable four times and continuous on (σ, ρ), and let $\left\|\mathcal{F}^{(4)}\right\|_\infty = \sup_{\eta\in(\sigma,\rho)} \left|\mathcal{F}^{(4)}(\eta)\right| < \infty$. Then, the subsequent inequality is valid:*

$$\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\rho}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\rho}{6}\right)\right]-\frac{1}{\rho-\sigma}\int_\sigma^\rho \mathcal{F}(\eta)d\eta\right| \le \frac{7}{51840}\left\|\mathcal{F}^{(4)}\right\|_\infty(\rho-\sigma)^4.$$

In the framework of differentiable convex functions, authors have examined several Euler-Maclaurin-type inequalities in [23]. Dedic et al. [13] reported a sequence of inequalities by employing the Euler-Maclaurin formulas, and these results were utilized to derive error estimates for Maclaurin quadrature rules. In [24], using the Riemann-Liouville fractional integrals, Hezenci has investigated some corrected Euler-Maclaurin-type inequalities. For a deeper understanding of these specific types of inequalities, interested readers are directed to [12, 15, 33] and the citations therein.
Motivated by ongoing investigations, we establish Euler-Maclaurin-type inequalities for several function classes concerning conformable fractional integrals. The study is divided into seven sections, with the introduction being the first, which includes fundamental definitions of fractional calculus and reviews related to research in the field. Section 2, we will prove an integral equality crucial for the main findings discussed. Section 3 presents various Euler-Maclaurin-type inequalities for differentiable convex functions using conformable fractional integrals. Section 4 examines Euler-Maclaurin-type inequalities for bounded functions through fractional integrals. Section 5 establishes fractional Euler-Maclaurin-type expressions for Lipschitzian functions. In Section 6, Euler-Maclaurin-type inequalities are proved via fractional integrals of bounded variation. Finally, in Section 7, we discuss our perspectives on Euler-Maclaurin-type inequalities and their potential consequences for future research areas.

2. Principal Outcome

Let us begin with the following evaluated integrals, which will be utilized in obtaining our key findings:

$$\mathcal{A}_1(\alpha,\beta) = \int_0^{\frac{2}{3}} \left|\left(\frac{1-(1-\eta)^\alpha}{\alpha}\right)^\beta - \frac{1}{4\alpha^\beta}\right| d\eta \tag{10}$$

$$= \begin{cases} \frac{1}{2\alpha^\beta}\left[C_1 - \frac{1}{3}\right] + \frac{1}{\alpha^{\beta+1}}\left[\mathcal{B}\left(\beta+1, \frac{1}{\alpha}, 1-\frac{1}{3^\alpha}\right) - 2\mathcal{B}\left(\beta+1, \frac{1}{\alpha}, 1-\left(1-\left(\frac{1}{4}\right)^{\frac{1}{\beta}}\right)\right)\right], & 0<\beta<\frac{\ln\left(\frac{1}{4}\right)}{\ln\left(1-\frac{1}{3^\alpha}\right)}, \\ \frac{1}{6\alpha^\beta} - \frac{1}{\alpha^{\beta+1}}\mathcal{B}\left(\beta+1, \frac{1}{\alpha}, 1-\frac{1}{3^\alpha}\right), & \beta > \frac{\ln\left(\frac{1}{4}\right)}{\ln\left(1-\frac{1}{3^\alpha}\right)}, \end{cases}$$

$$\mathcal{A}_2(\alpha,\beta) = \int_0^{\frac{2}{3}} \eta\left|\left(\frac{1-(1-\eta)^\alpha}{\alpha}\right)^\beta - \frac{1}{4\alpha^\beta}\right| d\eta$$

$$= \begin{cases} \frac{1}{4\alpha^\beta}\left[C_1^2 - \frac{2}{9}\right] + \frac{1}{\alpha^{\beta+1}}\left[2\mathcal{B}\left(\beta+1, \frac{2}{\alpha}, 1-\left(1-\left(\frac{1}{4}\right)^{\frac{1}{\beta}}\right)\right) - 2\mathcal{B}\left(\beta+1, \frac{1}{\alpha}, 1-\left(1-\left(\frac{1}{4}\right)^{\frac{1}{\beta}}\right)\right)\right. & \\ \left.-\mathcal{B}\left(\beta+1, \frac{2}{\alpha}, 1-\frac{1}{3^\alpha}\right) + \mathcal{B}\left(\beta+1, \frac{1}{\alpha}, 1-\frac{1}{3^\alpha}\right)\right], & 0<\beta<\frac{\ln\left(\frac{1}{4}\right)}{\ln\left(1-\frac{1}{3^\alpha}\right)}, \\ \frac{1}{18\alpha^\beta} + \frac{1}{\alpha^{\beta+1}}\left[\mathcal{B}\left(\beta+1, \frac{2}{\alpha}, 1-\frac{1}{3^\alpha}\right) - \mathcal{B}\left(\beta+1, \frac{1}{\alpha}, 1-\frac{1}{3^\alpha}\right)\right], & \beta > \frac{\ln\left(\frac{1}{4}\right)}{\ln\left(1-\frac{1}{3^\alpha}\right)}, \end{cases}$$

$$\mathcal{A}_3(\alpha,\beta) = \int_0^{\frac{2}{3}} \left(\frac{1+\eta}{2}\right)\left|\left(\frac{1-(1-\eta)^\alpha}{\alpha}\right)^\beta - \frac{1}{4\alpha^\beta}\right| d\eta$$

$$= \begin{cases} \frac{1}{4\alpha^{\beta}}\left[\frac{1}{2}C_1^2 + C_1 - \frac{2}{9}\right] + \frac{1}{\alpha^{\beta+1}}\left[\mathcal{B}\left(\beta+1, \frac{1}{\alpha}, 1-\frac{1}{3^{\alpha}}\right) - 2\mathcal{B}\left(\beta+1, \frac{1}{\alpha}, 1-\left(1-\left(\frac{1}{4}\right)^{\frac{1}{\beta}}\right)\right)\right. \\ \left. +\mathcal{B}\left(\beta+1, \frac{2}{\alpha}, 1-\left(1-\left(\frac{1}{4}\right)^{\frac{1}{\beta}}\right)\right) - \frac{1}{2}\mathcal{B}\left(\beta+1, \frac{2}{\alpha}, 1-\frac{1}{3^{\alpha}}\right)\right], & 0 < \beta < \frac{\ln\left(\frac{1}{4}\right)}{\ln\left(1-\frac{1}{3^{\alpha}}\right)}, \\ \frac{1}{9\alpha^{\beta}} + \frac{1}{\alpha^{\beta+1}}\left[\frac{1}{2}\mathcal{B}\left(\beta+1, \frac{2}{\alpha}, 1-\frac{1}{3^{\alpha}}\right) - \mathcal{B}\left(\beta+1, \frac{1}{\alpha}, 1-\frac{1}{3^{\alpha}}\right)\right], & \beta > \frac{\ln\left(\frac{1}{4}\right)}{\ln\left(1-\frac{1}{3^{\alpha}}\right)}, \end{cases}$$

$$\mathcal{A}_4(\alpha,\beta) = \int_0^{\frac{2}{3}} \left(\frac{1-\eta}{2}\right)\left|\left(\frac{1-(1-\eta)^{\alpha}}{\alpha}\right)^{\beta} - \frac{1}{4\alpha^{\beta}}\right| d\eta$$

$$= \begin{cases} \frac{1}{2\alpha^{\beta}}\left[-\frac{1}{2}C_1^2 + C_1 - \frac{1}{9}\right] + \frac{1}{\alpha^{\beta+1}}\left[\mathcal{B}\left(\beta+1, \frac{2}{\alpha}, 1-\frac{1}{3^{\alpha}}\right) - \mathcal{B}\left(\beta+1, \frac{2}{\alpha}, 1-\left(1-\left(\frac{1}{4}\right)^{\frac{1}{\beta}}\right)\right)\right], & 0 < \beta < \frac{\ln\left(\frac{1}{4}\right)}{\ln\left(1-\frac{1}{3^{\alpha}}\right)}, \\ \frac{1}{9\alpha^{\beta}} - \frac{1}{\alpha^{\beta+1}}\mathcal{B}\left(\beta+1, \frac{2}{\alpha}, 1-\frac{1}{3^{\alpha}}\right), & \beta > \frac{\ln\left(\frac{1}{4}\right)}{\ln\left(1-\frac{1}{3^{\alpha}}\right)}, \end{cases}$$

$$\mathcal{A}_5(\alpha,\beta) = \int_{\frac{2}{3}}^{1} \left(\frac{1}{\alpha^{\beta}} - \left(\frac{1-(1-\eta)^{\alpha}}{\alpha}\right)^{\beta}\right) d\eta = \frac{1}{3\alpha^{\beta}} + \frac{1}{\alpha^{\beta+1}}\left[\mathcal{B}\left(\beta+1, \frac{1}{\alpha}, 1-\frac{1}{3^{\alpha}}\right) - \mathfrak{B}\left(\beta+1, \frac{1}{\alpha}\right)\right],$$

$$\mathcal{A}_6(\alpha,\beta) = \int_{\frac{2}{3}}^{1} \eta\left(\frac{1}{\alpha^{\beta}} - \left(\frac{1-(1-\eta)^{\alpha}}{\alpha}\right)^{\beta}\right) d\eta$$

$$= \frac{5}{18\alpha^{\beta}} + \frac{1}{\alpha^{\beta+1}}\left[\mathfrak{B}\left(\beta+1, \frac{2}{\alpha}\right) - \mathcal{B}\left(\beta+1, \frac{2}{\alpha}, 1-\frac{1}{3^{\alpha}}\right) - \mathfrak{B}\left(\beta+1, \frac{1}{\alpha}\right) + \mathcal{B}\left(\beta+1, \frac{1}{\alpha}, 1-\frac{1}{3^{\alpha}}\right)\right],$$

$$\mathcal{A}_7(\alpha,\beta) = \int_{\frac{2}{3}}^{1} \left(\frac{1+\eta}{2}\right)\left(\frac{1}{\alpha^{\beta}} - \left(\frac{1-(1-\eta)^{\alpha}}{\alpha}\right)^{\beta}\right) d\eta$$

$$= \frac{11}{36\alpha^{\beta}} + \frac{1}{\alpha^{\beta+1}}\left[\mathcal{B}\left(\beta+1, \frac{1}{\alpha}, 1-\frac{1}{3^{\alpha}}\right) - \mathfrak{B}\left(\beta+1, \frac{1}{\alpha}\right) + \frac{1}{2}\mathfrak{B}\left(\beta+1, \frac{2}{\alpha}\right) - \frac{1}{2}\mathcal{B}\left(\beta+1, \frac{2}{\alpha}, 1-\frac{1}{3^{\alpha}}\right)\right],$$

$$\mathcal{A}_8(\alpha,\beta) = \int_{\frac{2}{3}}^{1} \left(\frac{1-\eta}{2}\right)\left(\frac{1}{\alpha^{\beta}} - \left(\frac{1-(1-\eta)^{\alpha}}{\alpha}\right)^{\beta}\right) d\eta = \frac{1}{36\alpha^{\beta}} + \frac{1}{\alpha^{\beta+1}}\left[\mathcal{B}\left(\beta+1, \frac{2}{\alpha}, 1-\frac{1}{3^{\alpha}}\right) - \mathfrak{B}\left(\beta+1, \frac{2}{\alpha}\right)\right].$$

Here, $C_1 = 1 - \left(1 - \left(\frac{1}{4}\right)^{\frac{1}{\beta}}\right)^{\frac{1}{\alpha}}$, the functions $\mathfrak{B}(\cdot,\cdot)$ and $\mathcal{B}(\cdot,\cdot,\cdot)$ are the Beta function and the incomplete Beta function defined as

$$\begin{cases} \mathfrak{B}(x,y) = \int\limits_0^1 \eta^{x-1}(1-\eta)^{y-1} d\eta, \\ \mathcal{B}(x,y,r) = \int\limits_0^r \eta^{x-1}(1-\eta)^{y-1} d\eta, \end{cases}$$

for $x, y > 0$ and $r \in [0,1]$. Now, we prove integral equality in order to illustrate the main conclusions of this study.

Lemma 2.1. *Let us consider that $\mathcal{F} : [\sigma,\rho] \to \mathbb{R}$ is an differentiable mapping (σ,ρ) and $\alpha,\beta > 0$ so that $\mathcal{F}' \in L_1([\sigma,\rho])$. Then, the subsequent equality holds:*

$$\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\rho}{6}\right) + 2\mathcal{F}\left(\frac{\sigma+\rho}{2}\right) + 3\mathcal{F}\left(\frac{\sigma+5\rho}{6}\right)\right] - \frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\rho-\sigma)^{\alpha\beta}}\left[{}^{\beta}J_{\sigma+}^{\alpha}\mathcal{F}\left(\frac{\sigma+\rho}{2}\right) + {}^{\beta}J_{\rho-}^{\alpha}\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)\right]$$
$$= \frac{\alpha^{\beta}(\rho-\sigma)}{4}\left[I_1 + I_2\right].$$

Here,

$$\begin{cases} I_1 = \int\limits_0^{\frac{2}{3}} \left(\left(\frac{1-(1-\eta)^\alpha}{\alpha}\right)^\beta - \frac{1}{4\alpha^\beta}\right)\left[\mathcal{F}'\left(\frac{1+\eta}{2}\rho + \frac{1-\eta}{2}\sigma\right) - \mathcal{F}'\left(\frac{1+\eta}{2}\sigma + \frac{1-\eta}{2}\rho\right)\right]d\eta, \\ I_2 = \int\limits_{\frac{2}{3}}^{1} \left(\left(\frac{1-(1-\eta)^\alpha}{\alpha}\right)^\beta - \frac{1}{\alpha^\beta}\right)\left[\mathcal{F}'\left(\frac{1+\eta}{2}\rho + \frac{1-\eta}{2}\sigma\right) - \mathcal{F}'\left(\frac{1+\eta}{2}\sigma + \frac{1-\eta}{2}\rho\right)\right]d\eta. \end{cases}$$

Proof. Using the integration by parts, we observe

$$\begin{aligned} I_1 &= \int\limits_0^{\frac{2}{3}} \left(\left(\frac{1-(1-\eta)^\alpha}{\alpha}\right)^\beta - \frac{1}{4\alpha^\beta}\right)\left[\mathcal{F}'\left(\frac{1+\eta}{2}\rho + \frac{1-\eta}{2}\sigma\right) - \mathcal{F}'\left(\frac{1+\eta}{2}\sigma + \frac{1-\eta}{2}\rho\right)\right]d\eta \qquad (11) \\ &= \frac{2}{\rho-\sigma}\left(\left(\frac{1-(1-\eta)^\alpha}{\alpha}\right)^\beta - \frac{1}{4\alpha^\beta}\right)\left[\mathcal{F}\left(\frac{1+\eta}{2}\rho + \frac{1-\eta}{2}\sigma\right) + \mathcal{F}\left(\frac{1+\eta}{2}\sigma + \frac{1-\eta}{2}\rho\right)\right]\Bigg|_0^{\frac{2}{3}} \\ &\quad - \frac{2\beta}{\rho-\sigma}\int_0^{\frac{2}{3}} \left(\frac{1-(1-\eta)^\alpha}{\alpha}\right)^{\beta-1}(1-\eta)^{\alpha-1}\left[\mathcal{F}\left(\frac{1+\eta}{2}\rho + \frac{1-\eta}{2}\sigma\right) + \mathcal{F}\left(\frac{1+\eta}{2}\sigma + \frac{1-\eta}{2}\rho\right)\right]d\eta \\ &= \frac{2}{\rho-\sigma}\left(\left(\frac{1-\frac{1}{3^\alpha}}{\alpha}\right)^\beta - \frac{1}{4\alpha^\beta}\right)\left[\mathcal{F}\left(\frac{5\rho+\sigma}{6}\right) + \mathcal{F}\left(\frac{5\sigma+\rho}{6}\right)\right] + \frac{4}{4\alpha^\beta(\rho-\sigma)}\mathcal{F}\left(\frac{\sigma+\rho}{2}\right) \\ &\quad - \frac{2\beta}{\rho-\sigma}\int_0^{\frac{2}{3}} \left(\frac{1-(1-\eta)^\alpha}{\alpha}\right)^{\beta-1}(1-\eta)^{\alpha-1}\left[\mathcal{F}\left(\frac{1+\eta}{2}\rho + \frac{1-\eta}{2}\sigma\right) + \mathcal{F}\left(\frac{1+\eta}{2}\sigma + \frac{1-\eta}{2}\rho\right)\right]d\eta. \end{aligned}$$

Similarly,

$$\begin{aligned} I_2 &= -\frac{2}{\rho-\sigma}\left(\left(\frac{1-\frac{1}{3^\alpha}}{\alpha}\right)^\beta - \frac{1}{\alpha^\beta}\right)\left[\mathcal{F}\left(\frac{5\rho+\sigma}{6}\right) + \mathcal{F}\left(\frac{5\sigma+\rho}{6}\right)\right] \qquad (12) \\ &\quad - \frac{2\beta}{\rho-\sigma}\int_{\frac{2}{3}}^1 \left(\frac{1-(1-\eta)^\alpha}{\alpha}\right)^{\beta-1}(1-\eta)^{\alpha-1}\left[\mathcal{F}\left(\frac{1+\eta}{2}\rho + \frac{1-\eta}{2}\sigma\right) + \mathcal{F}\left(\frac{1+\eta}{2}\sigma + \frac{1-\eta}{2}\rho\right)\right]d\eta. \end{aligned}$$

By substituting (11) and (12), then we achieve

$$\begin{aligned} I_1 + I_2 &= \frac{2}{4(\rho-\sigma)\alpha^\beta}\left[3\mathcal{F}\left(\frac{5\rho+\sigma}{6}\right) + 2\mathcal{F}\left(\frac{\sigma+\rho}{2}\right) + 3\mathcal{F}\left(\frac{5\sigma+\rho}{6}\right)\right] \qquad (13) \\ &\quad - \frac{2\beta}{\rho-\sigma}\left[\int_0^{\frac{2}{3}} \left(\frac{1-(1-\eta)^\alpha}{\alpha}\right)^{\beta-1}(1-\eta)^{\alpha-1}\left[\mathcal{F}\left(\frac{1+\eta}{2}\rho + \frac{1-\eta}{2}\sigma\right) + \mathcal{F}\left(\frac{1+\eta}{2}\sigma + \frac{1-\eta}{2}\rho\right)\right]d\eta\right. \\ &\quad \left. + \int_{\frac{2}{3}}^1 \left(\frac{1-(1-\eta)^\alpha}{\alpha}\right)^{\beta-1}(1-\eta)^{\alpha-1}\left[\mathcal{F}\left(\frac{1+\eta}{2}\rho + \frac{1-\eta}{2}\sigma\right) + \mathcal{F}\left(\frac{1+\eta}{2}\sigma + \frac{1-\eta}{2}\rho\right)\right]d\eta\right]. \end{aligned}$$

If we use the change of the variable $\upsilon = \frac{1+\eta}{2}\rho + \frac{1-\eta}{2}\sigma$ and $\upsilon = \frac{1+\eta}{2}\sigma + \frac{1-\eta}{2}\rho$ for $\eta \in [0,1]$, then the equality (13) can be rewritten as follows

$$\begin{aligned} I_1 + I_2 &= \frac{1}{2\alpha^\beta(\rho-\sigma)}\left[3\mathcal{F}\left(\frac{5\sigma+\rho}{6}\right) + 2\mathcal{F}\left(\frac{\sigma+\rho}{2}\right) + 3\mathcal{F}\left(\frac{\sigma+5\rho}{6}\right)\right] \qquad (14) \\ &\quad - \frac{2^{\alpha\beta+1}\Gamma(\beta+1)}{(\rho-\sigma)^{\alpha\beta+1}}\left[{}^\beta J^\alpha_{\sigma+}\mathcal{F}\left(\frac{\sigma+\rho}{2}\right) + {}^\beta J^\alpha_{\rho-}\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)\right]. \end{aligned}$$

Accordingly, multiplying both sides of (14) by $\frac{\alpha^\beta(\rho-\sigma)}{4}$, concludes the proof of Lemma 2.1. □

3. Convex functions: Fractional Euler-Maclaurin-type inequalities

Theorem 3.1. *Assume the conditions stipulated in Lemma 2.1 holds. If the function $|\mathcal{F}'|$ is convex on $[\sigma,\rho]$, then the subsequent inequality is valid:*

$$\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\rho}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\rho}{6}\right)\right]-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\rho-\sigma)^{\alpha\beta}}\left[{}^{\beta}J^{\alpha}_{\sigma+}\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)+{}^{\beta}J^{\alpha}_{\rho-}\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)\right]\right| \tag{15}$$
$$\leq \frac{\alpha^{\beta}(\rho-\sigma)}{4}\left(\mathcal{A}_1(\alpha,\beta)+\mathcal{A}_5(\alpha,\beta)\right)\left[|\mathcal{F}'(\sigma)|+\left|\mathcal{F}'(\rho)\right|\right],$$

where $\mathcal{A}_1(\alpha,\beta)$ and $\mathcal{A}_5(\alpha,\beta)$ are defined as in (10).

Proof. If we consider the absolute value in Lemma 2.1, we can directly get

$$\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\rho}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\rho}{6}\right)\right]-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\rho-\sigma)^{\alpha\beta}}\left[{}^{\beta}J^{\alpha}_{\sigma+}\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)+{}^{\beta}J^{\alpha}_{\rho-}\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)\right]\right| \tag{16}$$
$$\leq \frac{\alpha^{\beta}(\rho-\sigma)}{4}\left\{\int_0^{\frac{2}{3}}\left|\left(\frac{1-(1-\eta)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{4\alpha^{\beta}}\right|\left[\left|\mathcal{F}'\left(\frac{1+\eta}{2}\rho+\frac{1-\eta}{2}\sigma\right)\right|+\left|\mathcal{F}'\left(\frac{1+\eta}{2}\sigma+\frac{1-\eta}{2}\rho\right)\right|\right]d\eta\right.$$
$$\left.+\int_{\frac{2}{3}}^{1}\left|\left(\frac{1-(1-\eta)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{\alpha^{\beta}}\right|\left[\left|\mathcal{F}'\left(\frac{1+\eta}{2}\rho+\frac{1-\eta}{2}\sigma\right)\right|+\left|\mathcal{F}'\left(\frac{1+\eta}{2}\sigma+\frac{1-\eta}{2}\rho\right)\right|\right]d\eta\right\}.$$

Given that $|\mathcal{F}'|$ is convex, it becomes

$$\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\rho}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\rho}{6}\right)\right]-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\rho-\sigma)^{\alpha\beta}}\left[{}^{\beta}J^{\alpha}_{\sigma+}\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)+{}^{\beta}J^{\alpha}_{\rho-}\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)\right]$$
$$\leq \frac{\alpha^{\beta}(\rho-\sigma)}{4}\left\{\int_0^{\frac{2}{3}}\left|\left(\frac{1-(1-\eta)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{4\alpha^{\beta}}\right|\left[\frac{1+\eta}{2}\left|\mathcal{F}'(\rho)\right|+\frac{1-\eta}{2}|\mathcal{F}'(\sigma)|+\frac{1+\eta}{2}|\mathcal{F}'(\sigma)|+\frac{1-\eta}{2}\left|\mathcal{F}'(\rho)\right|\right]d\eta\right.$$
$$\left.+\int_{\frac{2}{3}}^{1}\left|\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\eta)^{\alpha}}{\alpha}\right)^{\beta}\right|\left[\frac{1+\eta}{2}\left|\mathcal{F}'(\rho)\right|+\frac{1-\eta}{2}|\mathcal{F}'(\sigma)|+\frac{1+\eta}{2}|\mathcal{F}'(\sigma)|+\frac{1-\eta}{2}\left|\mathcal{F}'(\rho)\right|\right]d\eta\right\}$$
$$=\frac{\alpha^{\beta}(\rho-\sigma)}{4}\left(\mathcal{A}_1(\alpha,\beta)+\mathcal{A}_5(\alpha,\beta)\right)\left[|\mathcal{F}'(\sigma)|+\left|\mathcal{F}'(\rho)\right|\right].$$

Therefore, the demonstration is completed. □

Remark 3.2. *If we take $\alpha=1$ in Theorem 3.1 leads to [25, Theorem 4].*

Remark 3.3. *If we choose $\alpha=\beta=1$ in Theorem 3.1, then we can obtain Euler-Maclaurin-type inequality*

$$\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\rho}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\rho}{6}\right)\right]-\frac{1}{\rho-\sigma}\int_{\sigma}^{\rho}\mathcal{F}(\eta)\,d\eta\right|\leq\frac{25(\rho-\sigma)}{576}\left[|\mathcal{F}'(\sigma)|+\left|\mathcal{F}'(\rho)\right|\right],$$

which is established in paper [23, Corollary 1].

Theorem 3.4. *Assume the conditions stipulated in Lemma 2.1 holds. If the function* $|\mathcal{F}'|^q$ *,* $q > 1$ *is convex on* $[\sigma, \rho]$*, then the subsequent inequality is valid:*

$$\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\rho}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\rho}{6}\right)\right]-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\rho-\sigma)^{\alpha\beta}}\left[{}^{\beta}J^{\alpha}_{\sigma+}\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)+{}^{\beta}J^{\alpha}_{\rho-}\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)\right]\right| \tag{17}$$

$$\leq \frac{\alpha^{\beta}(\rho-\sigma)}{4}\left\{\left(\int_0^{\frac{2}{3}}\left|\left(\frac{1-(1-\eta)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{4\alpha^{\beta}}\right|^{p}d\eta\right)^{\frac{1}{p}}\left[\left(\frac{4\left|\mathcal{F}'(\rho)\right|^{q}+2\left|\mathcal{F}'(\sigma)\right|^{q}}{9}\right)^{\frac{1}{q}}+\left(\frac{4\left|\mathcal{F}'(\sigma)\right|^{q}+2\left|\mathcal{F}'(\rho)\right|^{q}}{9}\right)^{\frac{1}{q}}\right]\right.$$

$$\left.+\left(\int_{\frac{2}{3}}^{1}\left|\left(\frac{1-(1-\eta)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{\alpha^{\beta}}\right|^{p}d\eta\right)^{\frac{1}{p}}\left[\left(\frac{11\left|\mathcal{F}'(\rho)\right|^{q}+\left|\mathcal{F}'(\sigma)\right|^{q}}{36}\right)^{\frac{1}{q}}+\left(\frac{11\left|\mathcal{F}'(\sigma)\right|^{q}+\left|\mathcal{F}'(\rho)\right|^{q}}{36}\right)^{\frac{1}{q}}\right]\right\}.$$

Here, $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Through utilizing Hölder's inequality to (16), we achieve

$$\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\rho}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\rho}{6}\right)\right]-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\rho-\sigma)^{\alpha\beta}}\left[{}^{\beta}J^{\alpha}_{\sigma+}\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)+{}^{\beta}J^{\alpha}_{\rho-}\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)\right]\right|$$

$$\leq \frac{\alpha^{\beta}(\rho-\sigma)}{4}\left\{\left(\int_0^{\frac{2}{3}}\left|\left(\frac{1-(1-\eta)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{4\alpha^{\beta}}\right|^{p}d\eta\right)^{\frac{1}{p}}\left(\int_0^{\frac{2}{3}}\left|\mathcal{F}'\left(\frac{1+\eta}{2}\rho+\frac{1-\eta}{2}\sigma\right)\right|^{q}d\eta\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^{\frac{2}{3}}\left|\left(\frac{1-(1-\eta)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{4\alpha^{\beta}}\right|^{p}d\eta\right)^{\frac{1}{p}}\left(\int_0^{\frac{2}{3}}\left|\mathcal{F}'\left(\frac{1+\eta}{2}\sigma+\frac{1-\eta}{2}\rho\right)\right|^{q}d\eta\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{2}{3}}^{1}\left|\left(\frac{1-(1-\eta)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{\alpha^{\beta}}\right|^{p}d\eta\right)^{\frac{1}{p}}\left(\int_{\frac{2}{3}}^{1}\left|\mathcal{F}'\left(\frac{1+\eta}{2}\rho+\frac{1-\eta}{2}\sigma\right)\right|^{q}d\eta\right)^{\frac{1}{q}}$$

$$\left.+\left(\int_{\frac{2}{3}}^{1}\left|\left(\frac{1-(1-\eta)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{\alpha^{\beta}}\right|^{p}d\eta\right)^{\frac{1}{p}}\left(\int_{\frac{2}{3}}^{1}\left|\mathcal{F}'\left(\frac{1+\eta}{2}\sigma+\frac{1-\eta}{2}\rho\right)\right|^{q}d\eta\right)^{\frac{1}{q}}\right\}.$$

By employing the convexity $|\mathcal{F}'|^q$, it yields

$$\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\rho}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\rho}{6}\right)\right]-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\rho-\sigma)^{\alpha\beta}}\left[{}^{\beta}J^{\alpha}_{\sigma+}\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)+{}^{\beta}J^{\alpha}_{\rho-}\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)\right]\right|$$

$$\leq \frac{\alpha^{\beta}(\rho-\sigma)}{4}\left\{\left(\int_0^{\frac{2}{3}}\left|\left(\frac{1-(1-\eta)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{4\alpha^{\beta}}\right|^{p}d\eta\right)^{\frac{1}{p}}\left(\int_0^{\frac{2}{3}}\left(\frac{1+\eta}{2}\left|\mathcal{F}'(\rho)\right|^{q}+\frac{1-\eta}{2}\left|\mathcal{F}'(\sigma)\right|^{q}\right)d\eta\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^{\frac{2}{3}}\left|\left(\frac{1-(1-\eta)^\alpha}{\alpha}\right)^\beta-\frac{1}{4\alpha^\beta}\right|^p d\eta\right)^{\frac{1}{p}}\left(\int_0^{\frac{2}{3}}\left(\frac{1+\eta}{2}\left|\mathcal{F}'(\sigma)\right|^q+\frac{1-\eta}{2}\left|\mathcal{F}'(\rho)\right|^q\right)d\eta\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{2}{3}}^1\left|\left(\frac{1-(1-\eta)^\alpha}{\alpha}\right)^\beta-\frac{1}{\alpha^\beta}\right|^p d\eta\right)^{\frac{1}{p}}\left(\int_{\frac{2}{3}}^1\left(\frac{1+\eta}{2}\left|\mathcal{F}'(\rho)\right|^q+\frac{1-\eta}{2}\left|\mathcal{F}'(\sigma)\right|^q\right)d\eta\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{2}{3}}^1\left|\left(\frac{1-(1-\eta)^\alpha}{\alpha}\right)^\beta-\frac{1}{\alpha^\beta}\right|^p d\eta\right)^{\frac{1}{p}}\left(\int_{\frac{2}{3}}^1\left(\frac{1+\eta}{2}\left|\mathcal{F}'(\sigma)\right|^q+\frac{1-\eta}{2}\left|\mathcal{F}'(\rho)\right|^q\right)d\eta\right)^{\frac{1}{q}}\Bigg\}$$

$$=\frac{\alpha^\beta(\rho-\sigma)}{4}\left\{\left(\int_0^{\frac{2}{3}}\left|\left(\frac{1-(1-\eta)^\alpha}{\alpha}\right)^\beta-\frac{1}{4\alpha^\beta}\right|^p d\eta\right)^{\frac{1}{p}}\left[\left(\frac{4\left|\mathcal{F}'(\rho)\right|^q+2\left|\mathcal{F}'(\sigma)\right|^q}{9}\right)^{\frac{1}{q}}+\left(\frac{4\left|\mathcal{F}'(\sigma)\right|^q+2\left|\mathcal{F}'(\rho)\right|^q}{9}\right)^{\frac{1}{q}}\right]\right.$$

$$\left.+\left(\int_{\frac{2}{3}}^1\left|\left(\frac{1-(1-\eta)^\alpha}{\alpha}\right)^\beta-\frac{1}{\alpha^\beta}\right|^p d\eta\right)^{\frac{1}{p}}\left[\left(\frac{11\left|\mathcal{F}'(\rho)\right|^q+\left|\mathcal{F}'(\sigma)\right|^q}{36}\right)^{\frac{1}{q}}+\left(\frac{11\left|\mathcal{F}'(\sigma)\right|^q+\left|\mathcal{F}'(\rho)\right|^q}{36}\right)^{\frac{1}{q}}\right]\right\}.$$

Hence, the proof is finished. □

Remark 3.5. *If we take $\alpha = 1$ in Theorem 3.4 leads to [25, Theorem 5].*

Remark 3.6. *If we choose $\alpha = \beta = 1$ in Theorem 3.4 leads to [25, Corollary 1].*

Theorem 3.7. *Assume the conditions stipulated in Lemma 2.1 holds. If the function $|\mathcal{F}'|^q$, $q \geq 1$ is convex on $[\sigma,\rho]$, then the subsequent inequality is valid:*

$$\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\rho}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\rho}{6}\right)\right]-\frac{2^{\alpha\beta-1}\alpha^\beta\Gamma(\beta+1)}{(\rho-\sigma)^{\alpha\beta}}\left[{}^\beta J^\alpha_{\sigma+}\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)+{}^\beta J^\alpha_{\rho-}\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)\right]\right| \tag{18}$$

$$\leq\frac{\alpha^\beta(\rho-\sigma)}{4}\left\{(\mathcal{A}_1(\alpha,\beta))^{1-\frac{1}{q}}\left[\left(\mathcal{A}_3(\alpha,\beta)\left|\mathcal{F}'(\rho)\right|^q+\mathcal{A}_4(\alpha,\beta)\left|\mathcal{F}'(\sigma)\right|^q\right)^{\frac{1}{q}}+\left(\mathcal{A}_3(\alpha,\beta)\left|\mathcal{F}'(\sigma)\right|^q+\mathcal{A}_4(\alpha,\beta)\left|\mathcal{F}'(\rho)\right|^q\right)^{\frac{1}{q}}\right]\right.$$

$$\left.+(\mathcal{A}_5(\alpha,\beta))^{1-\frac{1}{q}}\left[\left(\mathcal{A}_7(\alpha,\beta)\left|\mathcal{F}'(\rho)\right|^q+\mathcal{A}_8(\alpha,\beta)\left|\mathcal{F}'(\sigma)\right|^q\right)^{\frac{1}{q}}+\left(\mathcal{A}_7(\alpha,\beta)\left|\mathcal{F}'(\sigma)\right|^q+\mathcal{A}_8(\alpha,\beta)\left|\mathcal{F}'(\rho)\right|^q\right)^{\frac{1}{q}}\right]\right\}.$$

Here, $\mathcal{A}_1(\alpha,\beta)-\mathcal{A}_8(\alpha,\beta)$ are defined as in (10).

Proof. With the help of the Power-Mean inequality in (16), we acquire

$$\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\rho}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\rho}{6}\right)\right]-\frac{2^{\alpha\beta-1}\alpha^\beta\Gamma(\beta+1)}{(\rho-\sigma)^{\alpha\beta}}\left[{}^\beta J^\alpha_{\sigma+}\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)+{}^\beta J^\alpha_{\rho-}\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)\right]\right|$$

$$\leq\frac{\alpha^\beta(\rho-\sigma)}{4}\left\{\left(\int_0^{\frac{2}{3}}\left|\left(\frac{1-(1-\eta)^\alpha}{\alpha}\right)^\beta-\frac{1}{4\alpha^\beta}\right|d\eta\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{2}{3}}\left|\left(\frac{1-(1-\eta)^\alpha}{\alpha}\right)^\beta-\frac{1}{4\alpha^\beta}\right|\left|\mathcal{F}'\left(\frac{1+\eta}{2}\rho+\frac{1-\eta}{2}\sigma\right)\right|^q d\eta\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^{\frac{2}{3}}\left|\left(\frac{1-(1-\eta)^\alpha}{\alpha}\right)^\beta-\frac{1}{4\alpha^\beta}\right|d\eta\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{2}{3}}\left|\left(\frac{1-(1-\eta)^\alpha}{\alpha}\right)^\beta-\frac{1}{4\alpha^\beta}\right|\left|\mathcal{F}'\left(\frac{1+\eta}{2}\sigma+\frac{1-\eta}{2}\rho\right)\right|^q d\eta\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{2}{3}}^{1}\left(\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\eta)^{\alpha}}{\alpha}\right)^{\beta}\right)d\eta\right)^{1-\frac{1}{q}}\left(\int_{\frac{2}{3}}^{1}\left(\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\eta)^{\alpha}}{\alpha}\right)^{\beta}\right)\left|\mathcal{F}'\left(\frac{1+\eta}{2}\rho+\frac{1-\eta}{2}\sigma\right)\right|^{q}d\eta\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{2}{3}}^{1}\left(\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\eta)^{\alpha}}{\alpha}\right)^{\beta}\right)d\eta\right)^{1-\frac{1}{q}}\left(\int_{\frac{2}{3}}^{1}\left(\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\eta)^{\alpha}}{\alpha}\right)^{\beta}\right)\left|\mathcal{F}'\left(\frac{1+\eta}{2}\sigma+\frac{1-\eta}{2}\rho\right)\right|^{q}d\eta\right)^{\frac{1}{q}}\Bigg\}.$$

Utilizing the convexity of $|\mathcal{F}'|^q$, it gives

$$\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\rho}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\rho}{6}\right)\right]-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\rho-\sigma)^{\alpha\beta}}\left[{}^{\beta}J^{\alpha}_{\sigma+}\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)+{}^{\beta}J^{\alpha}_{\rho-}\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)\right]\right|$$

$$\leq\frac{\alpha^{\beta}(\rho-\sigma)}{4}\Bigg\{\left(\int_{0}^{\frac{2}{3}}\left|\left(\frac{1-(1-\eta)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{4\alpha^{\beta}}\right|d\eta\right)^{1-\frac{1}{q}}\left(\int_{0}^{\frac{2}{3}}\left|\left(\frac{1-(1-\eta)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{4\alpha^{\beta}}\right|\left[\frac{1+\eta}{2}\left|\mathcal{F}'(\rho)\right|^{q}+\frac{1-\eta}{2}\left|\mathcal{F}'(\sigma)\right|^{q}\right]d\eta\right)^{\frac{1}{q}}$$

$$+\left(\int_{0}^{\frac{2}{3}}\left|\left(\frac{1-(1-\eta)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{4\alpha^{\beta}}\right|d\eta\right)^{1-\frac{1}{q}}\left(\int_{0}^{\frac{2}{3}}\left|\left(\frac{1-(1-\eta)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{4\alpha^{\beta}}\right|\left[\frac{1+\eta}{2}\left|\mathcal{F}'(\sigma)\right|^{q}+\frac{1-\eta}{2}\left|\mathcal{F}'(\rho)\right|^{q}\right]d\eta\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{2}{3}}^{1}\left(\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\eta)^{\alpha}}{\alpha}\right)^{\beta}\right)d\eta\right)^{1-\frac{1}{q}}\left(\int_{\frac{2}{3}}^{1}\left(\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\eta)^{\alpha}}{\alpha}\right)^{\beta}\right)\left[\frac{1+\eta}{2}\left|\mathcal{F}'(\rho)\right|^{q}+\frac{1-\eta}{2}\left|\mathcal{F}'(\sigma)\right|^{q}\right]d\eta\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{2}{3}}^{1}\left(\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\eta)^{\alpha}}{\alpha}\right)^{\beta}\right)d\eta\right)^{1-\frac{1}{q}}\left(\int_{\frac{2}{3}}^{1}\left(\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\eta)^{\alpha}}{\alpha}\right)^{\beta}\right)\left[\frac{1+\eta}{2}\left|\mathcal{F}'(\sigma)\right|^{q}+\frac{1-\eta}{2}\left|\mathcal{F}'(\rho)\right|^{q}\right]d\eta\right)^{\frac{1}{q}}\Bigg\}$$

$$=\frac{\alpha^{\beta}(\rho-\sigma)}{4}\Big\{\left(\mathcal{A}_1(\alpha,\beta)\right)^{1-\frac{1}{q}}\left[\left(\mathcal{A}_3(\alpha,\beta)\left|\mathcal{F}'(\rho)\right|^{q}+\mathcal{A}_4(\alpha,\beta)\left|\mathcal{F}'(\sigma)\right|^{q}\right)^{\frac{1}{q}}+\left(\mathcal{A}_3(\alpha,\beta)\left|\mathcal{F}'(\sigma)\right|^{q}+\mathcal{A}_4(\alpha,\beta)\left|\mathcal{F}'(\rho)\right|^{q}\right)^{\frac{1}{q}}\right]$$

$$+\left(\mathcal{A}_5(\alpha,\beta)\right)^{1-\frac{1}{q}}\left[\left(\mathcal{A}_7(\alpha,\beta)\left|\mathcal{F}'(\rho)\right|^{q}+\mathcal{A}_8(\alpha,\beta)\left|\mathcal{F}'(\sigma)\right|^{q}\right)^{\frac{1}{q}}+\left(\mathcal{A}_7(\alpha,\beta)\left|\mathcal{F}'(\sigma)\right|^{q}+\mathcal{A}_8(\alpha,\beta)\left|\mathcal{F}'(\rho)\right|^{q}\right)^{\frac{1}{q}}\right]\Big\}.$$

□

Remark 3.8. *If we take $\alpha = 1$ in Theorem 3.7 leads to [25, Theorem 6].*

Remark 3.9. *If we choose $\alpha = \beta = 1$ in Theorem 3.7 leads to [23, Corollary 3].*

4. Bounded functions: Euler-Maclaurin-type inequalities with conformable fractional integrals

In this section, we deal with some Euler-Maclaurin-type inequalities for bounded functions via conformable fractional integrals.

Theorem 4.1. *Suppose that the conditions of Lemma 2.1 holds. If there exist $m, M \in \mathbb{R}$ such that $m \leq \mathcal{F}'(\eta) \leq M$ for $\eta \in [\sigma, \rho]$, then it follows:*

$$\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\rho}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\rho}{6}\right)\right]-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\rho-\sigma)^{\alpha\beta}}\left[{}^{\beta}J^{\alpha}_{\sigma+}\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)+{}^{\beta}J^{\alpha}_{\rho-}\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)\right]\right| \tag{19}$$

$$\leq \frac{\alpha^{\beta}(\rho-\sigma)}{4}\left[\mathcal{A}_1(\alpha,\beta)+\mathcal{A}_5(\alpha,\beta)\right](M-m),$$

where $\mathcal{A}_1(\alpha,\beta)$ *and* $\mathcal{A}_5(\alpha,\beta)$ *are defined as in (10).*

Proof. By using Lemma 2.1, we have

$$\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\rho}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\rho}{6}\right)\right]-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\rho-\sigma)^{\alpha\beta}}\left[{}^{\beta}J^{\alpha}_{\sigma+}\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)+{}^{\beta}J^{\alpha}_{\rho-}\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)\right] \tag{20}$$

$$\leq \frac{\alpha^{\beta}(\rho-\sigma)}{4}\left\{\int_0^{\frac{2}{3}}\left(\left(\frac{1-(1-\eta)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{4\alpha^{\beta}}\right)\left[\mathcal{F}'\left(\frac{1+\eta}{2}\rho+\frac{1-\eta}{2}\sigma\right)-\frac{m+M}{2}\right]d\eta\right.$$

$$+\int_0^{\frac{2}{3}}\left(\left(\frac{1-(1-\eta)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{4\alpha^{\beta}}\right)\left[\frac{m+M}{2}-\mathcal{F}'\left(\frac{1+\eta}{2}\sigma+\frac{1-\eta}{2}\rho\right)\right]d\eta$$

$$+\int_{\frac{2}{3}}^{1}\left(\left(\frac{1-(1-\eta)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{\alpha^{\beta}}\right)\left[\mathcal{F}'\left(\frac{1+\eta}{2}\rho+\frac{1-\eta}{2}\sigma\right)-\frac{m+M}{2}\right]d\eta$$

$$\left.+\int_{\frac{2}{3}}^{1}\left(\left(\frac{1-(1-\eta)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{\alpha^{\beta}}\right)\left[\frac{m+M}{2}-\mathcal{F}'\left(\frac{1+\eta}{2}\sigma+\frac{1-\eta}{2}\rho\right)\right]d\eta\right\}.$$

Through the absolute value of (20), we attain

$$\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\rho}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\rho}{6}\right)\right]-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\rho-\sigma)^{\alpha\beta}}\left[{}^{\beta}J^{\alpha}_{\sigma+}\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)+{}^{\beta}J^{\alpha}_{\rho-}\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)\right]\right| \tag{21}$$

$$\leq \frac{\alpha^{\beta}(\rho-\sigma)}{4}\left\{\int_0^{\frac{2}{3}}\left|\left(\frac{1-(1-\eta)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{4\alpha^{\beta}}\right|\left|\mathcal{F}'\left(\frac{1+\eta}{2}\rho+\frac{1-\eta}{2}\sigma\right)-\frac{m+M}{2}\right|d\eta\right.$$

$$+\int_0^{\frac{2}{3}}\left|\left(\frac{1-(1-\eta)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{4\alpha^{\beta}}\right|\left|\frac{m+M}{2}-\mathcal{F}'\left(\frac{1+\eta}{2}\sigma+\frac{1-\eta}{2}\rho\right)\right|d\eta$$

$$+\int_{\frac{2}{3}}^{1}\left|\left(\frac{1-(1-\eta)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{\alpha^{\beta}}\right|\left|\mathcal{F}'\left(\frac{1+\eta}{2}\rho+\frac{1-\eta}{2}\sigma\right)-\frac{m+M}{2}\right|d\eta$$

$$\left.+\int_{\frac{2}{3}}^{1}\left|\left(\frac{1-(1-\eta)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{\alpha^{\beta}}\right|\left|\frac{m+M}{2}-\mathcal{F}'\left(\frac{1+\eta}{2}\sigma+\frac{1-\eta}{2}\rho\right)\right|d\eta\right\}.$$

It is known that $m \leq \mathcal{F}'(\eta) \leq M$ for $\eta \in [\sigma,\rho]$. Then, we observe

$$\left|\mathcal{F}'\left(\frac{1+\eta}{2}\rho+\frac{1-\eta}{2}\sigma\right)-\frac{m+M}{2}\right| \leq \frac{M-m}{2}, \tag{22}$$

and

$$\left|\frac{m+M}{2}-\mathcal{F}'\left(\frac{1+\eta}{2}\sigma+\frac{1-\eta}{2}\rho\right)\right|\leq\frac{M-m}{2}. \tag{23}$$

With the help of the (22) and (23), we achieve

$$\begin{aligned}&\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\rho}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\rho}{6}\right)\right]-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\rho-\sigma)^{\alpha\beta}}\left[{}^{\beta}J^{\alpha}_{\sigma+}\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)+{}^{\beta}J^{\alpha}_{\rho-}\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)\right]\right|\\&\leq\frac{\alpha^{\beta}(\rho-\sigma)}{4}(M-m)\left\{\int_0^{\frac{2}{3}}\left|\left(\frac{1-(1-\eta)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{4\alpha^{\beta}}\right|d\eta+\int_{\frac{2}{3}}^{1}\left(\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\eta)^{\alpha}}{\alpha}\right)^{\beta}\right)d\eta\right\}\\&=\frac{\alpha^{\beta}(\rho-\sigma)}{4}\left[\mathcal{A}_1(\alpha,\beta)+\mathcal{A}_5(\alpha,\beta)\right](M-m).\end{aligned}$$

□

Remark 4.2. *If we take $\alpha=1$ in Theorem 4.1 leads to [25, Theorem 7].*

Remark 4.3. *If we choose $\alpha=\beta=1$ in Theorem 4.1 leads to [25, Corollary 2].*

5. Lipschitzian functions: Fractional Euler-Maclaurin-type inequalities

In this section, we give some fractional Euler-Maclaurin-type inequalities for Lipschitzian functions.

Theorem 5.1. *Assume the conditions stipulated in Lemma 2.1 holds. If $\mathcal{F}'$ is an L-Lipschitzian function on $[\sigma,\rho]$, then the subsequent inequality is valid:*

$$\begin{aligned}&\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\rho}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\rho}{6}\right)\right]-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\rho-\sigma)^{\alpha\beta}}\left[{}^{\beta}J^{\alpha}_{\sigma+}\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)+{}^{\beta}J^{\alpha}_{\rho-}\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)\right]\right|\\&\leq\frac{\alpha^{\beta}(\rho-\sigma)^2}{4}\left[\mathcal{A}_2(\alpha,\beta)+\mathcal{A}_6(\alpha,\beta)\right]L,\end{aligned} \tag{24}$$

where $\mathcal{A}_2(\alpha,\beta)$ and $\mathcal{A}_6(\alpha,\beta)$ are defined as in (10).

Proof. By utilizing Lemma 2.1, we attain

$$\begin{aligned}&\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\rho}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\rho}{6}\right)\right]-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\rho-\sigma)^{\alpha\beta}}\left[{}^{\beta}J^{\alpha}_{\sigma+}\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)+{}^{\beta}J^{\alpha}_{\rho-}\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)\right]\right|\\&\leq\frac{\alpha^{\beta}(\rho-\sigma)}{4}\left\{\int_0^{\frac{2}{3}}\left|\left(\frac{1-(1-\eta)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{4\alpha^{\beta}}\right|\left|\mathcal{F}'\left(\frac{1+\eta}{2}\rho+\frac{1-\eta}{2}\sigma\right)-\mathcal{F}'\left(\frac{1+\eta}{2}\sigma+\frac{1-\eta}{2}\rho\right)\right|d\eta\right.\\&\quad\left.+\int_{\frac{2}{3}}^{1}\left|\left(\frac{1-(1-\eta)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{\alpha^{\beta}}\right|\left|\mathcal{F}'\left(\frac{1+\eta}{2}\rho+\frac{1-\eta}{2}\sigma\right)-\mathcal{F}'\left(\frac{1+\eta}{2}\sigma+\frac{1-\eta}{2}\rho\right)\right|d\eta\right\}.\end{aligned}$$

As $|\mathcal{F}'|$ is L-Lipschitzian function, we can conclude

$$\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\rho}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\rho}{6}\right)\right]-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\rho-\sigma)^{\alpha\beta}}\left[{}^{\beta}J^{\alpha}_{\sigma+}\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)+{}^{\beta}J^{\alpha}_{\rho-}\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)\right]\right|$$

$$\leq \frac{\alpha^{\beta}(\rho-\sigma)}{4}\left\{\int_{0}^{\frac{2}{3}}\left|\left(\frac{1-(1-\eta)^{\alpha}}{\alpha}\right)^{\beta}-\frac{1}{4\alpha^{\beta}}\right|L\eta(\rho-\sigma)d\eta+\int_{\frac{2}{3}}^{1}\left(\frac{1}{\alpha^{\beta}}-\left(\frac{1-(1-\eta)^{\alpha}}{\alpha}\right)^{\beta}\right)L\eta(\rho-\sigma)d\eta\right\}$$

$$= \frac{\alpha^{\beta}(\rho-\sigma)^{2}}{4}\left[\mathcal{A}_{2}(\alpha,\beta)+\mathcal{A}_{6}(\alpha,\beta)\right]L.$$

□

Remark 5.2. *If we take $\alpha = 1$ in Theorem 5.1 leads to [25, Theorem 8].*

Remark 5.3. *If we choose $\alpha = \beta = 1$ in Theorem 5.1 reduces to [25, Corollary 5].*

6. Functions of bounded variation: Euler-Maclaurin-type inequalities via fractional integrals

In this section, we represent Euler-Maclaurin-type inequalities by fractional integrals of bounded variation.

Theorem 6.1. *Consider that $\mathcal{F} : [\sigma, \rho] \to \mathbb{R}$ is a function of bounded variation on $[\sigma, \rho]$. Then, we get*

$$\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\rho}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\rho}{6}\right)\right]-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\rho-\sigma)^{\alpha\beta}}\left[{}^{\beta}J_{\sigma+}^{\alpha}\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)+{}^{\beta}J_{\rho-}^{\alpha}\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)\right]\right|$$
$$\leq \left(\frac{\rho-\sigma}{2}\right)^{\alpha\beta}\max\left\{\frac{1}{4}, 1-\left(1-\left(\frac{1}{3}\right)^{\alpha}\right)^{\beta}, \left|\frac{1}{4}-\left(1-\left(\frac{1}{3}\right)^{\alpha}\right)^{\beta}\right|\right\}\bigvee_{\sigma}^{\rho}(\mathcal{F}).$$

Here, $\bigvee_{\sigma}^{\rho}(\mathcal{F})$ demonstrates the total variation of $\mathcal{F}$ on $[\sigma, \rho]$.

Proof. Define the function $K_{\alpha,\beta}(x)$ by

$$K_{\alpha,\beta}(x)=\begin{cases}\frac{(\rho-\sigma)^{\alpha\beta}}{2^{\alpha\beta}}-\left[\left(\frac{\rho-\sigma}{2}\right)^{\alpha}-(x-\sigma)^{\alpha}\right]^{\beta}, & \sigma\leq x<\frac{5\sigma+\rho}{6},\\ \frac{(\rho-\sigma)^{\alpha\beta}}{2^{\alpha\beta+2}}-\left[\left(\frac{\rho-\sigma}{2}\right)^{\alpha}-(x-\sigma)^{\alpha}\right]^{\beta}, & \frac{5\sigma+\rho}{6}\leq x<\frac{\sigma+\rho}{2},\\ \left[\left(\frac{\rho-\sigma}{2}\right)^{\alpha}-(\rho-x)^{\alpha}\right]^{\beta}-\frac{(\rho-\sigma)^{\alpha\beta}}{2^{\alpha\beta+2}}, & \frac{\sigma+\rho}{2}\leq x<\frac{\sigma+5\rho}{6},\\ \left[\left(\frac{\rho-\sigma}{2}\right)^{\alpha}-(\rho-x)^{\alpha}\right]^{\beta}-\frac{(\rho-\sigma)^{\alpha\beta}}{2^{\alpha\beta}}, & \frac{\sigma+5\rho}{6}\leq x\leq\rho.\end{cases}$$

With the help of the integrating by parts, we get

$$\int_{\sigma}^{\rho}K_{\alpha,\beta}(x)\,d\mathcal{F}(x)$$

$$=\int_{\sigma}^{\frac{5\sigma+\rho}{6}}\left(\frac{(\rho-\sigma)^{\alpha\beta}}{2^{\alpha\beta}}-\left[\left(\frac{\rho-\sigma}{2}\right)^{\alpha}-(x-\sigma)^{\alpha}\right]^{\beta}\right)d\mathcal{F}(x)+\int_{\frac{5\sigma+\rho}{6}}^{\frac{\sigma+\rho}{2}}\left(\frac{(\rho-\sigma)^{\alpha\beta}}{2^{\alpha\beta+2}}-\left[\left(\frac{\rho-\sigma}{2}\right)^{\alpha}-(x-\sigma)^{\alpha}\right]^{\beta}\right)d\mathcal{F}(x)$$

$$+\int_{\frac{\sigma+\rho}{2}}^{\frac{\sigma+5\rho}{6}}\left(\left[\left(\frac{\rho-\sigma}{2}\right)^{\alpha}-(\rho-x)^{\alpha}\right]^{\beta}-\frac{(\rho-\sigma)^{\alpha\beta}}{2^{\alpha\beta+2}}\right)d\mathcal{F}(x)+\int_{\frac{\sigma+5\rho}{6}}^{\rho}\left(\left[\left(\frac{\rho-\sigma}{2}\right)^{\alpha}-(\rho-x)^{\alpha}\right]^{\beta}-\frac{(\rho-\sigma)^{\alpha\beta}}{2^{\alpha\beta}}\right)d\mathcal{F}(x)$$

$$=\left(\frac{(\rho-\sigma)^{\alpha\beta}}{2^{\alpha\beta}}-\left[\left(\frac{\rho-\sigma}{2}\right)^{\alpha}-(x-\sigma)^{\alpha}\right]^{\beta}\right)\mathcal{F}(x)\Big|_{\sigma}^{\frac{5\sigma+\rho}{6}}-\alpha\beta\int_{\sigma}^{\frac{5\sigma+\rho}{6}}\left[\left(\frac{\rho-\sigma}{2}\right)^{\alpha}-(x-\sigma)^{\alpha}\right]^{\beta-1}(x-\sigma)^{\alpha-1}\mathcal{F}(x)dx$$

$$+\left(\frac{(\rho-\sigma)^{\alpha\beta}}{2^{\alpha\beta+2}}-\left[\left(\frac{\rho-\sigma}{2}\right)^{\alpha}-(x-\sigma)^{\alpha}\right]^{\beta}\right)\mathcal{F}(x)\Big|_{\frac{5\sigma+\rho}{6}}^{\frac{\sigma+\rho}{2}}-\alpha\beta\int_{\frac{5\sigma+\rho}{6}}^{\frac{\sigma+\rho}{2}}\left[\left(\frac{\rho-\sigma}{2}\right)^{\alpha}-(x-\sigma)^{\alpha}\right]^{\beta-1}(x-\sigma)^{\alpha-1}\mathcal{F}(x)dx$$

$$+\left(\left[\left(\frac{\rho-\sigma}{2}\right)^{\alpha}-(\rho-x)^{\alpha}\right]^{\beta}-\frac{(\rho-\sigma)^{\alpha\beta}}{2^{\alpha\beta+2}}\right)\mathcal{F}(x)\Big|_{\frac{\sigma+\rho}{2}}^{\frac{\sigma+5\rho}{6}}-\alpha\beta\int_{\frac{\sigma+\rho}{2}}^{\frac{\sigma+5\rho}{6}}\left[\left(\frac{\rho-\sigma}{2}\right)^{\alpha}-(\rho-x)^{\alpha}\right]^{\beta-1}(\rho-x)^{\alpha-1}\mathcal{F}(x)dx$$

$$+\left(\left[\left(\frac{\rho-\sigma}{2}\right)^{\alpha}-(\rho-x)^{\alpha}\right]^{\beta}-\frac{(\rho-\sigma)^{\alpha\beta}}{2^{\alpha\beta}}\right)\mathcal{F}(x)\Big|_{\frac{\sigma+5\rho}{6}}^{\rho}-\alpha\beta\int_{\frac{\sigma+5\rho}{6}}^{\rho}\left[\left(\frac{\rho-\sigma}{2}\right)^{\alpha}-(\rho-x)^{\alpha}\right]^{\beta-1}(\rho-x)^{\alpha-1}\mathcal{F}(x)dx$$

$$=\left(\frac{(\rho-\sigma)^{\alpha\beta}}{2^{\alpha\beta}}-\left[\left(\frac{\rho-\sigma}{2}\right)^{\alpha}-\left(\frac{\rho-\sigma}{6}\right)^{\alpha}\right]^{\beta}\right)\mathcal{F}\left(\frac{5\sigma+\rho}{6}\right)$$

$$+\frac{(\rho-\sigma)^{\alpha\beta}}{2^{\alpha\beta+2}}\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)-\left(\frac{(\rho-\sigma)^{\alpha\beta}}{2^{\alpha\beta+2}}-\left[\left(\frac{\rho-\sigma}{2}\right)^{\alpha}-\left(\frac{\rho-\sigma}{6}\right)^{\alpha}\right]^{\beta}\right)\mathcal{F}\left(\frac{5\sigma+\rho}{6}\right)$$

$$+\left(\left[\left(\frac{\rho-\sigma}{2}\right)^{\alpha}-\left(\frac{\rho-\sigma}{6}\right)^{\alpha}\right]^{\beta}-\frac{(\rho-\sigma)^{\alpha\beta}}{2^{\alpha\beta+2}}\right)\mathcal{F}\left(\frac{\sigma+5\rho}{6}\right)+\frac{(\rho-\sigma)^{\alpha\beta}}{2^{\alpha\beta+2}}\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)$$

$$-\left(\left[\left(\frac{\rho-\sigma}{2}\right)^{\alpha}-\left(\frac{\rho-\sigma}{6}\right)^{\alpha}\right]^{\beta}-\frac{(\rho-\sigma)^{\alpha\beta}}{2^{\alpha\beta}}\right)\mathcal{F}\left(\frac{\sigma+5\rho}{6}\right)$$

$$-\alpha\beta\int_{\sigma}^{\frac{\sigma+\rho}{2}}\left[\left(\frac{\rho-\sigma}{2}\right)^{\alpha}-(x-\sigma)^{\alpha}\right]^{\beta-1}(x-\sigma)^{\alpha-1}\mathcal{F}(x)dx-\alpha\beta\int_{\frac{\sigma+\rho}{2}}^{\rho}\left[\left(\frac{\rho-\sigma}{2}\right)^{\alpha}-(\rho-x)^{\alpha}\right]^{\beta-1}(\rho-x)^{\alpha-1}\mathcal{F}(x)dx$$

$$=\left(\frac{\rho-\sigma}{2}\right)^{\alpha\beta}\left[\frac{3}{4}\mathcal{F}\left(\frac{5\sigma+\rho}{6}\right)+\frac{1}{2}\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)+\frac{3}{4}\mathcal{F}\left(\frac{\sigma+5\rho}{6}\right)\right]-\alpha^{\beta}\Gamma(\beta+1)\left[{}^{\beta}J^{\alpha}_{\sigma+}\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)+{}^{\beta}J^{\alpha}_{\rho-}\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)\right].$$

This follows

$$\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\rho}{6}\right)+2\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)+3\mathcal{F}\left(\frac{\sigma+5\rho}{6}\right)\right]-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\rho-\sigma)^{\alpha\beta}}\left[{}^{\beta}J^{\alpha}_{\sigma+}\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)+{}^{\beta}J^{\alpha}_{\rho-}\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)\right]\right|$$
$$=\frac{2^{\alpha\beta-1}}{(\rho-\sigma)^{\alpha\beta}}\int_{\sigma}^{\rho}K_{\alpha,\beta}(x)d\mathcal{F}(x).$$

It is known that if $g,\mathcal{F}:[\sigma,\rho]\to\mathbb{R}$ are such that g is continuous on $[\sigma,\rho]$ and $\mathcal{F}$ is of bounded variation on

$[\sigma,\rho]$, then $\int_\sigma^\rho g(\eta)d\mathcal{F}(\eta)$ exist and

$$\left|\int_\sigma^\rho g(\eta)d\mathcal{F}(\eta)\right| \leq \sup_{\eta\in[\sigma,\rho]} \left|g(\eta)\right| \bigvee_\sigma^\rho(\mathcal{F}). \tag{25}$$

By using (25), we have

$$\begin{aligned}
&\left|\int_\sigma^\rho K_{\alpha,\beta}(x)d\mathcal{F}(x)\right| \\
&\leq \left|\int_\sigma^{\frac{5\sigma+\rho}{6}} \left(\frac{(\rho-\sigma)^{\alpha\beta}}{2^{\alpha\beta}} - \left[\left(\frac{\rho-\sigma}{2}\right)^\alpha - (x-\sigma)^\alpha\right]^\beta\right) d\mathcal{F}(x)\right| + \left|\int_{\frac{5\sigma+\rho}{6}}^{\frac{\sigma+\rho}{2}} \left(\frac{(\rho-\sigma)^{\alpha\beta}}{2^{\alpha\beta+2}} - \left[\left(\frac{\rho-\sigma}{2}\right)^\alpha - (x-\sigma)^\alpha\right]^\beta\right) d\mathcal{F}(x)\right| \\
&\quad + \left|\int_{\frac{\sigma+\rho}{2}}^{\frac{\sigma+5\rho}{6}} \left(\left[\left(\frac{\rho-\sigma}{2}\right)^\alpha - (\rho-x)^\alpha\right]^\beta - \frac{(\rho-\sigma)^{\alpha\beta}}{2^{\alpha\beta+2}}\right) d\mathcal{F}(x)\right| + \left|\int_{\frac{\sigma+5\rho}{6}}^{\rho} \left(\left[\left(\frac{\rho-\sigma}{2}\right)^\alpha - (\rho-x)^\alpha\right]^\beta - \frac{(\rho-\sigma)^{\alpha\beta}}{2^{\alpha\beta}}\right) d\mathcal{F}(x)\right| \\
&\leq \sup_{x\in\left[\sigma,\frac{5\sigma+\rho}{6}\right]} \left|\frac{(\rho-\sigma)^{\alpha\beta}}{2^{\alpha\beta}} - \left[\left(\frac{\rho-\sigma}{2}\right)^\alpha - (x-\sigma)^\alpha\right]^\beta\right| \bigvee_\sigma^{\frac{5\sigma+\rho}{6}}(\mathcal{F}) + \sup_{x\in\left[\frac{5\sigma+\rho}{6},\frac{\sigma+\rho}{2}\right]} \left|\frac{(\rho-\sigma)^{\alpha\beta}}{2^{\alpha\beta+2}} - \left[\left(\frac{\rho-\sigma}{2}\right)^\alpha - (x-\sigma)^\alpha\right]^\beta\right| \bigvee_{\frac{5\sigma+\rho}{6}}^{\frac{\sigma+\rho}{2}}(\mathcal{F}) \\
&\quad + \sup_{x\in\left[\frac{\sigma+\rho}{2},\frac{\sigma+5\rho}{6}\right]} \left|\left[\left(\frac{\rho-\sigma}{2}\right)^\alpha - (\rho-x)^\alpha\right]^\beta - \frac{(\rho-\sigma)^{\alpha\beta}}{2^{\alpha\beta+2}}\right| \bigvee_{\frac{\sigma+\rho}{2}}^{\frac{\sigma+5\rho}{6}}(\mathcal{F}) + \sup_{x\in\left[\frac{\sigma+5\rho}{6},\rho\right]} \left|\left[\left(\frac{\rho-\sigma}{2}\right)^\alpha - (\rho-x)^\alpha\right]^\beta - \frac{(\rho-\sigma)^{\alpha\beta}}{2^{\alpha\beta}}\right| \bigvee_{\frac{\sigma+5\rho}{6}}^{\rho}(\mathcal{F}) \\
&= \left(\frac{\rho-\sigma}{2}\right)^{\alpha\beta} \left\{\left[1-\left(1-\left(\frac{1}{3}\right)^\alpha\right)^\beta\right] \bigvee_\sigma^{\frac{5\sigma+\rho}{6}}(\mathcal{F}) + \max\left\{\frac{1}{4}, \left|\frac{1}{4}-\left(1-\left(\frac{1}{3}\right)^\alpha\right)^\beta\right|\right\} \bigvee_{\frac{5\sigma+\rho}{6}}^{\frac{\sigma+\rho}{2}}(\mathcal{F})\right. \\
&\quad \left. + \max\left\{\frac{1}{4}, \left|\frac{1}{4}-\left(1-\left(\frac{1}{3}\right)^\alpha\right)^\beta\right|\right\} \bigvee_{\frac{\sigma+\rho}{2}}^{\frac{\sigma+5\rho}{6}}(\mathcal{F}) + \left[1-\left(1-\left(\frac{1}{3}\right)^\alpha\right)^\beta\right] \bigvee_{\frac{\sigma+5\rho}{6}}^{\rho}(\mathcal{F})\right\} \\
&\leq \left(\frac{\rho-\sigma}{2}\right)^{\alpha\beta} \max\left\{\frac{1}{4}, 1-\left(1-\left(\frac{1}{3}\right)^\alpha\right)^\beta, \left|\frac{1}{4}-\left(1-\left(\frac{1}{3}\right)^\alpha\right)^\beta\right|\right\} \bigvee_\sigma^\rho(\mathcal{F}).
\end{aligned}$$

This follows

$$\begin{aligned}
&\left|\frac{1}{8}\left[3\mathcal{F}\left(\frac{5\sigma+\rho}{6}\right) + 2\mathcal{F}\left(\frac{\sigma+\rho}{2}\right) + 3\mathcal{F}\left(\frac{\sigma+5\rho}{6}\right)\right] - \frac{2^{\alpha\beta-1}\alpha^\beta\Gamma(\beta+1)}{(\rho-\sigma)^{\alpha\beta}}\left[{}^\beta J^\alpha_{\sigma+}\mathcal{F}\left(\frac{\sigma+\rho}{2}\right) + {}^\beta J^\alpha_{\rho-}\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)\right]\right| \\
&= \frac{2^{\alpha\beta-1}}{(\rho-\sigma)^{\alpha\beta}} \left|\int_\sigma^\rho K_{\alpha,\beta}(x)d\mathcal{F}(x)\right| \\
&\leq \frac{2^{\alpha\beta-1}}{(\rho-\sigma)^{\alpha\beta}} \left(\frac{\rho-\sigma}{2}\right)^{\alpha\beta} \max\left\{\frac{1}{4}, 1-\left(1-\left(\frac{1}{3}\right)^\alpha\right)^\beta\right\} \bigvee_\sigma^\rho(\mathcal{F})
\end{aligned}$$

$$= \frac{1}{2} \max \left\{ \frac{1}{4}, 1 - \left(1 - \left(\frac{1}{3} \right)^{\alpha} \right)^{\beta} \right\} \bigvee_{\sigma}^{\rho} (\mathcal{F}).$$

This completes the proof. □

Remark 6.2. *If we take $\alpha = 1$ in Theorem 6.1 leads to [25, Theorem 9].*

Remark 6.3. *If we choose $\alpha = \beta = 1$ in Theorem 6.1 leads to [25, Corollary 6].*

7. Conclusion

In this investigation, we aim to derive Euler-Maclaurin-type inequalities applicable to various classes of functions by utilizing conformable fractional integrals. Initially, we introduce an integral equality crucial for establishing the article's key findings. We employ conformable fractional integrals to investigate Euler-Maclaurin-type inequalities tailored for differentiable convex functions. Furthermore, we extend our investigation to Euler-Maclaurin-type inequalities applicable to bounded functions through fractional integrals. Moreover, we consider fractional Euler-Maclaurin-type inequalities applicable to Lipschitzian functions. Finally, we demonstrate the validity of Euler-Maclaurin-type inequalities through fractional integrals of bounded variation. Furthermore,these inequalities can be applied to various fractional integrals.

References

[1] A. Abdeldaim and M. Yakout, *On some new integral inequalities of Gronwall–Bellman–Pachpatte type*, Applied Mathematics and Computation, 217(20), 7887–7899, 2011.
[2] M. A. Ali, H. Kara, J. Tariboon, S. Asawasamrit, H. Budak, and F. Hezenci, *Some new Simpson's-formula-type inequalities for twice-differentiable convex functions via generalized fractional operators*, Symmetry, 13(12), 2249, 2021.
[3] M. A. Ali, C. S. Goodrich, and H. Budak, *Some new parameterized Newton-type inequalities for differentiable functions via fractional integrals*, Journal of Inequalities and Applications, (1), 1–17, 2023.
[4] G. AlNemer, M. Kenawy, M. Zakarya, C. Cesarano, and H. M. Rezk, *Generalizations of Hardy's type inequalities via conformable calculus*, Symmetry, 13(2), 242, 2021.
[5] H. Budak, F. Hezenci, and H. Kara, *On parametrized inequalities of Ostrowski and Simpson type for convex functions via generalized fractional integral*, Math. Methods Appl. Sci., 44(30), 12522–12536, 2021.
[6] S. I. Butt, A. O. Akdemir, J. Nasir, and F. Jarad, *Some Hermite-Jensen-Mercer like inequalities for convex functions through a certain generalized fractional integrals and related results*, Miskolc Math. Notes, 21, 689–715, 2020.
[7] H. Budak, H. Kara, M. Z. Sarikaya, and M. E. Kiris, *New extensions of the Hermite-Hadamard inequalities involving Riemann-Liouville fractional integrals*, Miskolc Mathematical Notes, 21(2), 665–678, 2020.
[8] H. Budak and P. Karagozoglu, *Fractional Milne type inequalities*, Acta Mathematica Universitatis Comenianae, 1–15, 2024.
[9] B. Celik, H. Budak, and E. Set, *On generalized Milne type inequalities for new conformable fractional integrals*, Filomat, 38(5), 1807–1823, 2024.
[10] S. S. Dragomir, *On Simpson's quadrature formula for mappings of bounded variation and applications*, Tamkang Journal of Mathematics, 30(1), 53–58, 1999.
[11] P. J. Davis and P. Rabinowitz, *Methods of numerical integration*, Academic Press, New York-San Francisco-London, 1975.
[12] L. Dedic, M. Matic, J. Pevcaric, and A. Vukelic, *On Euler-Simpson* 3/8 *formulae*, Nonlinear Studies, 18(1), 1–26, 2011.
[13] L. J. Dedic, M. Matic, J. Pecaric, *Euler Maclaurin formulae*, Mathematics Inequalities and Applications, 6(2), 247–275, 2003.
[14] F. Ertugral and M. Z. Sarikaya, *Simpson-type integral inequalities for generalized fractional integral*, Revista de la Real Academia de Ciencias Exactas, FÃ-sicas y Naturales. Serie A. Matematicas, 113(4), 3115–3124, 2019.
[15] I. Franjic, J. Pecaric, I. Peric, and A. Vukelic, *Euler integral identity, quadrature formulae and error estimations*, Element, 2011.
[16] J. W. Green, *Recent applications of convex functions*, The American Mathematical Monthly, 61(7P1), 449–454, 1954.
[17] A. A. Hyder and A. H. Soliman, *A new generalized θ-conformable calculus and its applications in mathematical physics*, Physica Scripta, 96(1), 015208, 2020.
[18] F. Hezenci, H. Budak, and P. Kösem, *A new version of Newton's inequalities for Riemann-Liouville fractional integrals*, Rocky Mountain Journal of Mathematics, 53(1), 49–64, 2023.
[19] A. A. Hyder, H. Budak, and A. A. Almoneef, *Further midpoint inequalities via generalized fractional operators in Riemann-Liouville sense*, Fractal and Fractional, 6(9), 496, 2022.
[20] F. Hezenci and H. Budak, *Simpson-type inequalities for conformable fractional operators with respect to twice-differentiable functions*, Journal of Mathematical Extension, 17, 2023.
[21] F. Hezenci, *Fractional Maclaurin-type inequalities for twice-differentiable functions*, Rocky Mountain Journal of Mathematics, 2023.
[22] W. Haider, H. Budak, A. Shehzadi, F. Hezenci, and H. Chen, *A comprehensive study on Milne-type inequalities with tempered fractional integrals*, Boundary Value Problems, 2024(1), 1–16, https://doi.org/10.1186/s13661-024-01855-1.

[23] F. Hezenci and H. Budak, *Maclaurin-type inequalities for Riemann-Liouville fractional integrals*, Annales Universitatis Mariae Curie-Sklodowska, sectio A Mathematica, 76(2), 15–32, 2023.
[24] F. Hezenci, *Fractional inequalities of corrected Euler-Maclaurin-type for twice-differentiable functions*, Computational and Applied Mathematics, 42(92), 1–15, 2023.
[25] F. Hezenci and H. Budak, *Fractional Euler-Maclaurin-type inequalities for various function classes*, Computational and Applied Mathematics, 43(4), 261, 2023.
[26] F. Jarad, E. Uğurlu, T. Abdeljawad, and D. Baleanu, *On a new class of fractional operators*, Advances in Difference Equations, 2017, 1–16.
[27] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier B.V., Amsterdam, The Netherlands, 2006.
[28] H. Kara, H. Budak, S. Etemad, S. Rezapour, H. Ahmad, and M. K. Kaabar, *A study on the new class of inequalities of midpoint-type and trapezoidal-type based on twice differentiable functions with conformable operators*, Journal of Function Spaces, 2023.
[29] R. T. Rockafellar and R. J. B. Wets, *Variational analysis*, Vol. 317, Springer Science & Business Media, 2009.
[30] G. Rahman, S. Nisar, and F. Qi, *Some new inequalities of the Grüss type for conformable fractional integrals*, AIMS Mathematics, 3(4), 575–583, 2018.
[31] G. Rahman, Z. Ullah, A. Khan, E. Set, and K. S. Nisar, *Certain Chebyshev-type inequalities involving fractional conformable integral operators*, Mathematics, 7(4), 364, 2019.
[32] S. Rashid, A. O. Akdemir, K. S. Nisar, T. Abdeljawad, and G. Rahman, *New generalized reverse Minkowski and related integral inequalities involving generalized fractional conformable integrals*, Journal of Inequalities and Applications, 2020, 1–15.
[33] J. E. Peajcariaac and Y. L. Tong, *Convex Functions, Partial Orderings, and Statistical Applications*, Academic Press, Boston, 1992.
[34] M. Z. Sarikaya and H. Budak, *Some Hermite-Hadamard type integral inequalities for twice differentiable mappings via fractional integrals*, Facta Universitatis, Series: Mathematics and Informatics, 29(4), 371–384, 2015.
[35] P. K. Sarpong, O. H. Andrew, and A. P. Joseph, *Applications of Convex Function and Concave Functions*, 2018.
[36] E. Set, I. Mumcu, and M. E. Özdemir, *Grüss type inequalities involving new conformable fractional integral operators*, AIP Conference Proceedings, Vol. 1991, No. 1, p. 020020, AIP Publishing LLC, 2018.
[37] E. Set, A. O. Akdemir, A. GÖzpinar, and F. Jarad, *Ostrowski type inequalities via new fractional conformable integrals*, AIMS Math, 4(6), 1684–1697, 2019.
[38] A. Shehzadi, H. Budak, W. Haider, and H. Chen, *Milne-type Inequalities for co-ordinated Convex Functions*, Filomat, Vol 38, No 23, 2024.
[39] C. Ünal, F. Hezenci, and H. Budak, *Conformable fractional Newton-type inequalities with respect to differentiable convex functions*, Journal of Inequalities and Applications, 2023(1), 85.
[40] X. You, F. Hezenci, H. Budak, and H. Kara, *New Simpson type inequalities for twice differentiable functions via generalized fractional integrals*, AIMS Math, 7(3), 3959–3971, 2022.
[41] D. Zhao and M. Luo, *General conformable fractional derivative and its physical interpretation*, Calcolo, 54, 903–917, 2017.