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Weighted generalized core-EP inverse in Banach *-algebras

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Abstract. In this paper, we introduce the weighted generalized core-EP inverse in the context of a Banach *-algebra. We extend certain properties of the weighted core-EP inverse for rectangular complex matrices and bounded linear operators on Hilbert spaces, and we also establish some new properties. Furthermore, we present properties of the weighted generalized core-EP orders for all elements that are weighted g-Drazin invertible within a Banach *-algebra.

1. Introduction

Let $\mathbb{C}^{n\times n}$ be the Banach algebra of all $n\times n$ complex matrices with conjugate transpose $*$. The core-EP inverse of a complex matrix was introduced by Prasad and Mohana (see [26]). Let $R(X)$ represent the range space of a complex matrix X. A matrix $A\in C^{n\times n}$ has core-EP inverse X if and only if

$$XAX = X, R(X) = R(X^*) = R(A^k),$$

where $k = ind(A)$ is the Drazin index of A. Such X is unique, and we denote it by $A^{\textcircled{\dagger}}$ (see [29]). Recently, Gao et al. extended the concept of the core-EP inverse and introduced the notion of weighted core-EP inverse for a complex matrix (see [12]). Let $A\in\mathbb{C}^{m\times n}, W\in\mathbb{C}^{n\times m}$ and $k = max\{ind(AW), ind(WA)\}$. The weighted core-EP inverse of A is the unique solution to the system:

$$WAWX = (WA)^k[(WA)^k]^{\dagger}, R(X)\subseteq R((AW)^k),$$

and we denote X by $A^{\textcircled{\dagger},W}$. Then Mosić introduced and studied weighted core-EP inverse for a bounded linear operator between two Hilbert spaces as a generalization of the weighted core-EP inverse of a rectangular matrix (see [21]). Let X and Y be Hilbert spaces. Denote by $\mathcal{B}(X,Y)$ the set of all bounded linear operators from X and Y. Set $\mathcal{B}(X) = \mathcal{B}(X,X)$. Let $W\in\mathcal{B}(Y,X)$. A Wg-Drazin invertible $A\in\mathcal{B}(X,Y)$ has the weighted core-EP inverse X if X is the unique solution to the system:

$$WAWX = P_{R(WA)^d}, R(X)\subseteq R((AW)^d),$$

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and we denote X by $A^{\circledD,W}$ (see [21]). Here, $P_{R(WA)^d}$ is a projection on $R(WA)^d$. Since every bounded linear operator in $\mathcal{B}(X,Y)$ can be regarded as a restriction of an operator in $\mathcal{B}(X\bigoplus Y)$, it could be seen as an element in a C^*-algebra. Recently, Mosić extended the weighted core-EP inverse of bounded linear operators on Hilbert spaces to elements of a C^*-algebra and the weighted core-EP inverse in a C^*-algebra was characterized by means of range projections (see [23]).

A Banach algebra is called a Banach *-algebra if there exists an involution $* : x \to x^*$ satisfying $(x+y)^* = x^* + y^*, (\lambda x)^* = \overline{\lambda}x^*, (xy)^* = y^*x^*, (x^*)^* = x$. Let $\mathcal{A}$ be a Banach *-algebra, and let $w \in \mathcal{A}$. Following Mosić (see [24]), an element $a \in \mathcal{A}$ has wg-Drazin inverse x if there exists a unique $x \in \mathcal{A}$ such that

$$awx = xwa, xwawx = x \text{ and } a - awxwa \in \mathcal{A}^{qnil}.$$

Here, $\mathcal{A}^{qnil} = \{z \in \mathcal{A} \mid \lim\limits_{n\to\infty} \parallel z^n \parallel^{\frac{1}{n}} = 0\}$. We denote the preceding x by $a^{d,w}$ (see [6]). In the case that $w = 1$, $a^d = a^{d,w}$ is the g-Drazin inverse of a (i.e., generalized Drazin inverse). Evidently, we have the connection between weighted g-Drazin inverse $a^{d,w}$ and g-Drazin inverse:

$$a^{d,w} = [(aw)^d]^2a = a[(wa)^d]^2 = (aw)^d a(wa)^d.$$

The objective of this paper is to broaden the scope of the weighted core-EP inverse, initially defined for rectangular matrices, bounded linear operators, and elements within a C^*-algebra, by extending its application to elements in a Banach *-algebra. Furthermore, we aim to investigate the additional properties of this generalized inverse.

Definition 1.1. *An element a in a Banach *-algebra $\mathcal{A}$ has a w-weighted generalized core-EP inverse if there exists $x \in \mathcal{A}$ such that*

$$a(wx)^2 = x, (wawx)^* = wawx, \lim_{n\to\infty} \|(aw)^n - (xw)(aw)^{n+1}\|^{\frac{1}{n}} = 0.$$

The preceding x is unique if exists, and we denote it by $a^{\circledD,w}$. Interestingly, we shall demonstrate that the w-weighted generalized core-EP inverse and the weighted core-EP inverses for a complex matrix and a linear operator mentioned above are identical. In other words, this weighted generalized inverse naturally extends the definitions of those weighted generalized inverses provided in [12, 21, 23]. We say that a has generalized core-EP inverse if it has generalized 1-core-EP inverse and denote $a^{\circledD,1}$ by $a^{\circledD}$. Many properties of generalized core-EP inverse are presented in [4, Theorem 1.2], which can be proved by choosing the weight $e = 1$ in [2, 3].

In Section 2, we introduce weighted generalized core-EP inverse for an element in a Banach *-algebra. We give a characterization of weighted generalized core-EP inverse $a^{\circledD,w}$ in terms of generalized core-EP inverse and generalized Drazin inverse.

In Section 3, we establish the relationships between the weighted generalized core-EP inverse and the weighted g-Drazin inverse for an element within a Banach *-algebra. We prove that $a \in \mathcal{A}$ has w-weighted generalized core-EP inverse if and only if $a \in \mathcal{A}$ has weighted g-Drazin inverse and there exists $x \in \mathcal{A}$ such that $wawx = p_{im(wa)^d}, im(x) \subseteq im(aw)^d$.

In Section 4, a connection of a weighted generalized core-EP inverse $a^{\circledD,w}$ and Mary inverse (Theorem 4.1); (b,c)-inverse (Theorem 4.3) and 2-inverse (Theorem 4.5) is shown.

Finally, a new kind of order, called the weighted generalized core-EP order, is presented and proved that it is a pre-order. Also, there are some results showing the equivalent conditions for two elements which are in that pre-order.

Throughout the paper, all Banach *-algebras are complex with an identity. $\mathcal{A}^d, \mathcal{A}^{\circledD}, \mathcal{A}^{d,w}$ and $\mathcal{A}^{\circledD,w}$ denote the sets of all g-Drazin invertible, generalized core-EP invertible, wg-Drazin invertible and w-weighted generalized core-EP invertible elements in $\mathcal{A}$, respectively. Let $a \in \mathcal{A}$. Set $im(a) = \{ax \mid x \in \mathcal{A}\}$ and $ker(a) = \{x \in \mathcal{A} \mid ax = 0\}$. We use $p_{im(a)}$ to denote the projection p such that $im(p) = im(a)$.

2. Equivalent characterizations

This section aims to introduce the weighted generalized core-EP inverse in a Banach *-algebra and subsequently establish its equivalent characterizations. We start by

Theorem 2.1. *Let $a, w \in \mathcal{A}$. Then the following are equivalent:*

(1) There exists $x \in \mathcal{A}$ such that

$$a(wx)^2 = x, (dawx)^* = wawx, \lim_{n\to\infty} \|(aw)^n - (xw)(aw)^{n+1}\|^{\frac{1}{n}} = 0.$$

(2) $wa \in \mathcal{A}^{\textcircled{d}}$.

In this case, $x = a[(wa)^{\textcircled{d}}]^2$.

Proof. (1) ⇒ (2) By hypothesis, we have

$$a(wx)^2 = x, (wawx)^* = wawx, \lim_{n\to\infty} \|(aw)^n - (xw)(aw)^{n+1}\|^{\frac{1}{n}} = 0.$$

Then

$$(wa)(wx)^2 = wx, [(wa)(wx)]^* = (wa)(wx).$$

Moreover, we have

$$\begin{array}{rl} & \|(wa)^{n+1} - (wx)(wa)^{n+2}\|^{\frac{1}{n+1}} \\ = & \|w(aw)^n a - wxw(aw)^{n+1}a\|^{\frac{1}{n+1}} \\ = & \|w[(aw)^n - xw(aw)^{n+1}]a\|^{\frac{1}{n+1}} \\ \leq & \|w\|^{\frac{1}{n+1}}[\|(aw)^n - xw(aw)^{n+1}\|^{\frac{1}{n}}]^{\frac{n}{n+1}}\|a\|^{\frac{1}{n+1}}. \end{array}$$

Therefore

$$\lim_{n\to\infty} \|(wa)^{n+1} - (wx)(wa)^{n+2}\|^{\frac{1}{n+1}} = 0.$$

This implies that

$$wa \in \mathcal{A}^{\textcircled{d}} \text{ and } (wa)^{\textcircled{d}} = wx.$$

Accordingly,

$$x = a(wx)^2 = a[(wa)^{\textcircled{d}}]^2,$$

as required.

(2) ⇒ (1) Let $x = a[(wa)^{\textcircled{d}}]^2$. Then we check that

$$\begin{array}{rcl} a(wx)^2 & = & awa[(wa)^{\textcircled{d}}]^2 wa[(wa)^{\textcircled{d}}]^2 \\ & = & a[(wa)^{\textcircled{d}}]^2 = x, \\ (wawx)^* & = & [wawa((wa)^{\textcircled{d}})^2]^* = [wa(wa)^{\textcircled{d}}]^* \\ & = & wa(wa)^{\textcircled{d}} = wawx. \end{array}$$

Observing that

$$\begin{array}{rcl} (xw)(aw)^{n+1} & = & a[(wa)^{\textcircled{d}}]^2 w(aw)^{n+1} \\ & = & a(wa)^{\textcircled{d}}[(wa)^{\textcircled{d}}(wa)^{n+1}]w \\ & = & a(wa)^{\textcircled{d}}(wa)^n w - a(wa)^{\textcircled{d}}[(wa)^n - (wa)^{\textcircled{d}}(wa)^{n+1}]w \\ & = & a(wa)^{n-1}w - a[(wa)^{n-1} - (wa)^{\textcircled{d}}(wa)^n)]w \\ & - & a(wa)^{\textcircled{d}}[(wa)^n - (wa)^{\textcircled{d}}(wa)^{n+1}]w \\ & = & (aw)^n - a[(wa)^{n-1} - (wa)^{\textcircled{d}}(wa)^n]w \\ & - & a(wa)^{\textcircled{d}}[(wa)^n - (wa)^{\textcircled{d}}(wa)^{n+1}]w \end{array}$$

Hence, we have

$$\begin{array}{rl} & \|(aw)^n - (xw)(aw)^{n+1}\|^{\frac{1}{n}} \\ \leq & \|a\|^{\frac{1}{n}}\|(wa)^{n-1} - (wa)^{\circledD}(wa)^n\|^{\frac{1}{n}}\|w\|^{\frac{1}{n}} \\ + & \|a(wa)^{\circledD}\|^{\frac{1}{n}}\|(wa)^n - (wa)^{\circledD}(wa)^{n+1}\|^{\frac{1}{n}}\|w\|^{\frac{1}{n}}. \end{array}$$

Thus,

$$\lim_{n\to\infty}\|(aw)^n - (xw)(aw)^{n+1}\|^{\frac{1}{n}} = 0.$$

This completes the proof. □

Corollary 2.2. *Let $a, w \in \mathcal{A}$. Then the following are equivalent:*

(1) The system of conditions

$$a(wx)^2 = x, (wawx)^* = wawx, \lim_{n\to\infty}\|(aw)^n - (xw)(aw)^{n+1}\|^{\frac{1}{n}} = 0$$

is consistent and it has the unique solution given by $x = a[(wa)^{\circledD}]^2$.

(2) $wa \in \mathcal{A}^{\circledD}$.

Proof. In view of Theorem 2.1, $x = a[(wa)^{\circledD}]^2$. By virtue of [4, Theorem 1.2], the uniqueness is proved. □

The unique solution x above is called the w-weighted generalized core-EP inverse of a, and denote it by $a^{\circledD,w}$. That is, we have

$$a^{\circledD,w} = a[(wa)^{\circledD}]^2.$$

The set of all w-weighted generalized core-EP invertible elements will be denoted by $\mathcal{A}^{\circledD,w}$.

An element a in $\mathcal{A}$ has Moore-Penrose inverse x if it satisfies the following equations:

$$(1)\ axa = a, (2)\ xax = x, (3)\ (ax)^* = ax, (4)\ (xa)^* = xa.$$

Such x is unique if it exists, and we denote it by $a^{\dagger}$. We say that x is the $(1,3)$-inverse of a if x satisfies the equations (1) and (3). By $\mathcal{A}^{\dagger}$ and $\mathcal{A}^{(1,3)}$, we denote the set of all Moore-Penrose invertible and $(1,3)$-invertible elements in $\mathcal{A}$, respectively.

Theorem 2.3. *Let $a \in \mathcal{A}^{\circledD,w}$. Then*

$$a^{\circledD,w} = [(aw)^d]^2 a[w(aw)^d a]^{(1,3)}.$$

Proof. In view of [4, Theorem 1.2], we have

$$(wa)^{\circledD} = (wa)^d[(wa)(wa)^d]^{(1,3)}.$$

By virtue of Cline's formula (see [15, Theorem 2.1]), $aw \in \mathcal{A}^d$ and $(wa)^d = w((aw)^d)^2 a$. In light of Theorem 2.1, we check that

$$\begin{array}{rcl} a^{\circledD,w} & = & a[(wa)^{\circledD}]^2 \\ & = & a[(wa)^d((wa)(wa)^d)^{(1,3)}]^2 \\ & = & a[(wa)^d((wa)(wa)^d)^{(1,3)}][(wa)^d((wa)(wa)^d)^{(1,3)}] \\ & = & a(wa)^d[(wa)(wa)^d]((wa)(wa)^d)^{(1,3)}(wa)[(wa)^d(wa)^d]((wa)(wa)^d)^{(1,3)} \\ & = & a(wa)^d(wa)[(wa)^d]^2[(wa)(wa)^d]^{(1,3)}. \\ & = & a[(wa)^d]^2[(wa)(wa)^d]^{(1,3)} \\ & = & a(wa)^d(wa)^d[(wa)w((aw)^d)^2a]^{(1,3)} \\ & = & (aw)((aw)^d)^2(aw)((aw)^d)^2a[(wa)w((aw)^d)^2a]^{(1,3)} \\ & = & ((aw)^d)^2a[w(aw)^da]^{(1,3)}. \end{array}$$

□

As an immediate consequence, we provide a new formula of the weighted core-EP inverse which also extends [12, Theorem 2.10].

Corollary 2.4. *Let $A \in \mathbb{C}^{m\times n}, W \in \mathbb{C}^{n\times m}$. Then*

$$A^{\textcircled{\dag},W} = [(AW)^D]^2 A[W(AW)^D A]^{\dagger}.$$

Proof. By adding some zero columns and rows, we may regard A and W as square matrices. Then the result is obtained by Theorem 2.3. □

Theorem 2.5. *Let $a \in \mathcal{A}^{\textcircled{d},w}$. Then the following are equivalent:*

(1) $a(wa)^{\textcircled{d}}w \in \mathcal{A}^{(1,3)}$.
(2) $aw \in \mathcal{A}^{\textcircled{d}}$ *and*

$$a^{\textcircled{d},w} = (aw)^{\textcircled{d}}a(wa)^{\textcircled{d}}.$$

Proof. Since $a \in \mathcal{A}^{\textcircled{d},w}$, it follows by Theorem 2.1 that $wa \in \mathcal{A}^{\textcircled{d}}$.

(1) ⇒ (2) Choose the weight $e = 1$ in [2, Corollary 4.6], we show that $aw \in \mathcal{A}^{\textcircled{d}}$. Then we derive

$$\begin{array}{cl} & [1-(aw)^{\textcircled{d}}aw]a[(wa)^{\textcircled{d}}]^2 \\ = & [1-((aw)^d)^2((aw)^d)^{(1,3)}aw]a(wa)^{\textcircled{d}}(wa)^{\textcircled{d}} \\ = & [1-((aw)^d)^2((aw)^d)^{(1,3)}aw]a(wa)^d(wa)^d((wa)^d)^{(1,3)}(wa)^{\textcircled{d}} \\ = & [1-((aw)^d)^2((aw)^d)^{(1,3)}aw]aw((aw)^d)^2a(wa)^d((wa)^d)^{(1,3)}(wa)^{\bigcirc d} \\ = & [1-((aw)^d)^2((aw)^d)^{(1,3)}aw](aw)^da(wa)^d((wa)^d)^{(1,3)}(wa)^{\textcircled{d}} \\ = & [(aw)^d-((aw)^d)^2((aw)^d)^{(1,3)}(aw)^daw]a(wa)^d((wa)^d)^{(1,3)}(wa)^{\textcircled{d}} \\ = & [(aw)^d-((aw)^d)^2aw]a(wa)^d((wa)^d)^{(1,3)}(wa)^{\textcircled{d}} \\ = & 0. \end{array}$$

Hence,

$$a[(wa)^{\textcircled{d}}]^2 = (aw)^{\textcircled{d}}awa[(wa)^{\textcircled{d}}]^2.$$

In light of Theorem 2.1, we have

$$\begin{array}{rcl} a^{\textcircled{d},w} & = & a[(wa)^{\textcircled{d}}]^2 \\ & = & (aw)^{\textcircled{d}}awa[(wa)^{\textcircled{d}}]^2 \\ & = & (aw)^{\textcircled{d}}a(wa)[(wa)^{\textcircled{d}}]^2 \\ & = & (aw)^{\textcircled{d}}a(wa)^{\textcircled{d}}, \end{array}$$

as desired.

(2) ⇒ (1) This is proved by choosing the weight $e = 1$ in [2, Corollary 4.6]. □

Corollary 2.6. *Let $A \in \mathbb{C}^{m\times n}, W \in \mathbb{C}^{n\times m}$. Then*

$$A^{\textcircled{\dag},W} = (AW)^{\textcircled{\dag}}A(WA)^{\textcircled{\dag}}.$$

Proof. This is obvious by Theorem 2.5. □

3. Relations with weighted g-Drazin inverses

This section primarily aims to introduce new properties of the weighted core-EP inverse related to weighted g-Drazin inverses within the context of Banach *-algebras. Hereafter, we proceed to derive these properties.

Theorem 3.1. *Let $a, w \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}^{\textcircled{d},w}$.

(2) $a \in \mathcal{A}^{d,w}$ and there exists $x \in \mathcal{A}$ such that

$$xwawx = x, im(xw) = im(aw)^d \text{ and } im(wx)^* = im(wa)^d.$$

In this case, $a^{\textcircled{d},w} = a(wx)^2$.

Proof. (1) ⇒ (2) Set $x = a^{\textcircled{d},w}$. Then $x = a[(wa)^{\textcircled{d}}]^2$ by Theorem 2.1. Hence, $wx = wa[(wa)^{\textcircled{d}}]^2 = (wa)^{\textcircled{d}}$. In view of [4, Theorem 1.2], we have

$$wxwawx = wx, im(wx) = im(wa)^d \text{ and } im(wx)^* = im(wa)^d.$$

By virtue of [4, Theorem 1.2], we have

$$xwawx = a[(wa)^{\textcircled{d}}]^2(wa)^2[(wa)^{\textcircled{d}}]^2 = a[(wa)^{\textcircled{d}}]^2(wa)(wa)^{\textcircled{d}} = a[(wa)^{\textcircled{d}}]^2 = x.$$

Moreover, we have

$$\begin{array}{rcl} xw & = & a[(wa)^{\textcircled{d}}]^2w \\ & = & [a(wa)^{\textcircled{d}}][(wa)^{\textcircled{d}}w] \\ & = & [a(wa)(wa)^d(wa)^{\textcircled{d}}][(wa)^{\textcircled{d}}w] \\ & = & a(aw)^d(wa)[(wa)^{\textcircled{d}}]^2w \\ & = & a(wa)^d[(wa)^{\textcircled{d}}w]. \end{array}$$

Set $q = (wa)^{\textcircled{d}}w$. By using Cline's formula, we derive $xw = a(wa)^dq = aw[(aw)^d]^2aq$, and so $im(xw) \subseteq im(aw)^d$. In view of [4, Theorem 1.2], $(wa)^d \in \mathcal{A}^{(1,3)}$ and $(wa)^{\textcircled{d}} = [(wa)^d]^2[(wa)^d]^{(1,3)}$. Then we check that

$$\begin{array}{rcl} (aw)^d & = & a[(wa)^d]^2w = a(wa)^d(wa)^d[(wa)^d]^{(1,3)}(wa)^dw \\ & = & a[(wa)^d]^2[(wa)^d]^{(1,3)}(wa)^dw \\ & = & a[(wa)^d]^2[(wa)^d]^{(1,3)}[(wa)^d]^2(wa)w \\ & = & a[(wa)^d]^2[(wa)^d]^{(1,3)}(wa)^d[(wa)^d((wa)^d)^{(1,3)}(wa)^d]waw \\ & = & a[(wa)^d]^2[(wa)^d]^{(1,3)}[(wa)^d]^2[(wa)^d]^{(1,3)}(wa)^dwaw \\ & = & a[(wa)^{\textcircled{d}}]^2(wa)^dwaw \\ & = & xwa(wa)^dw. \end{array}$$

Hence, $im(aw)^d \subseteq im(xw)$. Therefore $im(xw) = im(aw)^d$, as desired.

(2) ⇒ (1) By hypothesis, there exists $x \in \mathcal{A}$ such that

$$xwawx = x, im(xw) = im(aw)^d \text{ and } im(wx)^* = im(wa)^d.$$

Then $xw = (aw)^dy$ for some $y \in \mathcal{A}$. By using Cline's formula, we have $wx = wxwawx = w(xw)awx = w(aw)^dyawx = wa[(wa)^d]^2wyawx$. This implies that $wx\mathcal{A} \subseteq (wa)^d\mathcal{A}$. On the other hand, we can find some $z \in \mathcal{A}$ such that $(wa)^d = w[(aw)^d]^2a \in wxw\mathcal{A}(aw)^da$, and then $(wa)^d\mathcal{A} \subseteq (wx)\mathcal{A}$. Therefore $im(wx) = im(wa)^d$. Obviously, $wx(wa)wx = wx$. According to [4, Theorem 1.2], $wa \in \mathcal{A}^{\textcircled{d}}$ and $wx = (wa)^{\textcircled{d}}$. In light of Theorem 2.1, $a \in \mathcal{A}^{\textcircled{d},w}$. Moreover, $a^{\textcircled{d},w} = a[(wa)^{\textcircled{d}}]^2 = a(wx)^2$, as asserted. □

Corollary 3.2. *Let $a \in \mathcal{A}^{\textcircled{d},w}$ and $x \in \mathcal{A}$. Then the following are equivalent:*

(1) $a^{\textcircled{d},w} = x$.

(2) The system of conditions

$$wawx = p_{im(wa)^d}, im(x) \subseteq im(aw)^d$$

is consistent and it has the unique solution x.

Proof. (1) ⇒ (2) Let $x = a^{\textcircled{d},w}$. By virtue of Theorem 2.1, $x = a[(wa)^{\textcircled{d}}]^2$. Then $wawx = wawa[(wa)^{\textcircled{d}}]^2 = wa(wa)^{\textcircled{d}}$. Set $p = (wa)(wx)$. Then $p^2 = p = p^*$, i.e., $p \in \mathcal{A}$ is a projection. Clearly, $im(p) = im(wa)^d$; whence, $wawx = p_{im(wa)^d}$. In light Theorem 3.1, we have $im(x) \subseteq im(aw)^d$.

Assume that there exists $y \in \mathcal{A}$ such that

$$wawy = p_{im(wa)^d}, im(y) \subseteq im(aw)^d.$$

Then we check that

$$waw(x - y) = 0,$$

and so $im(x - y) \subseteq ker(waw) \subseteq ker(aw)^d$. Since $im(x), im(y) \subseteq im(aw)^d$, we have $im(x - y) \subseteq im(aw)^d$. Therefore $im(x - y) \subseteq ker(aw)^d \bigcap im(aw)^d = 0$, and so $x = y$. The uniqueness is proved.

(2) $\Rightarrow$ (1) By the preceding discussion, the system of conditions

$$wawx = p_{im(wa)^d}, im(x) \subseteq im(aw)^d$$

has the solution $a^{\textcircled{d},w}$. Therefore $a^{\textcircled{d},w} = x$ by the uniqueness. $\square$

We note that every bounded linear operator on a Hilbert space can be regarded as a restriction of an element in a Banach *-algebra. The preceding theorem shows that the weighted core-EP inverse for a Hilbert space operator coincides with the weighted generalized core-EP inverse in a Banach *-algebra (see [21, Theorem 2.3]). An element a in $\mathcal{A}$ has core inverse if there exists some $x \in \mathcal{A}$ such that $ax^2 = x, (ax)^* = ax, a = xa^2$. Such x is unique if it exists and is denoted by $a^{\textcircled{\#}}$ (see [5, 30]). For subsequent use, we derive

Lemma 3.3. *Let $a \in \mathcal{A}^{\textcircled{d},w}$. Then*

(1) $a^{\textcircled{d},w} = a^{d,w} p_{(wa)^d}$.
(2) $a^{d,w} = a^{\textcircled{d},w} p_{im(wa)^d, ker(wa)^d}$.

Proof. (1) In view of [4, Theorem 1.2], we have $(wa)^{\textcircled{d}} = [(wa)^d]^2[(wa)^d]^{\textcircled{\#}}$. According to Theorem 2.1, we have

$$\begin{array}{rcl} a^{\textcircled{d},w} & = & a[(wa)^{\textcircled{d}}]^2 \\ & = & a[(wa)^d]^2[(wa)^d]^{\textcircled{\#}}[(wa)^d]^2[(wa)^d]^{\textcircled{\#}} \\ & = & a[(wa)^d]^3[(wa)^d]^{\textcircled{\#}} \\ & = & a[(wa)^d]^2(wa)^d[(wa)^d]^{\textcircled{\#}} \\ & = & a^{d,w} p_{(wa)^d}, \end{array}$$

where $p_{(wa)^d} = (wa)^d[(wa)^d]^{\textcircled{\#}}$.

(2) By the preceding discussion, we have $a^{\textcircled{d},w} = a[(wa)^d]^2(wa)^d[(wa)^d]^{\textcircled{\#}}$; hence, $a^{\textcircled{d},w}[(wa)^d]^2 = a[(wa)^d]^2[(wa)^d]^2$. Thus, we have

$$\begin{array}{rcl} a^{d,w} & = & a[(wa)^d]^2 \\ & = & a[(wa)^d]^2[(wa)^d]^2(wa)^2 \\ & = & a^{\textcircled{d},w}[(wa)^d]^2(wa)^2 \\ & = & a^{\textcircled{d},w} p_{im(wa)^d, ker(wa)^d}. \end{array}$$

Here, $p_{im(wa)^d, ker(wa)^d} = wa(wa)^d$, as required. $\square$

Let $a \in \mathcal{A}$. Set $a^{(1)} = \{x \in \mathcal{A} \mid axa = a\}$ and $a^{(3)} = \{x \in \mathcal{A} \mid (ax)^* = ax\}$. If $a \in \mathcal{A}^{d,w}$, then $(wawa^{d,w}waw)a^{d,w}(wawa^{d,w}waw) = wawa^{d,w}waw$; hence, $(wawa^{d,w}waw)^{(1)} \neq \emptyset$. We now derive

Theorem 3.4. *Let $a \in \mathcal{A}^{\textcircled{d},w}$ and $s \in (wawa^{d,w}waw)^{(1)}$. Then the following are equivalent:*

(1) $a^{\textcircled{d},w} = q(wawa^{d,w}waw)s$.
(2) $a^{d,w} = qwawa^{d,w}$ and $s \in (wawa^{d,w}waw)^{(3)}$.
(3) $q = a^{d,w} + y(1 - wa^{d,w}wa)$ and $s = a^{d,w}wa^{d,w}(wa^{d,w})^{(1,3)} + (1 - a^{d,w}waw)z$ for some $y, z \in \mathcal{A}$.

Proof. (1) ⇒ (2) Clearly, $wawa^{d,w}waw = (wa)^2(wa)^dw$. In view of Lemma 3.3, we have

$$\begin{array}{rcl} a^{d,w} & = & a^{\circledast,w}p_{im(wa)^d,ker(wa)^d} = a^{\circledast,w}wa(wa)^d \\ & = & q(wawa^{d,w}waw)swa(wa)^d \\ & = & q[wawa^{d,w}waw]s[(wa)^2(wa)^dw]a[(wa)^d]^2 \\ & = & q[(wa)^2(wa)^dw]a[(wa)^d]^2 \\ & = & qwawa[(wa)^d]^2 \\ & = & qwawa^{d,w}. \end{array}$$

Hence,

$$wawa^{\circledast,w} = waw[qwawa^{d,w}]waws = wawa^{d,w}waws.$$

In light of Corollary 3.2,

$$(wawa^{d,w}waw)s = wawa^{\circledast,w} = p_{im(wa)^d}.$$

Then $[(wawa^{d,w}waw)s]^* = (wawa^{d,w}waw)s$, and so $s \in (wawa^{d,w}waw)^{(3)}$.

(2) ⇒ (3) Since $a^{d,w} = qwawa^{d,w}$, we see that $a^{d,w} = qwawa^{d,w} = qwa(wa)^d$. Then $q = qwa(wa)^d + q(1 - wa(wa)^d) = a^{d,w} + q(1 - wa^{d,w}wa)$.

By hypothesis, we have $[(wa)^2(wa)^dw]s[(wa)^2(wa)^dw] = (wa)^2(wa)^dw$. Then

$$\begin{array}{rcl} \Big(wa^{d,w}\Big)\Big((wa)^2ws\Big) & = & (wa)^2(wa)^dws = wawa^{d,w}waws, \\ \Big((wa^{d,w})((wa)^2ws)\Big)^* & = & (wa^{d,w})((wa)^2ws), \\ \Big(wa^{d,w}\Big)\Big((wa)^2ws\Big)\Big(wa^{d,w}\Big) & = & [(wa)^2(wa)^dw]s[(wa)^2(wa)^dw][a((wa)^d)^3] \\ & = & [(wa)^2(wa)^dw][a((wa)^d)^3] = wa^{d,w}. \end{array}$$

Hence $(wa)^2ws = (wa^{d,w})^{(1,3)}$. Clearly, $a^{d,w}wa^{d,w} = a[(wa)^d]^2wa[(wa)^d]^2 = a[(wa)^d]^3$, and so

$$\begin{array}{rcl} a^{d,w}waws & = & a[(wa)^d]^2waws = a(wa)^dws = a[(wa)^d]^3(wa)^2ws \\ & = & [a^{d,w}wa^{d,w}][(wa)^2ws] = a^{d,w}wa^{d,w}(wa^{d,w})^{(1,3)}. \end{array}$$

Therefore $s = a^{d,w}waws + (1 - a^{d,w}waw)s = a^{d,w}wa^{d,w}(wa^{d,w})^{(1,3)} + (1 - a^{d,w}waw)s$, as desired.

(3) ⇒ (1) By hypothesis, we can find some $y, z \in \mathcal{A}$ such that $q = a^{d,w} + y(1 - wa^{d,w}wa)$ and $s = a^{d,w}wa^{d,w}(wa^{d,w})^{(1,3)} + (1 - a^{d,w}waw)z$. In view of [4, Theorem 1.2], we have $[(wa)^d]^2[(wa)^d]^{(1,3)} = (wa)^{\circledast}$. Since $wa^{d,w} = (wa)^d$, one directly verifies that

$$\begin{array}{rl} & q(wawa^{d,w}waw)s \\ = & [a^{d,w} + y(1 - wa^{d,w}wa)][wawa^{d,w}waw][a^{d,w}wa^{d,w}(wa^{d,w})^{(1,3)} \\ + & (1 - a^{d,w}waw)z] \\ = & a^{d,w}[wawa^{d,w}waw][a^{d,w}wa^{d,w}(wa^{d,w})^{(1,3)} + (1 - a^{d,w}waw)z] \\ = & a^{d,w}[wawa^{d,w}waw][a^{d,w}wa^{d,w}(wa^{d,w})^{(1,3)}] \\ = & a[(wa)^d]^3[(wa)^d]^{(1,3)} = a(wa)^d[(wa)^d]^2[(wa)^d]^{(1,3)} \\ = & a(wa)^d(wa)^{\circledast} = a(wa)^d(wa)[(wa)^{\circledast}]^2 \\ = & a[(wa)^{\circledast}]^2 = a^{\circledast,w}. \end{array}$$

This completes the proof. □

Corollary 3.5. *Let $a \in \mathcal{A}^{\circledast}$ and $s \in (aa^da)^{(1)}$. Then the following are equivalent:*

(1) $a^{\circledast} = q(aa^da)s$.
(2) $a^d = qaa^d$ and $s \in (aa^da)^{(3)}$.
(3) $q = a^d + y(1 - a^da)$ and $s = (a^d)^2(a^d)^{(1,3)} + (1 - a^da)z$ for some $y, z \in \mathcal{A}$.

Proof. This is obvious by choosing $w = 1$ in Theorem 3.4. □

We are ready to extend [22, Theorem 3.2] as follows.

Theorem 3.6. *Let $a \in \mathcal{A}^{\textcircled{\#},w}$ and $s \in (a^{d,w}w)^{(1)}$. Then the following are equivalent:*

(1) $a^{\textcircled{\#},w} = (wawa^{d,w}ws)^{\dagger}$.
(2) $s \in (a^{d,w}w)^{(3)}$.
(3) $s = awp_{im(aw)^d} + (1 - a^{d,w}waw)z$ for some $z \in \mathcal{A}$.

Proof. (1) ⇒ (2) By using Cline's formula (see [15, Theorem 2.1]), we have $wawa^{d,w}ws = wawa[(wa)^d]^2ws = (wa)(wa)^dws$. In view of [4, Theorem 1.2], we see that

$$\begin{array}{rcl} a^{\textcircled{\#},w}(wa)(wa)^dws & = & a[(wa)^{\textcircled{\#}}]^2(wa)(wa)^dws \\ & = & a[((wa)^d)^2((wa)^d)^{(1,3)}]^2(wa)(wa)^dws \\ & = & a((wa)^d)^2ws = (aw)^ds. \end{array}$$

By hypothesis, $[(aw)^ds]^* = (aw)^ds$. By using Cline's formula again, we have $a^{d,w}w = a[(wa)^d]^2w = (aw)^d$. Then $[a^{d,w}ws]^* = a^{d,w}ws$, and therefore $s \in (a^{d,w}w)^{(3)}$.

(2) ⇒ (3) By Cline's formula , $(a^{d,w}w)s = a[(wa)^d]^2ws = (aw)^ds$. Let $p = (aw)^ds$. By hypothesis, $p = p^* = p^2$ and $im(p) = im(aw)^d$. Hence, $(a^{d,w}w)s = p_{im(aw)^d}$. Accordingly,

$$\begin{array}{rcl} s & = & a^{d,w}waws + (1 - a^{d,w}waw)s \\ & = & a[(wa)^d]^2waws + (1 - a^{d,w}waw)s \\ & = & aw(aw)^ds + +(1 - a^{d,w}waw)s \\ & = & awp_{im(aw)^d} + (1 - a^{d,w}waw)s, \end{array}$$

as desired.

(3) ⇒ (1) Write $s = awp_{im(aw)^d} + (1 - a^{d,w}waw)z$ for some $z \in \mathcal{A}$. Then

$$\begin{array}{rcl} wawa^{d,w}ws & = & wawa^{d,w}w[awp_{im(aw)^d} + (1 - a^{d,w}waw)z] \\ & = & (wawa^{d,w}waw)p_{im(aw)^d} \\ & = & w(aw)^2(aw)^dp_{im(aw)^d} \\ & = & wawp_{im(aw)^d}. \end{array}$$

In view of [4, Theorem 1.2], $(wa)^{\textcircled{\#}} = [(wa)^d]^2[(wa)^d]^{(1,3)}$. Then we directly verify that

$$\begin{array}{rl} & a^{\textcircled{\#},w}wawp_{im(aw)^d} \\ = & a[(wa)^{\textcircled{\#}}]^2wawp_{im(aw)^d} \\ = & a[(wa)^d]^2[(wa)^d]^{(1,3)}[(wa)^d]^2[(wa)^d]^{(1,3)}w(aw)^2(aw)^dp_{im(aw)^d} \\ = & a[(wa)^d]^2[(wa)^d][(wa)^d]^{(1,3)}(wa)^dw(aw)^2p_{im(aw)^d} \\ = & a[(wa)^d]^2w(aw)^dawawp_{im(aw)^d} \\ = & a[(wa)^d]^2wawp_{im(aw)^d} \\ = & (aw)^dawp_{im(aw)^d} \\ = & p_{im(aw)^d}. \end{array}$$

Then $wawp_{im(aw)^d}a^{\textcircled{\#},w}wawp_{im(aw)^d} = waw[p_{im(aw)^d}]^2 = wawp_{im(aw)^d}$. Moreover, we check that

$$\begin{array}{rcl} wawp_{im(aw)^d}a^{\textcircled{\#},w} & = & wawa[(wa)^d]^2wsa[(wa)^{\textcircled{\#}}]^2 = wa(wa)^dwsa[(wa)^{\textcircled{\#}}]^2 \\ & = & wawa[(wa)^d]^2wsa[(wa)^{\textcircled{\#}}]^2 \\ & = & waw(aw)^dsa[(wa)^d]^2wawa[(wa)^{\textcircled{\#}}]^2 \\ & = & waw[(aw)^ds(aw)^d]awa[(wa)^{\textcircled{\#}}]^2 \\ & = & wawa(wa)(wa)^d[(wa)^{\textcircled{\#}}]^2 \\ & = & (wa)^2[(wa)^{\textcircled{\#}}]^2 = (wa)[(wa)((wa)^{\textcircled{\#}})^2] \\ & = & wa(wa)^{\textcircled{\#}}, \\ a^{\textcircled{\#},w}wawp_{im(aw)^d}a^{\textcircled{\#},w} & = & a^{\textcircled{\#},w}wa(wa)^{\textcircled{\#}} = a[(wa)^{\textcircled{\#}}]^2wa(wa)^{\textcircled{\#}} \\ & = & a[(wa)^{\textcircled{\#}}]^2 = a^{\textcircled{\#},w}, \\ (wawp_{im(aw)^d}a^{\textcircled{\#},w})^* & = & wawp_{im(aw)^d}a^{\textcircled{\#},w}. \end{array}$$

Therefore $a^{\textcircled{d},w} = (wawa^{d,w}ws)^{\dagger}$ and so the proof is complete. □

Now, it is straightforward to present the following corollary for the generalized core-EP inverse.

Corollary 3.7. *Let $a \in \mathcal{A}^{\textcircled{d}}$ and $s \in (a^d)^{(1)}$. Then the following are equivalent:*

(1) $a^{\textcircled{d}} = (aa^d s)^{\dagger}$.
(2) $s \in (a^d)^{(3)}$.
(3) $s = ap_{im(a^d)} + (1 - a^d a)z$ for some $z \in \mathcal{A}$.

4. Relations with other outer inverses

The primary focus of this section is to explore the relationships between the weighted generalized core-EP inverse and different types of outer inverses. Let $a, d \in \mathcal{A}$. We say that $x \in \mathcal{A}$ is the Mary inverse of a relatively to d provided that

$$xad = d = dax, x \in d\mathcal{A} \bigcap \mathcal{A}d.$$

We denote x by $a^{\| d}$ (see [16]). We now derive

Theorem 4.1. *Let $a \in \mathcal{A}^{\textcircled{d},w}$. Then $a^{\textcircled{d},w} = (waw)^{\| a(wa)^d((wa)^d)^*}$.*

Proof. Obviously, $a \in \mathcal{A}^{d,w}$. Let $x = a^{\textcircled{d},w}$. By virtue of Theorem 2.1, $x = a[(wa)^{\textcircled{d}}]^2$. In view of [4, Theorem 1.2], $(wa)^{\textcircled{d}} = [(wa)^d]^2\Big((wa)^d\Big)^{(1,3)}$. Then we check that

$$\begin{array}{rcl}
x(waw)a(wa)^d((wa)^d)^* & = & a[(wa)^{\textcircled{d}}]^2(wa)^2(wa)^d((wa)^d)^* \\
& = & a[[(wa)^d]^2\Big((wa)^d\Big)^{(1,3)}]^2(wa)^2(wa)^d((wa)^d)^* \\
& = & a[(wa)^d]^2\Big((wa)^d\Big)^{(1,3)}(wa)^d(wa)((wa)^d)^* \\
& = & a(wa)^d((wa)^d)^*, \\
a(wa)^d((wa)^d)^*(waw)x & = & a(wa)^d((wa)^d)^*(waw)a[(wa)^{\textcircled{d}}]^2 \\
& = & a(wa)^d((wa)^d)^*(wa)(wa)^{\textcircled{d}} \\
& = & a(wa)^d\Big((wa)(wa)^{\textcircled{d}}(wa)^d\Big)^* \\
& = & a(wa)^d\Big((wa)((wa)^d)^2((wa)^d)^{(1,3)}(wa)^d\Big)^* \\
& = & a(wa)^d((wa)^d)^*.
\end{array}$$

Furthermore, we have

$$\begin{array}{rcl}
x & = & a[(wa)^{\textcircled{d}}]^2 \\
& = & a(wa)^d[(wa)^d]^{(1,3)}(wa)^d(wa)[(wa)^{\textcircled{d}}]^2 \\
& = & \Big[a(wa)^d((wa)^d)^*\Big]\Big(((wa)^d)^{(1,3)}\Big)^*(wa)[(wa)^{\textcircled{d}}]^2, \\
& = & a[(wa)^d]^2(wa)^d)^{(1,3)}][(wa)^d]^2(wa)^d)^{(1,3)} \\
& = & a[(wa)^d]^2(wa)^d(wa)^d)^{(1,3)} \\
& = & a[(wa)^d]^2\Big((wa)^d)^{(1,3)}\Big)^*\Big((wa)^d\Big)^* \\
& = & a[(wa)^d]^2\Big((wa)^d)^{(1,3)}(wa)^d(wa)^d)^{(1,3)}\Big)^*\Big((wa)^d\Big)^* \\
& = & a[(wa)^d]^2\Big((wa)^d)^{(1,3)}\Big)^*((wa)^d)^{(1,3)}(wa)^d\Big((wa)^d\Big)^* \\
& = & a[(wa)^d]^2\Big((wa)^d)^{(1,3)}\Big)^*((wa)^d)^{(1,3)}(wa)^d w\Big[a(wa)^d\Big((wa)^d\Big)^*\Big]
\end{array}$$

Hence, $x \in \Big[a(wa)^d((wa)^d)^*\Big]\mathcal{A} \bigcap \mathcal{A}\Big[a(wa)^d((wa)^d)^*\Big]$. Therefore we conclude that $x = (waw)^{\| a(wa)^d((wa)^d)^*}$. □

Corollary 4.2. *Let $a \in \mathcal{A}^{\textcircled{d}}$. Then $a^{\textcircled{d}} = a^{\| aa^d(a^d)^*}$.*

Proof. This is obvious by choosing $w = 1$ in Theorem 4.1. □

Let $a, b, c \in \mathcal{A}$. An element a has (b, c)-inverse provide that there exists $x \in \mathcal{A}$ such that

$$xab = b, cax = c \text{ and } x \in b\mathcal{A}x \bigcap x\mathcal{A}c.$$

If such x exists, it is unique and denote it by $a^{(b,c)}$ (see [9]).

Theorem 4.3. *Let $a, w \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}^{\textcircled{d},w}$.
(2) $a \in \mathcal{A}^{d,w}$ and a has $((aw)^d, ((wa)^d)^*)$-inverse.

In this case, $a^{\textcircled{d},w} = a[(waw)^{((aw)^d,((wa)^d)^*)}]^2$.

Proof. (1) ⇒ (2) In view of Theorem 2.1, $a \in \mathcal{A}^{d,w}$. Let $x = a^{\textcircled{d},w}$. Set $q = aa^{\textcircled{d},w}$. We verify that

$$\begin{array}{rcl} x & = & a(wx)^2 = awa[(wa)^{\textcircled{d}}]^2wx \\ & = & awa[(wa)^d]^2\Big(wa(wa)^d\Big)^{(1,3)}(wa)^{\textcircled{d}}wx \\ & = & aw(aw)^d a(wa)^d\Big(wa(wa)^d\Big)^{(1,3)}(wa)^{\textcircled{d}}wx \in (aw)^d\mathcal{A}x, \\ x & = & xwawx = xwawa[(wa)^{\textcircled{d}}]^2 = xwawa(wa)^{\textcircled{d}}(wa)^d\Big(wa(wa)^d\Big)^{(1,3)} \\ & = & xwawa(wa)^{\textcircled{d}}(wa)^d(wa)(wa)^d\Big(wa(wa)^d\Big)^{(1,3)} \\ & = & xwawa(wa)^{\textcircled{d}}(wa)^d\Big((wa)(wa)^d\Big(wa(wa)^d\Big)^{(1,3)}\Big)^* \\ & = & xwawa(wa)^{\textcircled{d}}(wa)^d\Big((wa)\Big(wa(wa)^d\Big)^{(1,3)}\Big)^*\Big((wa)^d\Big)^* \in x\mathcal{A}(aw)^d, \end{array}$$

$$\begin{array}{rcl} x(waw)(aw)^d & = & a[(wa)^{\textcircled{d}}]^2(waw)(aw)^d \\ & = & a[(wa)^d]^2((wa)^d)^{(1,3)}[(wa)^d]^2((wa)^d)^{(1,3)}(waw)[(aw)^d]^2(aw) \\ & = & a[(wa)^d]^2((wa)^d)^{(1,3)}[(wa)^d]^2((wa)^d)^{(1,3)}wa(wa)^dw) \\ & = & a[(wa)^d]^2((wa)^d)^{(1,3)}[(wa)^d]^2waw) \\ & = & a[(wa)^d]^2w = (aw)^d, \\ ((wa)^d)^*wawx & = & ((wa)^d)^*wawa[(wa)^d]^2((wa)^d)^{(1,3)}[(wa)^d]^2((wa)^d)^{(1,3)} \\ & = & ((wa)^d)^*(wa)^d((wa)^d)^{(1,3)} = ((wa)^d)^*\Big((wa)^d((wa)^d)^{(1,3)}\Big)^* \\ & = & ((wa)^d)^*. \end{array}$$

Therefore waw has $((aw)^d, ((wa)^d)^*)$-inverse x, as desired.

(2) ⇒ (1) Let $x = (waw)^{((aw)^d,((wa)^d)^*)}$. Then we can find some $s, t \in \mathcal{A}$ such that

$$x = (aw)^d sx = xt((wa)^d)^*, xwaw(aw)^d = (aw)^d, ((wa)^d)^*wawx = ((wa)^d)^*.$$

Claim 1. $xwawx = x$. We verify that

$$\begin{array}{rcl} xwawx & = & [xt((wa)^d)^*]wawx \\ & = & xt[((wa)^d)^*wawx] \\ & = & xt((wa)^d)^* \\ & = & x. \end{array}$$

Claim 2. $im(xw) = im(aw)^d$. Since $xwaw(aw)^d = (aw)^d$, we have $im(aw)^d \subseteq im(xw)$. It follows from $x = (aw)^d sx$ that $im(xw) \subseteq im(aw)^d$; hence, $im(xw) = im(aw)^d$.

Claim 3. $im(wx)^* = im(wa)^d$. As $((wa)^d)^*wawx = ((wa)^d)^*$, we have $wa^d = (wx)^*(wa)^*(wa)^d \in im(wx)^*$, and so $im(wa)^d \subseteq im(wx)^*$. By hypothesis, we have $x = xt((wa)^d)^*$, and then $wx = wxt((wa)^d)^*$. This implies that $(wx)^* = (wa)^d(wxt)^*$; whence, $im(wx)^* \subseteq im(wa)^d$. Thus, $im(wx)^* = im(wa)^d$.

In light of Theorem 3.1, $a \in \mathcal{A}^{\textcircled{d},w}$ and $a^{\textcircled{d},w} = a(wx)^2$, as required. □

Consequently, we derive

Corollary 4.4. *Let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}^{\textcircled{d}}$.
(2) $a \in \mathcal{A}^d$ and a has $(a^d, (a^d)^*)$-inverse.

In this case, $a^{\textcircled{d}} = a\Big[a^{(a^d,(a^d)^*)}\Big]^2$.

Proof. This is obvious by choosing $w = 1$ in Theorem 4.3. □

Let $a \in \mathcal{A}$. We say that a has $\{2\}$-inverse x provided that $x = xax$. We denote $a^{(2)}_{T,S} = \{x \in \mathcal{A} \mid xax = x, im(a) = T, ker(a) = S\}$. We now proceed with the derivation.

Theorem 4.5. *Let $a \in \mathcal{A}^{\textcircled{d},w}$. Then*

$$a^{\textcircled{d},w} = (waw)^{(2)}_{im(aw)^d, ker((wa)^d)^{(1,3)}}.$$

Proof. Let $x = a^{\textcircled{d},w}$. In view of Theorem 3.1, we have $x = x(waw)x$.

Step 1. $im(x) = im(aw)^d$. In view of Theorem 2.1 and [4, Theorem 1.2], we have

$$\begin{array}{rcl} x & = & a[(wa)^{\textcircled{d}}]^2 = a[(wa)^d]^2((wa)^d)^{(1,3)}(wa)^{\textcircled{d}} \\ & = & \Big(a[(wa)^d]^2w\Big)a(wa)^d((wa)^d)^{(1,3)}(wa)^{\textcircled{d}} \\ & = & (aw)^d a(wa)^d((wa)^d)^{(1,3)}(wa)^{\textcircled{d}}. \end{array}$$

Hence, $im(x) \subseteq im(aw)^d$. It is easy to verify that

$$\begin{array}{rcl} (aw)^d & = & a[(wa)^d]^2w \\ & = & a[(wa)^d]^2((wa)^d)^{(1,3)}(wa)^d w \\ & = & a[(wa)^{\textcircled{d}}](wa)^d w \\ & = & a[(wa)^{\textcircled{d}}][(wa)^d]^2((wa)^d)^{(1,3)}(wa)^d waw \\ & = & a[(wa)^{\textcircled{d}}]^2(wa)^d waw \\ & = & x(wa)^d waw. \end{array}$$

Hence, $im(aw)^d \subseteq im(x)$, as required.

Step 2. $ker(x) = ker((wa)^d)^{(1,3)}$. Since $x = a(wa)^{\textcircled{d}}[(wa)^d]^2((wa)^d)^{(1,3)}$, we see that $ker(x) \subseteq ker((wa)^d)^{(1,3)}$. If $xz = 0$ for some $z \in \mathcal{A}$, then $a[(wa)^{\textcircled{d}}]^2z = 0$, and so $wa[(wa)^{\textcircled{d}}]^2z = 0$. This implies that $(wa)^{\textcircled{d}}z = 0$. In light of [4, Theorem 1.2], we have

$$\begin{array}{rcl} ((wa)^d)^{(1,3)}z & = & ((wa)^d)^{(1,3)}(wa)^d((wa)^d)^{(1,3)}z \\ & = & ((wa)^d)^{(1,3)}(wa)[(wa)^d]^2((wa)^d)^{(1,3)}z \\ & = & ((wa)^d)^{(1,3)}(wa)[(wa)^{\textcircled{d}}z] \\ & = & 0. \end{array}$$

We infer that $ker((wa)^d)^{(1,3)} \subseteq ker(x)$. This completes the proof. □

Corollary 4.6. *Let $a \in \mathcal{A}^{\textcircled{d}}$. Then*

$$a^{\textcircled{d}} = a^{(2)}_{im(a^d), ker(a^d)^{(1,3)}}.$$

We now provide a new properties of weighted core-EP inverse for a complex matrix.

Corollary 4.7. *Let $A \in \mathbb{C}^{m\times n}, W \in \mathbb{C}^{n\times m}$. Then*

$$\begin{array}{rcl} A^{\textcircled{\dagger},W} & = & A[(WAW)^{((AW)^D,((WA)^D)^*)}]^2 \\ & = & (WAW)^{(2)}_{R(AW)^D, N((WA)^D)^{\dagger}}. \end{array}$$

Proof. This is an immediate consequence of Theorem 4.3 and Theorem 4.5. □

5. Weighted Generalized core-EP orders

In 1978, Drazin [8] introduced the star partial order on a semigroup with involution. That is, if $a, b \in S^{\dagger}$, then a is below b under star partial order (written as $a <^* b$), if $aa^{\dagger} = ba^{\dagger}$ and $a^{\dagger}a = a^{\dagger}b$. After this publication, other partial orders such as minus, sharp, core and core-EP partial orders (see [1, 7, 10, 13, 17, 20, 27]), induced by specific generalized inverses, are defined and studied. Recently, in [28] Stanisev et al. studied a binary relation "$<^{||(b,c)}$", induced by the (b, c)-inverse, and proved that this relation is not a partial order, but an equivalence relation. Various weighted binary relations induced by specific weighted generalized inverses are studied. In [18], Mosić introduced and characterized new weighted pre-orders on the set of all bounded linear operators between two Banach spaces. In [11], Gao et al. investigated a binary relation on the set of rectangular complex matrices related to the weighted core-EP inverse. The weighted core-EP pre-order was introduced on the set of all weighted generalized Drazin invertible elements within a C^*-algebra (refer to [23]). For further reading on operator binary relations defined by various weighted generalized inverses, see [14, 19, 25, 31]. In this section, we investigate the weighted generalized core-EP order for two elements within a Banach *-algebra $\mathcal{A}$. Our exploration begins with the following definition.

Definition 5.1. *Let $a, b \in \mathcal{A}, w \in \mathcal{A} \setminus \{0\}$ and $a \in \mathcal{A}^{\textcircled{d},w}$. We define a binary relation "$\leq^{\textcircled{d},w}$" on $\mathcal{A}$ in the following way: $a \leq^{\textcircled{d},w} b$ if and only if*

$$(aw)a^{\textcircled{d},w} = (bw)a^{\textcircled{d},w}, a^{\textcircled{d},w}(wa) = a^{\textcircled{d},w}(wb).$$

Theorem 5.2. *The following statements are equivalent:*

(1) $a \leq^{\textcircled{d},w} b$.
(2) $a(wa)^d = b(wa)^d, (wa)^*(wa)^d = (wb)^*(wa)^d$.
(3) $a(wa)^{\textcircled{d}} = b(wa)^{\textcircled{d}}, (wa)^{\textcircled{d}}(wa) = (wa)^{\textcircled{d}}(wb)$.

Proof. (1) ⇒ (3) Since $a \leq^{\textcircled{d},w} b$, we have

$$(aw)a^{\textcircled{d},w} = (bw)a^{\textcircled{d},w}, a^{\textcircled{d},w}(wa) = a^{\textcircled{d},w}(wb).$$

In view of Theorem 2.1, $a^{\textcircled{d},w} = a[(wa)^{\textcircled{d}}]^2$. Hence,

$$(aw)a[(wa)^{\textcircled{d}}]^2 = (bw)a[(wa)^{\textcircled{d}}]^2.$$

As $wa[(wa)^{\textcircled{d}}]^2 = (wa)^{\textcircled{d}}$, we see that $a(wa)^{\textcircled{d}} = b(wa)^{\textcircled{d}}$. Similarly, we prove that $a^{\textcircled{d},w}(wa) = a^{\textcircled{d},w}(wb)$, as desired.

(3) ⇒ (2) In view of [4, Theorem 1.2], $(wa)^{\textcircled{d}} = ((wa)^d)^2((wa)^d)^{(1,3)}$. Since $(wa)^{\textcircled{d}}(wa) = (wa)^{\textcircled{d}}(wb)$, we have

$$[(wa)^d]^2((wa)^d)^{(1,3)}(wa) = [(wa)^d]^2((wa)^d)^{(1,3)}(wb).$$

As $(wa)^d = wa[(wa)^d]^2$, we get $(wa)^d((wa)^d)^{(1,3)}(wa) = (wa)^d((wa)^d)^{(1,3)}(wb)$. This implies that $(wa)^*(wa)^d((wa)^d)^{(1,3)} = (wb)^*(wa)^d((wa)^d)^{(1,3)}$. Accordingly, we have $(wa)^*(wa)^d = (wb)^*(wa)^d$, as required.

(2) ⇒ (1) Since $a(wa)^d = b(wa)^d$, by virtue of [4, Theorem 1.2], we have

$$\begin{aligned} a(wa)^{\textcircled{d}} &= a[(wa)^d]^2((wa)^d)^{(1,3)} \\ &= b[(wa)^d]^2((wa)^d)^{(1,3)} \\ &= b(wa)^{\textcircled{d}}. \end{aligned}$$

One easily checks that

$$wa^{\textcircled{d},w} = wa[(wa)^{\textcircled{d}}]^2 = (wa)^{\textcircled{d}},$$

and then $(aw)a^{\textcircled{d},w} = (bw)a^{\textcircled{d},w}$. Analogously, $a^{\textcircled{d},w}(wa) = a^{\textcircled{d},w}(wb)$. Therefore we complete the proof. □

We note that the relation $\leq^{\textcircled{d},w}$ is a pre-order as the following shows.

Corollary 5.3. *If $a \leq^{\textcircled{d},w} b$ and $b \leq^{\textcircled{d},w} c$, then $a \leq^{\textcircled{d},w} c$.*

Proof. In view of Theorem 5.2, we have

$$a(wa)^{\textcircled{\#}} = b(wa)^{\textcircled{\#}}, (wa)^{\textcircled{\#}}(wa) = (wa)^{\textcircled{\#}}(wb);$$
$$b(wb)^{\textcircled{\#}} = c(wb)^{\textcircled{\#}}, (wb)^{\textcircled{\#}}(wb) = (wb)^{\textcircled{\#}}(wc).$$

Then

$$\begin{array}{ll} & a(wa)^{\textcircled{\#}} \\ = & b(wa)^{\textcircled{\#}} = b[wa(wa)^{\textcircled{\#}}](wa)^{\textcircled{\#}} \\ = & b(wb)[(wa)^{\textcircled{\#}}]^2 = \cdots = b(wb)^n[(wa)^{\textcircled{\#}}]^{n+1} \\ = & b[(wb)^n - (wb)^d(wb)^{n+1}][(wa)^{\textcircled{\#}}]^{n+1} + b(wb)^d(wb)^{n+1}[(wa)^{\textcircled{\#}}]^{n+1}; \\ & c(wa)^{\textcircled{\#}} \\ = & c[wa(wa)^{\textcircled{\#}}](wa)^{\textcircled{\#}} = c[wb(wa)^{\textcircled{\#}}](wa)^{\textcircled{\#}} \\ = & cwb[(wa)^{\textcircled{\#}}]^2 = \cdots = c(wb)^n[(wa)^{\textcircled{\#}}]^{n+1} \\ = & c[(wb)^n - (wb)^d(wb)^{n+1}][(wa)^{\textcircled{\#}}]^{n+1} + c(wb)^d(wb)^{n+1}[(wa)^{\textcircled{\#}}]^{n+1}. \end{array}$$

Since $b(wb)^d = c(wb)^d$, we have

$$||a(wa)^{\textcircled{\#}} - c(wa)^{\textcircled{\#}}||^{\frac{1}{n}} \leq ||b-c||^{\frac{1}{n}}||(wb)^n - (wb)^d(wb)^{n+1}]||^{\frac{1}{n}}||(wa)^{\textcircled{\#}}||^{1+\frac{1}{n}}.$$

As $\lim_{n\to\infty} ||(wb)^n - (wb)^d(wb)^{n+1}||^{\frac{1}{n}} = 0$, we see that

$$\lim_{n\to\infty} ||a(wa)^{\textcircled{\#}} - c(wa)^{\textcircled{\#}}||^{\frac{1}{n}} = 0.$$

Therefore $a(wa)^{\textcircled{\#}} = c(wa)^{\textcircled{\#}}$. By a similar route, we check that $(wa)^{\textcircled{\#}}(wa) = (wa)^{\textcircled{\#}}(wc)$. Therefore $a \leq^{\textcircled{\#},w} c$, the corollary is true. □

We are now ready to prove the following.

Theorem 5.4. *Let $aw, wa \in \mathcal{A}^{\textcircled{\#}}$. Then the following statements are equivalent:*

(1) $a \leq^{\textcircled{\#},w} b$.
(2) $wa \leq^{\textcircled{\#}} wb$ and $aw \leq^{\textcircled{\#}} bw$.

Proof. (1) ⇒ (2) Since $a \leq^{\textcircled{\#},w} b$, we have

$$(aw)a^{\textcircled{\#},w} = (bw)a^{\textcircled{\#},w}, a^{\textcircled{\#},w}(wa) = a^{\textcircled{\#},w}(wb).$$

In view of Theorem 2.1,

$$a^{\textcircled{\#},w} = a[(wa)^{\textcircled{\#}}]^2.$$

Hence

$$\begin{array}{rcl} (wa)(wa)^{\textcircled{\#}} & = & w(aw)a[(wa)^{\textcircled{\#}}]^2 \\ & = & w(aw)a^{\textcircled{\#},w} \\ & = & w(bw)a^{\textcircled{\#},w} \\ & = & w(bw)a[(wa)^{\textcircled{\#}}]^2 \\ & = & (wb)(wa)^{\textcircled{\#}}. \end{array}$$

Furthermore, we check that

$$\begin{array}{rcl} (wa)^{\textcircled{\#}}(wa) & = & wa[(wa)^{\textcircled{\#}}]^2(wa) \\ & = & w[a((wa)^{\textcircled{\#}})^2](wa) \\ & = & w[a^{\textcircled{\#},w}(wb)] \\ & = & w[a[(wa)^{\textcircled{\#}}]^2(wb)] \\ & = & [wa((wa)^{\textcircled{\#}})^2](wb) \\ & = & (wa)^{\textcircled{\#}}(wb). \end{array}$$

Therefore $wa \leq^{\textcircled{\#}} wb$. Analogously, we prove that $aw \leq^{\textcircled{\#}} bw$, as required.

(2) ⇒ (1) By hypothesis, we have

$$\begin{aligned}(wa)(wa)^{\textcircled{d}} = (wb)(wa)^{\textcircled{d}}, (wa)^{\textcircled{d}}(wa) = (wa)^{\textcircled{d}}(wb);\\ (aw)(aw)^{\textcircled{d}} = (bw)(aw)^{\textcircled{d}}, (aw)^{\textcircled{d}}(aw) = (aw)^{\textcircled{d}}(bw).\end{aligned}$$

In view of [4, Theorem 1.2],

$$\begin{aligned}(aw)(aw)^d[(aw)(aw)^d]^{(1,3)} &= (aw)(aw)^{\textcircled{d}}\\ &= (bw)(aw)^{\textcircled{d}}\\ &= (bw)(aw)^d[(aw)(aw)^d]^{(1,3)},\end{aligned}$$

and so

$$\begin{aligned}&(aw)[(aw)^d]^2(aw)(aw)^d[(aw)(aw)^d]^{(1,3)}(aw)(aw)^d\\ =\ &(bw)[(aw)^d]^2(aw)(aw)^d[(aw)(aw)^d]^{(1,3)}(aw)(aw)^d.\end{aligned}$$

We infer that $aw[(aw)^d]^2(aw)(aw)^d = bw[(aw)^d]^2(aw)(aw)^d$, hence $aw(aw)^d = bw(aw)^d$. In view of Cline's formula and [4, Theorem 1.2], we derive

$$\begin{aligned}a(wa)^{\textcircled{d}} &= a(wa)^d[(wa)(wa)^d]^{(1,3)}\\ &= aw[(aw)^d]^2a[(wa)(wa)^d]^{(1,3)}\\ &= [aw(aw)^d](aw)^da[(wa)(wa)^d]^{(1,3)}\\ &= [bw(aw)^d](aw)^da[(wa)(wa)^d]^{(1,3)}\\ &= bw[(aw)^d]^2a[(wa)(wa)^d]^{(1,3)}\\ &= b(wa)^d[(wa)(wa)^d]^{(1,3)}\\ &= b(wa)^{\textcircled{d}}.\end{aligned}$$

Since $a^{\textcircled{d},w} = a[(wa)^{\textcircled{d}}]^2$, we verify that

$$\begin{aligned}(aw)a^{\textcircled{d},w} &= (aw)a[(wa)^{\textcircled{d}}]^2\\ &= a[wa((wa)^{\textcircled{d}})^2]\\ &= a(wa)^{\textcircled{d}}\\ &= b(wa)^{\textcircled{d}}\\ &= (bw)a[(wa)^{\textcircled{d}}]^2\\ &= (bw)a^{\textcircled{d},w}.\end{aligned}$$

Likewise, we prove that

$$a^{\textcircled{d},w}(wa) = a^{\textcircled{d},w}(wb).$$

Therefore $a \leq^{\textcircled{d},w} b$, as asserted. □

Corollary 5.5. *Let $aw, wa \in \mathcal{A}^{\textcircled{d}}$. Then the following statements are equivalent:*

(1) $a \leq^{\textcircled{d},w} b$.
(2)

$$\begin{aligned}(wa)(wa)^d = (wb)(wa)^d, (wa)^*(wa)^d = (wb)^*(wa)^d;\\ (aw)(aw)^d = (bw)(aw)^d, (aw)^*(aw)^d = (bw)^*(aw)^d.\end{aligned}$$

Proof. (1) ⇒ (2) In view of Theorem 5.4, $aw \in \mathcal{A}^{\textcircled{d}}$ and $aw \leq^{\textcircled{d}} bw$. Moreover, we have

$$(aw)(aw)^{\textcircled{d}} = (bw)(aw)^{\textcircled{d}}, (aw)^{\textcircled{d}}(aw) = (aw)^{\textcircled{d}}(bw).$$

As in the proof of Theorem 5.2, we prove that $(aw)(aw)^d = (bw)(aw)^d$. Since $(aw)^{\textcircled{d}}(aw) = (aw)^{\textcircled{d}}(bw)$, we have

$$(aw)(aw)^{\textcircled{d}}(aw) = (aw)(aw)^{\textcircled{d}}(bw).$$

Then

$$[(aw)(aw)^{\textcircled{d}}]^*(aw) = [(aw)(aw)^{\textcircled{d}}]^*(bw).$$

This implies that
$$(aw)^*(aw)(aw)^{\textcircled{d}} = (bw)^*(aw)(aw)^{\textcircled{d}}.$$

Obviously,
$$(aw)(aw)^{\textcircled{d}}[(aw)(aw)^d](aw)^d = (aw)^d;$$
hence, $(aw)^*(aw)^d = (bw)^*(aw)^d$. Similarly, we prove that $(wa)(wa)^d = (wb)(wa)^d, (wa)^*(wa)^d = (wb)^*(wa)^d$, as desired.

(2) $\Rightarrow$ (1) by hypothesis, $wa, aw \in \mathcal{A}^d$. Since $(aw)(aw)^d = (bw)(aw)^d$, we see that
$$(aw)(aw)^d[(aw)(aw)^d]^{(1,3)} = (bw)(aw)^d[(aw)(aw)^d]^{(1,3)},$$
and then $(aw)(aw)^{\textcircled{d}} = (bw)(aw)^{\textcircled{d}}$.

By hypothesis, $(aw)^*(aw)^d = (bw)^*(aw)^d$. Then
$$(aw)^*(aw)[(aw)^d((aw)(aw)^d)^{(1,3)}] = (bw)^*(aw)[(aw)^d((aw)(aw)^d)^{(1,3)}].$$

This implies that
$$(aw)^*(aw)(aw)^{\textcircled{d}} = (bw)^*(aw)(aw)^{\textcircled{d}}.$$

Then
$$(aw)(aw)^{\textcircled{d}}(aw) = (aw)(aw)^{\textcircled{d}}(bw).$$

We infer that
$$(aw)^{\textcircled{d}}(aw) = (aw)^{\textcircled{d}}(bw).$$

Accordingly, $aw \leq^{\textcircled{d}} bw$. Likewise, $wa \leq^{\textcircled{d}} wb$. This completes the proof. $\square$

Let $p, q \in \mathcal{A}$ be idempotents. Then for any $x \in \mathcal{A}$, we have $x = pxq + px(1-p) + (1-p)xq + (1-p)x(1-q)$. Thus x can be represented in the matrix form $x = \begin{pmatrix} pxp & px(1-p) \\ (1-p)xp & (1-p)x(1-q) \end{pmatrix}_{(p,q)}$. We possess all the information required to substantiate the following.

Theorem 5.6. *Let $aw, wa \in \mathcal{A}^{\textcircled{d}}$. Then the following are equivalent:*

(1) $a \leq^{\textcircled{d},w} b$.

(2) a, w and b are represented as
$$a = \begin{pmatrix} a_1 & a_{12} \\ 0 & a_2 \end{pmatrix}_{p\times q}, w = \begin{pmatrix} w_1 & w_{12} \\ 0 & w_2 \end{pmatrix}_{q\times p}, b = \begin{pmatrix} a_1 & b_{12} \\ 0 & b_2 \end{pmatrix}_{p\times q},$$
where
$$p = (aw)(aw)^{\textcircled{d}}, q = (wa)(wa)^{\textcircled{d}}, w_1a_{12} + w_{12}a_2 = w_1b_{12} + w_{12}b_2,$$
$$a_1w_1 \in (p\mathcal{A}p)^{-1}, a_2w_2 \in ((1-p)\mathcal{A}(1-p))^{qnil}.$$

Proof. (1) $\Rightarrow$ (2) Let $p = (aw)(aw)^{\textcircled{d}}$ and $q = (wa)(wa)^{\textcircled{d}}$. Then $[1-(aw)(aw)^{\textcircled{d}}](aw)^d = 0$ and $[1-(wa)(wa)^{\textcircled{d}}](wa)^d = 0$.
$$\begin{aligned}
(1-p)aq &= [1-(aw)(aw)^{\textcircled{d}}]a(wa)(wa)^{\textcircled{d}} \\
&= [1-(aw)(aw)^{\textcircled{d}}](aw)a(wa)(wa)^d(wa)^{\textcircled{d}} \\
&= [1-(aw)(aw)^{\textcircled{d}}](aw)^3[(aw)^d]^2a(wa)^{\textcircled{d}} \\
&= [1-(aw)(aw)^{\textcircled{d}}](aw)^d(aw)^2a(wa)^{\textcircled{d}} \\
&= 0; \\
(1-q)wp &= [1-(wa)(wa)^{\textcircled{d}}]w(aw)(aw)^{\textcircled{d}} \\
&= [1-(wa)(wa)^{\textcircled{d}}](wa)^d(wa)^2w(aw)^{\textcircled{d}} \\
&= 0.
\end{aligned}$$

We verify that

$$\begin{array}{rl} & \|a(wa)^{\textcircled{d}} - (aw)^{\textcircled{d}}(aw)]a(wa)^{\textcircled{d}}\|^{\frac{1}{n}} \\ = & \|a(wa)^n((wa)^{\textcircled{d}})^{n+1} - (aw)^{\textcircled{d}}(aw)]a(wa)^n((wa)^{\textcircled{d}})^{n+1}\|^{\frac{1}{n}} \\ = & \|[(aw)^n - (aw)^{\textcircled{d}}(aw)^{n+1}]a((wa)^{\textcircled{d}})^{n+1}\|^{\frac{1}{n}} \\ = & \|(aw)^n - (aw)^{\textcircled{d}}(aw)^{n+1}\|^{\frac{1}{n}}\|a\|^{\frac{1}{n}}\|(wa)^{\textcircled{d}}\|^{1+\frac{1}{n}}. \end{array}$$

Hence,

$$\lim_{n\to\infty} \|a(wa)^{\textcircled{d}} - (aw)^{\textcircled{d}}(aw)]a(wa)^{\textcircled{d}}\|^{\frac{1}{n}} = 0.$$

This implies that

$$[(aw)^{\textcircled{d}}(aw)]a(wa)^{\textcircled{d}} = a(wa)^{\textcircled{d}}.$$

Similarly, we have

$$[(wa)^{\textcircled{d}}wa](wa)w(aw)^{\textcircled{d}} = (wa)w(aw)^{\textcircled{d}}.$$

Thus we deduce that

$$\begin{array}{rcl} a_1w_1 & = & (aw)(aw)^{\textcircled{d}}a(wa)(wa)^{\textcircled{d}}w(aw)(aw)^{\textcircled{d}} \\ & = & (aw)[(aw)^{\textcircled{d}}(aw)]a(wa)^{\textcircled{d}}w(aw)(aw)^{\textcircled{d}} \\ & = & awa[(wa)^{\textcircled{d}}wa](wa)w[(aw)^{\textcircled{d}}]^2 \\ & = & a(wa)^2w[(aw)^{\textcircled{d}}]^2 = aw(aw)^2[(aw)^{\textcircled{d}}]^2 \\ & = & (aw)^2(aw)^{\textcircled{d}}. \end{array}$$

Therefore

$$[(aw)^2(aw)^{\textcircled{d}}](aw)^{\textcircled{d}} = (aw)^{\textcircled{d}}[(aw)^2(aw)^{\textcircled{d}}] = (aw)(aw)^{\textcircled{d}} = p.$$

Hence $a_1w_1 = [(aw)^{\textcircled{d}}]^{-1} \in (p\mathcal{A}p)^{-1}$.

Moreover, we have

$$\begin{array}{rcl} a_2w_2 & = & [1-(aw)(aw)^{\textcircled{d}}]a[1-(wa)(wa)^{\textcircled{d}}]w[1-(aw)(aw)^{\textcircled{d}}] \\ & = & [1-(aw)(aw)^{\textcircled{d}}]aw[1-a(wa)^{\textcircled{d}}w][1-(aw)(aw)^{\textcircled{d}}] \\ & = & aw[1-(aw)^{\textcircled{d}}aw][1-(aw)(aw)^{\textcircled{d}} - a(wa)^{\textcircled{d}}w + a((wa)^{\textcircled{d}}wa)w(aw)^{\textcircled{d}}] \\ & = & aw[1-(aw)^{\textcircled{d}}aw][1-a(wa)^{\textcircled{d}}w] \\ & = & aw[1-a(wa)^{\textcircled{d}}w - (aw)^{\textcircled{d}}aw + ((aw)^{\textcircled{d}}aw)a(wa)^{\textcircled{d}}w] \\ & = & aw[1-(aw)^{\textcircled{d}}aw]. \end{array}$$

Since $[1-(aw)^{\textcircled{d}}aw]aw = aw - (aw)^{\textcircled{d}}(aw)^2 \in \mathcal{A}^{qnil}$, by Cline's formula, $a_2w_2 \in \mathcal{A}^{qnil}$. Accordingly, $a_2w_2 \in [(1-p)\mathcal{A}(1-p)]^{qnil}$.

We check that $w_1a_1w_1 = w(aw)^2(aw)^{\textcircled{d}}$ and $(w_1a_1w_1)^{-1} = (aw)^{\textcircled{d}}a(wa)^{\textcircled{d}}$. Furthermore, we have

$$a^{\textcircled{d},w} = \begin{pmatrix} (w_1a_1w_1)^{-1} & 0 \\ 0 & 0 \end{pmatrix}_{p\times q}.$$

Write $b = \begin{pmatrix} b_1 & b_{12} \\ b_{21} & b_2 \end{pmatrix}_{p\times q}$. Since $a \le^{\textcircled{d},w} b$, we get

$$\begin{array}{rl} & \begin{pmatrix} a_1w_1(w_1a_1w_1)^{-1} & 0 \\ 0 & 0 \end{pmatrix}_{p\times q} \\ = & \begin{pmatrix} a_1 & a_{12} \\ 0 & a_2 \end{pmatrix}_{p\times q} \begin{pmatrix} w_1 & w_{12} \\ 0 & w_2 \end{pmatrix}_{q\times p} \begin{pmatrix} (w_1a_1w_1)^{-1} & 0 \\ 0 & 0 \end{pmatrix}_{p\times q} \\ = & (aw)a^{\textcircled{d},w} = (bw)a^{\textcircled{d},w} \\ = & \begin{pmatrix} b_1 & b_{12} \\ b_{21} & b_2 \end{pmatrix}_{p\times q} \begin{pmatrix} w_1 & w_{12} \\ 0 & w_2 \end{pmatrix}_{q\times p} \begin{pmatrix} (w_1a_1w_1)^{-1} & 0 \\ 0 & 0 \end{pmatrix}_{p\times p} \\ = & \begin{pmatrix} b_1w_1(w_1a_1w_1)^{-1} & 0 \\ b_{21}w_1(w_1a_1w_1)^{-1} & 0 \end{pmatrix}_{p\times q}. \end{array}$$

Then $a_1w_1(w_1a_1w_1)^{-1} = b_1w_1(w_1a_1w_1)^{-1}, b_{21}w_1(w_1a_1w_1)^{-1} = 0$. Hence, $a_1w_1 = b_1w_1$, and then $a_1 = (a_1w_1)(a_1w_1)(w_1a_1w_1)^{-1} = (b_1w_1)(a_1w_1)(w_1a_1w_1)^{-1} = b_1$. Also we have $b_{21}w_1 = 0$, and so $b_{21} = (b_{21}w_1)a_1w_1(w_1a_1w_1)^{-1} = 0$. Moreover, we obtain

$$\begin{aligned}
&\begin{pmatrix} (w_1a_1w_1)^{-1}w_1a_1 & (w_1a_1w_1)^{-1}w_1a_{12} + (w_1a_1w_1)^{-1}w_{12}a_2 \\ 0 & 0 \end{pmatrix}_{p\times q} \\
= &\begin{pmatrix} (w_1a_1w_1)^{-1} & 0 \\ 0 & 0 \end{pmatrix}_{p\times q} \begin{pmatrix} w_1 & w_{12} \\ 0 & w_2 \end{pmatrix}_{q\times p} \begin{pmatrix} a_1 & a_{12} \\ 0 & a_2 \end{pmatrix}_{p\times q} \\
= &\ a^{\circledR,w}(wa) = a^{\circledR,w}(wb) \\
= &\begin{pmatrix} (w_1a_1w_1)^{-1} & 0 \\ 0 & 0 \end{pmatrix}_{p\times q} \begin{pmatrix} w_1 & w_{12} \\ 0 & w_2 \end{pmatrix}_{q\times p} \begin{pmatrix} a_1 & b_{12} \\ 0 & b_2 \end{pmatrix}_{p\times q} \\
= &\begin{pmatrix} (w_1a_1w_1)^{-1}w_1a_1 & (w_1a_1w_1)^{-1}w_1b_{12} + (w_1a_1w_1)^{-1}w_{12}b_2 \\ 0 & 0 \end{pmatrix}_{p\times q}.
\end{aligned}$$

Thus, we derive

$$(w_1a_1w_1)^{-1}w_1a_{12} + (w_1a_1w_1)^{-1}w_{12}a_2 = (w_1a_1w_1)^{-1}w_1b_{12} + (w_1a_1w_1)^{-1}w_{12}b_2.$$

Consequently,

$$w_1a_{12} + w_{12}a_2 = w_1b_{12} + w_{12}b_2,$$

as desired.

(2) ⇒ (1) By hypothesis, we have

$$a^{\circledR,w} = \begin{pmatrix} (w_1a_1w_1)^{-1} & 0 \\ 0 & 0 \end{pmatrix}_{p\times q}.$$

Moreover, we derive

$$aw = \begin{pmatrix} a_1w_1 & a_1w_{12} + a_{12}w_2 \\ 0 & a_2w_2 \end{pmatrix}_{p\times p}, bw = \begin{pmatrix} a_1w_1 & a_1w_{12} + b_{12}w_2 \\ 0 & b_2w_2 \end{pmatrix}_{p\times p}.$$

Then

$$(aw)a^{\circledR,w} = \begin{pmatrix} a_1w_1(w_1a_1w_1)^{-1} & 0 \\ 0 & 0 \end{pmatrix}_{p\times p} = (bw)a^{\circledR,w}.$$

Likewise, we prove that

$$a^{\circledR,w}(wa) = a^{\circledR,w}(wb).$$

Therefore $a \leq^{\circledR,w} b$, as asserted. □

Corollary 5.7. *Let $a, b \in \mathcal{A}^{\circledR}$. Then the following are equivalent:*

(1) $a \leq^{\circledR} b$.
(2) a and b are represented as

$$a = \begin{pmatrix} a_1 & a_{12} \\ 0 & a_2 \end{pmatrix}_{p\times p}, b = \begin{pmatrix} a_1 & a_{12} \\ 0 & b_2 \end{pmatrix}_{p\times p},$$

where

$$p = aa^{\circledR}, a_1 \in (p\mathcal{A}p)^{-1}, a_2 \in ((1-p)\mathcal{A}(1-p))^{qnil}.$$

Proof. This is evident by setting $w = 1$ in Theorem 5.6. □

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