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Generalized w-core inverse in Banach algebras with involution

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Abstract. In this paper, we present the concept of the generalized w-core inverse within the context of a Banach *-algebra. We delineate the characteristics of this new generalized inverse by using the generalized weighted core decomposition and elucidate its representations via the weighted g-Drazin inverse. Additionally, we delve into an examination of the generalized w-core orders. This work broadens the scope of the weighted core inverse and pseudo core inverse, previously defined for complex matrices and linear bounded operators, to a more general setting.

1. Introduction

A Banach algebra $\mathcal{A}$ is called a Banach *-algebra if there exists an involution $* : x \to x^*$ satisfying $(x+y)^* = x^* + y^*, (\lambda x)^* = \overline{\lambda} x^*, (xy)^* = y^* x^*, (x^*)^* = x$. A C^*-algebra is a Banach *-algebra that satisfies the C^*-identity: $\|x^*x\| = \|x\|^2$ for all $x \in \mathcal{A}$. All C^*-algebras are Banach *-algebras, but the converse is not true.

Rakić et al. generalized the core inverse of a complex matrix to the case of an element in a ring (see [21]). An element a in a Banach *-algebra $\mathcal{A}$ has core inverse if there exists $x \in \mathcal{A}$ such that

$$ax^2 = x, (ax)^* = ax, xa^2 = a.$$

If such x exists, it is unique, and denote it by $a^{\textcircled{\#}}$ (see [1, 6]).

Zhu et al. introduced and studied w-core inverse for a ring element (see [30]). Let $a, w \in \mathcal{A}$. An element $a \in \mathcal{A}$ has w-core inverse if there exists $x \in \mathcal{A}$ such that

$$awx^2 = x, (awx)^* = awx, xawa = a.$$

If such x exists, it is unique, and denote it by $a_w^{\textcircled{\#}}$. Let $\mathcal{A}_w^{\textcircled{\#}}$ denote the set of all w-core invertible elements in $\mathcal{A}$. The w-core inverse was studied by many authors, e.g., [7, 10–12, 24, 25, 30, 32]. As is well known,

$$a \in \mathcal{A}_w^{\textcircled{\#}} \Leftrightarrow awx^2 = x, (awx)^* = awx, xawa = a, awxa = a, xawx = x$$

for some $x \in \mathcal{A}$ (see [30, Theorem 2.13]).

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Prasad et al. extended the concept of the core inverse and introduced the notion of core-EP inverse (i.e., pseudo core inverse) (see [8, 19]). An element $a \in \mathcal{A}$ has core-EP inverse (i.e., pseudo core inverse) if there exist $x \in \mathcal{A}$ and $k \in \mathbb{N}$ such that

$$ax^2 = x, (ax)^* = ax, xa^{k+1} = a^k.$$

If such x exists, it is unique, and denote it by $a^{\textcircled{\dagger}}$. The core-EP inverse has been investigated from many different views, e.g., [2, 8, 9, 13–16, 19, 22, 27, 28].

The motivation of this paper is to introduce and examine a new type of generalized weighted inverse, which serves as a logical extension of the generalized inverses mentioned above. Let

$$\mathcal{A}^{qnil} = \{x \in \mathcal{A} \mid \lim_{n\to\infty} \| x^n \|^{\frac{1}{n}} = 0\}.$$

As is well known, $x \in \mathcal{A}^{qnil}$ if and only if $1 + \lambda x \in \mathcal{A}$ is invertible for any $\lambda \in \mathbb{C}$. Set $\mathcal{A}_w^{qnil} = \{x \in \mathcal{A} \mid xw \in \mathcal{A}^{qnil}\}$. In Section 2, we introduce the generalized weighted core inverse in terms of a new kind of generalized weighted core decomposition. Many new properties of the w-core inverse and the core-EP inverse are thereby obtained.

Definition 1.1. *An element $a \in \mathcal{A}$ has generalized w-core decomposition if there exist $x, y \in \mathcal{A}$ such that*

$$a = x + y, x^*y = ywx = 0, x \in \mathcal{A}_w^{\textcircled{\#}}, y \in \mathcal{A}_w^{qnil}.$$

Let $a, w \in \mathcal{A}$. We prove that an element $a \in \mathcal{A}$ admits a generalized w-core decomposition if and only if there exists a unique element $x \in \mathcal{A}$ such that

$$x = awx^2, (awx)^* = awx, x(aw)^2x = awx, \lim_{n\to\infty} \|(aw)^n - awx(aw)^n\|^{\frac{1}{n}} = 0.$$

Recall that an element $a \in \mathcal{A}$ has g-Drazin inverse (i.e., generalized Drazin inverse) if there exists some $x \in \mathcal{A}$ such that

$$ax^2 = x, ax = xa, a - a^2x \in \mathcal{A}^{qnil}.$$

Such x is unique, if exists, and denote it by a^d. A fundamental result in operator theory states that a bounded linear operator on a Hilbert space possesses a g-Drazin inverse if and only if it admits an EP-quasinilpotent decomposition (see [18, Theorem 1]). The g-Drazin inverse is of great importance in matrix and operator theory (see [3]). An element $a \in \mathcal{A}$ has generalized w-Drazin inverse x if there exists a unique $x \in \mathcal{A}$ such that

$$awx = xwa, xwawx = x \text{ and } a - awxwa \in \mathcal{A}^{qnil}.$$

We denote such x by $a^{d,w}$ (see [17]). Evidently, $a^{d,w} = [(aw)^d]^2a = a[(wa)^d]^2 = (aw)^d a(wa)^d$. In Section 3, we establish equivalences between the generalized w-core inverse and the weighted g-Drazin inverse for elements in a Banach algebra, utilizing involved image characterizations.

The aim of Section 4 is to characterize the generalized weighted core inverse of an element in a Banach *-algebra in terms of other related generalized inverses, such as the weighted core inverse.

Finally, in Section 5, we introduce a new pre-order derived from the generalized w-core inverse in a Banach algebra. This relation provides a novel framework for analyzing the connections between generalized w-core invertible elements by extending the established w-core pre-order.

Throughout the paper, all Banach *-algebras are complex with an identity. $\mathcal{A}^{d,w}$ and $\mathcal{A}_w^{\textcircled{\#}}$ denote the sets of all weighted g-Drazin and w-core invertible elements in $\mathcal{A}$, respectively. Let $\mathbb{C}^{n\times n}$ be the Banach algebra of all $n \times n$ complex matrices with conjugate transpose $*$.

2. Generalized w-core decomposition

The objective of this section is to introduce the concept of the generalized w-core inverse within the framework of a Banach *-algebra. We begin with

Theorem 2.1. *Let $a, w \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}$ has generalized w-core decomposition.
(2) There exists $x \in \mathcal{A}$ such that

$$x = awx^2, (awx)^* = awx, x(aw)^2x = awx, \lim_{n\to\infty} \|(aw)^n - awx(aw)^n\|^{\frac{1}{n}} = 0.$$

Proof. (1) ⇒ (2) By hypothesis, there exist $z, y \in \mathcal{A}$ such that

$$a = z + y, z^*y = ywz = 0, z \in \mathcal{A}_w^{\textcircled{\#}}, y \in \mathcal{A}_w^{qnil}.$$

Set $x = z_w^{\textcircled{\#}}$. Then

$$\begin{array}{rcl} awx & = & (z+y)wz_w^{\textcircled{\#}} = zwz_w^{\textcircled{\#}}, \\ (awx)^* & = & (zwz_w^{\textcircled{\#}})^* = zwz_w^{\textcircled{\#}} = awx, \\ awx^2 & = & (awx)x = (zwz_w^{\textcircled{\#}})z_w^{\textcircled{\#}} = zw(z_w^{\textcircled{\#}})^2 = x. \end{array}$$

It is easy to verify that

$$\begin{array}{rcl} xawx & = & z_w^{\textcircled{\#}}(zwz_w^{\textcircled{\#}}) = z_w^{\textcircled{\#}} = x, \\ x(aw)^2x & = & (xaw)(awx) = z_w^{\textcircled{\#}}(z+y)wzwz_w^{\textcircled{\#}} = z_w^{\textcircled{\#}}zwzwz_w^{\textcircled{\#}} \\ & = & z_w^{\textcircled{\#}}zwzwz_w^{\textcircled{\#}} = zwz_w^{\textcircled{\#}} = awx. \end{array}$$

Moreover, we have

$$awxa = (awx)a = zwz_w^{\textcircled{\#}}(z+y) = zwz_w^{\textcircled{\#}}z = z,$$

and so

$$a - awxa = a - z = y \in \mathcal{A}_w^{qnil}.$$

Then

$$\begin{array}{rcl} \|(aw)^n - awx(aw)^n\| & = & \|(a - awxa)w(aw)^{n-1}\| \\ & = & \|yw(aw)^{n-1}\| = \|ywaw(aw)^{n-2}\| \\ & = & \|yw(z+y)w(aw)^{n-2}\| = \|(yw)^2(aw)^{n-2}\| \\ & = & \cdots = \|(yw)^n\|. \end{array}$$

Since $y \in \mathcal{A}_w^{qnil}$, we see that $\lim_{n\to\infty} \|(yw)^n\|^{\frac{1}{n}} = 0$. Therefore

$$\lim_{n\to\infty} \|(aw)^n - awx(aw)^n\|^{\frac{1}{n}} = 0,$$

as required.

(2) ⇒ (1) By hypotheses, there exists $x \in \mathcal{A}$ such that

$$x = awx^2, (awx)^* = awx, x(aw)^2x = awx, \lim_{n\to\infty} \|(aw)^n - awx(aw)^n\|^{\frac{1}{n}} = 0.$$

Then we check that

$$xawx = xaw(awx^2) = [x(aw)^2x]x = awx^2 = x.$$

Set $z = awxa$ and $y = a - awxa$. We verify that

$$\begin{array}{rcl} ywz & = & (a - awxa)wawxa = awawxa - awx(aw)^2xa \\ & = & awawxa - aw(awx)a = 0, \\ z^*y & = & (awxa)^*y = a^*(awx)y = a^*(awx)(a - awxa) \\ & = & a^*aw(xa - xawxa) = 0. \end{array}$$

We claim that $z \in \mathcal{A}_w^{\textcircled{\#}}$ and $z_w^{\textcircled{\#}} = x$.

Claim 1. $x = zwx^2$. We verify that

$$zwx^2 = awx(awx^2) = awx^2 = x.$$

Claim 2. $(zwx)^* = zwx$. Clearly, we have $zwx = aw(xawx) = awx$, and then $(zwx)^* = (awx)^* = awx = zwx$.
Claim 3. $xzwz = z$. One checks that

$$xzwz = (xawx)awawxa = x(aw)^2xa = awxa = z.$$

Therefore $z \in \mathcal{A}_w^{\circledast}$. Moreover, we see that

$$\begin{array}{lll} \|(aw)^n - awx(aw)^n\| & = & \|(a - awxa)w(aw)^{n-1}\| \\ & = & \|yw(aw)^{n-1}\| = \|ywaw(aw)^{n-2}\| \\ & = & \|yw(z + y)w(aw)^{n-2}\| = \|(yw)^2(aw)^{n-2}\| \\ & = & \cdots = \|(yw)^n\|. \end{array}$$

Therefore

$$\lim_{n\to\infty} \|(yw)^n\|^{\frac{1}{n}} = 0,$$

and then $y \in \mathcal{A}_w^{qnil}$. This completes the proof. $\square$

Corollary 2.2. *Let $a, w \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}$ has generalized w-core decomposition.
(2) There exist unique elements $x, y \in \mathcal{A}$ such that

$$a = x + y, x^*y = ywx = 0, x \in \mathcal{A}_w^{\circledast}, y \in \mathcal{A}_w^{qnil}.$$

Proof. (1) ⇒ (2) In view of Theorem 2.1, there exists $v \in \mathcal{A}$ such that

$$v = awv^2, (awv)^* = awv, v(aw)^2v = awv, \lim_{n\to\infty} \|(aw)^n - awv(aw)^n\|^{\frac{1}{n}} = 0.$$

Set $z = awva$ and $y = a - awva$. As in the proof of Theorem 2.1, we have

$$a = z + y, z^*y = ywz = 0, z \in \mathcal{A}_w^{\circledast}, y \in \mathcal{A}_w^{qnil}.$$

Suppose that there exist $b, c \in \mathcal{A}$ such that

$$a = b + c, b^*c = cwb = 0, b \in \mathcal{A}_w^{\circledast}, c \in \mathcal{A}_w^{qnil}.$$

Obviously, $awv = aw(awv^2) = (aw)^2v^2 = \cdots = (aw)^nv^n$. Since $a = b + c$, we have $aw = bw + cw$. As $(cw)(bw) = (cwb)w = 0$, we have

$$(aw)^n = \sum_{i=0}^{n}(bw)^i(cw)^{n-i} = (cw)^n + \sum_{i=1}^{n}(bw)^i(cw)^{n-i}.$$

Hence,

$$[(aw)^n]^*c = [(cw)^n]^*c.$$

Since $cwb = 0$, we see that $(aw)^nb = (aw)^{n-1}(bw + cw)b = (aw)^{n-1}bwb = \cdots = (bw)^nb$, and then $(aw)^nbw = (bw)^nbw = (bw)^{n+1}$. This implies that

$$(aw)^nbw[(bw)^{\circledast}]^{n+1} = (bw)^{n+1}[(bw)^{\circledast}]^{n+1} = (bw)(bw)^{\circledast}.$$

Therefore

$$\begin{aligned} b &= bwb_w^{\circledast}b = (bw)(b_w^{\circledast}b) \\ &= [(aw)^n bw[(bw)^{\circledast}]^{n+1}bw](b_w^{\circledast}b) \\ &= (aw)^n bw[(bw)^{\circledast}]^{n+1}[bwb_w^{\circledast}b] \\ &= (aw)^n bw[(bw)^{\circledast}]^{n+1}b \\ &= (aw)^n s, \end{aligned}$$

where $s = bw[(bw)^{\circledast}]^{n+1}b$. Accordingly,

$$\begin{aligned} b - z &= b - awva = b - awv(b + c) = b - awvb - awvc \\ &= b - (aw)vb - [(aw)^n v^n]^* c \\ &= b - (aw)vb - (v^n)^*((aw)^n)^* c \\ &= (aw)^n s - (aw)v(aw)^n s - (v^n)^*((cw)^n)^* c \\ &= [(aw)^n - (aw)v(aw)^n]s - (v^n)^*((cw)^n)^* c \end{aligned}$$

Hence,

$$\|b - z\|^{\frac{1}{n}} \leq \|(aw)^n - (aw)v(aw)^n\|^{\frac{1}{n}}\|s\|^{\frac{1}{n}} + \|(v^n)^*\|^{\frac{1}{n}}\|((cw)^n)^*\|^{\frac{1}{n}}\|c\|^{\frac{1}{n}}.$$

Since $cw \in \mathcal{A}^{qnil}$, we have $1 - \overline{\lambda}cw \in \mathcal{A}^{-1}$; hence, $1 - \lambda(cw)^* \in \mathcal{A}^{-1}$. We infer that $(cw)^* \in \mathcal{A}^{qnil}$. Thus, we prove that $\lim\limits_{n\to\infty} \|((cw)^n)^*\|^{\frac{1}{n}} = 0$. This yields that

$$\lim_{n\to\infty} \|b - z\|^{\frac{1}{n}} = 0.$$

Therefore $b = z$, and then $c = a - b = a - z = y$, as required.

(2) ⇒ (1) This is trivial. □

Theorem 2.3. *Let $a, w \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}$ has generalized w-core decomposition.

(2) There exists a unique $x \in \mathcal{A}$ such that

$$x = awx^2, (awx)^* = awx, x(aw)^2x = awx, \lim_{n\to\infty} \|(aw)^n - awx(aw)^n\|^{\frac{1}{n}} = 0.$$

Proof. (2) ⇒ (1) This is obvious by Theorem 2.1.

(1) ⇒ (2) By hypothesis, there exists $x \in \mathcal{A}$ such that

$$x = awx^2, (awx)^* = awx, x(aw)^2x = awx, \lim_{n\to\infty} \|(aw)^n - awx(aw)^n\|^{\frac{1}{n}} = 0.$$

Assume that there exists $y \in \mathcal{A}$ such that

$$y = awy^2, (awy)^* = awy, y(aw)^2y = awy, \lim_{n\to\infty} \|(aw)^n - awy(aw)^n\|^{\frac{1}{n}} = 0.$$

Set $a_1 = awxa, a_2 = a - a_1$ and $b_1 = awya, b_2 = a - b_1$. As in the proof of Theorem 2.1, we prove that

$$\begin{aligned} a_1^* a_2 = a_2 w a_1 = 0, a_1 \in \mathcal{A}_w^{\circledast}, a_2 \in \mathcal{A}_w^{qnil}, \\ b_1^* b_2 = b_2 w b_1 = 0, b_1 \in \mathcal{A}_w^{\circledast}, b_2 \in \mathcal{A}_w^{qnil}. \end{aligned}$$

As in the proof of Corollary 2.2 , we verify that $awxa = a_1 = b_1 = awya$. Therefore

$$x = (awxa)_w^{\circledast} = (a_1)_w^{\circledast} = (b_1)_w^{\circledast} = (awya)_w^{\circledast} = y.$$

Accordingly, $x = y$, the result follows. □

We denote x in Theorem 2.3 by $a_w^{\circledcirc}$, and call it the generalized w-core inverse of a. Let $\mathcal{A}_w^{\circledcirc}$ stand for the set of all elements in $\mathcal{A}$ that have a generalized w-core inverse.

Corollary 2.4. *Let $a = x + y$ be the generalized w-core decomposition of $a \in \mathcal{A}$. Then $a^{\textcircled{d}}_w = x^{\textcircled{\#}}_w$.*

Proof. Let $a = x + y$ be the generalized w-core decomposition of $a \in \mathcal{A}$. Similarly to the proof of Theorem 2.1, $x^{\textcircled{\#}}_w$ is the generalized w-core inverse of a. So the theorem is true. □

Theorem 2.5. *Let $a, w \in \mathcal{A}$. Then $a \in \mathcal{A}^{\textcircled{d}}_w$ if and only if there exists a projection $p \in \mathcal{A}$ such that*

(1) $(1-p)a \in (1-p)aw\mathcal{A}$ and $(1-p)aw \in \mathcal{A}^{\#}$;
(2) $aw + p \in \mathcal{A}$ is right invertible and $paw = pawp \in \mathcal{A}^{qnil}$.

Proof. (1) ⇒ (2) Since $a \in \mathcal{A}^{\textcircled{d}}_w$, by using Theorem 2.3, there exist $x, y \in \mathcal{A}$ such that

$$a = x + y, x^*y = ywx = 0, x \in \mathcal{A}^{\textcircled{\#}}_w, y \in \mathcal{A}^{qnil}_w.$$

Since $x \in \mathcal{A}^{\textcircled{\#}}_w$, we have

$$x^{\textcircled{\#}}_w = xw(x^{\textcircled{\#}}_w)^2, (xwx^{\textcircled{\#}}_w)^* = xwx^{\textcircled{\#}}_w, x^{\textcircled{\#}}_w xwx^{\textcircled{\#}}_w = x^{\textcircled{\#}}_w, x^{\textcircled{\#}}_w xwx = x, x = xwx^{\textcircled{\#}}_w x.$$

Let $p = 1 - xwx^{\textcircled{\#}}_w$. Then $p^2 = p = p^*$ and $px = 0$. We directly check that

$$(xw + 1 - xwx^{\textcircled{\#}}_w)(x^{\textcircled{\#}}_w + 1 - xwx^{\textcircled{\#}}_w) = 1 + x(1 - xwx^{\textcircled{\#}}_w) \in \mathcal{A}^{-1}.$$

Let $q = [x^{\textcircled{\#}}_w + 1 - xwx^{\textcircled{\#}}_w][1 + x(1 - xwx^{\textcircled{\#}}_w)]^{-1}$. Then $(xw + p)q = 1$. This implies that $xw + p \in \mathcal{A}$ is right invertible. Moreover, we have

$$\begin{array}{rcl} 1 + ywq & = & 1 + [ywxw(x^{\textcircled{\#}}_w)^2 + yw - (ywx)wx^{\textcircled{\#}}_w][1 + x(1 - xwx^{\textcircled{\#}}_w)]^{-1} \\ & = & 1 + yw[1 + x(1 - xwx^{\textcircled{\#}}_w)]^{-1} = 1 + yw \in \mathcal{A}^{-1}. \end{array}$$

By using Jacobson's lemma, $1 + qyw \in \mathcal{A}^{-1}$. Therefore, we check that

$$\begin{array}{rcl} pa & = & p(x + y) = py = (1 - xwx^{\textcircled{\#}}_w)y = y \in \mathcal{A}^{qnil}_w, \\ paw(1-p) & = & ywxwx^{\textcircled{\#}}_w = 0, \\ aw + p & = & xw + yw + p = (xw + p)[1 + qyw] \in \mathcal{A} \text{ is right invertible.} \end{array}$$

Since $(1-p)a = xwx^{\textcircled{\#}}_w(x + y) = xwx^{\textcircled{\#}}_w x = x \in \mathcal{A}^{\textcircled{\#}}_w$, it follows by [30, Theorem 2.10] that $(1-p)aw \in \mathcal{A}^{\#}$ and $(1-p)a \in (1-p)aw\mathcal{A}$, as required.

(2) ⇒ (1) By hypothesis, there exists a projection $p \in \mathcal{A}$ such that $(1-p)a \in \mathcal{A}^{\textcircled{\#}}_w$;

$$aw + p \in \mathcal{A} \text{ right invertible}, paw(1-p) = 0, pa \in \mathcal{A}^{qnil}_w.$$

Set $x = (1-p)a$ and $y = pa$. Then

$$\begin{array}{rcl} x^*y & = & [a^*(1-p)^*]pa = 0, \\ ywx & = & paw(1-p)a = 0, \\ y & = & pa \in \mathcal{A}^{qnil}_w. \end{array}$$

Write $(aw+p)q = 1$ for some $q \in \mathcal{A}$. Then $(1-p)awq = (1-p)(aw+p)q = 1-p$, and so $(1-p)awq(1-p)a = (1-p)a$ and $[(1-p)awq]^* = (1-p)^* = 1-p = (1-p)awq$. Hence, $(1-p)a \in \mathcal{A}^{(1,3)}$.

By hypothesis, $(1-p)a \in (1-p)aw\mathcal{A}$ and $(1-p)aw \in \mathcal{A}^{\#}$. In light of [30, Lemma 2.8], $w \in \mathcal{A}^{\|(1-p)a}$. According to [30, Theorem 2.6], $(1-p)a \in \mathcal{A}^{\textcircled{\#}}_w$. That is, $x \in \mathcal{A}^{\textcircled{\#}}_w$. Therefore $a \in \mathcal{A}^{\textcircled{d}}_w$. □

We use $\mathcal{A}^{\textcircled{d}}$ to denote $\mathcal{A}^{\textcircled{d}}_w$ in the case where $w = 1$. That is, $a \in \mathcal{A}^{\textcircled{d}}$ if and only if there exist $x, y \in \mathcal{A}$ such that $a = x + y, x^*y = yx = 0, x \in \mathcal{A}^{\textcircled{\#}}, y \in \mathcal{A}^{qnil}$. We denote $a^{\textcircled{d}}_1$ by $a^{\textcircled{d}}$. Evidently, $a \in \mathcal{A}^{\textcircled{d}}$ if and only if $a \in \mathcal{A}^{e,\textcircled{d}}$ in the case where $e = 1$ (see [4, Theorem 2.3]).

Corollary 2.6. *Let $a, w \in \mathcal{A}$. Then $a \in \mathcal{A}^{\textcircled{d}}$ if and only if there exists a projection $p \in \mathcal{A}$ such that*

(1) $(1-p)a \in \mathcal{A}^{\#}$;
(2) $a + p \in \mathcal{A}$ is right invertible and $pa = pap \in \mathcal{A}^{qnil}$.

Proof. This is obvious by choosing $w = 1$ in Theorem 2.5. □

3. Characterizations by weighted g-Drazin inverses

The goal of this section is to elucidate the generalized w-core inverse through the lens of the image associated with the weighted g-Drazin inverse. Evidently, $a_w^{\#} = (aw)^{\#}$. But $aw \in \mathcal{A}^{\#}$ do not imply $a \in \mathcal{A}_w^{\#}$. For instance, letting $a = \begin{pmatrix} -i & 1 \\ 0 & i \end{pmatrix}, w = \begin{pmatrix} i & -1 \\ 0 & 0 \end{pmatrix}$. Then $aw \in \mathbb{C}^{2\times 2}$ has the core-inverse $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, while a has not w-core inverse. Contract to this observation, we now derive the following result which enable us to investigate the generalized w-core inverse by using the weighted g-Drazin inverse.

Lemma 3.1. *Let $a, w \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}$ has generalized w-core decomposition.
(2) $aw \in \mathcal{A}^{\circledD}$.
(3) $a \in \mathcal{A}^{d,w}$ and there exists a unique $x \in \mathcal{A}$ such that

$$x = awx^2, (awx)^* = awx, \lim_{n\to\infty} \|(aw)^n - awx(aw)^n\|^{\frac{1}{n}} = 0.$$

In this case, $a_w^{\circledD} = x = (aw)^{\circledD}$.

Proof. (1) $\Rightarrow$ (2) Since $a \in \mathcal{A}_w^{\circledD}$, there exist $x, y \in \mathcal{A}$ such that

$$a = x + y, x^*y = ywx = 0, x \in \mathcal{A}_w^{\#}, y \in \mathcal{A}_w^{qnil}.$$

Hence, $aw = xw + yw$. Since $x \in \mathcal{A}_w^{\#}$, we have

$$xw(x_w^{\#})^2 = x_w^{\#}, (xwx_w^{\#})^* = xwx_w^{\#}, x_w^{\#}xwx = x.$$

Then

$$xw(x_w^{\#})^2 = x_w^{\#}, (xwx_w^{\#})^* = xwx_w^{\#}, x_w^{\#}(xw)^2 = xw.$$

This shows that $xw \in \mathcal{A}^{\#}$. Obviously, $yw \in \mathcal{A}^{qnil}$. Moreover, we check that

$$(xw)^*(yw) = w^*(x^*y)w = 0, (yw)(xw) = (ywx)w = 0.$$

In light of [4, Corollary 2.2], $aw \in \mathcal{A}^{\circledD}$. Moreover, we have $a_w^{\circledD} = x_w^{\#} = (xw)^{\#} = (aw)^{\circledD}$.

(2) $\Rightarrow$ (1) Let $x = (aw)^{\circledD}$. Then $aw \in \mathcal{A}^d$. In view of [4, Theorem 2.1], we have

$$x = awx^2, (awx)^* = awx, \lim_{n\to\infty} \|(aw)^n - x(aw)^{n+1}\|^{\frac{1}{n}} = 0.$$

We easily check that

$$\begin{aligned} \|awx - x(aw)^2x\|^{\frac{1}{n}} &= \|awx - x(aw)[(aw)x]\|^{\frac{1}{n}} \\ &= \|awx - x(aw)[(aw)^n x^n]\|^{\frac{1}{n}} \\ &= \|(aw)^n x^n - x(aw)^{n+1}x^n\|^{\frac{1}{n}} \\ &\le \|(aw)^n - x(aw)^{n+1}\|^{\frac{1}{n}} \|x\|. \end{aligned}$$

Hence,

$$\lim_{n\to\infty} \|awx - x(aw)^2x\|^{\frac{1}{n}} = 0.$$

Therefore $x(aw)^2x = awx$, and then $a \in \mathcal{A}_w^{\circledD}$.

(2) $\Rightarrow$ (3) In view of [4, Theorem 2.5], $a \in \mathcal{A}^{d,w}$ and there exists $x \in \mathcal{A}$ such that

$$x = awx^2, (awx)^* = awx, \lim_{n\to\infty} \|(aw)^n - awx(aw)^n\|^{\frac{1}{n}} = 0.$$

Moreover, we have $(aw)(aw)^d x = (aw)^{\circledS}$. Since $x = awx^2$, by induction, we have $x = (aw)^n x^{n+1}$ for any $n \in \mathbb{N}$. Then

$$\begin{aligned} \|x - (aw)(aw)^d x\|^{\frac{1}{n}} &= \|[1 - (aw)(aw)^d](aw)^n x^{n+1}\|^{\frac{1}{n}} \\ &= \|(aw)^n - (aw)^d (aw)^{n+1}\|^{\frac{1}{n}} \|x\|^{1+\frac{1}{n}}. \end{aligned}$$

As $\lim\limits_{n\to\infty} \|(aw)^n - (aw)^d (aw)^{n+1}\|^{\frac{1}{n}} = 0$, we see that

$$\lim_{n\to\infty} \|x - (aw)(aw)^d x\|^{\frac{1}{n}} = 0,$$

and therefore $x = (aw)(aw)^d x = (aw)^{\circledS}$, as required.

(3) ⇒ (2) Since $a \in \mathcal{A}^{d,w}$, we have $aw \in \mathcal{A}^d$. Therefore $aw \in \mathcal{A}^{\circledS}$ by [4, Theorem 2.5]. □

We are ready to prove:

Theorem 3.2. *Let $a, w \in \mathcal{A}$. Then $a \in \mathcal{A}^{\circledS}_w$ if and only if*

(1) $a \in \mathcal{A}^{d,w}$;
(2) There exists $x \in \mathcal{A}$ such that

$$xawx = x, x\mathcal{A} = a^{d,w}\mathcal{A}, \mathcal{A}x = \mathcal{A}(a^{d,w})^*.$$

In this case, $a^{\circledS}_w = x$.

Proof. $\Longrightarrow$ Choose $x = a^{\circledS}_w$. Then $aw \in \mathcal{A}^{\circledS}$ and $x = (aw)^{\circledS}$. In view of [4, Theorem 3.3], $aw \in \mathcal{A}^d$ and

$$x(aw)x = x, x\mathcal{A} = (aw)^d\mathcal{A}, \mathcal{A}x = \mathcal{A}((aw)^d)^*.$$

Since $a^{d,w} = [(aw)^d]^2 a = a[(wa)^d]^2 = (aw)^d a (wa)^d$, we easily check that $(aw)^d = [(aw)^d]^2 aw = a^{d,w}w$, and then

$$(aw)^d\mathcal{A} = a^{d,w}\mathcal{A}.$$

On the other hand, $(a^{d,w})^* = [(aw)^d a]^* [(aw)^d]^*$ and $[(aw)^d]^* = [((aw)^d)^2 aw]^* = w^*(a^{d,w})^*$. Thus, $\mathcal{A}[(aw)^d]^* = \mathcal{A}(a^{d,w})^*$. Therefore

$$x\mathcal{A} = a^{d,w}\mathcal{A}, \mathcal{A}x = \mathcal{A}(a^{d,w})^*$$

.

$\Longleftarrow$ By hypothesis, there exists $x \in \mathcal{A}$ such that

$$xawx = x, x\mathcal{A} = a^{d,w}\mathcal{A}, \mathcal{A}x = \mathcal{A}(a^{d,w})^*.$$

As the argument above, we have

$$(aw)^d\mathcal{A} = a^{d,w}\mathcal{A}, \mathcal{A}[(aw)^d]^* = \mathcal{A}(a^{d,w})^*.$$

Therefore we have

$$xawx = x, x\mathcal{A} = (aw)^d\mathcal{A}, \mathcal{A}x = \mathcal{A}((aw)^d)^*.$$

In light of [4, Theorem 3.3], $aw \in \mathcal{A}^{\circledS}$. According to Lemma 3.1, $a^{\circledS}_w = x$. □

An element $a \in \mathcal{A}$ has pseudo w-core inverse if there exist $x, y \in \mathcal{A}$ such that

$$a = x + y, x^* y = ywx = 0, x \in \mathcal{A}^{\oplus}_w, y \in \mathcal{A}^{nil}_w.$$

The preceding x is unique if it exists, and we denote its w-core inverse by the pseudo w-core inverse of a, i.e., $a^{\textcircled{D}}_w = x^{\oplus}_w$. Evidently, $a^{\textcircled{D}}_w = z$ if and only if there exists $n \in \mathbb{N}$ such that

$$z = awz^2, (awz)^* = awz, z(aw)^2 z = awz \text{ and } (aw)^n = awz(aw)^n.$$

In this case, $a^{\textcircled{D}}_w = z$.

Corollary 3.3. *Let $a, w \in \mathcal{A}$. Then a has pseudo w-core inverse if and only if*

(1) $a \in \mathcal{A}_w^{\textcircled{d}}$.
(2) aw has Drazin inverse.

Proof. $\Longrightarrow$ Obviously, $a \in \mathcal{A}_w^{\textcircled{d}}$ and $aw \in \mathcal{A}^{\textcircled{D}}$. In view of [8, Theorem 2.3], aw has Drazin inverse, as desired.
$\Longrightarrow$ In view of Theorem 3.2, there exists $x \in \mathcal{A}$ such that

$$xawx = x, x\mathcal{A} = a^{d,w}\mathcal{A}, \mathcal{A}x = \mathcal{A}(a^{d,w})^*.$$

Since aw has Drazin inverse, we have $a^{d,w} = [(aw)^d]^2 a = [(aw)^D]^2 a = a^{D,w}$. Hence,

$$x(aw)x = x, x\mathcal{A} = (aw)^D\mathcal{A}, \mathcal{A}x = \mathcal{A}((aw)^D)^*.$$

Analogously to Theorem 3.2, we prove that $aw \in \mathcal{A}^{\textcircled{D}}$. Similarly to Lemma 3.1, a has pseudo w-core inverse. □

Let $\mathcal{R}(X)$ represent the range space of a complex matrix X and X^* be the conjugate transpose of X. We improve [2, Theorem 3.5] and provide a new characterizations of pseudo W-core inverse for any complex matrix.

Corollary 3.4. *Let $A, X, W \in \mathbb{C}^{n\times n}$. Then the following are equivalent:*

(1) X is the pseudo W-core inverse of A.
(2) $XAWX = X, \mathcal{R}(X) = \mathcal{R}(X^*) = \mathcal{R}((AW)^D)$.

Proof. Obviously, $AW \in \mathbb{C}^{n\times n}$ has Drazin inverse. Since $A^{D,W} = [(AW)^D]^2 A$ and $(AW)^D = [(AW)^D]^2 AW = A^{D,W}W$, we see that $\mathcal{R}(A^{D,W}) = \mathcal{R}((AW)^D)$. Therefore we obtain the result by Theorem 3.2 and Corollary 3.3. □

Recall that an element x is a $(1,3)$-inverse of a if $a = axa$ and $(ax)^* = ax$. Such an element x is denoted by $a^{(1,3)}$ (see [8]). Let $\mathcal{A}^{(1,3)}$ denote the set of all elements in $\mathcal{A}$ that have a $(1,3)$-inverse. We now derive

Theorem 3.5. *Let $a, w \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}_w^{\textcircled{d}}$.
(2) $a \in \mathcal{A}^{d,w}$ and $a^{d,w} \in \mathcal{A}^{(1,3)}$.
(3) $a \in \mathcal{A}^{d,w}$ and there exists a projection $q \in \mathcal{A}$ such that $a^{d,w}\mathcal{A} = q\mathcal{A}$.

In this case, $a_w^{\textcircled{d}} = a^{d,w}wa^{d,w}(a^{d,w})^{(1,3)} = a^{d,w}wq$.

Proof. (1) ⇒ (2) In view of Theorem 3.2, $a \in \mathcal{A}^{d,w}$. Let $x = a_w^{\textcircled{d}}$. By virtue of Theorem 2.1, there exists $x \in \mathcal{A}$ such that

$$x = awx^2, (awx)^* = awx, x(aw)^2 x = awx, \lim_{n\to\infty} \|(aw)^n - awx(aw)^n\|^{\frac{1}{n}} = 0.$$

Let $z = (wa)^2 wx$. Then

$$\begin{array}{rcl} a^{d,w}z & = & [(aw)^d]^2 a[(wa)^2 wx] = (aw)^2(aw)^d x \\ & = & (aw)^2(aw)^d(aw)^{\textcircled{d}} = (aw)(aw)^{\textcircled{d}}. \end{array}$$

Therefore $(a^{d,w}z)^* = [(aw)(aw)^{\textcircled{d}}]^* = (aw)(aw)^{\textcircled{d}} = a^{d,w}z$. Moreover, we verify that

$$\begin{array}{rcl} a^{d,w}za^{d,w} & = & (aw)(aw)^{\textcircled{d}}a^{d,w} \\ & = & (aw)(aw)^{\textcircled{d}}[(aw)^d]^2 a \\ & = & (aw)(aw)^{\textcircled{d}}(aw)^2[(aw)^d]^4 a \\ & = & (aw)^2[(aw)^d]^4 a = [(aw)^d]^2 a = a^{d,w}. \end{array}$$

Accordingly, $a^{d,w} \in \mathcal{A}^{(1,3)}$, as required.

(2) ⇒ (1) Let $x = a^{d,w}wa^{d,w}(a^{d,w})^{(1,3)}$. Then we check that

$$\begin{array}{rcl} xawx & = & a^{d,w}wa^{d,w}(a^{d,w})^{(1,3)}(aw)a^{d,w}wa^{d,w}(a^{d,w})^{(1,3)} \\ & = & a^{d,w}w[a^{d,w}(a^{d,w})^{(1,3)}a^{d,w}]w(aw)a^{d,w}(a^{d,w})^{(1,3)} \\ & = & a^{d,w}wa^{d,w}w(awa^{d,w})(a^{d,w})^{(1,3)} \\ & = & [a^{d,w}wawa^{d,w}]wa^{d,w}(a^{d,w})^{(1,3)} \\ & = & a^{d,w}wa^{d,w}(a^{d,w})^{(1,3)} = x. \end{array}$$

Clearly, $x\mathcal{A} \subseteq a^{d,w}\mathcal{A}$. Also we see that

$$a^{d,w} = (a^{d,w}w)^2a = a^{d,w}wa^{d,w}(a^{d,w})^{(1,3)}a^{d,w}wa = xa^{d,w}wa;$$

hence, $a^{d,w}\mathcal{A} \subseteq x\mathcal{A}$. Thus $x\mathcal{A} = a^{d,w}\mathcal{A}$.

We easily verify that

$$\begin{array}{rcl} x & = & a^{d,w}wa^{d,w}(a^{d,w})^{(1,3)} = a^{d,w}w[a^{d,w}(a^{d,w})^{(1,3)}] \\ & = & a^{d,w}w[a^{d,w}(a^{d,w})^{(1,3)}]^* = a^{d,w}w[(a^{d,w})^{(1,3)}]^*[a^{d,w}]^*; \end{array}$$

and then, $\mathcal{A}x \subseteq \mathcal{A}(a^{d,w})^*$. Also we check that

$$\begin{array}{rcl} [a^{d,w}]^* & = & [a^{d,w}(a^{d,w})^{(1,3)}a^{d,w}]^* = [(a^{d,w}(a^{d,w})^{(1,3)})^*a^{d,w}]^* \\ & = & [a^{d,w}]^*a^{d,w}(a^{d,w})^{(1,3)} \\ & = & [a^{d,w}]^*[a^{d,w}wawa^{d,w}](a^{d,w})^{(1,3)} \\ & = & [a^{d,w}]^*aw[a^{d,w}wa^{d,w}(a^{d,w})^{(1,3)}] \\ & = & [a^{d,w}]^*awx, \end{array}$$

and then $\mathcal{A}(a^{d,w})^* \subseteq \mathcal{A}x$. Hence $\mathcal{A}x = \mathcal{A}(a^{d,w})^*$. Accordingly, $a \in \mathcal{A}^{\circledcirc}_w$ by Theorem 3.2.

(2) ⇒ (3) By hypothesis, $a^{d,w} \in \mathcal{A}^{(1,3)}$, and so $a^{d,w} = a^{d,w}(a^{d,w})^{(1,3)}a^{d,w}$ and $[a^{d,w}(a^{d,w})^{(1,3)}]^* = a^{d,w}(a^{d,w})^{(1,3)}$. Let $q = a^{d,w}(a^{d,w})^{(1,3)}$. Then $a^{d,w}\mathcal{A} = q\mathcal{A}, q^2 = q = q^*$, as required.

(3) ⇒ (2) Let $x = a^{d,w}wq$. Then $awx = awa^{d,w}wq = aw[(aw)^d]^2awq = aw(aw)^dq = q$, and so $(awx)^* = q^* = q = awx$. Moreover, we have

$$awx^2 = (awx)x = qa^{d,w}wq = a^{d,w}wq = x.$$

Obviously, $a^{d,w}w(aw) = (aw)a^{d,w}w$, and then we verify that

$$\begin{array}{rl} & ||(aw)^n - x(aw)^{n+1}|| \\ = & ||[(aw)^n - (a^{d,w}wq)a^{d,w}w(aw)^{n+2}] - [x((aw)^{n+1} - a^{d,w}w(aw)^{n+2})]|| \\ \leq & ||(aw)^n - a^{d,w}w(aw)^{n+1}|| + ||x||||(aw)^{n+1} - a^{d,w}w(aw)^{n+2}|| \\ \leq & \Big(1 + ||x||||aw||\Big)||(aw)^n - a^{d,w}w(aw)^{n+1}|| \\ = & \Big(1 + ||x||||aw||\Big)||(aw)^n(1 - a^{d,w}waw)^n|| \\ = & \Big(1 + ||x||||aw||\Big)||(aw - a^{d,w}w(aw)^2)^n||. \end{array}$$

This implies that

$$||(aw)^n - x(aw)^{n+1}||^{\frac{1}{n}} \leq \Big(1 + ||x||||aw||\Big)^{\frac{1}{n}}||(aw - a^{d,w}w(aw)^2)^n||^{\frac{1}{n}}.$$

Since $aw - a^{d,w}w(aw)^2 = aw - (aw)^d(aw)^2 \in \mathcal{A}^{qnil}$, we have

$$\lim_{n\to\infty} ||(aw - a^{d,w}w(aw)^2)^n||^{\frac{1}{n}} = 0.$$

Therefore

$$\lim_{n\to\infty} ||(aw)^n - x(aw)^{n+1}||^{\frac{1}{n}} = 0.$$

Then $x = (aw)^{\circledcirc}$. In view of Theorem 2.1, $a \in \mathcal{A}^{\circledcirc}_w$. In this case, $a^{\circledcirc}_w = x = a^{d,w}wq = a^{d,w}wa^{d,w}(a^{d,w})^{(1,3)}$. □

Corollary 3.6. *Let $a, w \in \mathcal{A}$. Then the following are equivalent:*

(1) a has pseudo w-core inverse.
(2) $a \in \mathcal{A}^{D,w}$ and $a^{D,w} \in \mathcal{A}^{(1,3)}$.
(3) $a \in \mathcal{A}^{D,w}$ and there exists a projection $q \in \mathcal{A}$ such that $a^{D,w}\mathcal{A} = q\mathcal{A}$.

In this case, $a_w^{\textcircled{D}} = a^{D,w}wa^{D,w}(a^{D,w})^{(1,3)} = a^{D,w}wq$.

Proof. As $a \in \mathcal{A}^D$, we have $a^d = a^D$. Therefore we complete the proof by Theorem 3.5. □

4. Relations with weighted core inverses

The objective of this section is to delineate the connections between the generalized weighted core inverse and other types of weighted generalized inverses. We now proceed to the demonstration that forms the crux of this section's development.

Theorem 4.1. *Let $a, w \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}_w^{\textcircled{d}}$.
(2) $a \in \mathcal{A}^{d,w}$ and $a^{d,w} \in \mathcal{A}_w^{\textcircled{\#}}$.

In this case,

$$a_w^{\textcircled{d}} = [a^{d,w}w]^2(a^{d,w})_w^{\textcircled{\#}}.$$

Proof. (1) ⇒ (2) In view of Theorem 3.2, $a \in \mathcal{A}^{d,w}$. Let $x = a_w^{\textcircled{d}}$. Then we have

$$x = awx^2, (awx)^* = awx, x(aw)^2x = awx, \lim_{n\to\infty} \|(aw)^n - awx(aw)^n\|^{\frac{1}{n}} = 0.$$

We verify that

$$\begin{array}{rcl} \|aw(aw)^d - awx(aw)(aw)^d\| & = & \|(aw)^n[(aw)^d]^n - awx(aw)^n[(aw)^d]^n\| \\ & \leq & \|(aw)^n - awx(aw)^n\|\|[(aw)^d]^n\|. \end{array}$$

Since $\lim_{n\to\infty} \|(aw)^n - awx(aw)^n\|^{\frac{1}{n}} = 0$, we deduce that

$$\lim_{n\to\infty} \|aw(aw)^d - awx(aw)(aw)^d\|^{\frac{1}{n}} = 0.$$

Hence, $awx(aw)(aw)^d = aw(aw)^d$. Since $awx - aw(aw)^d(awx) = (aw)^nx^n - (aw)^d(aw)^{n+1}x^n = [(aw)^n - (aw)^d(aw)^{n+1}]x^n$, we deduce that

$$\|awx - aw(aw)^d(awx)\|^{\frac{1}{n}} \leq \|(aw)^n - (aw)^d(aw)^{n+1}\|^{\frac{1}{n}}\|x\|.$$

This implies that $\lim_{n\to\infty} \|awx - aw(aw)^d(awx)\|^{\frac{1}{n}} = 0$, and then $aw(aw)^d(awx) = awx$. Let $z = (aw)^2x$. Then

$$\begin{array}{rcl} a^{d,w}wz & = & a^{d,w}w(aw)^2x = [(aw)^d]^2aw(aw)^2x \\ & = & aw(aw)^d(awx) = awx, \\ a^{d,w}wz^2 & = & (awx)z = (awx)(aw)^2x = aw[x(aw)^2x] = (aw)^2x = z, \\ (a^{d,w}wz)^* & = & (awx)^* = awx = a^{d,w}wz, \\ za^{d,w}wa^{d,w} & = & (aw)^2xa^{d,w}wa^{d,w} = aw[awx(aw)(aw)^d]a^{d,w}wa^{d,w} \\ & = & aw[(aw)(aw)^d]a^{d,w}wa^{d,w} = aw[(aw)(aw)^d][(aw)^d]^2awa^{d,w} \\ & = & aw(aw)^da^{d,w} = a^{d,w}. \end{array}$$

Accordingly, $a^{d,w}wz^2 = z, (a^{d,w}wz)^* = a^{d,w}wz, za^{d,w}wa^{d,w} = a^{d,w}$. Then $a^{d,w} \in \mathcal{A}_w^{\textcircled{\#}}$ and $(a^{d,w})_w^{\textcircled{\#}} = z = (aw)^2a_w^{\textcircled{d}}$, as desired.

(2) ⇒ (1) Set $x = (a^{d,w})_w^{\textcircled{\#}}$. Then we have $a^{d,w}wxa^{d,w} = a^{d,w}, [a^{d,w}wx]^* = a^{d,w}wx$. Hence, $a^{d,w} \in \mathcal{A}^{(1,3)}$. According to Theorem 3.5, $a \in \mathcal{A}_w^{\textcircled{d}}$. Moreover, we have

$$a_w^{\textcircled{d}} = a^{d,w}wa^{d,w}(wx) = a^{d,w}wa^{d,w}w(a^{d,w})_w^{\textcircled{\#}} = [a^{d,w}w]^2(a^{d,w})_w^{\textcircled{\#}}.$$

□

As an immediate consequence, we present formulas for the generalized weighted core inverse of a complex matrix.

Corollary 4.2. *Let $A, W \in \mathbb{C}^{n\times n}$. Then*

$$\begin{array}{rcl} A^{\textcircled{D}}_W & = & [A^{D,W}W]^2(A^{D,W})^{\textcircled{\#}}_W \\ & = & (AW)^k A[(AW)^{k+1}A]^{\dagger}, \end{array}$$

where $k = max\{ind(AW), ind(WA)\}$.

Proof. By virtue of Theorem 4.1, $A^{\textcircled{D}}_W = [A^{D,W}W]^2(A^{D,W})^{\textcircled{\#}}_W$. In view of [9, Theorem 2.10],

$$W^{\textcircled{\dagger},A} = (WA)^k[A(WA)^{k+1}]^{\dagger}.$$

According to Lemma 3.1, we get

$$\begin{array}{rcl} A^{\textcircled{D}}_W & = & (AW)^{\textcircled{D}} = A[W\big((AW)^{\textcircled{D}}\big)^2] = A[W^{\textcircled{\dagger},A}] \\ & = & A[(WA)^k[A(WA)^{k+1}]^{\dagger}] = (AW)^kA[(AW)^{k+1}A]^{\dagger}, \end{array}$$

as asserted. $\square$

Example 4.3.

Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & -1 \end{pmatrix}, W = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in \mathbb{C}^{3\times 3}$. We take the involution on $\mathbb{C}^{4\times 4}$ as the conjugate transpose. Then

$$AW = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, WA = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

Hence, $max\{ind(AW), ind(WA)\} = 1$. Moreover, we have

$$\begin{array}{rcl} A^{\textcircled{D}}_W & = & AWA[(AW)^2A]^{\dagger} \\ & = & \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^{\dagger} \\ & = & \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ & = & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{array}$$

Evidently, we check that

$$AW(A^{\textcircled{D}}_W)^2 = A^{\textcircled{D}}_W, (AWA^{\textcircled{D}}_W)^* = AWA^{\textcircled{D}}_W, AW = AWA^{\textcircled{D}}_W AW.$$

Let $a, x \in \mathcal{A}$. x is called $(1,4)$-inverse of a and is denoted by $a^{(1,4)}$ provided that $axa = a$ and $(xa)^* = xa$. We use $\mathcal{A}^{(1,4)}$ to stand for sets of all $(1,4)$ invertible elements in $\mathcal{A}$.

Lemma 4.4. *Let $f \in \mathcal{A}$ be an idempotent. Then the following are equivalent:*

(1) $f \in \mathcal{A}^{(1,3)}$.
(2) $f^{\pi} \in \mathcal{A}^{(1,4)}$.

Proof. See [20, Lemma 3.2]. □

Theorem 4.5. *Let $a, w \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}_w^{\textcircled{d}}$.
(2) $a \in \mathcal{A}^{d,w}$ and $awa^{d,w}w \in \mathcal{A}^{(1,3)}$.
(3) $a \in \mathcal{A}^{d,w}$ and $(aw)^{\pi} \in \mathcal{A}^{(1,4)}$.

In this case, $a_w^{\textcircled{d}} = a^{d,w}w(awa^{d,w}w)^{(1,3)} = a^{d,w}w[1 - (a^{\pi})^{(1,4)}a^{\pi}]$.

Proof. (1) ⇒ (2) In view of Theorem 3.2, $a \in \mathcal{A}^{d,w}$. For any $m \in \mathbb{N}$, we check that

$$\begin{array}{rcl} \|awa^{d,w}w - awa_w^{\textcircled{d}}awa^{d,w}w\| & = & \|(aw)^m(a^{d,w}w)^m - awa_w^{\textcircled{d}}(aw)^m(a^{d,w}w)^m\| \\ & \leq & \|(aw)^m - awa_w^{\textcircled{d}}(aw)^m\|\|(a^{d,w}w)^m\|. \end{array}$$

Since

$$\lim_{m\to\infty} \|(aw)^m - awa_w^{\textcircled{d}}(aw)^m\|^{\frac{1}{m}} = 0,$$

we have

$$\lim_{m\to\infty} \|awa^{d,w}w - awa_w^{\textcircled{d}}awa^{d,w}w\|^{\frac{1}{m}} = 0.$$

Hence $awa_w^{\textcircled{d}}awa^{d,w}w = awa^{d,w}w$, and then

$$\begin{array}{rcl} [awa^{d,w}w][awa_w^{\textcircled{d}}] & = & aw[(aw)^d]^2awawa_w^{\textcircled{d}} = awa_w^{\textcircled{d}}, \\ ((awa^{d,w}w)(awa_w^{\textcircled{d}}))^* & = & (awa_w^{\textcircled{d}})^* = awa_w^{\textcircled{d}} = (awa^{d,w}w)(awa_w^{\textcircled{d}}), \\ (awa^{d,w}w)(awa_w^{\textcircled{d}})(awa^{d,w}w) & = & awa_w^{\textcircled{d}}awa^{d,w}w = awa^{d,w}w. \end{array}$$

Accordingly, $awa^{d,w}w \in \mathcal{A}^{(1,3)}$, as desired.

(2) ⇒ (1) Let $x = a^{d,w}w(awa^{d,w}w)^{(1,3)}$. Then we verify that

$$\begin{array}{rcl} awx & = & awa^{d,w}w(awa^{d,w}w)^{(1,3)} = aw(aw)^d(aw(aw)^d)^{(1,3)}, \\ (awx)^* & = & awx, \\ awx^2 & = & aw(aw)^d(aw(aw)^d)^{(1,3)}a^{d,w}w(awa^{d,w}w)^{(1,3)} \\ & = & aw(aw)^d(aw(aw)^d)^{(1,3)}aw[(aw)^d]^2(awa^{d,w}w)^{(1,3)} \\ & = & (aw)^d(awa^{d,w}w)^{(1,3)} = x, \end{array}$$

$$\begin{array}{rl} & \|(aw)^n - awx(aw)^n\| \\ = & \|(aw)^n - aw(aw)^d(aw(aw)^d)^{(1,3)}(aw)^n\| \\ \leq & \|(aw)^n - (aw)^d(aw)^{n+1}\| + \|(aw)(aw)^d(aw)^n - aw(aw)^d(aw(aw)^d)^{(1,3)}(aw)^n\| \\ \leq & \|(aw)^n - (aw)^d(aw)^{n+1}\| + \|(aw)(aw)^d(aw(aw)^d)^{(1,3)}(aw)(aw)^d(aw)^n \\ - & aw(aw)^d(aw(aw)^d)^{(1,3)}(aw)^n\| \\ \leq & \|(aw)^n - (aw)^d(aw)^{n+1}\| + \|(aw)(aw)^d(aw(aw)^d)^{(1,3)}\|\|(aw)^d(aw)^{n+1} - (aw)^n\| \\ = & \|(aw)^n - (aw)^d(aw)^{n+1}\|[1 + \|(aw)(aw)^d(aw(aw)^d)^{(1,3)}\|]. \end{array}$$

Hence,

$$\lim_{n\to\infty} \|(aw)^n - awx(aw)^n\|^{\frac{1}{n}} = 0.$$

Therefore $a \in \mathcal{A}_w^{\textcircled{d}}$. In this case, $a_w^{\textcircled{d}} = a^{d,w}w(awa^{d,w}w)^{(1,3)}$.

(2) ⇔ (3) In view of Lemma 4.4, $aw(aw)^d \in \mathcal{A}^{(1,3)}$ if and only if $(aw)^{\pi} = 1 - (aw)(aw)^d \in \mathcal{A}^{(1,4)}$, as desired. □

Corollary 4.6. *Let $a, w \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}^{\textcircled{d}}$.
(2) $a \in \mathcal{A}^d$ and $aa^d \in \mathcal{A}^{(1,3)}$.
(3) $a \in \mathcal{A}^d$ and $a^\pi \in \mathcal{A}^{(1,4)}$.

In this case, $a^{\textcircled{d}} = a^d(aa^d)^{(1,3)} = a^d[1 - (a^\pi)^{(1,4)}a^\pi]$.

Proof. This is obvious by choosing $w = 1$ in Theorem 4.6. □

5. Generalized w-core orders

Let $w \in \mathcal{A}$ and $a, b \in \mathcal{A}_w^{\textcircled{d}}$. We define the relation $a \leq_w^{\textcircled{d}} b$ to hold if the following conditions are satisfied: $awa_w^{\textcircled{d}} = bwa_w^{\textcircled{d}}, a_w^{\textcircled{d}}a = a_w^{\textcircled{d}}b$. This section investigates such generalized weighted core orders for pairs of elements in a Banach *-algebra. The subsequent result will serve as a fundamental tool for analyzing the properties of these orders.

Lemma 5.1. *Let $a, b \in \mathcal{A}_w^{\textcircled{d}}$. Then the following are equivalent:*

(1) $a \leq_w^{\textcircled{d}} b$.
(2) $awa^{d,w} = bwa^{d,w}, a^*a^{d,w} = b^*a^{d,w}$.

Proof. (1) ⇒ (2) Since $a \leq_w^{\textcircled{d}} b$, we have

$$awa_w^{\textcircled{d}} = bwa_w^{\textcircled{d}}, a_w^{\textcircled{d}}a = a_w^{\textcircled{d}}b.$$

By virtue of Theorem 4.5, $a_w^{\textcircled{d}} = a^{d,w}w(awa^{d,w}w)^{(1,3)}$. Then

$$awa^{d,w}w(awa^{d,w}w)^{(1,3)} = bwa^{d,w}w(awa^{d,w}w)^{(1,3)},$$

and so

$$awa^{d,w}w(awa^{d,w}w)^{(1,3)} = bw(aw)^d[awa^{d,w}w(awa^{d,w}w)^{(1,3)}],$$

Thus, we have

$$awa^{d,w}w = bw(aw)^d awa^{d,w}w = bwa^{d,w}w.$$

Since $a^{d,w} = (a^{d,w}w)^2a$, we have $awa^{d,w} = bwa^{d,w}$.

As $a_w^{\textcircled{d}}a = a_w^{\textcircled{d}}b$, we have

$$a^{d,w}wa^{d,w}(a^{d,w})^{(1,3)}a = a^{d,w}wa^{d,w}(a^{d,w})^{(1,3)}b.$$

It follows from $a^{d,w} = a(wa^{d,w})^2$ that $a^{d,w}(a^{d,w})^{(1,3)}a = a^{d,w}(a^{d,w})^{(1,3)}b$. This implies that $a^*a^{d,w}(a^{d,w})^{(1,3)} = b^*a^{d,w}(a^{d,w})^{(1,3)}$. Therefore $a^*a^{d,w} = b^*a^{d,w}$, as required.

(2) ⇒ (1) Since $awa^{d,w} = bwa^{d,w}$, by virtue of Theorem 3.5, we have

$$\begin{array}{rcl} awa_w^{\textcircled{d}} & = & awa^{d,w}wa^{d,w}(a^{d,w})^{(1,3)} \\ & = & bwa^{d,w}wa^{d,w}(a^{d,w})^{(1,3)} = bwa_w^{\textcircled{d}}. \end{array}$$

Since $a^*a^{d,w} = b^*a^{d,w}$, we have $a^*a^{d,w}(a^{d,w})^{(1,3)} = b^*a^{d,w}(a^{d,w})^{(1,3)}$, and then $a^{d,w}(a^{d,w})^{(1,3)}a = a^{d,w}(a^{d,w})^{(1,3)}b$. Therefore we derive

$$\begin{array}{rcl} a_w^{\textcircled{d}}a & = & a^{d,w}w[a^{d,w}(a^{d,w})^{(1,3)}a] \\ & = & a^{d,w}w[a^{d,w}(a^{d,w})^{(1,3)}b] = a_w^{\textcircled{d}}b, \end{array}$$

thus yielding the result. □

Let $a \in \mathcal{A}_w^{\#}, b \in \mathcal{A}$. Recall that $a \leq_w^{\#} b$ if $awa_w^{\#} = bwa_w^{\#}$ and $a_w^{\#}a = a_w^{\#}b$ (see [32]). We are now ready to prove:

Theorem 5.2. *Let $a, b \in \mathcal{A}_w^{\textcircled{d}}$. If $a = a_1 + a_2, b = b_1 + b_2$ are generalized w-core decompositions of a and b. Then the following are equivalent:*

(1) $a \leq_w^{\textcircled{d}} b$.
(2) $a_1 \leq_w^{\#} b_1$.

Proof. (1) ⇒ (2) Since $a \leq_w^{\textcircled{d}} b$, we have $awa_w^{\textcircled{d}} = bwa_w^{\textcircled{d}}$ and $a_w^{\textcircled{d}}a = a_w^{\textcircled{d}}b$. For any $m \in \mathbb{N}$, we derive

$$\begin{array}{rcl} a_1w(a_1)_w^{\#} & = & (a_1+a_2)w(a_1)_w^{\#} = awa_w^{\textcircled{d}} = bwa_w^{\textcircled{d}} \\ & = & bwaw(a_w^{\textcircled{d}})^2 = bw[awa_w^{\textcircled{d}}]a_w^{\textcircled{d}} = bw[bwa_w^{\textcircled{d}}]a_w^{\textcircled{d}} \\ & = & (bw)^2(a_w^{\textcircled{d}})^2 = \cdots = (bw)^m(a_w^{\textcircled{d}})^m, \\ b_1w(a_1)_w^{\#} & = & bwb_w^{\textcircled{d}}bwa_w^{\textcircled{d}} = bwb_w^{\textcircled{d}}bwaw(a_w^{\textcircled{d}})^2 = bwb_w^{\textcircled{d}}(bw)^2(a_w^{\textcircled{d}})^2 \\ & = & \cdots = bwb_w^{\textcircled{d}}(bw)^m(a_w^{\textcircled{d}})^m. \end{array}$$

Thus, we have
$$\|a_1w(a_1)_w^{\#} - b_1w(a_1)_w^{\#}\| \leq \|(bw)^m - bwb_w^{\textcircled{d}}(bw)^m\|\|(a_w^{\textcircled{d}})^m\|.$$

In view of Theorem 2.1,
$$\lim_{m\to\infty} \|(bw)^m - bwb_w^{\textcircled{d}}(bw)^m\|^{\frac{1}{m}} = 0.$$

Hence,
$$\lim_{m\to\infty} \|a_1w(a_1)_w^{\#} - b_1w(a_1)_w^{\#}\|^{\frac{1}{m}} = 0.$$

Therefore $a_1w(a_1)_w^{\#} = b_1w(a_1)_w^{\#}$.

Since $b_1 = bwb_w^{\textcircled{d}}b$, we verify that

$$awa_w^{\textcircled{d}} = a_1w(a_1)_w^{\#} = b_1w(a_1)_w^{\#} = bwb_w^{\textcircled{d}}bwa_w^{\textcircled{d}} = bwb_w^{\textcircled{d}}awa_w^{\textcircled{d}}.$$

Thus,
$$[awa_w^{\textcircled{d}}]^* = [bwb_w^{\textcircled{d}}awa_w^{\textcircled{d}}]^*,$$

and so
$$awa_w^{d} = awa_w^{\textcircled{d}}bwb_w^{\textcircled{d}}.$$

Then we see that
$$\begin{array}{rcl} (a_1)_w^{\#}a_1 & = & a_w^{\textcircled{d}}(awa_w^{\textcircled{d}}a) \\ & = & a_w^{\textcircled{d}}(awa_w^{\textcircled{d}})b \\ & = & a_w^{\textcircled{d}}(awa_w^{\textcircled{d}}bwb_w^{\textcircled{d}})b \\ & = & (a_w^{\textcircled{d}}awa_w^{\textcircled{d}})bwb_w^{\textcircled{d}})b \\ & = & a_w^{\textcircled{d}}(bwb_w^{\textcircled{d}}b) = (a_1)_w^{\#}b_1. \end{array}$$

Therefore $a_1 \leq_w^{\#} b_1$.

(2) ⇒ (1) Obviously, we have

$$awa_w^{\textcircled{d}} = (a_1+a_2)wa_1^{\#} = a_1wa_1^{\#} = b_1wa_1^{\#} = bwb_w^{\textcircled{d}}bwa_w^{\textcircled{d}}.$$

Then
$$a_w^{\textcircled{d}} = aw(a_w^{\textcircled{d}})^2 = bwb_w^{\textcircled{d}}bw(a_w^{\textcircled{d}})^2.$$

Since $\lim\limits_{n\to\infty} \|(bw)^n - bwb_w^{\textcircled{d}}(bw)^n\|^{\frac{1}{n}} = 0$, we deduce that

$$bwb_w^{\textcircled{d}}bwa_w^{\textcircled{d}} = bwa_w^{\textcircled{d}}.$$

This implies that $awa_w^{\textcircled{d}} = bwa_w^{\textcircled{d}}$.

Clearly, $a_w^{\textcircled{d}}a_2 = (a_1)_w^{\#}a_2 = (a_1)_w^{\#}a_1w(a_1)_w^{\#}a_2 = (a_1)_w^{\#}(a_1w(a_1)_w^{\#})^*a_2 = (a_1)_w^{\#}[w(a_1)_w^{\#}]^*(a_1)^*a_2 = 0$.

Moreover, we have
$$\begin{array}{rcl} awa_w^{\textcircled{d}} & = & bwb_w^{\textcircled{d}}bwa_w^{\textcircled{d}} \\ & = & (bwb_w^{\textcircled{d}})(awa_w^{\textcircled{d}}). \end{array}$$

Then

$$\begin{array}{rcl} awa_w^{\circledS} & = & (awa_w^{\circledS})^* \\ & = & (awa_w^{\circledS})^*(bwb_w^{\circledS})^* \\ & = & awa_w^{\circledS}bwb_w^{\circledS}. \end{array}$$

Hence, $a_w^{\circledS} = a_w^{\circledS}awa_w^{\circledS} = a_w^{\circledS}awa_w^{\circledS}bwb_w^{\circledS} = a_w^{\circledS}bwb_w^{\circledS}$. Accordingly, $a_w^{\circledS}b = a_w^{\circledS}bwb_w^{\circledS}b = (a_1)_w^{\#}b_1 = (a_1)_w^{\#}a_1 = a_w^{\circledS}(a_1 + a_2) = a_w^{\circledS}a$, thus yielding the result. □

Corollary 5.3. *The relation $\leq_w^{\circledS}$ is a pre-order on the set of all generalized w-core invertible elements of $\mathcal{A}$.*

Proof. Step 1. $a \leq_w^{\circledS} a$. Let $a = a_1 + a_2$ be the generalized w-core decomposition. In view of [32, Theorem 2.3], $a_1 \leq_w^{\#} a_1$. By using Theorem 5.1, $a \leq_w^{\circledS} a$.

Step 2. Assume that $a \leq_w^{\circledS} b$ and $b \leq_w^{\circledS} c$. We claim that $a \leq_w^{\circledS} c$. Let $a = a_1 + a_2, b = b_1 + b_2$ and $c = c_1 + c_2$ be the generalized w-core decompositions of a, b and c, respectively. By virtue of Lemma 5.1, we have $a_1 \leq_w^{\#} b_1$ and $b_1 \leq_w^{\#} c_1$. In view of [32, Theorem 2.3], we have $a_1 \leq_w^{\#} c_1$. By using Lemma 5.1 again, $a \leq_w^{\circledS} c$.

Therefore the relation $\leq_w^{\circledS}$ for generalized w-core invertible elements is a pre-order. □

The relation $\leq_w^{\circledS}$ for generalized w-core invertible elements is a pre-order, while it is not partial order as the following shows.

Example 5.4.

Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, W = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathbb{C}^{3\times 3}$. We take the involution on $\mathbb{C}^{3\times 3}$ as the conjugate transpose. Then $A^{D,W} = [(AW)^D]^2A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = [(BW)^D]^2B = B^{D,W}$. By using Lemma 5.1, we directly verify that $A \leq_W^{\circledS} B$ and $B \leq_W^{\circledS} A$. But $A \neq B$.

Theorem 5.5. *Let $a, b \in \mathcal{A}_w^{\circledS}$. Then the following are equivalent:*

(1) $a \leq_w^{\circledS} b$.
(2) a and b are represented as

$$a = \begin{pmatrix} a_1 & a_{12} \\ 0 & a_2 \end{pmatrix}_{(p,q)}, b = \begin{pmatrix} a_1 & a_{12} \\ 0 & a_2 + (b-a) \end{pmatrix}_{(p,q)},$$

where $p = awa_w^{\circledS}$ and $q = (wa)(wa)^d$.

Proof. (1) ⇒ (2) Let $p = awa_w^{\circledS}$ and $q = (wa)(wa)^d$. Then $p^2 = p = p^* \in \mathcal{A}$ and $q^2 = q \in \mathcal{A}$. We check that

$$\begin{array}{rcl} (1-p)aq & = & [1 - awa_w^{\circledS}]a(wa)(wa)^d \\ & = & [1 - awa_w^{\circledS}]awa(wa)^n[(wa)^d]^{n+1} \\ & = & [1 - aw(aw)^{\circledS}](aw)^{n+1}a[(wa)^d]^{n+1} \\ & = & [(aw)^{n+1} - (aw)a_w^{\circledS}(aw)^{n+1}]a[(wa)^d]^{n+1}. \end{array}$$

Hence,

$$||(1-p)aq||^{\frac{1}{n+1}} \leq ||(aw)^{n+1} - (aw)a_w^{\circledS}(aw)^{n+1}||^{\frac{1}{n+1}}||a||^{\frac{1}{n+1}}||(wa)^d||.$$

This implies that $\lim\limits_{n\to\infty} ||(1-p)aq||^{\frac{1}{n+1}} = 0$, and then $(1-p)ap = 0$. By virtue of Lemma 5.1, we have $awa^{d,w} = bwa^{d,w}$. Then we verify that

$$\begin{array}{rcl} (1-p)bq & = & [1 - awa_w^{\circledS}]b(wa)(wa)^d \\ & = & [1 - awa_w^{\circledS}]bwa^{d,w}wa \\ & = & [1 - awa_w^{\circledS}]awa^{d,w}wa = 0. \end{array}$$

Write $b = \begin{pmatrix} b_1 & b_{12} \\ 0 & b_2 \end{pmatrix}_p$. Clearly, we have

$$\begin{array}{rcl} pbq & = & awa_w^{\textcircled{d}}b(wa)(wa)^d \\ & = & awa_w^{\textcircled{d}}bwa^{d,w}wa \\ & = & awa_w^{\textcircled{d}}awa^{d,w}wa \\ & = & awa_w^{\textcircled{d}}a(wa)(wa)^d \\ & = & paq, \end{array}$$

and so $a_1 = b_1$.

Also we have

$$\begin{array}{rcl} pb(1-q) & = & awa_w^{\textcircled{d}}b[1-(wa)(wa)^d] \\ & = & aw(a_w^{\textcircled{d}}b)[1-(wa)(wa)^d] \\ & = & aw(a_w^{\textcircled{d}}a)[1-(wa)(wa)^d] \\ & = & pa(1-p). \end{array}$$

Moreover, $(1-p)b(1-q) = (1-p)b = b-pb = b-aw(a_w^{\textcircled{d}}b) = b-aw(a_w^{\textcircled{d}}a) = b-pa = (1-p)a+(b-a) = a_2+(b-a)$, as desired.

(2) $\Rightarrow$ (1) By hypothesis, $paq = pbq$ and $pa(1-q) = pb(1-q)$. Then $pa = pb$. Hence, $awa_w^{\textcircled{d}}a = awa_w^{\textcircled{d}}b$. This implies that $a_w^{\textcircled{d}}a = a_w^{\textcircled{d}}b$.

Moreover, we have $(1-p)aq = 0 = (1-p)bq$. As $paq = pbq$, we have $aq = bq$, and so $a(wa)(wa)^d = b(wa)(wa)^d$. Then $awa_w^{\textcircled{d}} = bwa_w^{\textcircled{d}}$. In light of Lemma 5.1, $a \leq_w^{\textcircled{d}} b$, as asserted. □

The generalized core-EP inverse for a Banach algebra element was introduced in [5]. $a \leq^{\textcircled{d}} b$ if and only if $aa^{\textcircled{d}} = ba^{\textcircled{d}}$ and $a^{\textcircled{d}}a = a^{\textcircled{d}}b$. As an immediate consequence of Theorem 5.5, we derive

Corollary 5.6. *Let $a, b \in \mathcal{A}^{\textcircled{d}}$. Then the following are equivalent:*

(1) $a \leq^{\textcircled{d}} b$.

(2) a and b are represented as

$$a = \begin{pmatrix} a_1 & a_{12} \\ 0 & a_2 \end{pmatrix}_{(p,q)}, b = \begin{pmatrix} a_1 & a_{12} \\ 0 & a_2 + (b-a) \end{pmatrix}_{(p,q)},$$

where $p = aa^{\textcircled{d}}$ and $q = aa^d$.

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