Filomat 39:30 (2025), 10761–10769
https://doi.org/10.2298/FIL2530761Z

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

The common Hermitian solution to the system of matrix equations $B_iXB_i^* = A_i, i = 1, 2, 3$

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Abstract. In this paper, a new solvability condition and the explicit expression of the common Hermitian solutions (CHSs) to the system of matrix equations $B_iXB_i^* = A_i, i = 1, 2, 3$ are deduced by using the Moore-Penrose inverses and some matrix decompositions. Finally, a numerical example is provided to validate the accuracy of the obtained results.

1. Introduction

Throughout this paper, we denote the complex $m\times n$ matrix space by $\mathbb{C}^{m\times n}$, the complex $n\times n$ Hermitian matrix space by $\mathbb{C}_{\mathrm{H}}^{n}$, and denote the conjugate transpose, the Moore-Penrose inverse, the rank, the range space and the null space of a complex matrix $A \in \mathbb{C}^{m\times n}$ by A^*, $A^\dagger$, $r(A)$, $\mathcal{R}(A)$ and $\mathcal{N}(A)$, respectively. I_n represents the identity matrix of size n. $P_{\mathcal{L}}$ stands for the orthogonal projector on the subspace $\mathcal{L} \subset \mathbb{C}^n$. Also, the symbols E_A and F_A stand for the two orthogonal projectors: $E_A = I_m - AA^\dagger$, $F_A = I_n - A^\dagger A$, where $A \in \mathbb{C}^{m\times n}$.

There has been a lot of interest in the matrix equation $AXA^* = B$ and some results were established [1, 5–7, 9, 10, 13, 16]. In 2001, Navarra et al. [8] provided the general common solution to the linear matrix equations $A_1XB_1 = C_1$ and $A_2XB_2 = C_2$. In 2004, Zhang [17] gave the general common Hermitian Nonnegative-definite solution to the matrix equations $AXA^* = B$ and $CXC^* = D$ by using the Moore-Penrose generalized inverse of matrices and some matrix decompositions. In 2011, Zhang et al. [18] studied the common Hermitian least squares solutions of the matrix equations $A_1XA_1^* = B_1$ and $A_2XA_2^* = B_2$ subject to inequality restrictions. When it comes to three matrix equations, Tian [11] proposed the solvability conditions of the matrix equations $A_iXB_i = C_i, i = 1, 2, 3$ by using the ranks of matrices. In particular, Tian [12] discussed the solvability of the matrix equations

$$B_1XB_1^* = A_1,\ B_2XB_2^* = A_2,\ B_3XB_3^* = A_3 \tag{1}$$

2020 *Mathematics Subject Classification*. Primary 15A09; Secondary 15A24.
Keywords. matrix equation; generalized inverse; generalized singular value decomposition; common Hermitian solution.
Received: 09 April 2025; Revised: 20 September 2025; Accepted: 22 September 2025
Communicated by Dijana Mosić
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to have a common Hermitian solution (CHS), where $B_i \in \mathbb{C}^{m_i \times n}$, $A_i \in \mathbb{C}_{\mathrm{H}}^{m_i}$, $i = 1, 2, 3$ are given matrices and $X \in \mathbb{C}_{\mathrm{H}}^{n}$ is an unknown matrix to be determined, by using the maximal and minimal ranks and inertias of the Hermitian matrix-valued function $A_1 - B_1 X B_1^*$. It is worth mentioning that the explicit expression for the general CHS to (1) was not given, as was pointed out by Tian in [12] that: " A challenging open problem on the triple matrix equations in (1) is to give a parametric form for their general CHS." To the best of our knowledge, it seems difficult to propose an explicit expression of the CHSs to (1) by the methods in [12]. So, how to obtain the explicit expression for the general CHS of (1)? This paper provides the answer.

In this paper, we propose an alternative approach to presenting an explicit expression for the general CHS of (1). The new solvability conditions for the existence of the CHSs and the explicit expression for the general CHS of (1) are provided by means of the Moore-Penrose inverses, orthogonal projectors and some matrix decompositions. The numerical example verifies the correctness of the obtained results.

2. Preliminaries

The following lemmas are needed in what follows.

Lemma 2.1. *[12] Let $B_i \in \mathbb{C}^{m_i \times n}$, $A_i \in \mathbb{C}_{\mathrm{H}}^{m_i}$, $i = 2, 3$. The pair of matrix equations*

$$B_2 X B_2^* = A_2, \ \ B_3 X B_3^* = A_3 \tag{2}$$

have a CHS if and only if

$$B_2 B_2^\dagger A_2 = A_2, \ \ B_3 B_3^\dagger A_3 = A_3, \ \ r\begin{bmatrix} A_2 & 0 & B_2 \\ 0 & -A_3 & B_3 \\ B_2^* & B_3^* & 0 \end{bmatrix} = 2r\begin{bmatrix} B_2 \\ B_3 \end{bmatrix}. \tag{3}$$

In this case, the general CHS of Eqs. (2) *is*

$$X = X_0 + V F_B + F_B V^* + F_{B_2} U F_{B_3} + F_{B_3} U^* F_{B_2}, \tag{4}$$

where X_0 is a special CHS to the pair of equations, $B = \begin{bmatrix} B_2 \\ B_3 \end{bmatrix}$, and $U, V \in \mathbb{C}^{n \times n}$ are arbitrary matrices.

From above, we can see that Tian [12] mentioned a special CHS X_0. Regrettably, the explicit expression of X_0 has not been given, although X_0 plays an important role in proposing an explicit expression for the general CHS of (1). To solve this problem, we attempt to give a concrete expression of X_0 by the method in [14] as follows.

Suppose that the generalized singular value decomposition (GSVD) of the matrix pair $[B_2^*, B_3^*]$ is

$$B_2^* = M \Sigma_1 P^*, \quad B_3^* = M \Sigma_2 Q^*, \tag{5}$$

where $M \in \mathbb{C}^{n \times n}$ are nonsingular matrices and $P \in \mathbb{C}^{m_2 \times m_2}$, $Q \in \mathbb{C}^{m_3 \times m_3}$ are unitary matrices, and

$$\Sigma_1 = \begin{matrix} \begin{bmatrix} I & 0 & 0 \\ 0 & \Lambda & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{matrix} r-s \\ s \\ k-r \\ n-k \end{matrix} \\ \begin{matrix} r-s & s & m_2 - r \end{matrix} & \end{matrix}, \tag{6}$$

$$\Sigma_2 = \begin{matrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix} & \begin{matrix} r-s \\ s \\ k-r \\ n-k \end{matrix} \\ \begin{matrix} g & s & k-r \end{matrix} & \end{matrix}, \tag{7}$$

$k = \text{rank}\left([B_2^*, B_3^*]\right) = m_3 + r - s - g$, and

$$\Lambda = \text{diag}(\lambda_1, \cdots, \lambda_s), \;\; \Delta = \text{diag}(\delta_1, \cdots, \delta_s)$$

with

$$1 > \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s > 0, \;\; 0 < \delta_1 \leq \delta_2 \leq \cdots \leq \delta_s < 1,$$
$$\lambda_i^2 + \delta_i^2 = 1, \; i = 1, \cdots, s.$$

Partition the matrices M^*XM, P^*A_2P and Q^*A_3Q into the following forms:

$$M^*XM = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} & Y_{14} \\ Y_{21} & Y_{22} & Y_{23} & Y_{24} \\ Y_{31} & Y_{32} & Y_{33} & Y_{34} \\ Y_{41} & Y_{42} & Y_{43} & Y_{44} \end{bmatrix} \begin{matrix} r \\ s \\ k-r \\ n-k \end{matrix}, \tag{8}$$
$$\qquad\qquad \begin{matrix} r & s & k-r & n-k \end{matrix}$$

$$P^*A_2P = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{12}^* & A_{22} & A_{23} \\ A_{13}^* & A_{23}^* & A_{33} \end{bmatrix} \begin{matrix} r-s \\ s \\ m_2-r \end{matrix}, \tag{9}$$
$$\qquad\qquad \begin{matrix} r-s & s & m_2-r \end{matrix}$$

$$Q^*A_3Q = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{A}_{13} \\ \tilde{A}_{12}^* & \tilde{A}_{22} & \tilde{A}_{23} \\ \tilde{A}_{13}^* & \tilde{A}_{23}^* & \tilde{A}_{33} \end{bmatrix} \begin{matrix} g \\ s \\ k-r \end{matrix}, \tag{10}$$
$$\qquad\qquad \begin{matrix} g & s & k-r \end{matrix}$$

then a special CHS X_0 to Eqs. (2) can be expressed as:

$$X_0 = (M^*)^{-1} \begin{bmatrix} A_{11} & A_{12}\Lambda^{-1} & 0 & 0 \\ \Lambda^{-1}A_{12}^* & \Delta^{-1}\tilde{A}_{22}\Delta^{-1} & \Delta^{-1}\tilde{A}_{23} & 0 \\ 0 & \tilde{A}_{23}^*\Delta^{-1} & \tilde{A}_{33} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} M^{-1}. \tag{11}$$

In this way, we obtain the concrete expression of a special CHS X_0 for Eqs. (2). To simplify the proof of Theorem 3.1 in the following, Lemma 2.2 is needed.

Lemma 2.2. *Let $G_1 \in \mathbb{C}^{m_1 \times n}, G_2 \in \mathbb{C}^{n \times m_1}$ and $W \in \mathbb{C}^{m_1 \times m_1}$, and let $H = F_{G_2}G_1$. If $\mathcal{R}(G_2^*) \subset \mathcal{R}(G_1)$, then the matrix equation*

$$G_1VG_2 + (G_1VG_2)^* = W, \tag{12}$$

has a solution $V \in \mathbb{C}^{n \times n}$ if and only if

$$W = W^*, \;\; G_1G_1^\dagger WG_2^\dagger G_2 + G_2^\dagger G_2WG_1G_1^\dagger - G_2^\dagger G_2WG_2^\dagger G_2 = W, \tag{13}$$

in this case, a particular solution of Eq. (12) *is*

$$V_0 = \frac{1}{2}G_1^\dagger WG_2^\dagger - \frac{1}{2}G_1^\dagger G_2^\dagger G_2W\left((G_2^*)^\dagger G_1H^\dagger\right)^* + \frac{1}{2}H^\dagger WG_2^\dagger, \tag{14}$$

and the general solution of Eq. (12) *is*

$$\begin{aligned} V = {} & V_0 + 2Q - G_1^\dagger G_1QG_2G_2^\dagger - G_1^\dagger G_2^*Q^*G_1^*G_2^\dagger + G_1^\dagger G_2^\dagger G_2G_1Q\left((G_2^*)^\dagger G_1H^\dagger G_2^*\right)^* \\ & + G_1^\dagger G_2^*Q^*\left((G_2^*)^\dagger G_1H^\dagger G_1\right)^* - H^\dagger G_2^*Q^*G_1^*G_2^\dagger - H^\dagger G_1QG_2G_2^\dagger, \end{aligned} \tag{15}$$

where $Q \in \mathbb{C}^{n \times n}$ is an arbitrary matrix.

Proof.

$$\begin{aligned}\mathcal{R}(H) &= P_{\mathcal{N}(G_2)}\mathcal{R}(G_1)\\ &= \mathcal{N}(G_2) \cap \left(\mathcal{N}(G_2) \cap \mathcal{N}(G_1^*)\right)^{\perp}\\ &= \mathcal{N}(G_2) \cap \left(\mathcal{R}(G_2^*) + \mathcal{R}(G_1)\right)\\ &= \mathcal{N}(G_2) \cap \mathcal{R}(G_1) \quad \text{(see [3])},\end{aligned}$$

which leads to $\mathcal{R}(H)^{\perp} = \mathcal{R}(G_2^*) + \mathcal{N}(G_1^*)$. From $\mathcal{R}(G_2^*) \subset \mathcal{R}(G_1)$, we have $G_1G_1^{\dagger}G_2^{\dagger}G_2 = G_2^{\dagger}G_2$, which implies that $E_{G_1}G_2^{\dagger}G_2 = G_2^{\dagger}G_2E_{G_1} = 0$, then we can obtain that

$$(E_{G_1} + G_2^{\dagger}G_2)^* = E_{G_1} + G_2^{\dagger}G_2 = (E_{G_1} + G_2^{\dagger}G_2)^2.$$

Namely, $E_{G_1} + G_2^{\dagger}G_2$ is the orthogonal projector (see [2, p.81, Ex.68]), concretely, $E_{G_1} + G_2^{\dagger}G_2 = P_{\mathcal{R}(G_2^*)+\mathcal{N}(G_1^*)} = P_{\mathcal{R}(H)^{\perp}}$. Combined with $P_{\mathcal{R}(H)^{\perp}} + P_{\mathcal{R}(H)} = I$, we can see that

$$HH^{\dagger} = G_1G_1^{\dagger} - G_2^{\dagger}G_2. \tag{16}$$

On the other hand, as shown in [15]:

$$G_1G_1^{\dagger}WG_2^{\dagger}G_2 + G_2^{\dagger}G_2WG_1G_1^{\dagger} + G_2^{\dagger}G_2WHH^{\dagger} + HH^{\dagger}WG_2^{\dagger}G_2 = 2W,$$

according to (16), we have

$$G_1G_1^{\dagger}WG_2^{\dagger}G_2 + G_2^{\dagger}G_2WG_1G_1^{\dagger} + G_2^{\dagger}G_2W(G_1G_1^{\dagger} - G_2^{\dagger}G_2) + (G_1G_1^{\dagger} - G_2^{\dagger}G_2)WG_2^{\dagger}G_2 = 2W.$$

By simple calculation, we can deduce that

$$G_1G_1^{\dagger}WG_2^{\dagger}G_2 + G_2^{\dagger}G_2WG_1G_1^{\dagger} - G_2^{\dagger}G_2WG_2^{\dagger}G_2 = W,$$

which is the second condition of (13). □

Lemma 2.3. *[4] Let $\tilde{G}_1 \in \mathbb{C}^{m_1\times n}, \tilde{G}_2 \in \mathbb{C}^{n\times m_1}$ and $\tilde{W} = \tilde{W}^* \in \mathbb{C}^{m_1\times m_1}$. Then the matrix equation*

$$\tilde{G}_1U\tilde{G}_2 + (\tilde{G}_1U\tilde{G}_2)^* = \tilde{W} \tag{17}$$

has a solution $U \in \mathbb{C}^{n\times n}$ if and only if

$$E_{\tilde{G}_1}\tilde{W}E_{\tilde{G}_1} = 0, \;\; F_{\tilde{G}_2}\tilde{W}F_{\tilde{G}_2} = 0, \;\; E_LF_{\tilde{G}_2}\tilde{W}\tilde{G}_1\tilde{G}_1^{\dagger} = 0, \tag{18}$$

where $L = F_{\tilde{G}_2}\tilde{G}_1\tilde{G}_1^{\dagger}$. In this case, all the solutions $U \in \mathbb{C}^{n\times n}$ satisfying Eq. (17) are given by

$$U = \tilde{G}_1^{\dagger}\left(\Theta + F_LS_UF_L\tilde{G}_1\tilde{G}_1^{\dagger}\right)\tilde{G}_2^{\dagger} + M - \tilde{G}_1^{\dagger}\tilde{G}_1M\tilde{G}_2\tilde{G}_2^{\dagger}, \tag{19}$$

where $S_U \in \mathbb{C}^{m_1\times m_1}, M \in \mathbb{C}^{n\times n}$ are arbitrary matrices with $S_U^* = -S_U$ and Θ is given by

$$\Theta = \frac{1}{2}\tilde{W}(2I - \tilde{G}_1\tilde{G}_1^{\dagger}) + \frac{1}{2}(\Psi - \Psi^*)\tilde{G}_1\tilde{G}_1^{\dagger},$$

with $\Psi = 2L^{\dagger}F_{\tilde{G}_2}\tilde{W} + \left(I - L^{\dagger}F_{\tilde{G}_2}\right)\tilde{W}L^{\dagger}L$.

3. Main results

In this section, we will present the main results of this paper.

Theorem 3.1. *Let $B_i \in \mathbb{C}^{m_i\times n}$, $A_i \in \mathbb{C}_{\mathrm{H}}^{m_i}$, $i = 1, 2, 3$, and denote $G_1 = B_1$, $B = \begin{bmatrix} B_2 \\ B_3 \end{bmatrix}$, $G_2 = F_B B_1^*$, $G_3 = B_1 F_{B_2}$, $G_4 = F_{B_3} B_1^*$. Then, Eqs. (1) has a CHS X if and only if*

$$\begin{aligned} &B_2B_2^\dagger A_2 = A_2,\ \ B_3B_3^\dagger A_3 = A_3,\ \ r\begin{bmatrix} A_2 & 0 & B_2 \\ 0 & -A_3 & B_3 \\ B_2^* & B_3^* & 0 \end{bmatrix} = 2r\begin{bmatrix} B_2 \\ B_3 \end{bmatrix}, \\ &E_{\tilde{G}_1}\tilde{W}E_{\tilde{G}_1} = 0,\ \ F_{\tilde{G}_2}\tilde{W}F_{\tilde{G}_2} = 0,\ \ E_L F_{\tilde{G}_2}\tilde{W}\tilde{G}_1\tilde{G}_1^{\dagger} = 0, \end{aligned} \tag{20}$$

where $\tilde{G}_1 = F_{G_2}G_3$, $\tilde{G}_2 = G_4F_{G_2}$, X_0 is given by (11), $A = A_1 - G_1X_0G_1^*$, $\tilde{W} = A - G_1G_1^\dagger AG_2^\dagger G_2 - G_2^\dagger G_2AG_1G_1^\dagger + G_2^\dagger G_2AG_2^\dagger G_2$, $L = F_{\tilde{G}_2}\tilde{G}_1\tilde{G}_1^\dagger$. In this case, let the GSVD of the matrix pair $[B_2^*, B_3^*]$ is given by (5). Then, the general CHS to (1) is given by

$$\begin{aligned} X = {} & (M^*)^{-1}\begin{bmatrix} A_{11} & A_{12}\Lambda^{-1} & 0 & 0 \\ \Lambda^{-1}A_{12}^* & \Delta^{-1}\tilde{A}_{22}\Delta^{-1} & \Delta^{-1}\tilde{A}_{23} & 0 \\ 0 & \tilde{A}_{23}^*\Delta^{-1} & \tilde{A}_{33} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} M^{-1} \\ & +(\tfrac{1}{2}G_1^\dagger(A-\tilde{G})G_2^\dagger - \tfrac{1}{2}G_1^\dagger G_2^\dagger G_2(A-\tilde{G})\left((G_2^*)^\dagger G_1H^\dagger\right)^* + \tfrac{1}{2}H^\dagger(A-\tilde{G})G_2^\dagger \\ & +2Q - G_1^\dagger G_1QG_2G_2^\dagger - G_1^\dagger G_2^*Q^*G_1^*G_2^\dagger + G_1^\dagger G_2^\dagger G_2G_1Q\left((G_2^*)^\dagger G_1H^\dagger G_2^*\right)^* \\ & +G_1^\dagger G_2^*Q^*\left((G_2^*)^\dagger G_1H^\dagger G_1\right)^* - H^\dagger G_2^*Q^*G_1^*G_2^\dagger - H^\dagger G_1QG_2G_2^\dagger)F_B \\ & +F_B(\tfrac{1}{2}G_1^\dagger(A-\tilde{G})G_2^\dagger - \tfrac{1}{2}G_1^\dagger G_2^\dagger G_2(A-\tilde{G})\left((G_2^*)^\dagger G_1H^\dagger\right)^* + \tfrac{1}{2}H^\dagger(A-\tilde{G})G_2^\dagger \\ & +2Q - G_1^\dagger G_1QG_2G_2^\dagger - G_1^\dagger G_2^*Q^*G_1^*G_2^\dagger + G_1^\dagger G_2^\dagger G_2G_1Q\left((G_2^*)^\dagger G_1H^\dagger G_2^*\right)^* \\ & +G_1^\dagger G_2^*Q^*\left((G_2^*)^\dagger G_1H^\dagger G_1\right)^* - H^\dagger G_2^*Q^*G_1^*G_2^\dagger - H^\dagger G_1QG_2G_2^\dagger)^* \\ & +F_{B_2}(\tilde{G}_1^\dagger\left(\Theta + F_LS_UF_L\tilde{G}_1\tilde{G}_1^\dagger\right)\tilde{G}_2^\dagger + M - \tilde{G}_1^\dagger\tilde{G}_1M\tilde{G}_2\tilde{G}_2^\dagger)F_{B_3} \\ & +F_{B_3}(\tilde{G}_1^\dagger\left(\Theta + F_LS_UF_L\tilde{G}_1\tilde{G}_1^\dagger\right)\tilde{G}_2^\dagger + M - \tilde{G}_1^\dagger\tilde{G}_1M\tilde{G}_2\tilde{G}_2^\dagger)^*F_{B_2}, \end{aligned} \tag{21}$$

where M is given by (5), Λ, Δ are given by (6) and (7), A_{11}, A_{12} are given by (9), $\tilde{A}_{22}, \tilde{A}_{23}, \tilde{A}_{33}$ are given by (10), $S_U \in \mathbb{C}^{m_1\times m_1}$, $M \in \mathbb{C}^{n\times n}$ are arbitrary matrices with $S_U^* = -S_U$ and Θ is given by $\Theta = \frac{1}{2}\tilde{W}(2I - \tilde{G}_1\tilde{G}_1^\dagger) + \frac{1}{2}(\Psi - \Psi^*)\tilde{G}_1\tilde{G}_1^\dagger$ with $\Psi = 2L^\dagger F_{\tilde{G}_2}\tilde{W} + \left(I - L^\dagger F_{\tilde{G}_2}\right)\tilde{W}L^\dagger L$, U is given by (19), $\tilde{G} = G_3UG_4 + (G_3UG_4)^*$, $H = F_{G_2}G_1$, $Q \in \mathbb{C}^{n\times n}$ is an arbitrary matrix.

Proof. By Lemma 2.1, if the conditions of (3) hold, then Eqs. (2) has a CHS, which is given by (4). Substituting (4) into $B_1XB_1^* = A_1$ yields

$$G_1VG_2 + (G_1VG_2)^* = A - \tilde{G}, \tag{22}$$

where $G_1 = B_1$, $G_2 = F_BB_1^*$, $G_3 = B_1F_{B_2}$, $G_4 = F_{B_3}B_1^*$, $A = A_1 - G_1X_0G_1^*$, $\tilde{G} = G_3UG_4 + (G_3UG_4)^*$. Note that $\mathcal{R}(G_2^*) \subset \mathcal{R}(G_1)$, according to Lemma 2.2, Eq. (22) has a solution with respect to V if and only if

$$G_1G_1^\dagger(A-\tilde{G})G_2^\dagger G_2 + G_2^\dagger G_2(A-\tilde{G})G_1G_1^\dagger - G_2^\dagger G_2(A-\tilde{G})G_2^\dagger G_2 = A - \tilde{G}. \tag{23}$$

Note that

$$G_1G_1^\dagger\tilde{G}G_2^\dagger G_2 = \tilde{G}G_2^\dagger G_2,\ \ G_2^\dagger G_2\tilde{G}G_1G_1^\dagger = G_2^\dagger G_2\tilde{G},$$

then Eq. (23) can be simplified to

$$F_{G_2}\tilde{G}F_{G_2} = A - G_1G_1^\dagger AG_2^\dagger G_2 - G_2^\dagger G_2AG_1G_1^\dagger + G_2^\dagger G_2AG_2^\dagger G_2 \triangleq \tilde{W}, \tag{24}$$

which is equivalent to

$$\tilde{G}_1 U \tilde{G}_2 + (\tilde{G}_1 U \tilde{G}_2)^* = \tilde{W}, \tag{25}$$

where $\tilde{G}_1 = F_{G_2} G_3$, $\tilde{G}_2 = G_4 F_{G_2}$. By Lemma 2.3, we know that (25) has a solution U if and only if

$$E_{\tilde{G}_1} \tilde{W} E_{\tilde{G}_1} = 0, \ \ F_{\tilde{G}_2} \tilde{W} F_{\tilde{G}_2} = 0, \ \ E_L F_{\tilde{G}_2} \tilde{W} \tilde{G}_1 \tilde{G}_1^{\dagger} = 0, \tag{26}$$

where $L = F_{\tilde{G}_2} \tilde{G}_1 \tilde{G}_1^{\dagger}$. In this case, all the solutions $U \in \mathbb{C}^{n\times n}$ satisfying Eq. (25) are given by

$$U = \tilde{G}_1^{\dagger} \left(\Theta + F_L S_U F_L \tilde{G}_1 \tilde{G}_1^{\dagger} \right) \tilde{G}_2^{\dagger} + M - \tilde{G}_1^{\dagger} \tilde{G}_1 M \tilde{G}_2 \tilde{G}_2^{\dagger}, \tag{27}$$

where $S_U \in \mathbb{C}^{m_1 \times m_1}, M \in \mathbb{C}^{n\times n}$ are arbitrary matrices with $S_U^* = -S_U$ and Θ is given by $\Theta = \frac{1}{2}\tilde{W}(2I - \tilde{G}_1 \tilde{G}_1^{\dagger}) + \frac{1}{2}(\Psi - \Psi^*)\tilde{G}_1 \tilde{G}_1^{\dagger}$ with $\Psi = 2L^{\dagger} F_{\tilde{G}_2} \tilde{W} + \left(I - L^{\dagger} F_{\tilde{G}_2} \right) \tilde{W} L^{\dagger} L$. In this case, a particular solution of Eq. (22) is

$$V_0 = \frac{1}{2} G_1^{\dagger} (A - \tilde{G}) G_2^{\dagger} - \frac{1}{2} G_1^{\dagger} G_2^{\dagger} G_2 (A - \tilde{G}) \left((G_2^*)^{\dagger} G_1 H^{\dagger} \right)^* + \frac{1}{2} H^{\dagger} (A - \tilde{G}) G_2^{\dagger}, \tag{28}$$

and the general solution of Eq. (22) is

$$\begin{aligned} V = {} & V_0 + 2Q - G_1^{\dagger} G_1 Q G_2 G_2^{\dagger} - G_1^{\dagger} G_2^* Q^* G_1^* G_2^{\dagger} + G_1^{\dagger} G_2^{\dagger} G_2 G_1 Q \left((G_2^*)^{\dagger} G_1 H^{\dagger} G_2^* \right)^* \\ & + G_1^{\dagger} G_2^* Q^* \left((G_2^*)^{\dagger} G_1 H^{\dagger} G_1 \right)^* - H^{\dagger} G_2^* Q^* G_1^* G_2^{\dagger} - H^{\dagger} G_1 Q G_2 G_2^{\dagger}, \end{aligned} \tag{29}$$

where $H = F_{G_2} G_1$, $Q \in \mathbb{C}^{n\times n}$ is an arbitrary matrix. Substituting (11), (27)–(29) into (4), we can obtain (21), which completes the proof of the theorem. □

4. Numerical algorithm and numerical example

According to Theorem 3.1, we have the following algorithm for solving the CHSs to the system of matrix equations $B_i X B_i^* = A_i, i = 1, 2, 3$.

Algorithm 4.1.

1) Input matrices $A_i, B_i, i = 1, 2, 3$.

2) Compute the GSVD of the matrix pair $[B_2^*, B_3^*]$ *by* (5).

3) Compute the matrices Λ, Δ *by* (6), (7) *respectively.*

4) Compute the matrices A_{11}, A_{12} *by* (9) *and* $\tilde{A}_{22}, \tilde{A}_{23}, \tilde{A}_{33}$ *by* (10), *respectively.*

5) Compute the matrix X_0 *by* (11).

6) Set and compute $B = \begin{bmatrix} B_2 \\ B_3 \end{bmatrix}, G_1 = B_1, G_2 = F_B B_1^*, G_3 = B_1 F_{B_2}, G_4 = F_{B_3} B_1^*$, *respectively.*

7) Set and compute $\tilde{G}_1 = F_{G_2} G_3, \tilde{G}_2 = G_4 F_{G_2}, A = A_1 - G_1 X_0 G_1^*, \tilde{W} = A - G_1 G_1^{\dagger} A G_2^{\dagger} G_2 - G_2^{\dagger} G_2 A G_1 G_1^{\dagger} + G_2^{\dagger} G_2 A G_2^{\dagger} G_2, L = F_{\tilde{G}_2} \tilde{G}_1 \tilde{G}_1^{\dagger}$, *respectively.*

8) If the conditions (20) *are satisfied, go to 9); otherwise, Eqs.* (1) *has no solution, and stop.*

9) Compute the matrices $\Psi = 2L^{\dagger} F_{\tilde{G}_2} \tilde{W} + \left(I - L^{\dagger} F_{\tilde{G}_2} \right) \tilde{W} L^{\dagger} L$ *and* $\Theta = \frac{1}{2}\tilde{W}(2I - \tilde{G}_1 \tilde{G}_1^{\dagger}) + \frac{1}{2}(\Psi - \Psi^*)\tilde{G}_1 \tilde{G}_1^{\dagger}$.

10) Choose an anti-Hermitian matrix S_U *and an arbitrary matrix* M, *and compute the matrix* U *by* (27).

11) Set $\tilde{G} = G_3 U G_4 + (G_3 U G_4)^*$, $H = F_{G_2} G_1$ *and compute* V_0 *by* (28).

12) Randomly choose a matrix Q and compute the matrix V by (29).

13) Compute the matrix X by (21).

Example 4.2. *Let $m_1 = 9, m_2 = 6, m_3 = 7, n = 8$, and the matrices $A_i, B_i, i = 1, 2, 3$ be given by*

$$A_1 = \begin{bmatrix} 11.5522 & -1.6915 & -4.8264 & 2.9655 & 3.6532 & 4.5762 & -8.1947 & 3.1956 & -5.9452 \\ -1.6915 & 15.3901 & -1.9604 & -0.7541 & -5.3244 & -4.8430 & -7.1907 & 9.5113 & -2.2223 \\ -4.8264 & -1.9604 & 6.7358 & 0.0925 & 1.8210 & -0.3171 & 7.6910 & -5.4922 & 3.4358 \\ 2.9655 & -0.7541 & 0.0925 & 3.7798 & 1.0617 & 3.1507 & -4.3743 & 1.3239 & -2.5993 \\ 3.6532 & -5.3244 & 1.8210 & 1.0617 & 10.1027 & 6.2921 & 6.5884 & -6.4899 & -1.6964 \\ 4.5762 & -4.8430 & -0.3171 & 3.1507 & 6.2921 & 6.5933 & -0.5179 & -2.8882 & -2.7979 \\ -8.1947 & -7.1907 & 7.6910 & -4.3743 & 6.5884 & -0.5179 & 19.9964 & -12.5359 & 6.4704 \\ 3.1956 & 9.5113 & -5.4922 & 1.3239 & -6.4899 & -2.8882 & -12.5359 & 10.9659 & -4.2259 \\ -5.9452 & -2.2223 & 3.4358 & -2.5993 & -1.6964 & -2.7979 & 6.4704 & -4.2259 & 4.9131 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 7.6328 & -1.7408 & -8.3158 & 17.8663 & -9.2566 & 6.4950 \\ -1.7408 & 1.0226 & 2.8591 & -1.3057 & 0.5694 & -0.8097 \\ -8.3158 & 2.8591 & 10.9606 & -13.2918 & 6.6678 & -5.4796 \\ 17.8663 & -1.3057 & -13.2918 & 64.1307 & -33.7562 & 20.6961 \\ -9.2566 & 0.5694 & 6.6678 & -33.7562 & 17.8161 & -10.9168 \\ 6.4950 & -0.8097 & -5.4796 & 20.6961 & -10.9168 & 7.0053 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 12.5190 & 12.2033 & -3.0146 & -5.2901 & -8.1796 & -6.8860 & -0.3867 \\ 12.2033 & 12.1214 & -3.3136 & -5.7012 & -8.2843 & -8.5243 & -0.7792 \\ -3.0146 & -3.3136 & 1.9657 & 2.9562 & 4.0307 & 5.3581 & 1.6654 \\ -5.2901 & -5.7012 & 2.9562 & 4.7468 & 6.2315 & 6.8062 & 1.5921 \\ -8.1796 & -8.2843 & 4.0307 & 6.2315 & 9.6677 & 8.2436 & 2.8276 \\ -6.8860 & -8.5243 & 5.3581 & 6.8062 & 8.2436 & 27.4260 & 8.6728 \\ -0.3867 & -0.7792 & 1.6654 & 1.5921 & 2.8276 & 8.6728 & 4.1935 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.0975 & -0.8003 & 0.9340 & 0.2769 & 0.3816 & 0.2760 & 0.2238 & 0.1493 \\ 0.2785 & 0.1419 & -0.6787 & 0.0462 & 0.7655 & -0.6797 & 0.7513 & 0.2575 \\ 0.5469 & 0.4218 & 0.7577 & 0.0971 & -0.7952 & -0.6551 & 0.2551 & -0.8407 \\ -0.9575 & 0.9157 & 0.7431 & 0.8235 & 0.1869 & 0.1626 & -0.5060 & 0.2543 \\ 0.9649 & 0.7922 & 0.3922 & -0.6948 & 0.4898 & 0.1190 & -0.6991 & -0.8143 \\ -0.1576 & 0.9595 & 0.6555 & 0.3171 & 0.4456 & 0.4984 & -0.8909 & 0.2435 \\ 0.9706 & 0.6557 & 0.1712 & -0.9502 & -0.6463 & -0.9597 & -0.9593 & -0.9293 \\ -0.9572 & 0.0357 & -0.7060 & 0.0344 & 0.7094 & 0.3404 & 0.5472 & 0.3500 \\ 0.4854 & -0.8491 & 0.0318 & 0.4387 & -0.7547 & -0.5853 & 0.1386 & 0.1966 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0.2064 & -0.3903 & 0.2976 & -0.5322 & 0.3890 & 0.0676 & 0.3861 & 0.4498 \\ 0.1129 & 0.1476 & -0.1766 & 0.0349 & -0.1084 & -0.1816 & 0.0653 & 0.0051 \\ -0.3136 & 0.7396 & -0.4826 & 0.5711 & -0.3454 & -0.4335 & -0.2835 & -0.4740 \\ -0.3839 & 0.5890 & 0.4376 & -0.8247 & 1.2958 & -0.7176 & 1.3766 & 1.1075 \\ 0.2077 & -0.3126 & -0.1834 & 0.4599 & -0.6852 & 0.4612 & -0.6919 & -0.5360 \\ -0.1004 & 0.0241 & 0.1421 & -0.3871 & 0.4565 & -0.2722 & 0.3521 & 0.2886 \end{bmatrix},$$

$$B_3 = \begin{bmatrix} 0.2406 & -0.7790 & 0.4905 & -0.8191 & -0.0210 & -0.0776 & -0.4255 & -0.5349 \\ 0.0239 & -0.8572 & 0.3299 & -0.5801 & 0.1987 & -0.0091 & -0.5429 & -0.3670 \\ 0.1204 & 0.4323 & 0.1236 & -0.1114 & -0.4536 & 0.1623 & 0.3338 & -0.3960 \\ 0.3766 & 0.8046 & 0.4311 & -0.3777 & -0.8256 & 0.3281 & 0.8347 & -0.5687 \\ -0.2304 & 0.8978 & -0.0255 & 0.2753 & -0.5915 & 0.6439 & 0.6157 & -0.5651 \\ 0.2110 & 0.4656 & -0.8597 & 0.4009 & -0.6987 & -1.0448 & -0.7130 & -0.7249 \\ -0.4696 & -0.1073 & -0.6575 & 0.5483 & 0.0571 & -0.1682 & -0.7437 & -0.3882 \end{bmatrix}.$$

It is easy to numerically verify that the conditions (20) *hold* $\Big($*in fact,* $\|B_2B_2^{\dagger}A_2 - A_2\| = 4.7231 \times 10^{-14}, \|B_3B_3^{\dagger}A_3 -$

$A_3\| = 2.3648 \times 10^{-14}$, $r\begin{bmatrix} A_2 & 0 & B_2 \\ 0 & -A_3 & B_3 \\ B_2^* & B_3^* & 0 \end{bmatrix} = 2r\begin{bmatrix} B_2 \\ B_3 \end{bmatrix} = 12$, $\|E_{\tilde{G_1}}\tilde{W}E_{\tilde{G_1}}\| = 7.3353 \times 10^{-11}$, $\|F_{\tilde{G_2}}\tilde{W}F_{\tilde{G_2}}\| = 6.1053 \times 10^{-11}$, $\|E_L F_{\tilde{G_2}}\tilde{W}\tilde{G_1}\tilde{G_1}^{\dagger}\| = 1.7398 \times 10^{-11}\big)$. *According to Algorithm 4.1, if choose* $Q = 0, S_U = 0, M = 0$, *we can obtain a CHS* X *of Eqs.* (1) *as follows:*

$$X = \begin{bmatrix} 3.0611 & 1.3587 & -0.7805 & 1.4306 & -1.6005 & 0.9157 & -1.5652 & 0.0215 \\ 1.3587 & 2.6879 & -1.8897 & 2.1478 & -1.1223 & -0.7139 & 1.3046 & 0.2548 \\ -0.7805 & -1.8897 & 3.9024 & -1.2384 & 3.8051 & 1.5041 & -0.3445 & -0.9902 \\ 1.4306 & 2.1478 & -1.2384 & 3.5793 & -1.8638 & 1.2363 & 0.5353 & -0.9739 \\ -1.6005 & -1.1223 & 3.8051 & -1.8638 & 7.7615 & -0.4786 & 2.7100 & 0.4343 \\ 0.9157 & -0.7139 & 1.5041 & 1.2363 & -0.4786 & 4.3347 & -1.9066 & -0.5330 \\ -1.5652 & 1.3046 & -0.3445 & 0.5353 & 2.7100 & -1.9066 & 4.9403 & 1.3409 \\ 0.0215 & 0.2548 & -0.9902 & -0.9739 & 0.4343 & -0.5330 & 1.3409 & 2.4195 \end{bmatrix}.$$

The absolute errors are estimated by

$$\|B_1XB_1^* - A_1\| = 3.4443 \times 10^{-11}, \quad \|B_2XB_2^* - A_2\| = 7.1347 \times 10^{-13},$$

$$\|B_3XB_3^* - A_3\| = 8.4669 \times 10^{-13}, \quad \|X^* - X\| = 3.1567 \times 10^{-12},$$

which implies that X *is a CHS to* (1).

5. Conclusions

As is well known, Tian [12] discussed the solvability of (1) to have a CHS, but the explicit expression for the general CHS of (1) has not yet been given. In this paper, we not only provide some new necessary and sufficient conditions for (1) to have a CHS, but also give the explicit expression for the general CHS of (1). To obtain the Theorem 3.1, a special CHS X_0 (11) in Lemma 2.1 and an improved result (13) in Lemma 2.2 are provided, both of which paly a crucial role in deriving the explicit expression for the general CHS of (1). On this basis, the explicit expression for the general CHS of (1) is proposed for the first time using the generalized inverses and some matrix decompositions.

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