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Scaled neutralized entropy on subsets

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Abstract. In this manuscript, we introduce the notion of scaled neutralized entropy on subsets by neutralized Bowen open ball, and establish variational principles for scaled neutralized entropy on subsets in terms of scaled neutralized Brin-Katok local entropy and scaled neutralized Katok's entropy.

1. Introduction

In this manuscript, a topological dynamical system (TDS, for brevity) is a pair (X, T), where X is a compact metric space equipped with a metric d and T is a continuous map from X to X. The sets of natural, nonnegative integers and real numbers are denoted by $\mathbb{N}$, $\mathbb{Z}^+$ and $\mathbb{R}$, respectively. We denote by $\mathcal{M}(X)$ the set of all Borel probability measures on X.

Topological entropy plays a pivotal role in ergodic theory. The topological entropy introduced by Adler, Konheim and McAndrew [1] is a crucial topological invariant to capture the topological complexity of dynamical systems. Subsequently, Bowen [4] provided equivalent definitions for topological entropy using spanning sets and separated sets. Bowen [5] in his profound paper further revealed that the entropy can be viewed as the analog of dimension from the perspective of dimension theory, which led to fruitful results in dimension theory, ergodic theory, multifractal analysis and other fields of dynamical systems [2, 12, 13].

The classical variational principle for topological entropy [15] is well-known for revealing the intricate relationship between measure-theoretic entropy and topological entropy, thereby bridging dynamical systems and ergodic theory. More precisely,

$$h_{top}(T, X) = \sup_{\rho \in \mathcal{M}(X,T)} h_\rho(T)$$

where $h_{top}(T, X)$ is the topological entropy of X, and $h_\rho(T)$ represents the measure-theoretic entropy of invariant measure ρ [9]. Later, Feng and Huang [8] established a variational principle for Bowen topological entropy of compact subsets. Specifically, they derived

$$h_{top}^B(T, K) = \sup\{\underline{h}_\rho^{BK}(T) : \rho \in \mathcal{M}(X), \rho(K) = 1\},$$

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where $h_{top}^{B}(T, K)$ is the Bowen topological entropy of compact subset K, and $\underline{h}_{\rho}^{BK}(T)$ represents the lower Brin-Katok local entropy of ρ.

To investigate systems with sub-exponential (or super-exponentia) divergence of orbits, it is often necessary to introduce a more appropriate scale in which the scaled entropy, introduced in [18], should be computed. This approach enables the assessment of the complexity level of systems with zero or infinite topological entropy. Recently, Chen and Li [7] introduced the scaled packing topological entropy for the amenable group actions and establish a corresponding variational principle.

Recently, Ovadia et al. [3] introduced a significant concept in dynamical systems: the neutralized Bowen open ball. This concept is defined as follows:

$$B_n(x, e^{-n\epsilon}) = \{y \in X : d(T^j x, T^j y) < e^{-n\epsilon}, \forall 0 \leq j \leq n-1\}.$$

By leveraging this concept, Ovadia et al. introduced neutralized Brin-Katok local entropy to estimate the asymptotic measure of sets with a distinctive geometric shape via neutralizing the sub-exponential effects. Subsequently, Yang et al. [17] utilized the neutralized Bowen open ball to give a notion of the neutralized Bowen topological entropy on subsets by replacing the usual Bowen ball by neutralized Bowen open ball. The neutralized Bowen topological entropy can be seen as the topological counterpart of the neutralized local entropy in [3]. Further generalizing these results, Nazarian Sarkooh et al. [10, 11] generalized the results in [17] to non-autonomous dynamical systems and free semigroup actions, respectively. Recently, Qu et al. [14] generalized the notion of neutralized Bowen topological entropy in [17] to the non-additive neutralized Bowen topological pressure, and established the variational principle for non-additive potentials with tempered distortion. In summary, the neutralized Bowen open ball has emerged as a powerful tool in the study of dynamical systems. By neutralizing sub-exponential effects and providing a more nuanced description of neighborhoods under dynamics, it has enabled researchers to develop some new entropies measures and generalize existing results to more complex systems.

Motivated by [5, 7, 8, 12, 18], a natural question arises concerning the variational principle for scaled neutralized entropy on non-empty compact subsets when considering neutralized Bowen balls. To address this question, we introduce the notions of scaled neutralized Brin-Katok local entropy and scaled neutralized Katok's entropy of Borel probability measure. Our primary objective is to establish a variational principle that relates these scaled neutralized entropy measures to the topological entropy defined using neutralized Bowen balls. Specifically, we aim to demonstrate that the supremum of the scaled neutralized Katok's entropy of all Borel probability measures supported on a given compact subset is equal to the topological entropy of that subset, as defined using neutralized Bowen balls.

This paper is organized as follows. In Section 2, we introduce the notions of scaled neutralized entropy of subsets, scaled neutralized Brin-Katok local entropy, and scaled neutralized Katok's entropy of Borel probability measures. In Section 3, we state our main results, that is, the variational principle for the scaled neutralized entropy of subsets. In Section 4, we shall discuss the relationship among the scaled neutralized entropy with respect to equivalent scaled sequences.

2. Preliminaries

In this section, we recall the notions of neutralized Bowen open ball and scaled sequences, then we use them to propose some necessary concepts named scaled neutralized entropy of subsets, scaled neutralized neutralized Brin-Katok local entropy, scaled neutralized Katok's entropy of Borel probability measures and scaled neutralized entropy of Borel probability measures in the sense of Katok.

A positive sequence $\mathbf{a} = a(n)_{n\in\mathbb{N}}$ is called a scaled sequence, if it monotonically increases to infinity. We denote by $\mathcal{SS}$ the set of all scaled sequences. Throughout the paper, with referring to [7, Condition 2.8], we also assume that $\mathbf{a} = a(n)_{n\in\mathbb{N}}$ satisfy the following condition, if there are no additional instructions.

Condition 2.1. *Let* $\mathbf{a} \in \mathcal{SS}$ *and*

$$\liminf_{n\to\infty} \frac{a(n)}{\log n} = +\infty.$$

2.1. Neutralized topological entropy of subsets

Given $n \in \mathbb{N}$, $x, y \in X$, we define the Bowen metric d_n on X as

$$d_n(x, y) := \max_{0 \le k \le n-1} d(T^k(x), T^k(y)).$$

Then *Bowen open ball* of radius ϵ and order n in the metric d_n around x is defined by

$$B_n(x, \epsilon) := \{y \in X : d_n(x, y) < \epsilon\}.$$

Ovadia et al. [3] proposed the neutralized Bowen open ball by replacing the radius ϵ in the classical Bowen ball with $e^{-n\epsilon}$. The *neutralized Bowen open ball* of radius ϵ and order n in the metric d_n around x is given by

$$B_n(x, e^{-n\epsilon}) := \{y \in X : d_n(x, y) < e^{-n\epsilon}\}.$$

Following the idea of [5, 8, 12], Yang et al. [17] used the neutralized Bowen balls to give a notion of the *neutralized Bowen topological entropy* of subsets. Let $Z \subset X$ be a non-empty subset, $\epsilon > 0$, $N \in \mathbb{N}$ and $s \ge 0$. Put

$$M^s_{N,\epsilon}(Z) := \inf \sum_{i \in I} e^{-n_i s},$$

where the infimum is taken over all finite or countable covers $\{B_{n_i}(x_i, e^{-n_i \epsilon})\}_{i \in I}$ of Z with $n_i \ge N, x_i \in X$.

Clearly, the limit $M^s_\epsilon(Z) := \lim_{N \to \infty} M^s_{N,\epsilon}(Z)$ exists. The quantity $M^s_\epsilon(Z)$ has a critical value of parameter s jumping from ∞ to 0, which we denote by

$$M_\epsilon(Z) := \inf\{s : M^s_\epsilon(Z) = 0\} = \sup\{s : M^s_\epsilon(Z) = \infty\}.$$

Definition 2.2. ([17]) Given a non-empty subset $Z \subset X$, the neutralized Bowen topological entropy of T on the set Z is defined by

$$h^{\widetilde{B}}_{top}(T, Z) := \lim_{\epsilon \to 0} M_\epsilon(Z) = \inf_{\epsilon > 0} M_\epsilon(Z).$$

2.2. Scaled neutralized entropy of subsets

Let $Z \subset X$ be a non-empty subset, $\mathbf{a} \in \mathcal{SS}$, $\epsilon > 0$, $N \in \mathbb{N}$ and $s \ge 0$. Put

$$\widehat{M}^s_{N,\epsilon}(Z, \mathbf{a}) := \inf \sum_{i \in I} \exp(-sa(n_i)),$$

where the infimum is taken over all finite or countable covers $\{B_{n_i}(x_i, e^{-n_i \epsilon})\}_{i \in I}$ of Z with $n_i \ge N, x_i \in X$.

Obviously, as $\widehat{M}^s_{N,\epsilon}(Z, \mathbf{a})$ is non-decreasing when N increases, then the limit $\widehat{M}^s_\epsilon(Z, \mathbf{a}) := \lim_{N \to \infty} \widehat{M}^s_{N,\epsilon}(Z, \mathbf{a})$ exists.

Proposition 2.3. *If there exists s such that $\widehat{M}^s_\epsilon(Z, \mathbf{a}) < \infty$, then $\widehat{M}^t_\epsilon(Z, \mathbf{a}) = 0$ for any $t > s$; If $\widehat{M}^s_\epsilon(Z, \mathbf{a}) > 0$ for some s, then $\widehat{M}^t_\epsilon(Z, \mathbf{a}) = \infty$ for any $t < s$.*

Proof. It suffices to show the first statement. Assume that $c := \widehat{M}^s_\epsilon(Z, \mathbf{a}) < \infty$. For sufficiently large N, there is a finite or countable covers $\{B_{n_i}(x_i, e^{-n_i \epsilon})\}_{i \in I}$ of Z with $n_i \ge N, x_i \in X$ so that $\sum_{i \in I} \exp(-sa(n_i)) < c + 1$. Fix $t > s$. Then

$$\widehat{M}^t_{N,\epsilon}(Z, \mathbf{a}) \le \sum_{i \in I} \exp(-ta(n_i)) \le \exp((s - t)a(N)) \sum_{i \in I} \exp(-sa(n_i)),$$

which implies that $\widehat{M}^t_\epsilon(Z, \mathbf{a}) = 0$ by letting $N \to \infty$. □

By Proposition 2.3, the quantity $\widehat{M}^s_\epsilon(Z, \mathbf{a})$ has a critical value of parameter s jumping from ∞ to 0, which we denote by

$$\widehat{M}_\epsilon(Z, \mathbf{a}) := \inf\{s : \widehat{M}^s_\epsilon(Z, \mathbf{a}) = 0\} = \sup\{s : \widehat{M}^s_\epsilon(Z, \mathbf{a}) = \infty\}.$$

Definition 2.4. Given a non-empty subset $Z \subset X$, $\mathbf{a} \in \mathcal{SS}$, the scaled neutralized entropy of T on the set Z with respect to the scaled sequence $\mathbf{a}$ is defined by

$$\widetilde{h_{top}^{SB}}(T, Z, \mathbf{a}) := \lim_{\epsilon \to 0} \widehat{M}_\epsilon(Z, \mathbf{a}) = \inf_{\epsilon > 0} \widehat{M}_\epsilon(Z, \mathbf{a}).$$

Actually, if we take $\mathbf{a} = \{n\}_{n \in \mathbb{N}} = \mathbb{N}$, Definition 2.4 coincides with Definition 2.2.

The next proposition presents some basic properties related to scaled neutralized entropy.

Proposition 2.5. *(1) For every $Z \subset X$, the value of $\widetilde{h_{top}^{SB}}(T, Z, \mathbf{a})$ is independent of the choice of metrics on X.*

(2) If $Z_1 \subset Z_2 \subset X$, then $\widetilde{h_{top}^{SB}}(T, Z_1, \mathbf{a}) \le \widetilde{h_{top}^{SB}}(T, Z_2, \mathbf{a})$.

(3) If $Z = \bigcup_{i \ge 1} Z_i$ is a union of sets $Z_i \subset X$, then $\widehat{M}_\epsilon(Z, \mathbf{a}) = \sup_{i \ge 1} \widehat{M}_\epsilon(Z_i, \mathbf{a})$.

(4) If $Z = \bigcup_{i=1}^{k} Z_i$ is a union of sets $Z_i \subset X$, then $\widetilde{h_{top}^{SB}}(T, Z, \mathbf{a}) = \max_{k \ge i \ge 1} \widetilde{h_{top}^{SB}}(T, Z_i, \mathbf{a})$.

Proof. The proofs of (2)-(4) are standard, one can refer to [12]. We only need to show the case (1). To see it, let d_1, d_2 be two compatible metrics on X. Then for every $\epsilon' > 0$ there exists $\delta' > 0$ such that for all $x, y \in X$ with $d_1(x, y) < \delta'$, one has $d_2(x, y) < \epsilon'$. Now, fix $\epsilon > 0$ and $0 < \delta < \epsilon$. For every $N \ge 1$, there exists $n \ge N$ (depending on ϵ, δ and N) such that for any $x, y \in X$ with $d_1(x, y) < e^{-n\delta}$, then $d_2(x, y) < e^{-N\epsilon}$. Hence, we have $\widehat{M}^s_{N,\epsilon,d_2}(Z, \mathbf{a}) \le \widehat{M}^s_{n,\delta,d_1}(Z, \mathbf{a}) \le \widehat{M}^s_{\delta,d_1}(Z, \mathbf{a})$ and $\widehat{M}^s_{\epsilon,d_2}(Z, \mathbf{a}) \le \widehat{M}^s_{\delta,d_1}(Z, \mathbf{a})$. This implies that $\widehat{M}_{\epsilon,d_2}(Z, \mathbf{a}) \le \widehat{M}_{\delta,d_1}(Z, \mathbf{a})$. Letting $\epsilon \to 0$, one has $\widetilde{h_{top}^{SB}}(T, Z, \mathbf{a}, d_2) \le \widetilde{h_{top}^{SB}}(T, Z, \mathbf{a}, d_1)$. Then we swap the roles of d_1 and d_2, and we get the opposite inequality. □

2.3. Scaled neutralized entropy of the subset Z of X by spanning sets and separated sets

Analogous to the classical topological entropy [15], Yang et al. [17] also gave the notion of *neutralized topological entropy of the subset Z of X* by spanning sets and separated sets. Now, we introduce the notion of *scaled neutralized topological entropy of the subset Z of X* via spanning sets and separated sets.

Given $\epsilon > 0$, $n \in \mathbb{N}$, $\mathbf{a} \in \mathcal{SS}$ and a non-empty subset Z of X, a set $E \subset X$ is *an neutralized (n, ϵ)-spanning set of Z* if for any $x \in Z$, there exists $y \in E$ such that $d_n(x, y) < e^{-n\epsilon}$. We denote the smallest cardinality of neutralized (n, ϵ)-spanning sets of Z by $r_n(Z, \epsilon)$.

A set $F \subset Z$ is *an neutralized (n, ϵ)-separated set of Z* if for any $x, y \in F$ with $x \ne y$, one has $d_n(x, y) \ge e^{-n\epsilon}$. We denote the largest cardinality of neutralized (n, ϵ)-separated sets of Z by $s_n(Z, \epsilon)$. Put

$$r(Z, \mathbf{a}, \epsilon) := \limsup_{n \to \infty} \frac{1}{a(n)} \log r_n(Z, \epsilon), \quad s(Z, \mathbf{a}, \epsilon) := \limsup_{n \to \infty} \frac{1}{a(n)} \log s_n(Z, \epsilon).$$

Definition 2.6. The scaled neutralized topological entropy of Z with respect to the scaled sequence $\mathbf{a}$ is defined by

$$\widetilde{h^S_{top}}(T, Z, \mathbf{a}) := \lim_{\epsilon \to 0} r(Z, \mathbf{a}, \epsilon) = \inf_{\epsilon > 0} r(Z, \mathbf{a}, \epsilon).$$

The following proposition shows that scaled neutralized topological entropy of Z can be equivalently expressed by separated sets.

Proposition 2.7. *Let (X, T) be a TDS. Then for any non-empty subset $Z \subset X$, $\mathbf{a} \in \mathcal{SS}$,*

$$\widetilde{h^S_{top}}(T, Z, \mathbf{a}) = \lim_{\epsilon \to 0} s(Z, \mathbf{a}, \epsilon).$$

Proof. The inequality $r(Z, \mathbf{a}, \epsilon) \le s(Z, \mathbf{a}, \epsilon)$ follows by using the fact of an neutralized (n, ϵ)-separated set of Z with largest cardinality is an neutralized (n, ϵ)-spanning set of Z.

Conversely, fix $\epsilon > 0$ and choose n_0 such that $2e^{-\frac{\epsilon}{2}n} < 1$ for all $n \ge n_0$. Fix $n \ge n_0$. Let E be an neutralized (n, ϵ)-spanning set of Z, and F be an neutralized $(n, \frac{\epsilon}{2})$-separated set of Z. Define a map $\psi : F \to E$ by choosing a fixed $\psi(x) \in E$ such that $d_n(x, \psi(x)) < e^{-n\epsilon}$ for every $x \in F$. Then ψ is injective. Hence, $s_n(Z, \frac{\epsilon}{2}) \le r_n(Z, \epsilon)$, which implies that $s(Z, \mathbf{a}, \frac{\epsilon}{2}) \le r(Z, \mathbf{a}, \epsilon)$. □

Proposition 2.8. *Let (X, T) be a TDS. Then for any non-empty subset $Z \subset X$,*

$$h^{\widetilde{SB}}_{top}(T, Z, \mathbf{a}) \le \widetilde{h}^{S}_{top}(T, Z, \mathbf{a}).$$

Proof. Suppose that $\widetilde{h}^{S}_{top}(T, Z) < \infty$. Let $\widetilde{h}^{S}_{top}(T, Z) < s$. Then there exists $\epsilon > 0$ satisfying for every sufficiently large n, there is an neutralized (n, ϵ)-spanning set E of Z such that $\#E < \exp(a(n)s)$. Notice that $Z \subset \bigcup_{x\in E} B_n(x, e^{-n\epsilon})$. Then

$$\widehat{M}^{s}_{n,\epsilon}(Z, \mathbf{a}) \le \sum_{x\in E} \exp(-a(n)s) < 1,$$

this yields that $\widetilde{h}^{SB}_{top}(T, Z, \mathbf{a}) \le \widehat{M}_{\epsilon}(Z, \mathbf{a}) \le s$. Letting $s \to \widetilde{h}^{S}_{top}(T, Z, \mathbf{a})$, then $h^{\widetilde{SB}}_{top}(T, Z, \mathbf{a}) \le \widetilde{h}^{S}_{top}(T, Z, \mathbf{a})$. □

Remark 2.9. Denote by $h^{SB}_{top}(T, Z, \mathbf{a})$, $h^{S}_{top}(T, Z, \mathbf{a})$ the scaled topological entropy of T on the set Z, scalded (upper capacity) entropy of T on the set Z defined by Bowen open balls $\{B_{n_i}(x_i, \epsilon)\}_{i\in I}$. Notice that for given $\epsilon > 0$, $B_n(x, e^{-n\epsilon}) \subset B_n(x, \epsilon)$ for sufficiently large n, hence we deduce that $h^{SB}_{top}(T, Z, \mathbf{a}) \le h^{\widetilde{SB}}_{top}(T, Z, \mathbf{a})$ and $h^{S}_{top}(T, Z, \mathbf{a}) \le \widetilde{h}^{S}_{top}(T, Z, \mathbf{a})$.

Motivated by the work of [12, 15, 19], in the next two subsections, we shall introduce the notions of lower scaled neutralized Brin-Katok local entropy, the scaled neutralized Katok entropy of Borel probability measures, and the scaled neutralized Katok entropy of Borel probability measures in the sense of Katok, and then utilize them to establish variational principle for scaled neutralized entropy.

2.4. *Lower scaled neutralized Brin-Katok local entropy*

Analogously, following the idea of [6, 8, 17], we use the neutralized Bowen ball to define the lower scaled neutralized Brin-Katok local entropy.

Definition 2.10. Given $\rho \in \mathcal{M}(X)$, $\mathbf{a} \in \mathcal{SS}$ and $\epsilon > 0$, let

$$\underline{h}^{\widetilde{BK}_S}_{\rho}(T, \epsilon, \mathbf{a}) := \int \underline{h}^{\widetilde{BK}_S}_{\rho}(T, x, \epsilon, \mathbf{a}) d\rho,$$

where

$$\underline{h}^{\widetilde{BK}_S}_{\rho}(T, x, \epsilon, \mathbf{a}) := \liminf_{n\to\infty} -\frac{\log \rho(B_n(x, e^{-n\epsilon}))}{a(n)}.$$

The lower scaled neutralized Brin-Katok local entropy of ρ with respect to the scaled sequence $\mathbf{a}$ is defined by

$$\underline{h}^{\widetilde{BK}_S}_{\rho}(T, \mathbf{a}) := \lim_{\epsilon\to 0} \underline{h}^{\widetilde{BK}_S}_{\rho}(T, \epsilon, \mathbf{a}).$$

2.5. *Scaled neutralized entropy of Katok's sense*

In this subsection, we follow the works in [19], and then we introduce the concepts of scaled neutralized Katok's entropy of Borel probability measure and scaled neutralized entropy of Borel probability measure in the sense of Katok.

Let $\epsilon > 0, s \ge 0, N \in \mathbb{N}, \rho \in \mathcal{M}(X)$, $\mathbf{a} \in \mathcal{SS}$ and $\delta \in (0, 1)$. Put

$$\Xi^{s}_{N,\epsilon}(\rho, \delta, \mathbf{a}) := \inf \sum_{i\in I} \exp(-sa(n_i)),$$

where the infimum is taken over all finite or countable covers $\{B_{n_i}(x_i, e^{-n_i\epsilon})\}_{i\in I}$ so that $\rho(\bigcup_{i\in I} B_{n_i}(x_i, e^{-n_i\epsilon})) > 1 - \delta$ with $n_i \ge N, x_i \in X$.

Let $\Xi^{s}_{\epsilon}(\rho, \delta, \mathbf{a}) := \lim_{N\to\infty} \Xi^{s}_{N,\epsilon}(\rho, \delta, \mathbf{a})$. There is a critical value of parameter s for $\Xi^{s}_{\epsilon}(\rho, \delta, \mathbf{a})$ jumping from ∞ to 0. The critical value is defined by

$$\Xi_{\epsilon}(\rho, \delta, \mathbf{a}) := \inf\{s : \Xi^{s}_{\epsilon}(\rho, \delta, \mathbf{a}) = 0\} = \sup\{s : \Xi^{s}_{\epsilon}(\rho, \delta, \mathbf{a}) = \infty\}.$$

Define $h^{\widetilde{K}_S}_{\rho}(T, \epsilon, \mathbf{a}) := \lim_{\delta\to 0} \Xi_{\epsilon}(\rho, \delta, \mathbf{a})$.

Definition 2.11. The scaled neutralized entropy of ρ with respect to the scaled sequence $\mathbf{a}$ in the sense of Katok is defined by

$$h_{\rho}^{\widetilde{K_S}}(T,\mathbf{a}) := \lim_{\epsilon\to 0} h_{\rho}^{\widetilde{K_S}}(T,\epsilon,\mathbf{a}).$$

Proposition 2.12. *Let $\rho \in \mathcal{M}(X)$ and $\mathbf{a} \in \mathcal{SS}$. Then for every $\epsilon > 0$, one has*

$$\underline{h}_{\rho}^{\widetilde{BK_S}}(T,\frac{\epsilon}{2},\mathbf{a}) \le h_{\rho}^{\widetilde{K_S}}(T,\epsilon,\mathbf{a}).$$

Moreover, we obtain $\underline{h}_{\rho}^{\widetilde{BK_S}}(T,\mathbf{a}) \le h_{\rho}^{\widetilde{K_S}}(T,\mathbf{a})$.

Proof. Assume that $\underline{h}_{\rho}^{\widetilde{BK_S}}(T,\frac{\epsilon}{2},\mathbf{a}) > 0$. We define

$$E_N = \{x \in X : \rho(B_n(x,e^{-\frac{n\epsilon}{2}})) < \exp(-sa(n)) \text{ for all } n \ge N\}.$$

Let $s < \underline{h}_{\rho}^{\widetilde{BK_S}}(T,\frac{\epsilon}{2},\mathbf{a})$. Then there exists N_0 with $e^{\frac{N_0}{2}\epsilon} > 2$ so that $\rho(E_{N_0}) > 0$. Fix $\delta_0 = \frac{1}{2}\rho(E_{N_0}) > 0$. Let $\{B_{n_i}(x_i,e^{-n_i\epsilon})\}_{i\in I}$ be a finite or countable cover so that $\rho(\bigcup_{i\in I} B_{n_i}(x_i,e^{-n_i\epsilon})) > 1-\delta_0$ with $n_i \ge N_0, x_i \in X$. Then $\rho(E_{N_0} \cap \bigcup_{i\in I} B_{n_i}(x_i,e^{-n_i\epsilon})) \ge \frac{1}{2}\rho(E_{N_0}) > 0$. Denote by $I_1 = \{i \in I : E_{N_0} \cap B_{n_i}(x_i,e^{-n_i\epsilon}) \ne \emptyset\}$. For every $i \in I_1$, we choose $y_i \in E_{N_0} \cap B_{n_i}(x_i,e^{-n_i\epsilon})$ such that

$$E_{N_0} \cap B_{n_i}(x_i,e^{-n_i\epsilon}) \subset B_{n_i}(y_i,2e^{-n_i\epsilon}) \subset B_{n_i}(y_i,e^{-\frac{n_i}{2}\epsilon}).$$

Hence,

$$\sum_{i\in I}\exp(-sa(n_i)) \ge \sum_{i\in I_1}\exp(-sa(n_i)) \ge \sum_{i\in I_1}\rho(B_{n_i}(y_i,e^{-\frac{n_i\epsilon}{2}})) \ge \frac{\rho(E_{N_0})}{2} > 0.$$

This implies that $\Xi_{\epsilon}^{s}(\rho,\delta_0,\mathbf{a}) \ge \Xi_{N_0,\epsilon}^{s}(\rho,\delta_0,\mathbf{a}) > 0$ and hence $\Xi_{\epsilon}(\rho,\delta_0,\mathbf{a}) \ge s$. Consequently, $h_{\rho}^{\widetilde{K_S}}(T,\epsilon,\mathbf{a}) \ge s$. Letting $s \to \underline{h}_{\rho}^{\widetilde{BK_S}}(T,\frac{\epsilon}{2},\mathbf{a})$ this finishes the proof. □

Definition 2.13. Furthermore, the scaled neutralized entropy of ρ with respect to the scaled sequence $\mathbf{a}$ is defined by

$$h_{\rho}^{\widetilde{B_S}}(T,\mathbf{a}) := \lim_{\epsilon\to 0} h_{\rho}^{\widetilde{B_S}}(T,\epsilon,\mathbf{a}),$$

where $h_{\rho}^{\widetilde{B_S}}(T,\epsilon,\mathbf{a}) := \lim_{\delta\to 0}\inf\{\widehat{M}_{\epsilon}(Z,\mathbf{a}) : \rho(Z) \ge 1-\delta\}$, and $\widehat{M}_{\epsilon}(Z,\mathbf{a})$ is given in subsection 2.2.

The next theorem proves an equivalence between scaled neutralized entropy of ρ and scaled neutralized entropy of ρ in the sense of Katok.

Theorem 2.14. *Let (X,T) be a TDS and $\rho \in \mathcal{M}(X)$, $\mathbf{a} \in \mathcal{SS}$. Then for any $\epsilon > 0$, one has*

$$h_{\rho}^{\widetilde{K_S}}(T,\epsilon,\mathbf{a}) = h_{\rho}^{\widetilde{B_S}}(T,\epsilon,\mathbf{a}).$$

Furthermore, we have

$$h_{\rho}^{\widetilde{B_S}}(T,\mathbf{a}) = h_{\rho}^{\widetilde{K_S}}(T,\mathbf{a}).$$

Proof. First, we show that $h_{\rho}^{\widetilde{K_S}}(T,\epsilon,\mathbf{a}) \le h_{\rho}^{\widetilde{B_S}}(T,\epsilon,\mathbf{a})$. For any $N \in \mathbb{N}, 0 < \epsilon, 0 < \delta < 1$, and $Z \subset X$ with $\rho(Z) \ge 1-\delta$, then one has

$$\Xi_{N,\epsilon}^{s}(\rho,\delta,\mathbf{a}) \le \widehat{M}_{N,\epsilon}^{s}(Z,\mathbf{a}).$$

Letting $N \to \infty$, we deduce

$$\Xi_{\epsilon}^{s}(\rho,\delta,\mathbf{a}) \le \widehat{M}_{\epsilon}^{s}(Z,\mathbf{a}).$$

This indicates that

$$\Xi_\epsilon(\rho,\delta,\mathbf{a}) \le \widehat{M}_\epsilon(Z,\mathbf{a}).$$

and then

$$\Xi_\epsilon(\rho,\delta,\mathbf{a}) \le \inf\left\{\widehat{M}_\epsilon(Z,\mathbf{a}) : \rho(Z) \ge 1-\delta\right\}.$$

Hence, taking $\delta \to 0$, we deduce that $h_\rho^{\widetilde{K}_S}(T,\epsilon,\mathbf{a}) \le h_\rho^{\widetilde{B}_S}(T,\epsilon,\mathbf{a})$.

Conversely, given $\epsilon > 0$, we are to prove $h_\rho^{\widetilde{K}_S}(T,\epsilon,\mathbf{a}) \ge h_\rho^{\widetilde{B}_S}(T,\epsilon,\mathbf{a})$, let $\zeta = h_\rho^{\widetilde{K}_S}(T,\epsilon,\mathbf{a})$. For any $s > 0$, there exists δ_s so that

$$\Xi_\epsilon(\rho,\delta,\mathbf{a}) < \zeta + s, \forall \delta < \delta_s,$$

which yields that $\Xi_\epsilon^{\zeta+s}(\rho,\delta,\mathbf{a}) = 0$. For any $N \in \mathbb{N}$, we can find a sequence of $\delta_{N,m}$ with $\lim_{m\to\infty} \delta_{N,m} = 0$ and $\{B_{n_i}(x_i, e^{-n_i\epsilon})\}_{i\in I_{N,m}}$ such that $x_i \in X, n_i \ge N, \rho\left(\bigcup_{i\in I_{N,m}} B_{n_i}(x_i, e^{-n_i\epsilon})\right) \ge 1-\delta_{N,m}$, and

$$\sum_{i\in I_{N,m}} e^{-a(n_i)\cdot(\zeta+s)} \le \frac{1}{2^m}.$$

Let

$$Z_N = \bigcup_{m\in\mathbb{N}} \bigcup_{i\in I_{N,m}} B_{n_i}(x_i, e^{-n_i\epsilon}).$$

Then $\rho(Z_N) = 1$ and $\widehat{M}_{N,\epsilon}^{\zeta+s}(Z_N,\mathbf{a}) \le 1$. Let $Z_s = \cap_{N\in\mathbb{N}} Z_N$. Thus $\rho(Z_s) = 1$ and

$$\widehat{M}_{N,\epsilon}^{\zeta+s}(Z_s,\mathbf{a}) \le \widehat{M}_{N,\epsilon}^{\zeta+s}(Z_N,\mathbf{a}) \le 1, \forall N \in \mathbb{N}.$$

This indicates that

$$\lim_{\delta\to 0} \inf\{\widehat{M}_\epsilon(Z,\mathbf{a}) : \rho(Z) \ge 1-\delta\} \le \zeta + s,$$

Since s is arbitrary, then we have $h_\rho^{\widetilde{B}_S}(T,\epsilon,\mathbf{a}) \le \zeta = h_\rho^{\widetilde{K}_S}(T,\epsilon,\mathbf{a})$. The proof is completed. □

3. Proofs of main results

In this section, we offer the proofs of main results. Motivated by [8, 17], we introduce the notion of *scaled neutralized weighted entropy with respect to a scaled sequence* **a**, which can help us connect the *scaled neutralized entropy* to the *lower scaled neutralized Brin-Katok local entropy*.

3.1. Scaled neutralized weighted entropy

Let $f : X \to \mathbb{R}$ be a bounded real-valued function on X. Let $s \ge 0$, $N \in \mathbb{N}$, $\mathbf{a} \in \mathcal{SS}$ and $\epsilon > 0$. Define

$$\widehat{W}_{N,\epsilon}^s(f,\mathbf{a}) := \inf \sum_{i\in I} c_i \exp(-sa(n_i)),$$

where the infimum is taken over all finite or countable families $\{(B_{n_i}(x_i, e^{-n_i\epsilon}), c_i)\}_{i\in I}$ with $0 < c_i < \infty$, $x_i \in X$, $n_i \ge N$ so that

$$\sum_{i\in I} c_i \chi_{B_{n_i}(x_i, e^{-n_i\epsilon})} \ge f,$$

where χ_A denotes the characteristic function of A.

Let $Z \subset X$ be a non-empty subset. We set $\widehat{W}_{N,\epsilon}^s(Z,\mathbf{a}) := \widehat{W}_{N,\epsilon}^s(\chi_Z,\mathbf{a})$. Let $\widehat{W}_\epsilon^s(Z,\mathbf{a}) := \lim_{N\to\infty} \widehat{W}_{N,\epsilon}^s(Z,\mathbf{a})$. There is a critical value of s such that the quantity $\widehat{W}_\epsilon^s(Z,\mathbf{a})$ jumps from ∞ to 0. Then such critical value is defined by

$$\widehat{W}_\epsilon(Z,\mathbf{a}) := \inf\{s : \widehat{W}_\epsilon^s(Z,\mathbf{a}) = 0\} = \sup\{s : \widehat{W}_\epsilon^s(Z,\mathbf{a}) = \infty\}.$$

Definition 3.1. Given a non-empty subset $Z \subset X$, $\mathbf{a} \in \mathcal{SS}$, the scaled neutralized weighted entropy of T on the set Z with respect to the scaled sequence $\mathbf{a}$ is defined by

$$h_S^{\widetilde{WB}}(T, Z, \mathbf{a}) := \lim_{\epsilon \to 0} \widehat{W}_\epsilon(Z, \mathbf{a}).$$

Lemma 3.2. ([16]) *Let (X, d) be a compact metric space. Let $r > 0$ and $\mathcal{B} = \{B(x_i, r)\}_{i \in I}$ be a family of open balls of X. Define*

$$I(i) := \{j \in I : B(x_j, r) \cap B(x_i, r) \neq \emptyset\}.$$

Then there exists a finite index subset $J \subset I$ so that for any $i, j \in J$ with $i \neq j$, $I(i) \cap I(j) = \emptyset$ and

$$\bigcup_{i \in I} B(x_i, r) \subset \bigcup_{j \in J} B(x_j, 5r).$$

Proposition 3.3. *Let Z be a non-empty subset of X and $\epsilon > 0, s > 0, \theta > 0$, $\mathbf{a} \in \mathcal{SS}$. Then for sufficiently large N, one has*

$$\widehat{M}^{s+\theta}_{N,\frac{\epsilon}{2}}(Z, \mathbf{a}) \leq \widehat{W}^s_{N,\epsilon}(Z, \mathbf{a}) \leq \widehat{M}^s_{N,\epsilon}(Z, \mathbf{a}).$$

Moreover, we deduce that

$$h_{top}^{\widetilde{SB}}(T, Z, \mathbf{a}) = h_S^{\widetilde{WB}}(T, Z, \mathbf{a}).$$

Proof. It is easy to check that $\widehat{W}^s_{N,\epsilon}(Z, \mathbf{a}) \leq \widehat{M}^s_{N,\epsilon}(Z, \mathbf{a})$ by the definitions. Next we are to show $\widehat{M}^{s+\theta}_{N,\frac{\epsilon}{2}}(Z, \mathbf{a}) \leq \widehat{W}^s_{N,\epsilon}(Z, \mathbf{a})$.

Let N be a sufficiently large integer such that $e^{\frac{n\epsilon}{2}} > 5$ and $\frac{n^2}{\exp(a(n)\theta)} < 1$ holds for all $n \geq N$. Let $\{(B_{n_i}(x_i, e^{-n_i\epsilon}), c_i)\}_{i \in I}$ with $0 < c_i < \infty$, $x_i \in X$, $n_i \geq N$ be a finite or countable family satisfying $\sum_{i \in I} c_i \chi_{B_{n_i}(x_i, e^{-n_i\epsilon})} \geq \chi_Z$. Define $I_n = \{i \in I : n_i = n\}$, where $n \geq N$. Let $t > 0$ and $n \geq N$. We define

$$Z_{n,t} = \{z \in Z : \sum_{i \in I_n} c_i \chi_{B_n(x_i, e^{-n\epsilon})}(z) > t\}$$

and

$$I_n^t = \{i \in I_n : B_n(x_i, e^{-n\epsilon}) \cap Z_{n,t} \neq \emptyset\}.$$

Then $Z_{n,t} \subset \bigcup_{i \in I_n^t} B_n(x_i, e^{-n\epsilon})$. Let $\mathcal{B} = \{B_n(x_i, e^{-n\epsilon})\}_{i \in I_n^t}$. By Lemma 3.2, there exists a finite index subset $J \subset I_n^t$ such that

$$\bigcup_{i \in I_n^t} B_n(x_i, e^{-n\epsilon}) \subset \bigcup_{j \in J} B_n(x_j, 5e^{-n\epsilon}) \subset \bigcup_{j \in J} B_n(x_j, e^{-\frac{n\epsilon}{2}})$$

and for any $i, j \in J$ with $i \neq j$, $I_n^t(i) \cap I_n^t(j) = \emptyset$, where $I_n^t(i) = \{j \in I_n^t : B_n(x_j, e^{-n\epsilon}) \cap B_n(x_i, e^{-n\epsilon}) \neq \emptyset\}$.

For each $j \in J$, we choose $y_j \in B_n(x_j, e^{-n\epsilon}) \cap Z_{n,t}$. Then $\sum_{i \in I_n^t} c_i \chi_{B_n(x_i, e^{-n\epsilon})}(y_j) > t$ and so $\sum_{i \in I_n^t(j)} c_i > t$. Summing for $j \in J$, we have

$$\#J < \frac{1}{t} \sum_{j \in J} \sum_{i \in I_n^t(j)} c_i \leq \frac{1}{t} \sum_{i \in I_n^t} c_i.$$

It follows that $\widehat{M}^{s+\theta}_{N,\frac{\epsilon}{2}}(Z_{n,t}, \mathbf{a}) \leq \#J \cdot \exp(-a(n)(s+\theta)) \leq \frac{1}{n^2 t} \sum_{i \in I_n} c_i \exp(-a(n)s)$. Note that for any $t \in (0, 1)$, $Z = \bigcup_{n \geq N} Z_{n, \frac{1}{n^2}t}$. Hence

$$\widehat{M}^{s+\theta}_{N,\frac{\epsilon}{2}}(Z, \mathbf{a}) \leq \sum_{n \geq N} \widehat{M}^{s+\theta}_{N,\frac{\epsilon}{2}}(Z_{n,\frac{1}{n^2}t}, \mathbf{a}) \leq \frac{1}{t} \sum_{i \in I} c_i \exp(-a(n_i)s).$$

Taking $t \to 1$, one has $\widehat{M}^{s+\theta}_{N,\frac{\epsilon}{2}}(Z,\mathbf{a}) \le \widehat{W}^{s}_{N,\epsilon}(Z,\mathbf{a})$. Furthermore,

$$h^{\widetilde{SB}}_{top}(T,Z,\mathbf{a}) = h^{\widetilde{WB}}_{S}(T,Z,\mathbf{a}).$$

The proof is completed. □

Lemma 3.4 (Frostman's lemma)**.** *Let Z be a non-empty compact subset of X and $s \ge 0$, $N \in \mathbb{N}$, $\epsilon > 0$. Set $c := \widehat{W}^{s}_{N,\epsilon}(Z,\mathbf{a}) > 0$. Then there exists a Borel probability measure ρ on X such that $\rho(Z) = 1$ and for any $x \in X$, $n \ge N$,*

$$\rho(B_n(x, e^{-n\epsilon})) \le \frac{1}{c}\exp(-sa(n)).$$

Proof. It is not hard to check that when we replace $B_n(x,\epsilon)$ by $B_n(x,e^{-n\epsilon})$, the proof of [8, Lemma 3.4] is still valid for Lemma 3.4. Here we leave it out. □

Theorem 3.5. *Let (X,T) be a TDS, $\mathbf{a}$ be a scaled sequence and Z be a non-empty compact subset of X. Then*

$$\begin{aligned} h^{\widetilde{SB}}_{top}(T,Z,\mathbf{a}) &= \lim_{\epsilon\to 0}\sup\{\underline{h}^{\widetilde{BK}_S}_{\rho}(T,\epsilon,\mathbf{a}) : \rho\in\mathcal{M}(X), \rho(Z)=1\} \\ &= \lim_{\epsilon\to 0}\sup\{h^{\widetilde{K}_S}_{\rho}(T,\epsilon,\mathbf{a}) : \rho\in\mathcal{M}(X), \rho(Z)=1\} \\ &= \lim_{\epsilon\to 0}\sup\{h^{\widetilde{B}_S}_{\rho}(T,\epsilon,\mathbf{a}) : \rho\in\mathcal{M}(X), \rho(Z)=1\}. \end{aligned}$$

Proof. Notice that for every $\rho \in \mathcal{M}(X)$ with $\rho(Z)=1$ and $\epsilon > 0$, one has $h^{\widetilde{K}_S}_{\rho}(T,\epsilon,\mathbf{a}) \le \widehat{M}_{\epsilon}(Z,\mathbf{a})$. By Proposition 2.12, we have

$$\begin{aligned} &\lim_{\epsilon\to 0}\sup\{\underline{h}^{\widetilde{BK}_S}_{\rho}(T,\epsilon,\mathbf{a}) : \rho\in\mathcal{M}(X), \rho(Z)=1\} \\ \le &\lim_{\epsilon\to 0}\sup\{h^{\widetilde{K}_S}_{\rho}(T,\epsilon,\mathbf{a}) : \rho\in\mathcal{M}(X), \rho(Z)=1\} \\ \le &h^{\widetilde{SB}}_{top}(T,Z,\mathbf{a}). \end{aligned}$$

On the other hand, fix $\epsilon > 0$ and assume that $\widehat{M}_{\epsilon}(Z,\mathbf{a}) > 0$. Let $s < \widehat{M}_{\epsilon}(Z,\mathbf{a})$. Then one has $\widehat{W}^{s}_{2\epsilon}(Z,\mathbf{a}) > 0$ by Proposition 3.3. So there exists N such that $c := \widehat{W}^{s}_{N,2\epsilon}(Z,\mathbf{a}) > 0$. By Lemma 3.4, there exists a Borel probability measure ρ on X such that $\rho(Z)=1$ and for any $x \in X$, $n \ge N$,

$$\rho(B_n(x, e^{-2n\epsilon})) \le \frac{1}{c}\exp(-sa(n)).$$

This yields that $\underline{h}^{\widetilde{BK}_S}_{\rho}(T,2\epsilon,\mathbf{a}) \ge s$. Letting $s \to \widehat{M}_{\epsilon}(Z,\mathbf{a})$, we derive that

$$\widehat{M}_{\epsilon}(Z,\mathbf{a}) \le \underline{h}^{\widetilde{BK}_S}_{\rho}(T,2\epsilon,\mathbf{a}) \le \sup\{\underline{h}^{\widetilde{BK}_S}_{\rho}(T,2\epsilon,\mathbf{a}) : \rho\in\mathcal{M}(X), \rho(Z)=1\}.$$

It follows that

$$\lim_{\epsilon\to 0}\sup\{\underline{h}^{\widetilde{BK}_S}_{\rho}(T,\epsilon,\mathbf{a}) : \rho\in\mathcal{M}(X), \rho(Z)=1\} \ge h^{\widetilde{SB}}_{top}(T,Z,\mathbf{a}).$$

Moreover, by Theorem 2.14, one has

$$\begin{aligned} h^{\widetilde{SB}}_{top}(T,Z,\mathbf{a}) &= \lim_{\epsilon\to 0}\sup\{\underline{h}^{\widetilde{BK}_S}_{\rho}(T,\epsilon,\mathbf{a}) : \rho\in\mathcal{M}(X), \rho(Z)=1\} \\ &= \lim_{\epsilon\to 0}\sup\{h^{\widetilde{K}_S}_{\rho}(T,\epsilon,\mathbf{a}) : \rho\in\mathcal{M}(X), \rho(Z)=1\} \\ &= \lim_{\epsilon\to 0}\sup\{h^{\widetilde{B}_S}_{\rho}(T,\epsilon,\mathbf{a}) : \rho\in\mathcal{M}(X), \rho(Z)=1\}. \end{aligned}$$

□

Theorem 3.6. *Let (X,T) be a TDS and* $\mathbf{a}$ *be a scaled sequence. Suppose ρ is a Borel probability measure, Z is a Borel set and $s \geq 0$. If $\rho(Z) > 0$ and $\underline{h}_{\rho}^{\widetilde{BKs}}(T,x,\epsilon,\mathbf{a}) \geq s$ for all $x \in Z$ and $\epsilon > 0$, then $h_{top}^{\widetilde{SB}}(T,Z,\mathbf{a}) \geq s$.*

Proof. Fix $\epsilon > 0$ and $\tau > 0$. For each $k \in \mathbb{N}$, set

$$E_k = \left\{ x \in Z : \frac{-\log \rho(B_n(x,e^{-n\epsilon}))}{a(n)} \geq s - \tau, \forall n \geq k \right\}.$$

Since $\underline{h}_{\rho}^{\widetilde{BKs}}(T,x,\epsilon,\mathbf{a}) \geq s$ for all $x \in E$, we have that $E_k \subset E_{k+1}$ and $\bigcup_{k=1}^{\infty} E_k = Z$. Hence, there is $k^* \in \mathbb{N}$ such that $\rho\left(E_{k^*}\right) > \rho(E)/2 > 0$. Denote by $E^* = E_{k^*}$. For $x \in E^*$ and $n \geq k^*$, we have

$$\rho(B_n(x,e^{-n\epsilon})) \leq \exp\left[-a(n)\cdot(s-\tau)\right].$$

For sufficiently large $k \geq k^*$ and each cover $\mathscr{F} = \left\{B_{n_i}\left(y_i, e^{-2n_i\epsilon}\right)\right\}_{i\in I}$ of E^* with $n_i \geq k$ for each $i \in I$. Without loss of generality, assume that $E^* \cap B_{n_i}\left(y_i, e^{-2n_i\epsilon}\right) \neq \emptyset$ for each $i \in I$. Thus there exists $x_i \in E^* \cap B_{n_i}\left(y_i, e^{-2n_i\epsilon}\right)$. It follows that $B_{n_i}\left(y_i, e^{-2n_i\epsilon}\right) \subset B_{n_i}\left(x_i, 2e^{-2n_i\epsilon}\right) \subset B_{n_i}\left(x_i, e^{-n_i\epsilon}\right)$, which indicates that $\left\{B_{n_i}\left(x_i, e^{-n_i\epsilon}\right)\right\}_{i\in I}$ is also a cover of E^*.

Hence,

$$\begin{aligned}\rho\left(E^*\right) &\leq \sum_{i\in I} \rho\left(B_{n_i}\left(x_i, e^{-n_i\epsilon}\right)\right) \\ &\leq \sum_{i\in I} \exp\left[-a(n_i)\cdot(s-r)\right].\end{aligned}$$

Therefore,

$$\widehat{M}_{k,2\epsilon}^{s-\tau}(E^*,\mathbf{a}) \geq \rho\left(E^*\right) > 0.$$

It follows that

$$\widehat{M}_{2\epsilon}^{s-\tau}(E^*,\mathbf{a}) \geq \rho\left(E^*\right) > 0.$$

This indicates that $\widehat{M}_{2\epsilon}(E^*,\mathbf{a}) \geq (s-\tau)$. Letting $\epsilon \to 0$, we deduce that $\widetilde{h}_{top}^{SB}(T,E^*,\mathbf{a}) \geq s-\tau$. Hence, $\widetilde{h}_{top}^{SB}(T,Z,\mathbf{a}) \geq \widetilde{h}_{top}^{SB}(T,E^*,\mathbf{a}) \geq s-\tau$. Moreover, one has

$$\widetilde{h}_{top}^{SB}(T,Z,\mathbf{a}) \geq s$$

by the arbitrariness of τ. The proof is finished. □

4. Equivalent scaled sequences of the scaled neutralized entropy

Following the idea of [18], we shall explore the property of the scaled neutralized entropy with respect to equivalent scaled sequence in this section.

Definition 4.1. ([18]) We say that two scaled sequences $\mathbf{a}, \mathbf{b} \in \mathcal{SS}$ are equivalent and denote by $\mathbf{a} \sim \mathbf{b}$, if

$$0 < \liminf_{n\to\infty} \frac{b(n)}{a(n)} \leq \limsup_{n\to\infty} \frac{b(n)}{a(n)} < \infty.$$

It is not hard to see that $\sim$ is an equivalence relation on $\mathcal{SS}$. For any $\mathbf{a} \in \mathcal{SS}$, the equivalent class is denoted as $[\mathbf{a}] := \{\mathbf{b} \in \mathcal{SS} : \mathbf{b} \sim \mathbf{a}\}$ and let $\mathcal{A} := \mathcal{SS}/\sim$. For two equivalent classes $[\mathbf{a}], [\mathbf{b}] \in \mathcal{A}$, we denote $[\mathbf{a}] \preceq [\mathbf{b}]$, if

$$\limsup_{n\to\infty} \frac{a(n)}{b(n)} = 0$$

Proposition 4.2. *For* $\mathbf{a}, \mathbf{b} \in \mathcal{SS}, Z \subset X$, *the following properties hold:*
(1) If $a(n) \le b(n)$ *for all sufficiently large* $n \in \mathbb{N}$, *then*

$$\widehat{M}^s_\epsilon(Z, \mathbf{a}) \ge \widehat{M}^s_\epsilon(Z, \mathbf{b})$$

and hence

$$h^{\widetilde{SB}}_{top}(T, Z, \mathbf{a}) \ge h^{\widetilde{SB}}_{top}(T, Z, \mathbf{b}).$$

(2) For each $m > 0$ *we have that*

$$m h^{\widetilde{SB}}_{top}(T, Z, m\mathbf{a}) = h^{\widetilde{SB}}_{top}(T, Z, \mathbf{a})$$

where $m\mathbf{a} = \{m \cdot a(n)\}$.
(3) If there exists a constant $C > 0$ *such that for any sufficiently large* $n \in \mathbb{N}$, $\frac{1}{C} b(n) \le a(n) \le C b(n)$, *then*

$$\frac{1}{C} h^{\widetilde{SB}}_{top}(T, Z, \mathbf{b}) \le h^{\widetilde{SB}}_{top}(T, Z, \mathbf{a}) \le C h^{\widetilde{SB}}_{top}(T, Z, \mathbf{b}).$$

Proof. The proofs follow directly from the definition of scaled neutralized entropy. □

Corollary 4.3. *If* $[\mathbf{b}] \le [\mathbf{a}]$, *then*

$$h^{\widetilde{SB}}_{top}(T, Z, \mathbf{a}) \le h^{\widetilde{SB}}_{top}(T, Z, \mathbf{b}).$$

Proposition 4.4. *If there exists* $[\mathbf{a}] \in \mathcal{A}$ *such that* $h^{\widetilde{SB}}_{top}(T, Z, \mathbf{a})$ *is positive and finite, then*

$$h^{\widetilde{SB}}_{top}(T, Z, \mathbf{b}) = \begin{cases} 0, & \text{if } [\mathbf{a}] \le [\mathbf{b}], \\ \infty, & \text{if } [\mathbf{b}] \le [\mathbf{a}]. \end{cases}$$

Particularly, there exists at most one element in $(\mathcal{A}, \le)$ *so that the scaled neutralized entropy is positive and finite.*

Proof. Assume that there exists $[\mathbf{a}] \in \mathcal{A}$ such that $h^{\widetilde{SB}}_{top}(T, Z, \mathbf{a})$ is positive and finite. Then, for any $[\mathbf{b}] \ge [\mathbf{a}]$, take $\mathbf{a_1} \in [\mathbf{a}]$ and $\mathbf{b}_1 \in [\mathbf{b}]$, we have

$$\limsup_{n\to+\infty} \frac{a_1(n)}{b_1(n)} = 0$$

Fix two scaled sequences $\mathbf{a}_1$ and $\mathbf{b}_1$ and give a sufficiently small number $m > 0$, we have

$$a_1(n) < m b_1(n),$$

whenever n is large enough. Therefore,

$$\widehat{M}^s_{N,\epsilon}(Z, \mathbf{a}_1) \ge \widehat{M}^s_{N,\epsilon}(Z, m\mathbf{b}_1).$$

This implies that

$$h^{\widetilde{SB}}_{top}(T, Z, \mathbf{a}_1) \ge h^{\widetilde{SB}}_{top}(T, Z, m\mathbf{b}_1) = \frac{1}{m} h^{\widetilde{SB}}_{top}(T, Z, \mathbf{b}_1),$$

thus,

$$m h^{\widetilde{SB}}_{top}(T, Z, \mathbf{a}_1) \ge h^{\widetilde{SB}}_{top}(T, Z, \mathbf{b}_1).$$

Since m is arbitrarily chosen, hence

$$h^{\widetilde{SB}}_{top}(T, Z, \mathbf{b}) = 0$$

Conversely, by the similar way, assume $[\mathbf{b}] \le [\mathbf{a}]$, take $\mathbf{a}_2 \in [\mathbf{a}]$ and $\mathbf{b}_2 \in [\mathbf{b}]$, we have

$$\limsup_{n\to\infty} \frac{b_2(n)}{a_2(n)} = 0$$

Similarly

$$\widehat{M}^s_{N,\epsilon}(Z,\mathbf{b}_2) \geq \widehat{M}^s_{N,\epsilon}(Z,m\mathbf{a}_2).$$

It follows that

$$h^{\widetilde{SB}}_{top}(T,Z,\mathbf{b}_2) > \frac{1}{m}h^{\widetilde{SB}}_{top}(T,Z,\mathbf{a}_2).$$

Hence,

$$h^{\widetilde{SB}}_{top}(T,Z,\mathbf{b}) = \infty.$$

□

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