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Statistical and uniformly statistical convergence of sequences of functions

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Abstract. In this paper, we provide an overview of statistical convergence by introducing the notion of statistical convergence for function sequences with values in a topological space. To define statistical uniform convergence, we further narrow the value set to a uniform space, highlighting that this concept serves to refine the previously defined one. Throughout the article, we examine the relationships between these concepts and their special cases.

1. Introduction

One of the joint research topics of analysis and topology is the convergence of sequences of functions. In topological and metric spaces, the convergence of function sequences is used to explore many properties and characterizations. By defining topologies based on the convergence of function sequences, the structure and properties of function spaces can be rigorously studied. This approach provides a solid foundation for learning continuity, compactness, and convergence in function spaces. Choosing a suitable topology for function spaces can significantly affect the concepts of convergence.

The usual convergence studied in metric, normed, and topological spaces is examined in detail to obtain new results and solve many topological problems related to convergence.

Based on this idea, new types of convergence, such as statistical convergence, have emerged. The concept of statistical convergence for sequences of real numbers was initially formulated by H. Fast in [4] and H. Steinhaus in [9], drawing from the idea of the asymptotic density of a set $A \subset \mathbb{N}$. Although, the seeds of statistical convergence were first sown in the inaugural edition of Zygmund's renowned monograph [13] in 1935. This convergence has many applications in mathematical analysis and topology. In recent years, many papers have been written using the idea of statistical convergence (see [1], [3], [10], [11], [12]). For real-valued functions, the notions of statistical convergence (pointwise sense) and statistical Cauchy sequence were introduced and examined by Gökhan et al. [6]. Statistical convergence discussed in summability theory was introduced and studied by Di Maio and Kočinac [7] in topological spaces. Statistical convergence in function spaces is also analyzed in [2].

For sequences of functions, the topological structure of the value set is one of the main factors influencing convergence. From this perspective, we provide an overview of statistical convergence by introducing the

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notion of statistical convergence for function sequences with values in a topological space. To define statistical uniform convergence, we further narrow the value set to a uniform space, highlighting that this concept serves to refine the previously defined one. Throughout the article, we examine the relationships between these concepts and their special cases.

Throughout this paper, unless otherwise stated clearly, all spaces are assumed to be Hausdorff. For the real line with the natural topology we use $\mathbb{R}$.

2. Preliminaries

A uniformity on a set Y is a collection $\mathfrak{U}$ of subsets of $Y \times Y$ satisfying the following properties:

(U1) $\Delta \subseteq U$, for every $U \in \mathfrak{U}$, where $\Delta = \{(y, y) : y \in Y\}$.
(U2) If $U \in \mathfrak{U}$, then $U^{-1} \in \mathfrak{U}$, where $U^{-1} = \{(y_1, y_2) : (y_2, y_1) \in U\}$.
(U3) If $U \in \mathfrak{U}$ and $U \subseteq V \subseteq Y \times Y$, then $V \in \mathfrak{U}$.
(U4) If $U_1, U_2 \in \mathfrak{U}$, then $U_1 \cap U_2 \in \mathfrak{U}$.
(U5) For every $U \in \mathfrak{U}$ there exists $V \in \mathfrak{U}$ such that $V \circ V \subseteq U$, where $V \circ V = \{(y_1, y_2) :$ there exists $y \in Y$ such that $(y_1, y) \in V$ and $(y, y_2) \in V\}$.

A uniform space is a pair $(Y, \mathfrak{U})$ consisting of a set Y and a uniformity $\mathfrak{U}$ on the set Y. The elements of $\mathfrak{U}$ are called entourages. An entourage V is called symmetric if $V^{-1} = V$. For every $U \in \mathfrak{U}$ and $y_0 \in Y$ we use the following notation:

$$U[y_0] = \{y \in Y : (y_0, y) \in U\}.$$

For every uniform space $(Y, \mathfrak{U})$ the uniform topology $\tau_{\mathfrak{U}}$ on Y is the family consisting of the empty set and all subsets O of Y such that for each $y \in O$ there is $U \in \mathfrak{U}$ with $U[y] \subseteq O$.

A mapping f from a topological space X into a uniform space $(Y, \mathfrak{U})$ is called continuous at x_0 if for each $U \in \mathfrak{U}$ there exists an open neighbourhood O_{x_0} of x_0 such that $f(O_{x_0}) \subseteq U[f(O_{x_0})]$ or equivalently $(f(x_0), f(x)) \in U$, for every $x_0 \in O_{x_0}$. The mapping f is called continuous if it is continuous at every point of X.

If $A \subset \mathbb{N}$, then $A(n)$ denotes the set

$$A(n) = \{k \in A : k \leq n\}.$$

The natural (or asymptotic) density of A is given by

$$\delta(A) = \lim_{n\to\infty} \frac{|A(n)|}{n},$$

if it exists. Here the density is in $[0, 1]$. Clearly, finite subsets have natural density zero. A subset A of $\mathbb{N}$ is statistically dense if $\delta(A) = 1$[7]. We recall also that

$$\delta(\mathbb{N}\backslash A) = 1 - \delta(A)$$

for $A \subset \mathbb{N}$.

A sequence $(x_n)_{n\in\mathbb{N}}$ in a topological space X is said to converge statistically to $x \in X$, if for every neighborhood U of x,

$$\delta(\{n \in \mathbb{N} : x_n \notin U\}) = 0.$$

For $X = \mathbb{R}$, equivalently this definition says that there exists a subset A of $\mathbb{N}$ with $\delta(A) = 1$ such that the sequence $(x_n)_{n\in A}$ converges to x, i.e. for every neighborhood V of x there is $n_0 \in \mathbb{N}$ such that $n \geq n_0$ and $n \in A$ imply $x_n \in V$ [7].

Let Y^X ($C(X, Y)$) denote the set of all (continuous) functions from a topological space X to a topological space Y and $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in Y^X. We say that f_n converges pointwise to a function $f : X \to Y$ if for every $x \in X$ the sequence $f_n(x)$ converges to $f(x)$ in Y, i. e. for every $x \in X$ and each neighborhood V of $f(x)$ in Y, there is some positive integer $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies $f_n(x) \in V$.

In the next section, starting from the definitions of convergence of sequences in topological spaces and pointwise convergence of function sequences in Y^X, pointwise statistical convergence of function sequences in Y^X will be introduced.

3. Pointwise statistical convergence

Definition 3.1. A sequence $(f_n)_{n\in\mathbb{N}}$ in Y^X is said to be (pointwise) statistically convergent (or shortly, st-convergence) to f if for each $x \in X$ and for each neighborhood V of $f(x)$ in Y,

$$\delta(\{n \in \mathbb{N} : f_n(x) \notin V\}) = 0.$$

We denote it by

$$f_n \xrightarrow{st} f \text{ or } st - \lim_{n\to\infty} f_n(x) = f(x)$$

and we call f the statistical limit of (f_n).

By reducing the space Y to a uniform space, Definition 3.1 can be expressed as follows.

Definition 3.2. Let $(Y, \mathfrak{U})$ be a uniform space. A sequence $(f_n)_{n\in\mathbb{N}}$ in Y^X is said to be (pointwise) statistically convergent to f if for each $x \in X$ and for each $\mathcal{U} \in \mathfrak{U}$,

$$\delta(\{n \in \mathbb{N} : (f_n(x), f(x)) \notin \mathcal{U}\}) = 0.$$

In other words, for each $x \in X$ and for each $\mathcal{U} \in \mathfrak{U}$,

$$\delta(\{n \in \mathbb{N} : f_n(x) \notin \mathcal{U}[f(x)]\}) = 0.$$

In this definition, by assuming that the domain is a uniform space, we naturally connect pointwise statistical convergence with uniform statistical convergence and set the stage for utilizing the results obtained here in the next section.

Theorem 3.3. *If a sequence $(f_n)_{n\in\mathbb{N}}$ in Y^X is statistically convergent, then its statistical limit is unique.*

Proof. Suppose that

$$f_n \xrightarrow{st} f_1 \text{ and } f_n \xrightarrow{st} f_2$$

with $f_1 \neq f_2$. Let $x \in X$. Let U and V be neighborhood of $f_1(x)$ and $f_2(x)$ in Y, respectively, such that

$$U \cap V = \emptyset.$$

Since $f_n \xrightarrow{st} f_1$ and $f_n \xrightarrow{st} f_2$, then

$$\delta(K_1) = \delta(\{n \in \mathbb{N} : f_n(x) \notin U\}) = 0$$

and

$$\delta(K_2) = \delta(\{n \in \mathbb{N} : f_n(x) \notin V\}) = 0,$$

respectively. Now, let $K = K_1 \cup K_2$. Thus, $\delta(K) = 0$ which implies $\delta(\mathbb{N}\backslash K) = 1$. If $k \in \mathbb{N}\backslash K$, then $f_n(x) \in U$ and $f_n(x) \in V$. This contradicts the fact that $U \cap V = \emptyset$. Hence, $f_1 = f_2$. $\square$

Theorem 3.4. *If a sequence $(f_n)_{n\in\mathbb{N}}$ in Y^X is convergent, then it is statistically convergent.*

Proof. Assume that $f_n \longrightarrow f$. There is some positive integer $n_0 \in \mathbb{N}$ such that $n > n_0$ implies $f_n(x) \in V$, for each $x \in X$ and each neighborhood V of $f(x)$ in Y. Let

$$A = \{n \in \mathbb{N} : f_n(x) \notin V\} = \{1, 2, ..., n_0\}.$$

Since A is finite, its density is zero, i.e. $\delta(A) = 0$. Hence, $f_n \xrightarrow{st} f$. □

The converse of the previous theorem need not be true in general.

Example 3.5. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^X$ defined by

$$f_n(x) = \begin{cases} 1 & \text{if } n \text{ is prime,} \\ 0 & \text{otherwise.} \end{cases}$$

It is easily seen that $(f_n)_{n\in\mathbb{N}}$ is not convergent. But this sequence statistically converges to zero function f_0 since the set of prime numbers has density 0.

Definition 3.6. A subsequence $(f_{n_k})_{k\in\mathbb{N}}$ of a sequence $(f_n)_{n\in\mathbb{N}}$ is statistically dense in $(f_n)_{n\in\mathbb{N}}$ if the set of indices n_k is a statistically dense subset of $\mathbb{N}$.

Theorem 3.7. *A sequence $(f_n)_{n\in\mathbb{N}}$ is statistically convergent if and only if any of its statistically dense subsequences is statistically convergent.*

Proof. The sufficiency part is obvious, because each sequence is statistically dense in itself.

Necessity, let $(f_n)_{n\in\mathbb{N}}$ statistically converge to f and let $(f_{n_k})_{k\in\mathbb{N}}$ be a statistically dense subsequence of $(f_n)_{n\in\mathbb{N}}$. On the contrary, suppose that $(f_{n_k})_{k\in\mathbb{N}}$ is not statistically convergent. Which means that for each $x \in X$ and for each neighborhood V of $f(x)$ in Y,

$$\delta(\{n \in \mathbb{N} : f_{n_k}(x) \notin V\}) \neq 0.$$

Then, we have

$$\delta(\{n \in \mathbb{N} : f_n(x) \notin V\}) \geq \delta(\{n \in \mathbb{N} : f_{n_k}(x) \notin V\}) \neq 0$$

and consequently $(f_n)_{n\in\mathbb{N}}$ is not statistically convergent. A contradiction. □

Example 3.8. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^X$ defined by

$$f_n(x) = \begin{cases} n & \text{if } n \text{ is a square,} \\ \frac{1}{n} & \text{otherwise.} \end{cases}$$

Here, f_0 being the zero function, the sequence $f_n \xrightarrow{st} f_0$. Let's consider subsequence $(f_{n_k})_{k\in\mathbb{N}}$, k is a square. Since

$$\delta(A) = \delta(\{k^2 : k \in \mathbb{N}\}) = 0,$$

this subsequence of a statistically convergent sequence is not statistically convergent.

Definition 3.9. A sequence $(f_n)_{n\in\mathbb{N}}$ is said to be st*-convergent to f if there is $A \subset \mathbb{N}$ with $\delta(A) = 1$ such that

$$\lim_{\substack{m\to\infty \\ (m\in A)}} f_m(x) = f(x),$$

for each $x \in X$. We denote it by

$$f_n \xrightarrow{st^*} f \text{ or } st^* - \lim_{n\to\infty} f_n(x) = f(x).$$

The next result follows from Theorem 3.7.

Corollary 3.10. *If a sequence $(f_n)_{n\in\mathbb{N}}$ st^*-convergent to f, then there is a subsequence of $(f_n)_{n\in\mathbb{N}}$ converging to f.*

Theorem 3.11. *If a sequence $(f_n)_{n\in\mathbb{N}}$ in Y^X st^*-convergent to $f \in Y^X$, then (f_n) st-convergent to f.*

Proof. Let $x \in X$ and Let V be a neighborhood of $f(x)$ in Y. Since (f_n) st^*-convergent to f, there are $A \subset \mathbb{N}$ with $\delta(A) = 1$ and $m_0 \in \mathbb{N}$ such that $m > m_0$ and $m \in A$ imply $f_m(x) \in V$. Then,

$$\{m \in \mathbb{N} : f_m(x) \notin V\} \subset \{1, 2, ..., m_0\} \cup (\mathbb{N}\backslash A)$$

and since

$$\delta(\{1, 2, ..., m_0\} \cup (\mathbb{N}\backslash A)) = 0,$$

it follows (f_n) st-convergence to f. $\square$

The converse holds if the topological space Y is first countable.

Theorem 3.12. *Let Y be a first countable topological space. If a sequence $(f)_{n\in\mathbb{N}}$ in Y^X st-convergent to $f \in Y^X$, then $(f_n)_{n\in\mathbb{N}}$ st^*-convergent to f.*

Proof. If Y is first countable, then every point has a countable nested local basis. Let

$$V_1 \supset V_2 \supset V_3 \supset \ ...$$

be a countable nested local base at $f(x)$ and let set

$$N_i = \{n \in \mathbb{N} : f_n(x) \in V_i\},$$

for every $i \in \mathbb{N}$. Then, we have

$$N_1 \supset N_2 \supset N_3 \supset \ ... \text{ and } \delta(N_i) = 1,$$

for every $i \in \mathbb{N}$. Let $k_1 \in N_1$. There is $k_2 \in N_2$, $k_2 > k_1$, such that for every $n \geq k_2$ it holds

$$\frac{|N_2(n)|}{n} = \frac{|\{m \in N_2 : m \leq n\}|}{n} > \frac{1}{2},$$

since $\delta(N_2) = 1$ and so on. This way, we get $k_i \in N_i$, $k_1 < k_2 < ... < k_i$ such that for every $n \geq k_i$, one has

$$\frac{|N_i(n)|}{n} = \frac{|\{m \in N_i : m \leq n\}|}{n} > 1 - \frac{1}{i}.$$

Define the set $A \subset \mathbb{N}$ for each $k \leq k_1$, $k \in A$; if $i \geq 1$ and $k_i < k \leq k_{i+1}$, then $k \in A$ if and only if $k \in N_i$. Let

$$A = \{n_1 < n_2 < ...\}.$$

If $n \in \mathbb{N}$ is such that $k_i \leq n \leq k_{i+1}$, then

$$\frac{|A(n)|}{n} = \frac{|N_i(n)|}{n} > 1 - \frac{1}{i},$$

and hence $\delta(A) = 1$. We prove that

$$\lim_{n\to\infty, n\in A} f_n(x) = \lim_{i\to\infty} f_{n_i}(x) = f(x).$$

Let W be a neighborhood of $f(x)$ in Y and $V_i \subset W$. If $n \in A$, $n \geq k_i$, then there exist $j \geq i$ with $k_j \leq n \leq k_{j+1}$. Hence, by definition of A, $n \in A_j$. Therefore, for each $n \in A$, $n \geq k_i$ we have

$$f_n(x) \in V_j \subset V_i \subset W$$

and so

$$\lim_{i\to\infty} f_{n_i}(x) = f(x) \text{ or } f_n \xrightarrow{st^*} f.$$

$\square$

4. Uniform statistical convergence

In this section, the uniform statistical convergence of sequences of functions will be introduced by taking $(Y, \mathfrak{U})$ as a uniform space.

Definition 4.1. A sequence $(f_n)_{n\in\mathbb{N}}$ in Y^X is said to be uniformly st*-convergent to f if for each $\mathcal{U} \in \mathfrak{U}$, there is $A \subset \mathbb{N}$ with $\delta(A) = 1$ such that for each $x \in X$ and for each $n \in A$, $(f_m(x), f(x)) \in \mathcal{U}$. We denote it by

$$f_n \overset{st^*-u}{\longrightarrow} f.$$

The next Theorem follows from Theorem 3.4.

Theorem 4.2. *If a sequence $(f_n)_{n\in\mathbb{N}}$ in Y^X is uniformly convergent, then it is uniformly statistically convergent.*

Definition 4.3. A sequence $(f_n)_{n\in\mathbb{N}}$ in Y^X is said to be uniformly statistically convergent to a function f on X if for each $\mathcal{U} \in \mathfrak{U}$,

$$\delta(\{n \in \mathbb{N} : (f_n(x), f(x)) \notin \mathcal{U}, \text{ for all } x \in X\}) = 0.$$

In other words, for each $\mathcal{U} \in \mathfrak{U}$,

$$\delta(\{n \in \mathbb{N} : f_n(x) \notin \mathcal{U}[f(x)], \text{ for all } x \in X\}) = 0.$$

We denote it by

$$f_n \overset{st-u}{\longrightarrow} f.$$

By Definition 3.2 and Definition 4.3, we easily obtain the following theorem.

Theorem 4.4. *If a sequence $(f_n)_{n\in\mathbb{N}}$ in Y^X uniformly statistically convergent to $f \in Y^X$, then (f_n) statistically convergent to f.*

The next theorem follows from Definition 3.2 and Theorem 3.11.

Theorem 4.5. *If a sequence $(f_n)_{n\in\mathbb{N}}$ in Y^X uniformly st*-convergent to $f \in Y^X$, then (f_n) uniformly statistically convergent to f.*

Definition 4.1 above is given in [5], taking the space Y as semi-uniform. Considering Theorem 4.5, it can be seen that Definition 4.3 is more general than Definition 4.1.

Lemma 4.6. ([8])$(Y, \mathfrak{U})$ *uniform space and $\mathcal{U} \in \mathfrak{U}$. Then, there exists a symmetric entourage $\mathcal{V} \in \mathfrak{U}$ such that $\mathcal{V} \circ \mathcal{V} \circ \mathcal{V} \subseteq \mathcal{U}$.*

Theorem 4.7. *If a sequence $(f_n)_{n\in\mathbb{N}}$ in $C(X, Y)$ and $f_n \overset{st-u}{\longrightarrow} f$, then $f \in C(X, Y)$.*

Proof. Suppose that $f_n \overset{st-u}{\longrightarrow} f$ and let $x_0 \in X$. We prove that f is continuous at x_0. By Lemma 4.6, there exists a symmetric entourage $\mathcal{V} \in \mathfrak{U}$ such that $\mathcal{V} \circ \mathcal{V} \circ \mathcal{V} \subseteq \mathcal{U}$. Since $f_n \overset{st-u}{\longrightarrow} f$, the set

$$A = \{n \in \mathbb{N} : f_n(x) \notin \mathcal{V}[f(x)]\}$$

has a asymptotic density 0, for each $x \in X$. Let $n_0 \in \mathbb{N}$. Then,

$$f_{n_0}(x_0) \notin \mathcal{V}[f(x_0)]. \tag{1}$$

Since f_{n_0} is continuous at x_0, there exists an open neighbourhood $O_{x_0}$of x_0 such that

$$f_{n_0}(x) \notin \mathcal{V}[f_{n_0}(x_0)],$$

for each $x_0 \in O_{x_0}$. Let $x \in O_{x_0}$. Then,

$$f_{n_0}(x) \notin \mathcal{V}[f_{n_0}(x_0)] \tag{2}$$

and

$$f_{n_0}(x) \notin \mathcal{V}[f(x)]. \tag{3}$$

Therefore, using successively the relations (1), (2), and (3), we have

$$f(x) \notin \mathcal{V}[f_{n_0}(x)] \subseteq (\mathcal{V} \circ \mathcal{V})[f_{n_0}(x)] \subseteq (\mathcal{V} \circ \mathcal{V} \circ \mathcal{V})[f_{n_0}(x)]$$

and the continuity of f is proved. □

5. Conclusion

In this study, a more general definition has been presented for the pointwise statistical convergence of function sequences, based on the assumption that the value set is a topological space. Additionally, a more comprehensive approach has been developed for the concept of uniform statistical convergence using density. The relationship between two different types of statistical convergence has been highlighted, and the fundamental results have been presented in detail. Thus, by utilizing the possibilities provided by both topological and uniform structures, more general and robust results have been obtained regarding the statistical convergence of function sequences. These general definitions and the obtained results form a solid foundation for future work in areas such as functional analysis, topology, and convergence theory, opening doors to new research directions.

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