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A family of optimal linear codes of length 2^s over the ring $\frac{\mathbb{F}_2[u]}{\langle u^2 \rangle}$ with Lee distance 4

Sujata Bansal[a], **Pramod Kumar Kewat**[a,*]

[a]*Department of Mathematics and Computing, Indian Institute of Technology (Indian School of Mines), Dhanbad, 826004, India*

Abstract. Optimal linear codes over finite rings are of significant interest in coding theory. In this paper, we introduce a method for constructing an infinite class of Lee metric codes of type $4^{m_1}2^{m_2}$ with length 2^s over the ring $R = \frac{\mathbb{F}_2[u]}{\langle u^2 \rangle}$. These codes are constructed by taking a union of certain particular cosets of the ideal $I = \langle u(x+1) \rangle$ in the ring $S = \frac{R[x]}{\langle x^{2^s}+1 \rangle}$, for $s \geq 2$. We discuss their algebraic structure and demonstrate that these codes are linear with Lee distance 4 and are optimal with respect to the Lee sphere packing bound. Furthermore, we conclude that these codes are even codes with respect to the Lee metric and also provide the complete Lee weight distribution of these codes.

1. Introduction

Introduced by Lee in 1958 [17], the Lee metric has emerged as a significant alternative to the Hamming metric. It plays a crucial role in modern communication systems, particularly in channels employing phase modulation and multilevel quantized pulse amplitude modulation. In 1966, significant progress was made in the direction of the Lee metric when Berlekamp [1] introduced negacyclic codes over $GF(p)$ for the Lee metric. These codes are equipped with an efficient decoding algorithm in addition to having good error-correcting capabilities. Golomb and Welch [12] further advanced the field by introducing the notion of perfect Lee metric codes. They proposed conjectures and conditions regarding the existence and non-existence of these codes over specific alphabets. They also created perfect codes of length 2 over $\mathbb{Z}_e$, where $e = 2T^2 + 2T + 1$. Chiyang and Wolf [4] developed discrete, symmetric, memoryless channels matched to the Lee metric in 1971. This assists in identifying modulation schemes where the Lee metric codes perform better than other metrics. Furthermore, Satyanarayan [19] presented Lee metric codes over $\mathbb{Z}_m$. The construction of various families of linear codes over $\mathbb{Z}_4$ for the Lee metric in 1994 by Hammons et al. [14] marked a significant milestone in coding theory. Under appropriately defined gray maps, images of these linear codes represent optimal nonlinear binary codes. Since then, codes having finite rings as alphabets have attracted a lot of attention, but there hasn't been much work done on the Lee metric codes.

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* Corresponding author: Pramod Kumar Kewat
Email addresses: sujatabansal8@gmail.com (Sujata Bansal), pramodk@iitism.ac.in (Pramod Kumar Kewat)
ORCID iDs: https://orcid.org/0009-0009-6994-6412 (Sujata Bansal), https://orcid.org/0000-0002-2483-0960 (Pramod Kumar Kewat)

The Lee distance of several $\mathbb{Z}_4$-cyclic codes of length 2^e was calculated by the authors in [15]. The structure of the ring $R = \mathbb{F}_2 + u\mathbb{F}_2$, where $u^2 = 0$ presents an exciting fusion of properties from both $\mathbb{Z}_4$ and $\mathbb{F}_4$. In [3], A. Bonnecaze and P. Udaya showcased this by constructing optimal and extremal self-dual binary codes through the gray image of $\mathbb{F}_2 + u\mathbb{F}_2$-linear codes. Furthermore, for cyclic codes over $\mathbb{F}_2 + u\mathbb{F}_2$, they described a simple decoding algorithm [25]. Its decoding efficiency is the main benefit of studying these codes. Unlike $\mathbb{Z}_4$, the ring $R = \mathbb{F}_2 + u\mathbb{F}_2$ offers the advantage of easy decoding and implementation, making it a preferred choice for practical applications. The authors in [23] constructed infinite codes of two-Lee-weight over $\mathbb{F}_2 + u\mathbb{F}_2$, where $u^2 = 0$. In [2], the notion of the Lee weight over $\mathbb{F}_{2^m}$ and $\mathbb{F}_{2^m} + u\mathbb{F}_{2^m}$ was introduced. Dinh et al. [5] found the Lee distance of cyclic codes having 2^s length over $\mathbb{F}_{2^m}$ and $\mathbb{F}_{2^m} + u\mathbb{F}_{2^m}$. In recent decades, numerous infinite families of codes over rings have been the subject of study, utilizing the Lee metric (see [6], [7], [8], [10], [16], [18], [22], [23], [24]). However, the majority of these codes are optimal in the sense that their gray images are optimal over some finite field. In addition, numerous optimal linear codes over finite fields have been derived through various Gray maps defined over different finite rings (see [13], [20], [21], [26]).

In this paper, we design an infinite class of linear codes over the ring $R = \mathbb{F}_2 + u\mathbb{F}_2$, where $u^2 = 0$. In 1971, Goethals [11] devised the NR-code by puncturing a union of cosets from the extended binary Golay code in $\mathbb{F}_2^{24}$, resulting in an optimal binary non-linear code. Motivated by this approach, we specifically select cosets of the ideal $I = \langle u(x+1)\rangle$ of the quotient ring $S = \frac{R[x]}{\langle x^{2^s}+1\rangle}$ to construct codes over R. Normally, if we take a coset of a linear code, then it forms a non-linear code with the same parameters. However, our approach differs as we carefully select cosets to ensure that their union yields a linear code over R. This code possesses a length of 2^s, a minimum Lee distance of 4, and a size of $2^{2^{s+1}-s-2}$ for $s \geq 2$. We revisit Golomb's sphere packing bound and observe that it holds true for codes over $R = \mathbb{F}_2 + u\mathbb{F}_2, u^2 = 0$. We establish that, with respect to the Lee sphere packing bound, the constructed class of codes is optimal. We also compute the complete Lee weight distribution of the constructed codes.

The paper is organized as follows: Section 2 provides essential definitions and background information. In Section 3, we present our construction method and detail the process of constructing codes of length 2^s over the ring $R = \mathbb{F}_2 + u\mathbb{F}_2, u^2 = 0$. Section 4 examines the linearity and optimality of the code. Finally, in Section 5, we provide the complete Lee weight distribution of the codes.

2. Preliminaries

2.1. *Ring* $R = \frac{\mathbb{F}_2[u]}{\langle u^2\rangle} = \mathbb{F}_2 + u\mathbb{F}_2$, $u^2 = 0$

Let $\mathbb{F}_2$ be the binary field. We define the ring R as $R = \frac{\mathbb{F}_2[u]}{\langle u^2\rangle} = \mathbb{F}_2 + u\mathbb{F}_2$, where $u^2 = 0$. With the maximal ideal $\langle u\rangle$, R is a finite commutative chain ring. Every element $r \in R$ can be uniquely written as $r = a + ub$, where $a, b \in \mathbb{F}_2$ and $u^2 = 0$. Any polynomial $r(x)$ in $R[x]$ with degree at most $n-1$ has a unique expression:

$$r(x) = \sum_{i=0}^{n-1} r_i(x+1)^i + u\sum_{i=0}^{n-1} t_i(x+1)^i, \text{ where } r_i, t_i \in \mathbb{F}_2, 0 \leq i \leq n-1.$$

2.2. *Linear and cyclic codes over R*

Any non-empty subset of R^n is defined as a code of length n over R. A linear code C over R with length n is an R-submodule of R^n. Furthermore, C is cyclic if $(c_{n-1}, c_0, \ldots, c_{n-2}) \in C$, for $(c_0, c_1, \ldots, c_{n-1}) \in C$. It is possible to express any element $\mathbf{c} = (c_0, c_1, \ldots, c_{n-1}) \in C$ as a polynomial $c(x) = c_0 + c_1x + \ldots + c_{n-1}x^{n-1}$ in $\frac{R[x]}{\langle x^n-1\rangle}$. The following claim is quite evident.

Proposition 2.1. *A linear code C over R of length n is an R-submodule of the quotient ring $\frac{R[x]}{\langle x^n-1\rangle}$, and if C is cyclic, then it is an ideal of $\frac{R[x]}{\langle x^n-1\rangle}$.*

Any R-linear code C contains a set of $m_1 + m_2$ codewords $c_1, c_2, \dots, c_{m_1}, c_{m_1+1}, \dots, c_{m_1+m_2}$ such that every codeword of C can be represented uniquely as $\sum_{i=1}^{m_1} r_i c_i + \sum_{i=m_1+1}^{m_1+m_2} r_i c_i$, where $r_i \in R$ for $1 \le i \le m_1$ and $r_i \in \mathbb{F}_2$ for $m_1 + 1 \le i \le m_1 + m_2$. If we express $c_1, \dots, c_{m_1}, c_{m_1+1}, \dots, c_{m_1+m_2}$ as the rows of a matrix G, then G is called a generator matrix of C and every codeword of C can be written as $\begin{bmatrix} r_1 & \dots & r_{m_1} & r_{m_1+1} & \dots & r_{m_1+m_2} \end{bmatrix} G$ for some $r_1, \dots r_{m_1} \in R$, and $r_{m_1+1}, \dots r_{m_1+m_2} \in \mathbb{F}_2$. If $m_2 = 0$, the code C is a free R-module. The code C has $4^{m_1} 2^{m_2} = 2^{2m_1+m_2}$ codewords and is said to be of type $4^{m_1} 2^{m_2}$. The rank of the code C is defined as $m_1 + m_2$ (see [3], [9]).

Definition 2.2. *If every codeword in a code has even weight, the code is said to be even.*

We fix the notations $R = \frac{\mathbb{F}_2[u]}{\langle u^2 \rangle} = \mathbb{F}_2 + u\mathbb{F}_2$, where $u^2 = 0$ and $S = \frac{R[x]}{\langle x^{2^s} + 1 \rangle}, s \ge 2$ throughout the paper. The following lemma is easy to verify.

Lemma 2.3. *For any non-negative integer t, $(x+1)^{2^t} = x^{2^t} + 1$ in S. Particularly, $x + 1$ is nilpotent in S with nilpotency index 2^s.*

2.3. Lee weights over R

The Lee weight wt_L over $\mathbb{F}_2$ is defined as $wt_L(1) = 1$ and $wt_L(0) = 0$. For an element $a + ub \in R$, we define the Lee weight as:

$$wt_L(a + ub) = wt_L(a + b, b) = wt_L(a + b) + wt_L(b).$$

The Lee weight of a vector $v = (v_0, v_1, \dots, v_{n-1}) \in R^n$ is defined as:

$$wt_L(v) = \sum_{i=0}^{n-1} wt_L(v_i).$$

The Hamming weight of $v = (v_0, v_1, \dots, v_{n-1})$ is the count of its non-zero components. The Lee distance between two elements u and v in R^n is defined as the Lee weight of their difference.

$$d_L(u, v) = wt_L(u - v).$$

Definition 2.4. *The Lee distance of a code C over R is defined as:*

$$d_L(C) = \min\{d_L(x, y) \mid x, y \in C, x \ne y\}.$$

If C is linear, then $d_L(C)$ is the same as the minimum non-zero Lee weight of C.

Definition 2.5. *If P and Q are two subsets of R^n, then*

$$d_L(P, Q) = \min\{d_L(p, q) \mid p \in P, q \in Q\}.$$

Theorem 2.6. *Let $I = \langle u(x + 1) \rangle$ be an ideal of $S = \frac{R[x]}{\langle x^{2^s}+1 \rangle}$. The ideal I is an even code with respect to the Hamming weight and a doubly even code with respect to the Lee weight. Moreover, the number of codewords with Hamming weight $2k$ is $\binom{2^s}{2k}$ and the number of codewords with Lee weight $4k$ is $\binom{2^s}{2k}$, where $0 \le k \le 2^{s-1}$ and $d_L(I) = 4$.*

Proof. Since $(x + 1)^{2^s} = x^{2^s} + 1 = 0$ in $S = \frac{R[x]}{\langle x^{2^s}+1 \rangle}$ and $u^2 = 0$, a general element $a(x)$ of I can be expressed as:

$$\begin{aligned} a(x) &= u(x + 1)\left[a_0 + a_1(x + 1) + a_2(x + 1)^2 + \dots + a_{2^s-2}(x + 1)^{2^s-2}\right] \\ &= u\left[a_0(x + 1) + a_1(x + 1)^2 + \dots + a_{2^s-2}(x + 1)^{2^s-1}\right], \end{aligned}$$

where $a_i \in \mathbb{F}_2$ for each i. Consequently, $|I| = 2^{2^s-1}$. We can express $a(x)$ as:

$$\begin{aligned} &u\left[a_0(x+1) + a_1(x+1)^2 + \ldots + a_{2^s-2}(x+1)^{2^s-1}\right] \\ =&u\left[b_0 + b_1x + b_2x^2 + \ldots + b_{2^s-1}x^{2^s-1}\right], \end{aligned}$$

where $b_i \in \mathbb{F}_2$ for each $0 \le i \le 2^s - 1$. Substituting $x = 1$ gives

$$b_0 + b_1 + b_2 + \ldots + b_{2^s-1} \equiv 0 \pmod 2.$$

This implies $b_0 \equiv b_1 + b_2 + \ldots + b_{2^s-1} \pmod 2$. Therefore, any element of I can be written as:

$$u\left[(b_1 + b_2 + \ldots + b_{2^s-1}) + b_1x + b_2x^2 + b_3x^3 + \ldots + b_{2^s-1}x^{2^s-1}\right].$$

Suppose

$$r(x) = u\left[(b_1 + b_2 + \ldots + b_{2^s-1}) + b_1x + b_2x^2 + b_3x^3 + \ldots + b_{2^s-1}x^{2^s-1}\right] \in I,$$

for some $b_1, b_2, \ldots, b_{2^s-1} \in \mathbb{F}_2$. Clearly, if $wt_H(b_1, b_2, \ldots, b_{2^s-1}) = w$ is even, then

$$b_1 + b_2 + \ldots + b_{2^s-1} \equiv 0 \pmod 2 \text{ and } wt_H(r(x)) = w,$$

and if $wt_H(b_1, b_2, \ldots, b_{2^s-1}) = w$ is odd, then

$$b_1 + b_2 + \ldots + b_{2^s-1} \equiv 1 \pmod 2 \text{ and } wt_H(r(x)) = w + 1.$$

Hence, every codeword in I has even Hamming weight. Next, suppose $c(x) = u\Big[(c_1 + c_2 + \ldots + c_{2^s-1}) + c_1x + c_2x^2 + \ldots + c_{2^s-1}x^{2^s-1}\Big]$ is an element of I with Hamming weight $2k$, for some $0 \le k \le 2^{s-1}$. Then either $wt_H(c_1, c_2, \ldots, c_{2^s-1}) = 2k$ or $wt_H(c_1, c_2, \ldots, c_{2^s-1}) = 2k-1$. Hence, the number of elements in I with Hamming weight $2k$ is given by

$$\binom{2^s-1}{2k} + \binom{2^s-1}{2k-1} = \binom{2^s}{2k}.$$

Let $c(x) = u\left(c_0 + c_1x + c_2x^2 + \ldots + c_{2^s-1}x^{2^s-1}\right) \in I$, for some $c_0, c_1, c_2, \ldots, c_{2^s-1} \in \mathbb{F}_2$, with $wt_H(c(x)) = 2k$, where $0 \le k \le 2^{s-1}$. Since $wt_L(u) = 2$ and $wt_H(c(x)) = 2k$, thus $wt_L(c(x)) = 4k$. Hence, the number of elements in I with Lee weight $4k$ is $\binom{2^s}{2k}$. Finally, since I is a linear code over R of length 2^s and the minimum Lee weight of any nonzero element of I is 4, we have $d_L(I) = 4$. □

Remark 2.7. *Consider the ideal $J = \langle u \rangle$ of S. We have $|J| = 2^{2^s}$. Let J_e denote the subset of J consisting of all elements with even Hamming weight. Then $|J_e| = 2^{2^s-1}$. Since $I \subseteq J_e$ and $|I| = |J_e|$, it follows that $I = J_e$. Thus, I contains all elements of $\langle u \rangle$ with even Hamming weight.*

Lemma 2.8. *If $t(x) + I$ is a coset of I in $S = \frac{R[x]}{\langle x^{2^s}+1 \rangle}$ for some $t(x) \in S$, then $d_L(t(x) + I) = 4$.*

Proof. Let $t(x) + i_1(x)$ and $t(x) + i_2(x)$ be two distinct elements of the coset $t(x) + I$, where $i_1(x), i_2(x) \in I$ and $i_1(x) \neq i_2(x)$. The Lee distance between $t(x) + i_1(x)$ and $t(x) + i_2(x)$ is given by

$$\begin{aligned} d_L(t(x) + i_1(x),\, t(x) + i_2(x)) &= wt_L(t(x) + i_1(x) - (t(x) + i_2(x))) \\ &= wt_L(i_1(x) - i_2(x)), \end{aligned}$$

which is at least 4, since $i_1(x) - i_2(x) \in I$. Moreover, since $d_L(t(x),\, t(x) + u(x+1)) = 4$, we conclude that the Lee distance of the coset $t(x) + I$ is exactly 4, i.e., $d_L(t(x) + I) = 4$. □

Lemma 2.9. *Let $I = \langle u(x+1)\rangle$ be the ideal of $S = \frac{R[x]}{\langle x^{2^s}+1\rangle}$, and $\alpha(x)+I, \beta(x)+I$ be two distinct cosets of I in S, where $\alpha(x) = \sum_{i=0}^{2^s-1}\alpha_i x^i, \beta(x) = \sum_{i=0}^{2^s-1}\beta_i x^i \in \frac{R[x]}{\langle x^{2^s}+1\rangle}$, with $\alpha_i, \beta_i \in \mathbb{F}_2$. If $d_L(\alpha(x), \beta(x)) \geq 4$, then $d_L(\alpha(x)+I, \beta(x)+I) \geq 4$.*

Proof. Suppose $d_L(\alpha(x), \beta(x)) = d \geq 4$. Since $\alpha_i, \beta_i \in \mathbb{F}_2$, thus $d_H(\alpha(x), \beta(x)) = d_L(\alpha(x), \beta(x)) = d$. Suppose $\alpha(x)$ and $\beta(x)$ differ at the positions $i_1, i_2, \ldots, i_d$ positions, where $0 \leq i_1 < i_2 < \ldots < i_d \leq 2^s - 1$. Let $D = \{i_1, i_2, \ldots, i_d\}$ and let E be the subset of D such that $\alpha(x)$ is 0 and $\beta(x)$ is 1 on E, and vice-versa on $D \setminus E$.

Now let $\alpha(x)+r(x) \in \alpha(x)+I$ and $\beta(x)+t(x) \in \beta(x)+I$, where $r(x) = u\left[\sum_{i=0}^{2^s-1} r_i x^i\right], t(x) = u\left[\sum_{i=0}^{2^s-1} t_i x^i\right] \in I$, and $r_i, t_i \in \mathbb{F}_2$ for all i. Then

$$\begin{aligned}
d_L(\alpha(x)+r(x),\, \beta(x)+t(x)) &\geq \sum_{j\in D} d_L(\alpha_j + ur_j,\, \beta_j + ut_j) \\
&= \sum_{j\in E} d_L(\alpha_j + ur_j,\, \beta_j + ut_j) + \sum_{j\in D\setminus E} d_L(\alpha_j + ur_j,\, \beta_j + ut_j) \\
&= \sum_{j\in E} d_L(ur_j,\, 1 + ut_j) + \sum_{j\in D\setminus E} d_L(1 + ur_j,\, ut_j) \\
&= \sum_{j\in E} wt_L(1 + u(r_j + t_j)) + \sum_{j\in D\setminus E} wt_L(1 + u(r_j + t_j)) \\
&= \sum_{j\in D} wt_L(1 + u(r_j + t_j)) \\
&= \sum_{j\in D} wt_L(r_j + t_j, 1 + r_j + t_j) \\
&= \sum_{j\in D} 1 \\
&= d \geq 4.
\end{aligned}$$

This proves the result. □

3. Construction method for a family of optimal linear codes over R

In this section, we construct linear codes of length 2^s over R with the minimum Lee distance 4 for $s \geq 2$. To achieve this, we consider the ideal $I = \langle u(x+1)\rangle$ in $S = \frac{R[x]}{\langle x^{2^s}+1\rangle}$. Subsequently, we select certain cosets of I in S in a manner such that the union of these cosets forms a code over R with size $2^{2^{s+1}-s-2}$ and the minimum Lee distance 4. The following steps outline our construction process:

1. We take the ideal $I = \langle u(x+1)\rangle$ of size 2^{2^s-1} in the ring $S = \frac{R[x]}{\langle x^{2^s}+1\rangle}$.
2. **Construction of polynomials:** We establish the existence of 2^{2^s-s-1} polynomials over $\mathbb{F}_2$, each having degree at most $2^s - 1$ such that, for $s \geq 2$, the lowest possible Lee distance between any two different polynomials is at least 4. Throughout this paper, we consider the zero polynomial to have degree -1. For $s = 2$, we observe that the polynomials 0 and $1 + x + x^2 + x^3$ in $\mathbb{F}_2[x]$ meet these criteria, having the minimum Lee distance 4. We fix $\mathcal{F}_2 = \{0, 1 + x + x^2 + x^3\}$.
   Now, we apply the induction hypothesis. We assume the existence of $2^{2^{s-1}-(s-1)-1}$ such polynomials over $\mathbb{F}_2$ with degree at most $2^{s-1} - 1$. We denote the set of these polynomials by $\mathcal{F}_{s-1}$. We aim to construct 2^{2^s-s-1} such polynomials with degree at most $2^s - 1$.
   We define binomials $p_r^s(x) = \left(x^{2r} + x^{2r+1}\right)$ over $\mathbb{F}_2$ with degrees at most $2^s - 1$ for $0 \leq r \leq 2^{s-1} - 1$. Let $\Gamma = \{0, 1, \ldots, 2^{s-1} - 1\}$ and let Ψ and Ω be two distinct subsets of Γ with cardinalities $2i$ and $2j$, respectively, for some $0 \leq i, j \leq 2^{s-2}$. For the null set ϕ, we define the polynomial $\sum_{k\in\phi} p_k^s(x)$ to be the zero polynomial. Suppose $\alpha_1(x) = \sum_{k\in\psi} p_k^s(x)$ and $\alpha_2(x) = \sum_{k\in\Omega} p_k^s(x)$ are two polynomials over $\mathbb{F}_2$.

Claim 1: $d_L(\alpha_1(x), \alpha_2(x)) \geq 4$.

Since $\alpha_1(x) = \sum_{k\in\Psi} p_k^s(x) = \sum_{k\in\Psi}\left(x^{2k} + x^{2k+1}\right)$ and $\alpha_2(x) = \sum_{k\in\Omega} p_k^s(x) = \sum_{k\in\Omega}\left(x^{2k} + x^{2k+1}\right)$, we can express their sum as:

$$\begin{aligned}\alpha_1(x) + \alpha_2(x) &= \sum_{k\in\Psi}\left(x^{2k} + x^{2k+1}\right) + \sum_{k\in\Omega}\left(x^{2k} + x^{2k+1}\right)\\ &= \sum_{k\in\Psi\setminus\Omega}\left(x^{2k} + x^{2k+1}\right) + \sum_{k\in\Psi\cap\Omega}\left(x^{2k} + x^{2k+1}\right)\\ &\quad + \sum_{k\in\Omega\cap\Psi}\left(x^{2k} + x^{2k+1}\right) + \sum_{k\in\Omega\setminus\Psi}\left(x^{2k} + x^{2k+1}\right)\\ &= \sum_{k\in\Psi\setminus\Omega}\left(x^{2k} + x^{2k+1}\right) + \sum_{k\in\Omega\setminus\Psi}\left(x^{2k} + x^{2k+1}\right).\end{aligned}$$

It follows that

$$\begin{aligned}d_L(\alpha_1(x), \alpha_2(x)) &= wt_L(\alpha_1(x) + \alpha_2(x))\\ &= 2|\Psi \setminus \Omega| + 2|\Omega \setminus \Psi|.\end{aligned}$$

If $\Psi \subset \Omega$, then $\Psi \setminus \Omega = \phi$ and $|\Omega \setminus \Psi| = 2j - 2i \geq 2$. Consequently, $d_L(\alpha_1(x), \alpha_2(x)) \geq 4$. Similarly, if $\Omega \subset \Psi$, then also the result holds true. Now, let $\Psi \not\subseteq \Omega$, and $\Omega \not\subseteq \Psi$, then $|\Psi \setminus \Omega| \geq 1$ and $|\Omega \setminus \Psi| \geq 1$, which implies that $d_L(\alpha_1(x), \alpha_2(x)) \geq 4$.

This yields a collection of $\sum_{i=0}^{2^{s-2}} \binom{2^{s-1}}{2i} = 2^{2^{s-1}-1}$ polynomials over $\mathbb{F}_2$. We name these polynomials as α-type polynomials, which are the sums of certain binomials $p_r^s(x)$. We denote this set of α-type polynomials as $\mathcal{A}_s$.

Note that we have the set $\mathcal{F}_{s-1}$ of $2^{2^{s-1}-(s-1)-1}$ polynomials over $\mathbb{F}_2$ with degree at most $2^{s-1} - 1$ and the pairwise Lee distance at least 4. If we replace x by x^2 in any polynomial in $\mathcal{F}_{s-1}$, we get a polynomial of degree at most $2^s - 1$. We call such polynomials as β-type polynomials. We denote the set of β-type polynomials as $\mathcal{B}_s$. We have

$$\mathcal{B}_s = \left\{p(x^2) \mid p(x) \in \mathcal{F}_{s-1}\right\}.$$

It is obvious that the pairwise Lee distance between β-type polynomials is at least 4.

We define

$$\mathcal{F}_s = \mathcal{A}_s + \mathcal{B}_s = \{\alpha(x) + \beta(x) \mid \alpha(x) \in \mathcal{A}_s, \beta(x) \in \mathcal{B}_s\}, \quad s \geq 3.$$

We show that $|\mathcal{F}_s| = |\mathcal{A}_s||\mathcal{B}_s| = 2^{2^{s-1}-1} \times 2^{2^{s-1}-(s-1)-1} = 2^{2^s-s-1}$.

Consider two α-type polynomials $\alpha_1(x), \alpha_2(x) \in \mathcal{A}_s$, and two β-type polynomials $\beta_1(x), \beta_2(x) \in \mathcal{B}_s$, where either $\alpha_1(x) \neq \alpha_2(x)$ or $\beta_1(x) \neq \beta_2(x)$. We need to show that $\alpha_1(x) + \beta_1(x)$ and $\alpha_2(x) + \beta_2(x)$ are distinct, or equivalently, $\alpha_1(x) + \beta_1(x) + \alpha_2(x) + \beta_2(x) \neq 0$.

If $\beta_1(x) = \beta_2(x)$ and $\alpha_1(x) \neq \alpha_2(x)$, the claim is evident. Suppose $\beta_1(x) \neq \beta_2(x)$, then $wt_L(\beta_1(x) + \beta_2(x)) \geq 4$ and we can write $\beta_1(x) + \beta_2(x)$ as

$$\beta_1(x) + \beta_2(x) = \sum_{k\in\Omega} x^{2k}, \text{ for some } \Omega \subset \Gamma \text{ with } |\Omega| \geq 4.$$

Also, we can write

$$\alpha_1(x) + \alpha_2(x) = \sum_{k\in\Psi}\left(x^{2k} + x^{2k+1}\right), \text{ for some } \Psi \subseteq \Gamma.$$

Thus,

$$\begin{aligned}\alpha_1(x)+\alpha_2(x)+\beta_1(x)+\beta_2(x) &= \sum_{k\in\Psi}(x^{2k}+x^{2k+1})+\sum_{k\in\Omega}x^{2k}\\ &= \sum_{k\in\Psi\setminus\Omega}(x^{2k}+x^{2k+1})+\sum_{k\in\Psi\cap\Omega}(x^{2k}+x^{2k+1})\\ &\quad+\sum_{k\in\Psi\cap\Omega}x^{2k}+\sum_{k\in\Omega\setminus\Psi}x^{2k}\\ &= \sum_{k\in\Psi\setminus\Omega}(x^{2k}+x^{2k+1})+\sum_{k\in\Psi\cap\Omega}x^{2k+1}+\sum_{k\in\Omega\setminus\Psi}x^{2k}.\end{aligned}$$

Since, $\Psi\setminus\Omega, \Psi\cap\Omega, \Omega\setminus\Psi$ are three pairwise disjoint sets and $|\Omega|\neq\phi$, at least one between $\Psi\cap\Omega$ and $\Omega\setminus\Psi$ is non-empty. Consequently, $\alpha_1(x)+\beta_1(x)+\alpha_2(x)+\beta_2(x)$ is non-zero. Therefore, all the elements in $\mathcal{F}_s$ are distinct and

$$|\mathcal{F}_s| = |\mathcal{A}_s||\mathcal{B}_s| = 2^{2^{s-1}-1}\times 2^{2^{s-1}-(s-1)-1} = 2^{2^s-s-1}.$$

Thus, we have the set $\mathcal{F}_s$ of 2^{2^s-s-1} polynomials over $\mathbb{F}_2$ with degrees at most 2^s-1.
The crucial step is to show that the lowest possible Lee distance between these polynomials is at least 4. Let $\alpha_1(x)+\beta_1(x), \alpha_2(x)+\beta_2(x)$ be two distinct elements of $\mathcal{F}_s$.
Claim 2: $d_L\left(\alpha_1(x)+\beta_1(x), \alpha_2(x)+\beta_2(x)\right)\geq 4$.
If $\beta_1(x)=\beta_2(x)$, the claim is evident. Otherwise, as the above, we have

$$\alpha_1(x)+\beta_1(x)+\alpha_2(x)+\beta_2(x) = \sum_{k\in\Psi\setminus\Omega}(x^{2k}+x^{2k+1})+\sum_{k\in\Psi\cap\Omega}x^{2k+1}+\sum_{k\in\Omega\setminus\Psi}x^{2k},$$

and consequently,

$$\begin{aligned}wt_L\left[\alpha_1(x)+\beta_1(x)+\alpha_2(x)+\beta_2(x)\right] &= 2|\Psi\setminus\Omega|+|\Psi\cap\Omega|+|\Omega\setminus\Psi|\\ &= 2|\Psi\setminus\Omega|+|\Omega|\geq|\Omega|\geq 4.\end{aligned}$$

Hence,

$$d_L\left[\alpha_1(x)+\beta_1(x), \alpha_2(x)+\beta_2(x)\right] = wt_L\left[\alpha_1(x)+\beta_1(x)+\alpha_2(x)+\beta_2(x)\right]\geq 4.$$

Therefore, for $s\geq 2$, it is established that there exists 2^{2^s-s-1} polynomials over $\mathbb{F}_2$ with degrees at most 2^s-1 such that the lowest possible Lee distance between any two different polynomials is at least 4.

3. **Construction of cosets:** In this step, we construct 2^{2^s-s-1} cosets of $I=\langle u(x+1)\rangle$ in $S=\frac{R[x]}{\langle x^{2^s}+1\rangle}$ such that the Lee distance between any two distinct cosets is at least 4.
From the previous step, we have 2^{2^s-s-1} polynomials over $\mathbb{F}_2$ with degrees at most 2^s-1 and the lowest possible Lee distance between any two different polynomials is at least 4. By selecting these polynomials as coset representatives, Lemma 2.9 guarantees that the lowest possible Lee distance between two different cosets is at least 4. Furthermore, the distinctness of these cosets is established, as any two distinct polynomials $r_1(x)$ and $r_2(x)$ yield two distinct cosets of I as $r_1(x)+r_2(x)\notin I$, since $I\subset\langle u\rangle$.
Consequently, we have successfully constructed 2^{2^s-s-1} cosets of $I=\langle u(x+1)\rangle$ in $S=\frac{R[x]}{\langle x^{2^s}+1\rangle}$, ensuring that the Lee distance between any two distinct cosets is at least 4.
4. **Code formation:** Now, we unify all these cosets and denote the resulting set as C_s. The elements of $\mathcal{F}_s$ serve as coset representative polynomials for the cosets. Thus we define C_s as follows:

$$C_s = \cup_{r(x)\in\mathcal{F}_s}\left(r(x)+I\right).$$

It is well-known that distinct cosets share no common elements, and the cardinality of a coset is equal to the cardinality of the corresponding ideal. By Theorem 2.6, we have $|I| = 2^{2^s-1}$. This leads to the following:

$$|C_s| = 2^{2^s-s-1} \times 2^{2^s-1} = 2^{2^{s+1}-(s+1)-1}.$$

By Lemmas 2.8 and 2.9, it is clear that the Lee distance between any two distinct elements of C_s is at least 4. Thus, we have proved the following theorem.

Theorem 3.1. *The set C_s is a code of length 2^s over R with size $2^{2^{s+1}-(s+1)-1}$ and the minimum Lee distance 4.*

Remark 3.2. *It is obvious from the preceding construction that the polynomials in $\mathcal{A}_s$ have even Lee weight. Since $\mathcal{F}_2 = \{0, 1+x+x^2+x^3\}$. Each element in $\mathcal{F}_2$ have even Lee weight. For $s = 3$, both the α-type and β-type polynomials possess even Lee weight. Consider an α-type polynomial $\alpha(x) = \sum_{k\in\Psi}\left(x^{2k} + x^{2k+1}\right) \in \mathcal{A}_3$ and a β-type polynomial $\beta(x) = \sum_{k\in\Omega} x^{2k} \in \mathcal{B}_3$, where $\Psi, \Omega \subseteq \Gamma$. Then*

$$\alpha(x) + \beta(x) = \sum_{k\in\Psi\setminus\Omega}\left(x^{2k} + x^{2k+1}\right) + \sum_{k\in\Psi\cap\Omega} x^{2k+1} + \sum_{k\in\Omega\setminus\Psi} x^{2k},$$

and $wt_L(\alpha(x) + \beta(x)) = 2|\Psi \setminus \Omega| + |\Omega|$. This expression is always even, as $|\Omega|$ is even. Hence by induction, for $s \geq 2$, polynomials in $\mathcal{F}_s$ have even Lee weight only.

The following example serves as a clear illustration of the construction process.

Example 3.3. ***Construction of C_s for $s = 3$:***

- *Consider the ideal $I = \langle u(x+1)\rangle$ in the ring $S = \frac{R[x]}{\langle x^8+1\rangle}$. The size of I is $|I| = 2^{8-1} = 2^7$.*
- *We define the binomials $p_0^3(x) = 1 + x$, $p_1^3(x) = x^2 + x^3$, $p_2^3(x) = x^5 + x^4$ and $p_3^3(x) = x^7 + x^6$.*
- *We define the set $\mathcal{A}_3$ of α-type polynomials as follows:*

$$\mathcal{A}_3 = \Big\{0, x^2 + x^3 + 1 + x, x^5 + x^4 + 1 + x, x^7 + x^6 + 1 + x, x^5 + x^4 + x^2 + x^3, \\ x^7 + x^6 + x^2 + x^3, x^7 + x^6 + x^5 + x^4, x^7 + x^6 + x^5 + x^4 + x^2 + x^3 + 1 + x\Big\}.$$

- *Now, since $\mathcal{F}_2 = \left\{0, x^3 + x^2 + x + 1\right\}$, replacing x by x^2, we get the following set $\mathcal{B}_3$ of β-type polynomials:*

$$\mathcal{B}_3 = \left\{0, x^6 + x^4 + x^2 + 1\right\}.$$

- *The set of coset representative polynomials is:*

$$\mathcal{F}_3 = \mathcal{A}_3 + \mathcal{B}_3 = \Big\{0, x^6 + x^4 + x^2 + 1, x^2 + x^3 + 1 + x, x^3 + x + x^6 + x^4, 1 + x + x^5 \\ + x^4, x^7 + x^6 + x + 1, x^5 + x^4 + x^2 + x^3, x^7 + x^6 + x^3 + x^2, x^7 \\ + x^6 + x^5 + x^4, x^7 + x^6 + x^5 + x^4 + x^2 + x^3 + 1 + x, x^6 + x^5 \\ + x^2 + x, x^7 + x^4 + x^2 + x, x^6 + x^5 + x^3 + 1, x^7 + x^4 + x^3 + 1, \\ x^7 + x^5 + x^2 + 1, x^7 + x^5 + x^3 + x\Big\},$$

  where $|\mathcal{F}_3| = |\mathcal{A}_3||\mathcal{B}_3| = 8 \times 2 = 16 = 2^4$.
- *The set of cosets of I in S such that the lowest possible Lee distance between any two different cosets is 4, is $\left\{r(x) + I \mid r(x) \in \mathcal{F}_3\right\}$.*
- *The code $C_3 = \cup_{r(x)\in\mathcal{F}_3} r(x) + I$ is a code of length 8 over R with size $|C_3| = 2^4 \times 2^7 = 2^{11}$ and the lowest possible Lee distance 4.*

4. Linearity and Optimality of the code C_s over R

4.1. Linearity of the code C_s over R

In this subsection, we prove the linearity of the code C_s over the ring R. Since the ring S is an R-module and $C_s \subseteq S$, we claim that C_s is an R-submodule of S. First we prove two lemmas.

Lemma 4.1. *The sum of two α-type polynomials is also an α-type polynomial.*

Proof. Let $\alpha_1(x), \alpha_2(x) \in \mathcal{A}_s$, which can be expressed as:

$$\alpha_1(x) = \sum_{k\in\Psi} \left(x^{2k} + x^{2k+1}\right), \text{ where } \Psi \subseteq \Gamma \text{ and } |\Psi| \text{ is even,}$$

and

$$\alpha_2(x) = \sum_{k\in\Omega} \left(x^{2k} + x^{2k+1}\right), \text{ where } \Omega \subseteq \Gamma \text{ and } |\Omega| \text{ is even.}$$

Their sum is given by

$$\begin{aligned}
\alpha_1(x) + \alpha_2(x) &= \sum_{k\in\Psi} \left(x^{2k} + x^{2k+1}\right) + \sum_{k\in\Omega} \left(x^{2k} + x^{2k+1}\right) \\
&= \sum_{k\in\Psi\setminus\Omega} \left(x^{2k} + x^{2k+1}\right) + \sum_{k\in\Psi\cap\Omega} \left(x^{2k} + x^{2k+1}\right) \\
&\quad + \sum_{k\in\Omega\cap\Psi} \left(x^{2k} + x^{2k+1}\right) + \sum_{k\in\Omega\setminus\Psi} \left(x^{2k} + x^{2k+1}\right) \\
&= \sum_{k\in\Psi\setminus\Omega} \left(x^{2k} + x^{2k+1}\right) + \sum_{k\in\Omega\setminus\Psi} \left(x^{2k} + x^{2k+1}\right).
\end{aligned}$$

Since the sets $\Psi \setminus \Omega$ and $\Omega \setminus \Psi$ are disjoint, we can rewrite the sum as:

$$\alpha_1(x) + \alpha_2(x) = \sum_{k\in\Psi\setminus\Omega\cup\Omega\setminus\Psi} \left(x^{2k} + x^{2k+1}\right).$$

Here, $|\Psi \setminus \Omega \cup \Omega \setminus \Psi| = |\Psi \setminus \Omega| + |\Omega \setminus \Psi| = |\Psi| + |\Omega| - 2|\Psi \cap \Omega|$, which is even since $|\Psi|$ and $|\Omega|$ are even. Therefore, $\alpha_1(x) + \alpha_2(x)$ is also an α-type polynomial. □

Lemma 4.2. *Let $\mathcal{F}_2 = \{0, 1 + x + x^2 + x^3\}$ and for $s \geq 3$, let $\mathcal{F}_s = \mathcal{A}_s + \mathcal{B}_s$ be the set of polynomials as described in Section 3, where $\mathcal{A}_s$ is the set of α-type polynomials and $\mathcal{B}_s$ is the set of β-type polynomials. If $r_1(x), r_2(x) \in \mathcal{F}_s$, then $r_1(x) + r_2(x) \in \mathcal{F}_s$.*

Proof. For $\mathcal{F}_2$, the result is obvious. Now for $s = 3$, if $\alpha_1(x), \alpha_2(x) \in \mathcal{A}_3$, then by Lemma 4.1, $\alpha_1(x) + \alpha_2(x) \in \mathcal{A}_3$. Since $\mathcal{B}_3$ is obtained from $\mathcal{F}_2$ by replacing x from x^2, therefore for $\beta_1(x), \beta_2(x) \in \mathcal{B}_3, \beta_1(x) + \beta_2(x) \in \mathcal{B}_3$.

If $r_1(x) = \alpha_1(x) + \beta_1(x)$, $r_2(x) = \alpha_2(x) + \beta_2(x) \in \mathcal{F}_3$, where $\alpha_1(x), \alpha_2(x) \in \mathcal{A}_3$ and $\beta_1(x), \beta_2(x) \in \mathcal{B}_3$. Then

$$\begin{aligned}
r_1(x) + r_2(x) &= \alpha_1(x) + \beta_1(x) + \alpha_2(x) + \beta_2(x) \\
&= \big(\alpha_1(x) + \alpha_2(x)\big) + \big(\beta_1(x) + \beta_2(x)\big).
\end{aligned}$$

As $\alpha_1(x) + \alpha_2(x) \in \mathcal{A}_3$ and $\beta_1(x) + \beta_2(x) \in \mathcal{B}_3$, thus $r_1(x) + r_2(x) \in \mathcal{F}_3$. Similarily by induction, we can prove the result for $s \geq 4$. □

Theorem 4.3. *C_s is an R-submodule of S.*

Proof. We need to establish the following two conditions:
(i) If $c_1(x), c_2(x) \in C_s$, then $c_1(x) + c_2(x) \in C_s$.
(ii) If $c(x) \in C_s$, then $rc(x) \in C_s$ for every $r \in R$.

Suppose $c_1(x), c_2(x) \in C_s$. Then $c_1(x) \in r_1(x)+I$ and $c_2(x) \in r_2(x)+I$, for some $r_1(x), r_2(x) \in \mathcal{F}_s$. Consequently, $c_1(x) = r_1(x) + t_1(x), c_2(x) = r_2(x) + t_2(x)$, where $t_1(x), t_2(x) \in I$. Thus

$$c_1(x) + c_2(x) = r_1(x) + t_1(x) + r_2(x) + t_2(x) = r_1(x) + r_2(x) + t_1(x) + t_2(x).$$

By the previous lemma, $r_1(x) + r_2(x) \in \mathcal{F}_s$, and since I is an ideal of S, $t_1(x) + t_2(x) \in I$, which implies that $r_1(x) + r_2(x) + t_1(x) + t_2(x) \in C_s$. Hence, Condition (i) is satisfied.

By the definition, it is clear that $0 \in C_s$. Since R has three non-zero elements, namely $1, u, 1 + u$, for Condition (ii), we show that for every $c(x) \in C_s, uc(x), (1 + u)c(x) \in C_s$. Let $c(x) = r(x) + t(x) \in C_s$, where $r(x) = \sum_{j=0}^{2^s-1} r_j x^j \in \mathcal{F}_s$ and $t(x) = u \sum_{j=0}^{2^s-1} t_j x^j \in I$, for some $r_j, t_j \in \mathbb{F}_2$. Then

$$\begin{aligned} uc(x) &= ur(x) + ut(x) \\ &= u \sum_{j=0}^{2^s-1} r_j x^j + u^2 \sum_{j=0}^{2^s-1} t_j x^j \\ &= u \sum_{j=0}^{2^s-1} r_j x^j = ur(x). \end{aligned}$$

From Remark 3.2, $wt_L(r(x))$ is even, implying that $wt_H(ur(x))$ is even. By Remark 2.7, I contains all the elements of $\langle u \rangle$ with even Hamming weight, therefore $ur(x) \in I$. Since $I \subseteq C_s$, thus $ur(x) = uc(x) \in C_s$. Also, we have

$$\begin{aligned} (1 + u)c(x) &= (1 + u)\,(r(x) + t(x)) \\ &= (1 + u) \sum_{j=0}^{2^s-1} r_j x^j + (1 + u)u \sum_{j=0}^{2^s-1} t_j x^j \\ &= \sum_{j=0}^{2^s-1} r_j x^j + u \sum_{j=0}^{2^s-1} r_j x^j + u \sum_{j=0}^{2^s-1} t_j x^j \\ &= r(x) + ur(x) + t(x). \end{aligned}$$

Now, since $ur(x) \in I$ and $t(x) \in I$, this implies that $ur(x) + t(x) \in I$, as I is an ideal of S. Since, $r(x) \in \mathcal{F}_s$ and $ur(x) + t(x) \in I$, thus $r(x) + ur(x) + t(x) \in C_s$. Hence, for every $c(x) \in C_s, (1 + u)c(x) \in C_s$. Therefore, condition (ii) is satisfied, establishing that C_s is an R-submodule of the module $S = \frac{R[x]}{\langle x^{2^s}+1 \rangle}$ over the ring R. □

4.2. Optimality of the code C_s over R

First we recall the sphere-packing bound for the Lee metric given by Golomb and Welch (1970).

Theorem 4.4. *(Sphere packing bound [12]): If C is a code of length n over an alphabet of size q with r-error-correcting capability, where errors are measured in the Lee metric, then the size of C can be maximum*

$$\frac{q^n}{\sum_{k\geq 0} 2^k \binom{n}{k}\binom{r}{k}}.$$

If C is a linear code over R of type $4^{m_1} 2^{m_2}$ with length 2^s and the minimum Lee distance 4, then its error-correcting capability is $\left\lfloor \frac{4-1}{2} \right\rfloor = 1$. The rank of C is $m_1 + m_2$, and the following bound holds:

$$2^{2m_1+m_2} \leq \frac{4^{2^s}}{1 + 2 \times 2^s} = \frac{2^{2^{s+1}}}{1 + 2^{s+1}} < \frac{2^{2^{s+1}}}{2^{s+1}} = 2^{2^{s+1}-(s+1)}.$$

Consequently, we have $2m_1 + m_2 < 2^{s+1} - (s+1)$.

Since C_s is a linear code of length 2^s, size $2^{2^{s+1}-(s+2)}$ and the minimum Lee distance 4 over the ring R, thus it is an optimal linear code with respect to the sphere packing bound over R for every $s \geq 2$. We list parameters and generator of C_s for some values of s in Table 1. We summarize the above discussion in the following result.

Table 1: Parameters and generator matrices of C_s for some integers $s \geq 2$

s	length	size	Generator matrix	(m_1, m_2)
2	4	2^4	$\begin{bmatrix} 1 & 1 & 1 & 1 \\ u & u & 0 & 0 \\ u & 0 & u & 0 \end{bmatrix}$	$(1, 2)$
3	8	2^{11}	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ u & u & 0 & 0 & 0 & 0 & 0 & 0 \\ u & 0 & 0 & u & 0 & 0 & 0 & 0 \\ u & 0 & 0 & 0 & 0 & u & 0 & 0 \end{bmatrix}$	$(4, 3)$
4	16	2^{26}	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ u & u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ u & 0 & 0 & u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ u & 0 & 0 & 0 & 0 & u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ u & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	$(11, 4)$

Theorem 4.5. *The code C_s is an optimal linear code over R with length 2^s, size $2^{2^{s+1}-s-2}$, and Lee distance 4, with respect to the Lee sphere packing bound, for every $s \geq 2$.*

5. Lee weight distribution of C_s

Let the notations be same as in the previous section. In this section, we compute the Lee weight distribution of C_s. Since C_s is a code of length 2^s over R, and the minimum and the maximum Lee weight of elements of R are 0 and 2, respectively. The maximum possible Lee weight of any element of C_s is 2^{s+1}. Let $L_i(C_s)$ represent the number of elements in C_s with Lee weight i, where $0 \leq i \leq 2^{s+1}$. The set $\{L_i(C_s)\}_{i=0}^{i=2^{s+1}}$ represents the Lee weight distribution of C_s.

Now, $C_s = \cup_{r(x)\in\mathcal{F}_s} (r(x) + I)$ is a union of certain cosets of the ideal $I = \langle u(x+1)\rangle$ in the ring $S = \frac{R[x]}{\langle x^{2^s}+1\rangle}$, where the coset representatives are chosen as polynomials from the set $\mathcal{F}_s$. Our initial focus is on enumerating these polynomials according to their Lee weights. We denote the number of polynomials in $\mathcal{F}_s$ with Lee weight w by $N_w^{(s)}$. It follows from Remark 3.2 that these polynomials have even Lee weight. Hence, the Lee weights of these polynomials can be of the form 4λ or $4\lambda + 2$ for $0 \leq \lambda \leq 2^{s-2}$. These polynomials arise as the sum of α-type and β-type polynomials. Suppose $r(x) = \alpha(x) + \beta(x)$ is one such polynomial. Then we show that $wt_L(r(x)) \geq wt_L(\beta(x))$.

Let $\alpha(x) = \sum_{k\in\Psi}\left(x^{2k} + x^{2k+1}\right)$ and $\beta(x) = \sum_{k\in\Omega} x^{2k}$, where $\Psi, \Omega \subseteq \Gamma$ and $|\Psi|, |\Omega|$ are even. Then the polynomial $r(x)$ is:

$$\begin{aligned} r(x) &= \sum_{k\in\Psi}\left(x^{2k} + x^{2k+1}\right) + \sum_{k\in\Omega} x^{2k} \\ &= \sum_{k\in\Psi\setminus\Omega}\left(x^{2k} + x^{2k+1}\right) + \sum_{k\in\Psi\cap\Omega}\left(x^{2k} + x^{2k+1}\right) + \sum_{k\in\Psi\cap\Omega} x^{2k} + \sum_{k\in\Omega\setminus\Psi} x^{2k} \\ &= \sum_{k\in\Psi\setminus\Omega}\left(x^{2k} + x^{2k+1}\right) + \sum_{k\in\Psi\cap\Omega} x^{2k+1} + \sum_{k\in\Omega\setminus\Psi} x^{2k}. \end{aligned}$$

Consequently,

$$wt_L(r(x)) = 2|\Psi \setminus \Omega| + |\Psi \cap \Omega| + |\Omega \setminus \Psi| = 2|\Psi \setminus \Omega| + |\Omega| \geq wt_L(\beta(x)).$$

In the subsequent lemma, we compute the number of polynomials with Lee weight 4λ in $\mathcal{F}_s$ for $0 \leq \lambda \leq 2^{s-2}$.

Lemma 5.1. *The number of polynomials with Lee weight 4λ in $\mathcal{F}_s$, for $s \geq 3$ is given by*

$$N_{4\lambda}^{(s)} = \sum_{\mu=0}^{\lambda} N_{4\mu}^{(s-1)}\binom{2^{s-1} - 4\mu}{2\lambda - 2\mu}\sum_{j=0}^{2\mu}\binom{4\mu}{2j} + \sum_{\sigma=0}^{\lambda-1} N_{4\sigma+2}^{(s-1)}2^{4\sigma+1}\binom{2^{s-1} - (4\sigma + 2)}{2\lambda - (2\sigma + 1)},$$

where $0 \leq \lambda \leq 2^{s-2}$. Additionally, $N_0^{(2)} = 1$, $N_4^{(2)} = 1$, and $N_j^{(2)} = 0$ for other values of j.

Proof. Let $r(x) = \alpha(x) + \beta(x)$ be a polynomial in $\mathcal{F}_s$ with Lee weight 4λ for some $0 \leq \lambda \leq 2^{s-2}$, where $\alpha(x) = \sum_{k\in\Psi}\left(x^{2k} + x^{2k+1}\right) \in \mathcal{A}_s$ is an α-type polynomial for some $\Psi \subseteq \Gamma$ with even cardinality and $\beta(x)$ is a β-type polynomial. It is clear from the previous discussion that $\beta(x)$ has even Lee weight and $wt_L(\beta(x)) \leq wt_L(r(x))$. Thus $\beta(x)$ can have Lee weight 4μ for some $0 \leq \mu \leq \lambda$, or $4\sigma + 2$ for some $0 \leq \sigma \leq \lambda - 1$.

Now, if $wt_L(\beta(x)) = 4\mu$, where $\beta(x) = \sum_{k\in\Omega} x^{2k}$, $\Omega \subseteq \Gamma$ and $|\Omega| = 4\mu$, then

$$r(x) = \alpha(x) + \beta(x) = \sum_{k\in\Psi\setminus\Omega}\left(x^{2k} + x^{2k+1}\right) + \sum_{k\in\Psi\cap\Omega} x^{2k+1} + \sum_{k\in\Omega\setminus\Psi} x^{2k},$$

and

$$wt_L(r(x)) = 2|\Psi \setminus \Omega| + |\Omega| = 2|\Psi \setminus \Omega| + 4\mu = 4\lambda.$$

Hence, $|\Psi \setminus \Omega| = 2\lambda - 2\mu$.

Since, $|\Gamma| = 2^{s-1}$ and $|\Omega| = 4\mu$, there are $\binom{2^{s-1}-4\mu}{2\lambda-2\mu}$ choices for $\Psi \setminus \Omega$. Also, since $|\Psi|$ and $|\Psi \setminus \Omega|$ are even, $|\Psi \cap \Omega|$ must also be even. Thus, there are $\binom{4\mu}{0} + \binom{4\mu}{2} + \ldots + \binom{4\mu}{4\mu} = \sum_{j=0}^{2\mu}\binom{4\mu}{2j}$ choices for $\Psi \cap \Omega$. Therefore, for a fixed $\beta(x)$ of Lee weight 4μ, there are $\binom{2^{s-1}-4\mu}{2\lambda-2\mu}\sum_{j=0}^{2\mu}\binom{4\mu}{2j}$ choices for $\alpha(x)$.

Similarily, if $wt_L(\beta(x)) = 4\sigma+2$, where $\beta(x) = \sum_{k\in\Omega} x^{2k}$, $\Omega \subseteq \Gamma$ and $|\Omega| = 4\sigma+2$, then $wt_L(r(x)) = 2|\Psi\setminus\Omega|+|\Omega|$ implies $|\Psi \setminus \Omega| = 2\lambda - (2\sigma + 1)$, which is odd and there are $\binom{2^{s-1}-(4\sigma+2)}{2\lambda-(2\sigma+1)}$ choices for $\Psi \setminus \Omega$. Furthermore, $|\Psi \cap \Omega|$ must be odd. Therefore, there are $\binom{4\sigma+2}{1} + \binom{4\sigma+2}{3} + \ldots + \binom{4\sigma+2}{4\sigma+1} = 2^{4\sigma+1}$ choices for $\Psi \cap \Omega$. Thus, for a fixed $\beta(x)$ of Lee weight $4\sigma + 2$, there are $2^{4\sigma+1}\binom{2^{s-1}-(4\sigma+2)}{2\lambda-(2\sigma+1)}$ choices for $\alpha(x)$.

Now, since there are $N_{4\mu}^{(s-1)}$ β-type polynomials of Lee weight 4μ and $N_{4\sigma+2}^{(s-1)}$ β-type polynomials of Lee weight $4\sigma + 2$ in $\mathcal{B}_s$, the number of coset representative polynomials with Lee weight 4λ is given by

$$N_{4\lambda}^{(s)} = \sum_{\mu=0}^{\lambda} N_{4\mu}^{(s-1)}\binom{2^{s-1} - 4\mu}{2\lambda - 2\mu}\sum_{j=0}^{2\mu}\binom{4\mu}{2j} + \sum_{\sigma=0}^{\lambda-1} N_{4\sigma+2}^{(s-1)}2^{4\sigma+1}\binom{2^{s-1} - (4\sigma + 2)}{2\lambda - (2\sigma + 1)},$$

where $0 \leq \lambda \leq 2^{s-2}$. □

Similarily, we can find the number of polynomials with Lee weight $4\lambda + 2$ in $\mathcal{F}_s$. The proof of the next lemma is quite similar to the previous one and we skip the proof.

Lemma 5.2. *The number of polynomials with Lee weight $4\lambda + 2$ in $\mathcal{F}_s$, for $s \geq 3$ is given by*

$$N^{(s)}_{4\lambda+2} = \sum_{\mu=0}^{\lambda} N^{(s-1)}_{4\mu} \binom{2^{s-1} - 4\mu}{2\lambda - (2\mu - 1)} \sum_{j=0}^{2\mu-1} \binom{4\mu}{2j+1} + \sum_{\sigma=0}^{\lambda} N^{(s-1)}_{4\sigma+2} 2^{4\sigma+1} \binom{2^{s-1} - (4\sigma + 2)}{2\lambda - 2\sigma},$$

where $0 \leq \lambda \leq 2^{s-2} - 1$. Additionally, $N^{(2)}_0 = 1$, $N^{(2)}_4 = 1$, and $N^{(2)}_j = 0$ for other values of j.

Next, we determine the number of codewords of a particular weight in a coset $r(x)+\langle u(x+1)\rangle$ of $I = \langle u(x+1)\rangle$, where $r(x) \in \mathcal{F}_s$. We utilize Theorem 2.6 and Remark 3.2, which establish that elements of I have even Hamming weight and $r(x)$ has even Lee weight. Let $\Delta = \{0, 1, 2, \ldots, 2^s - 1\}$ and $r(x) = \sum_{j\in V} x^j$, where $V \subseteq \Delta$ and $|V|$ is even.

Consider an arbitrary element $r(x) + t(x) \in r(x) + \langle u(x + 1)\rangle$, where $t(x) = u \sum_{j\in W} x^j$, for some $W \subseteq \Delta$ and $|W|$ even. Then

$$\begin{aligned} r(x) + t(x) &= \sum_{j\in V} x^j + u \sum_{j\in W} x^j \\ &= \sum_{j\in V\setminus W} x^j + \sum_{j\in V\cap W} x^j + u \sum_{j\in V\cap W} x^j + u \sum_{j\in W\setminus V} x^j \\ &= \sum_{j\in V\setminus W} x^j + (1 + u) \sum_{j\in V\cap W} x^j + u \sum_{j\in W\setminus V} x^j. \end{aligned}$$

Since, $wt_L(1) = wt_L(1 + u) = 1$ and $wt_L(u) = 2$, we have

$$\begin{aligned} wt_L(r(x) + t(x)) &= |V \setminus W| + |V \cap W| + 2|W \setminus V| \\ &= |V| + 2|W \setminus V| \\ &\geq |V| = wt_L(r(x)). \end{aligned}$$

We note the above observation as the following remark.

Remark 5.3. *The Lee weight of any element of the coset $r(x) + \langle u(x + 1)\rangle$, where $r(x) \in \mathcal{F}_s$, is greater than or equal to the Lee weight of the polynomial $r(x)$ and also the Lee weight of any element of the coset is even.*

In the following lemma, we compute the Lee weight distribution of a particular coset of I.

Lemma 5.4. *Let $I_{r(x)} = r(x) + I$ be a coset of $I = \langle u(x + 1)\rangle$ in S, where $r(x) = \sum_{j\in V} x^j$, $V \subseteq \Delta$ and $|V| = 4\lambda$ for some $0 \leq \lambda \leq 2^{s-2}$. Then there are $\binom{2^s-4\lambda}{2w-2\lambda} \sum_{j=0}^{2\lambda} \binom{4\lambda}{2j}$ elements in $I_{r(x)}$ with Lee weight $4w$ and $\binom{2^s-4\lambda}{2w'-2\lambda+1} \sum_{j=0}^{2\lambda-1} \binom{4\lambda}{2j+1}$ elements with Lee weight $4w' + 2$, where $\lambda \leq w \leq 2^{s-1}$ and $\lambda \leq w' \leq 2^{s-1} - 1$.*

Proof. Consider an arbitrary element $r(x) + t(x)$ of $I_{r(x)}$ with Lee weight $4w$, where $t(x) \in I$. It is quite obvious from the previous discussion that $wt_L(r(x) + t(x)) \geq wt_L(r(x)) = 4\lambda$, thus $\lambda \leq w \leq 2^{s-1}$. From Theorem 2.6, we know that $t(x)$ has even Hamming weight. Thus, we can write $t(x) = u \sum_{j\in W} x^j$, where $W \subseteq \Delta$ and $|W|$ is even. Then

$$wt_L(r(x) + t(x)) = |V| + 2|W \setminus V| = 4\lambda + 2|W \setminus V| = 4w.$$

Thus, $|W \setminus V| = 2w - 2\lambda$. Clearly, there are $\binom{2^s-4\lambda}{2w-2\lambda}$ choices for $W \setminus V$ and $\binom{4\lambda}{0} + \binom{4\lambda}{2} + \ldots + \binom{4\lambda}{4\lambda} = \sum_{j=0}^{2\lambda} \binom{4\lambda}{2j}$ choices for $W \cap V$. Therefore, there are $\binom{2^s-4\lambda}{2w-2\lambda} \sum_{j=0}^{2\lambda} \binom{4\lambda}{2j}$ choices for W. Hence, the number of elements in $I_{r(x)}$ with Lee weight $4w$ is $\binom{2^s-4\lambda}{2w-2\lambda} \sum_{j=0}^{2\lambda} \binom{4\lambda}{2j}$.

Next, suppose $r(x) + h(x)$ is an element of $I_{r(x)}$ with Lee weight $4w' + 2$, where $h(x) = u \sum_{j \in Z} x^j$ for some $Z \subseteq \Delta$ and $|Z|$ is even. Then, as the above, we have

$$wt_L(r(x) + h(x)) = 4\lambda + 2|Z \setminus V| = 4w' + 2.$$

Thus, $|Z \setminus V| = 2w' - 2\lambda + 1$, which is odd. As $|Z|$ is even, $|Z \cap V|$ must also be odd. There are $\binom{2^s-4\lambda}{2w'-2\lambda+1}$ choices for $Z \setminus V$ and $\binom{4\lambda}{1} + \binom{4\lambda}{3} + \ldots + \binom{4\lambda}{4\lambda-1} = \sum_{j=0}^{2\lambda-1} \binom{4\lambda}{2j+1}$choices for $Z \cap V$. Hence, the number of elements in $I_{r(x)}$ with Lee weight $4w' + 2$ is $\binom{2^s-4\lambda}{2w'-2\lambda+1} \sum_{j=0}^{2\lambda-1} \binom{4\lambda}{2j+1}$. □

In a similar way as above, we get the following lemma and we skip the proof here.

Lemma 5.5. *Let $I_{r(x)} = r(x) + \langle u(x+1) \rangle$ be a coset of I in S, where $r(x) = \sum_{j \in V} x^j$, $V \subseteq \Delta$ and $|V| = 4\lambda + 2$ for some $0 \le \lambda \le 2^{s-2} - 1$, then there are $2^{4\lambda+1}\binom{2^s-(4\lambda+2)}{2w-(2\lambda+1)}$ elements in $I_{r(x)}$ with Lee weight $4w$, where $\lambda + 1 \le w \le 2^{s-1}$ and $2^{4\lambda+1}\binom{2^s-(4\lambda+2)}{2w'-2\lambda}$ elements with Lee weight $4w' + 2$, where $\lambda \le w' \le 2^{s-1} - 1$.*

Now we prove the main theorem of this section.

Theorem 5.6. *The Lee weight distribution of C_s is given by*

1. $L_1(C_s) = L_2(C_s) = L_3(C_s) = 0$.
2. $L_j(C_s) = 0\ \forall j > 2^{s+1}$.
3. $L_{4j+1}(C_s) = L_{4j+3}(C_s) = 0$, *for every* $j \in \mathbb{N}$.
4. $L_{4d}(C_s) = \sum_{l=0}^{d} N_{4l}^{(s)} \binom{2^s-4l}{2d-2l} \sum_{j=0}^{2l} \binom{4l}{2j} + \sum_{l'=0}^{d-1} N_{4l'+2}^{(s)} 2^{4l'+1} \binom{2^s-(4l'+2)}{2d-(2l'+1)}$, *where* $0 \le d \le 2^{s-1}$.
5. $L_{4d'+2}(C_s) = \sum_{l=0}^{d'} N_{4l}^{(s)} \binom{2^s-4l}{2d'-2l+1} \sum_{j=0}^{2l-1} \binom{4l}{2j+1} + \sum_{l'=0}^{d'} N_{4l'+2}^{(s)} 2^{4l'+1} \binom{2^s-(4l'+2)}{2d'-(2l')}$, *where* $1 \le d' \le 2^{s-1} - 1$.

Proof. From Lemmas 5.4 and 5.5, it is quite obvious that in each coset, every element has even Lee weight. Thus the Lee weight of every element of C_s will be even. Hence, $L_{4j+1}(C_s) = L_{4j+3}(C_s) = 0$, for every $j \in \mathbb{N}$. It is obvious from the construction that $d_L(C_s) = 4$, implying $L_1(C_s) = L_2(C_s) = L_3(C_s) = 0$. Also, since the maximum posssible Lee weight of any element of C_s is 2^{s+1}, thus $L_j(C_s) = 0\ \forall j > 2^{s+1}$. Our work is to compute $L_{4d}(C_s)$, for $0 \le d \le 2^{s-1}$ and $L_{4d'+2}(C_s)$, for $1 \le d' \le 2^{s-1} - 1$.

Suppose $r(x) + t(x)$ is an element in C_s with Lee weight $4d$, where $r(x) \in \mathcal{F}_s$ and $t(x) \in I$. By Remark 5.3, elements of Lee weight $4d$ in C_s lie in cosets of I having coset representative polynomials as elements of $\mathcal{F}_s$ with Lee weight at most $4d$. That means $r(x)$ can have Lee weight $4l$, for some $0 \le l \le d$ or $4l' + 2$, for some $0 \le l' \le d - 1$. Thus, by using Lemmas 5.4 and 5.5, the number of elements in C_s of Lee weight $4d$ is given by

$$L_{4d}(C_s) = \sum_{l=0}^{d} N_{4l}^{(s)} \binom{2^s - 4l}{2d - 2l} \sum_{j=0}^{2l} \binom{4l}{2j} + \sum_{l'=0}^{d-1} N_{4l'+2}^{(s)} 2^{4l'+1} \binom{2^s - (4l' + 2)}{2d - (2l' + 1)},$$

where $0 \le d \le 2^{s-1}$. Similarily, the number of codewords in C_s with Lee weight $4d' + 2$ is given by

$$L_{4d'+2}(C_s) = \sum_{l=0}^{d'} N_{4l}^{(s)} \binom{2^s - 4l}{2d' - 2l + 1} \sum_{j=0}^{2l-1} \binom{4l}{2j + 1} + \sum_{l'=0}^{d'} N_{4l'+2}^{(s)} 2^{4l'+1} \binom{2^s - (4l' + 2)}{2d' - (2l')},$$

where $1 \le d' \le 2^{s-1} - 1$. □

We illustrate the above theorem by the following example.

Example 5.7. ***Lee weight distribution of the code*** C_3***:*** *We have*

$$\mathcal{F}_2 = \{0, 1 + x + x^2 + x^3\}.$$

Thus,

$$N_0^{(2)} = 1, N_4^{(2)} = 1, \text{ and } N_j^{(2)} = 0, \text{ for other values of } j.$$

For computing the Lee weight distribution of the code C_3*, first we compute* $N_{4\lambda}^{(3)}$*, for* $0 \le \lambda \le 2$ *and* $N_{4\lambda'+2}^{(3)}$*, for* $0 \le \lambda' \le 1$.

From Lemma 5.1,

$$N_{4\lambda}^{(3)} = \sum_{\mu=0}^{\lambda} N_{4\mu}^{(2)} \binom{2^2 - 4\mu}{2\lambda - 2\mu} \sum_{j=0}^{2\mu} \binom{4\mu}{2j} + \sum_{\sigma=0}^{\lambda-1} N_{4\sigma+2}^{(2)} 2^{4\sigma+1} \binom{2^2 - (4\sigma+2)}{2\lambda - (2\sigma+1)}.$$

This gives,

$$N_0^{(3)} = 1, \quad N_4^{(3)} = 14, \text{ and } N_8^{(3)} = 1.$$

From Lemma 5.2,

$$N_{4\lambda'+2}^{(3)} = \sum_{\mu=0}^{\lambda'} N_{4\mu}^{(2)} \binom{2^2 - 4\mu}{2\lambda' - (2\mu - 1)} \sum_{j=0}^{2\mu-1} \binom{4\mu}{2j+1} + \sum_{\sigma=0}^{\lambda'} N_{4\sigma+2}^{(2)} 2^{4\sigma+1} \binom{2^2 - (4\sigma+2)}{2\lambda' - (2\sigma)}.$$

Consequently,

$$N_2^{(3)} = 0, \text{ and } N_6^{(3)} = 0.$$

Hence,

$$N_0^{(3)} = 1, \quad N_4^{(3)} = 14, \quad N_8^{(3)} = 1, \text{ and } N_j^{(3)} = 0, \text{ for other values of } j.$$

Now, by Theorem 5.6, we have

$$L_{4d}(C_3) = \sum_{l=0}^{d} N_{4l}^{(3)} \binom{2^3 - 4l}{2d - 2l} \sum_{j=0}^{2l} \binom{4l}{2j} + \sum_{l'=0}^{d-1} N_{4l'+2}^{(3)} 2^{4l'+1} \binom{2^3 - (4l'+2)}{2d - (2l'+1)},$$

where $0 \le d \le 4$ *and*

$$L_{4d'+2}(C_3) = \sum_{l=0}^{d'} N_{4l}^{(3)} \binom{2^3 - 4l}{2d' - 2l + 1} \sum_{j=0}^{2l-1} \binom{4l}{2j+1} + \sum_{l'=0}^{d'} N_{4l'+2}^{(3)} 2^{4l'+1} \binom{2^3 - (4l'+2)}{2d' - (2l')},$$

where $1 \le d' \le 3$*. This gives us*

$$L_0(C_3) = 1, \quad L_4(C_3) = 140, \quad L_8(C_3) = 870, \quad L_{12}(C_3) = 140, \quad L_{16}(C_3) = 1,$$

and

$$L_6(C_3) = 448, \quad L_{10}(C_3) = 448, \quad L_{14}(C_3) = 0.$$

Thus from Theorem 5.6, the Lee weight distribution of C_3 *is given by:*

1. $L_1(C_3) = L_2(C_3) = L_3(C_3) = 0$.
2. $L_j(C_3) = 0 \; \forall j > 16$.
3. $L_{4j+1}(C_3) = L_{4j+3}(C_3) = 0$, *for every* $j \in \mathbb{N}$.
4. $L_0(C_3) = 1, \quad L_4(C_3) = 140, \quad L_8(C_3) = 870, \quad L_{12}(C_3) = 140, \quad L_{16}(C_3) = 1$.
5. $L_6(C_3) = 448, \quad L_{10}(C_3) = 448, \quad L_{14}(C_3) = 0$.

6. Conclusion

In this work, we have constructed an infinite family of linear codes over $\mathbb{F}_2 + u\mathbb{F}_2, u^2 = 0$. We analyzed the sphere packing bound for these linear codes over $\mathbb{F}_2 + u\mathbb{F}_2$ and showed that our codes achieve optimality with respect to this bound. Moreover, we determined the complete Lee weight distribution of the codes and observed that they are even codes with respect to the Lee weight with the minimum non-zero weight of the code being 4. It would be intriguing to apply this technique to construct good codes over the rings of type $\mathbb{F}_p + u\mathbb{F}_p + \ldots + u^{k-1}\mathbb{F}_p, u^k = 0$ for different primes p and integer $k \geq 2$.

Declarations

Conflict of interest: The authors declare that they have no conflict of interest.

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