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On the elementary symmetric functions of $\{1, 1/2, \ldots, 1/n\}\backslash\{1/i\}$

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Abstract. In 1946, P. Erdős and I. Niven proved that there are only finitely many positive integers n for which one or more of the elementary symmetric functions of $1, 1/2, \ldots, 1/n$ are integers. In 2012, Y. Chen and M. Tang proved that if $n \geqslant 4$, then none of the elementary symmetric functions of $1, 1/2, \ldots, 1/n$ are integers. In this paper, we prove that if $n \geqslant 5$, then none of the elementary symmetric functions of $\{1, 1/2, \ldots, 1/n\}\backslash\{1/i\}$ are integers except for $n = i = 2$ and $n = i = 4$.

1. Introduction

In mathematics, the harmonic series is the infinite series formed by summing all positive unit fractions, that is,

$$\sum_{m=1}^{\infty} \frac{1}{m} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots,$$

which is a divergent series and plays an important role in analysis. Adding the first n terms of the harmonic series produces a partial sum, called a harmonic number and denoted H_n, that is,

$$H_n = \sum_{m=1}^{n} \frac{1}{m}.$$

In number theory, it is well-known that for any integer $n > 1$, the harmonic number H_n is not an integer (see [7, P33]).

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Let $T(n,k)$ denote the k-th elementary symmetric function of $1, 1/2, 1/3, \ldots, 1/n$. That is,

$$T(n,k) = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n} \frac{1}{i_1 i_2 \cdots i_k},$$

where k is an integer with $1 \leqslant k \leqslant n$. With this notation, we mention that $T(n,1) = H_n$. In 1946, P. Erdős and I. Niven [2] considered the integrality of $T(n,k)$ and proved that there is only a finite number of positive integers n and k with $1 \leqslant k \leqslant n$ such that $T(n,k)$ is an integer. In 2012, Y. Chen and M. Tang [1] proved that $T(n,k)$ is not an integer except for the following two cases:

$$T(1,1) = 1, \quad T(3,2) = 1 \times \frac{1}{2} + 1 \times \frac{1}{3} + \frac{1}{2} \times \frac{1}{3} = 1.$$

For more related problems on this topic, we refer the reader to see [3–6, 8, 10, 11].

Let $S(n,i,k)$ denote the k-th elementary symmetric function of $\{1, 1/2, \ldots, 1/n\} \backslash \{1/i\}$. That is,

$$S(n,i,k) = \sum_{\substack{1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n \\ i_j \neq i \text{ for } j=1,2,\ldots,k}} \frac{1}{i_1 i_2 \cdots i_k},$$

where n, i, k are integers with $1 \leqslant k < n$ and $1 \leqslant i \leqslant n$. Thus, for $n = 2$, we have

$$S(2,1,1) = \frac{1}{2}, \qquad S(2,2,1) = 1.$$

For $n = 3$, we have

$$S(3,1,1) = \frac{5}{6}, \qquad S(3,1,2) = \frac{1}{6},$$
$$S(3,2,1) = \frac{4}{3}, \qquad S(3,2,2) = \frac{1}{3},$$
$$S(3,3,1) = \frac{3}{2}, \qquad S(3,3,2) = \frac{1}{2}.$$

For $n = 4$, we have

$$S(4,1,1) = \frac{13}{12}, \qquad S(4,1,2) = \frac{3}{8}, \qquad S(4,1,3) = \frac{1}{24},$$
$$S(4,2,1) = \frac{19}{12}, \qquad S(4,2,2) = \frac{2}{3}, \qquad S(4,2,3) = \frac{1}{12},$$
$$S(4,3,1) = \frac{7}{4}, \qquad S(4,3,2) = \frac{7}{8}, \qquad S(4,3,3) = \frac{1}{8},$$
$$S(4,4,1) = \frac{11}{6}, \qquad S(4,4,2) = 1, \qquad S(4,4,3) = \frac{1}{6}.$$

In this paper, we show that $S(2,2,1) = 1$ and $S(4,4,2) = 1$ are the only cases for which $S(n,i,k)$ is an integer. We state this result as follows.

Theorem 1.1. *If n, i and k are three positive integers with $k < n$ and $i \leqslant n$, then $S(n,i,k)$ is not an integer unless $(n,i,k) = (2,2,1)$ or $(4,4,2)$.*

The paper is organized as follows. In section 2, we show several lemmas which are needed for the proof of Theorem 1.1. In section 3, we give the proof of Theorem 1.1 by using the approach developed in [1, 10].

As usual, we let $\lfloor x \rfloor$ denote the integer part of the real number x and we let θ be the Chebyshev function. Let p be a prime and v_p denote the p-adic valuation on the field $\mathbb{Q}$ of rational numbers, i.e., $v_p(a) = b$ if p^b divides a and p^{b+1} does not divide a.

2. Technical lemmas

For our proof of Theorem 1.1, we need the following three lemmas.

Lemma 2.1 (see [9, P359]). *For $x \geq 1429$, we have*

$$x - \frac{0.334x}{\ln x} < \theta(x) < x + \frac{0.021x}{\ln x}.$$

Lemma 2.2. *Let k and n be positive integers such that $1 \leqslant k < n$. Suppose that there exists a prime $p > \max\{(k+2)(k+3)/2, 3k+8\}$ satisfying that*

$$\frac{n}{k+3} < p \leqslant \frac{n}{k+1}.$$

Then $S(n,i,k)$ is not an integer for every $i = 1, 2, \dots, n$.

Proof. First of all, by the assumption, there is a prime $p > \max\{(k+2)(k+3)/2, 3k+8\}$ such that

$$\frac{n}{k+3} < p \leqslant \frac{n}{k+1},$$

that is,

$$k+1 \leqslant \frac{n}{p} < k+3.$$

Thus, $\lfloor n/p \rfloor = k+1$ or $k+2$.

Let $\mathcal{S}_p(n)$ denote the set of all integers from 1 to n that can be divisible by p, that is,

$$\mathcal{S}_p(n) = \left\{p, 2p, 3p, \dots, \left\lfloor \frac{n}{p} \right\rfloor \cdot p\right\}.$$

Now we distinguish two cases as follows.

Case 1: $i \notin \mathcal{S}_p(n)$.

Since $p \nmid i$, we can separate the terms in the sum $S(n,i,k)$ into two parts. In fact, The denominator can be divisible by p^k if it is the product of k integers in $\mathcal{S}_p(n)$, otherwise the denominator can only be divided by at most p^{k-1} because each denominator can only be divided by at most p^{k-1}, and the same applies after the common denominator. That is,

$$S(n,i,k) = \sum_{\substack{i_1,i_2,\dots,i_k \in \mathcal{S}_p(n),\\ i_1<i_2<\cdots<i_k}} \frac{1}{i_1 i_2 \cdots i_k} + \sum_{\substack{1\leqslant i_1<i_2<\cdots<i_k\leqslant n,\\ \forall i_j \neq i \text{ and } \exists i_t \notin \mathcal{S}_p(n)}} \frac{1}{i_1 i_2 \cdots i_k} = \frac{1}{p^k} T\left(\left\lfloor \frac{n}{p} \right\rfloor, k\right) + \frac{b}{p^{k-1}c},$$

where b and c are two positive integers with $p \nmid c$. Since $p > 3k+8$ and

$$T(k+1,k) = \frac{1}{(k+1)!}\left(\sum_{1\leqslant j\leqslant k+1} j\right) = \frac{k+2}{2k!},$$

$$T(k+2,k) = \frac{1}{(k+2)!}\left(\sum_{1\leqslant j<s\leqslant k+2} js\right) = \frac{(k+3)(3k+8)}{24k!},$$

we have $v_p(T(\lfloor n/p \rfloor, k)) = 0$, say

$$T\left(\left\lfloor \frac{n}{p} \right\rfloor, k\right) = \frac{d}{a} \text{ for } a, d \in \mathbb{Z} \text{ and } p \nmid ad,$$

where $\mathbb{Z}$ is the set of integers. Thus

$$S(n,i,k) = \frac{1}{p^k} T\left(\left\lfloor \frac{n}{p} \right\rfloor, k\right) + \frac{b}{p^{k-1}c} = \frac{dc + pab}{p^k ac}$$

is not an integer since $v_p(S(n,i,k)) = -k < 0$. This completes the proof of Case 1.

Case 2: $i \in \mathcal{S}_p(n)$.

In this case, we say $i = pi'$ for some $i' \in \{1, 2, \ldots, \lfloor n/p \rfloor\}$. We can also separate the terms in the sum $S(n,i,k)$ into two parts, depending on whether the denominator is the product of k integers in $\mathcal{S}_p(n) \backslash \{i\}$. That is,

$$S(n,i,k) = \sum_{\substack{i_1, i_2, \ldots, i_k \in \mathcal{S}_p(n)\backslash\{i\}, \\ i_1 < i_2 < \cdots < i_k}} \frac{1}{i_1 i_2 \cdots i_k} + \sum_{\substack{1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n, \\ \forall i_j \neq i \text{ and } \exists i_t \notin \mathcal{S}_p(n)\backslash\{i\}}} \frac{1}{i_1 i_2 \cdots i_k} = \frac{1}{p^k} S\left(\left\lfloor \frac{n}{p} \right\rfloor, i', k\right) + \frac{b}{p^{k-1}c},$$

where b and c are two positive integers with $p \nmid c$. Since $p > (k+2)(k+3)/2$ and

$$S(k+1, i', k) = \frac{1}{(k+1)!/i'},$$
$$S(k+2, i', k) = \frac{1}{(k+2)!/i'}\left(-i' + \sum_{1 \leq j \leq k+2} j\right) = \frac{(k+2)(k+3)/2 - i'}{(k+2)!/i'},$$

we have $v_p(S(\lfloor n/p \rfloor, i', k)) = 0$, say

$$S\left(\left\lfloor \frac{n}{p} \right\rfloor, i', k\right) = \frac{d}{a} \text{ for } a, d \in \mathbb{Z} \text{ and } p \nmid ad.$$

Thus

$$S(n,i,k) = \frac{1}{p^k} S\left(\left\lfloor \frac{n}{p} \right\rfloor, i', k\right) + \frac{b}{p^{k-1}c} = \frac{dc + pab}{p^k ac}$$

is not an integer since $v_p(S(n,i,k)) = -k < 0$. This completes the proof. □

Lemma 2.3. *If k and n are two integers with $n \geqslant 9$ and $n > k \geqslant \mathrm{e} \ln n + \mathrm{e}$, then $S(n,i,k)$ is not an integer for every $i = 1, 2, \ldots, n$.*

Proof. By the proof of Lemma 3 in [1], we have $T(n,k) < 1$ whenever $\mathrm{e} \ln n + \mathrm{e} \leqslant k < n$. Note that

$$T(n,k) = S(n,i,k) + \frac{1}{i} S(n,i,k-1) \text{ for } i = 1, 2, \ldots, n$$

Thus, $S(n,i,k) < T(n,k) < 1$. It follows that $S(n,i,k)$ is not an integer. □

3. Proofs

In this section, we give the proof of Theorem 1.1.

Proof. By Lemma 2.3, we may assume that $1 \leqslant k < \mathrm{e} \ln n + \mathrm{e}$. Now we distinguish three cases as follows.

Case 1: $n \geqslant 50217$.

Since $n \geqslant 50217$, we have $\frac{n}{k+1} > \frac{n}{k+3} > \frac{n}{e\ln n+e+3} \geqslant 1429$. By Lemma 2.1, we get

$$\begin{aligned}\theta\Big(\frac{n}{k+1}\Big)-\theta\Big(\frac{n}{k+3}\Big) &> \frac{n}{k+1}-\frac{0.334\cdot\frac{n}{k+1}}{\ln\left(\frac{n}{k+1}\right)}-\frac{n}{k+3}-\frac{0.021\cdot\frac{n}{k+3}}{\ln\left(\frac{n}{k+3}\right)}\\ &> \frac{n}{k+1}\Big(\frac{2}{k+3}-\frac{0.355}{\ln\left(\frac{n}{k+3}\right)}\Big)\\ &> \frac{n}{k+1}\Big(\frac{2}{\mathrm{e}\ln n+\mathrm{e}+3}-\frac{0.355}{\ln\left(\frac{n}{\mathrm{e}\ln n+\mathrm{e}+3}\right)}\Big)\\ &> 0. \qquad\qquad (\text{whenever } n \geqslant 50217)\end{aligned}$$

Hence there is a prime p such that $\frac{n}{k+3} < p \leqslant \frac{n}{k+1}$ as desired.

Since $n \geqslant 50217$, it follows that $n > (\mathrm{e}\ln n + \mathrm{e} + 3)(3\mathrm{e}\ln n + 3\mathrm{e} + 8) > (k+3)(3k+8)$ and $n > (\mathrm{e}\ln n + \mathrm{e} + 2)(\mathrm{e}\ln n + \mathrm{e} + 3)^2/2 > (k+2)(k+3)^2/2$. Thus, we have $p > \frac{n}{k+3} > \max\{(k+2)(k+3)/2, 3k+8\}$. The proof of Case 1 is completed by Lemma 2.2 immediately.

Case 2: $13543 \leqslant n \leqslant 50216$. After computer verification[1)], for any $1 \leqslant k < \mathrm{e}\ln n + \mathrm{e}$, there is a prime p, such that $\frac{n}{k+3} < p \leqslant \frac{n}{k+1}$ and $p > \max\{(k+2)(k+3)/2, 3k+8\}$.

Case 3: $2 \leqslant n \leqslant 13542$. Denote that $N = 13542$, $K = \lfloor \mathrm{e}\ln N + \mathrm{e}\rfloor = 28$.

Notice that we have the following recursive formulae for $T(n,k)$:

$$\begin{aligned}&T(1,1) = 1;\\ &T(n,1) = T(n-1,1) + \frac{1}{n}, && \text{for } 2 \leqslant n \leqslant N;\\ &T(n,k) = T(n-1,k) + \frac{1}{n}T(n-1,k-1), && \text{for } 2 \leqslant k \leqslant K,\ k < n \leqslant N;\end{aligned}$$

and

$$T(n,n) = \frac{1}{n}T(n-1,n-1), \qquad \text{for } 2 \leqslant n \leqslant N.$$

Also, let $S(1,1,1) := 0$, then we have the following recursive formulae for $S(n,i,k)$:

$$\begin{aligned}&S(n,i,1) = \begin{cases} S(n-1,i,1) + \frac{1}{n}, & \text{if } 1 \leqslant i < n,\\ T(n-1,1), & \text{if } i = n,\end{cases} && \text{for } 2 \leqslant n \leqslant N;\\ &S(n,i,k) = T(n,k) - \frac{1}{i}S(n,i,k-1), && \text{for } 2 \leqslant n \leqslant N,\ 1 \leqslant i \leqslant n,\ 2 \leqslant k < \min(n, \mathrm{e}\ln n + \mathrm{e}).\end{aligned}$$

In particular, we have

$$S(n,n,k) = T(n-1,k), \quad \text{for } 2 \leqslant n \leqslant N,\ 1 \leqslant k < \min(n, \mathrm{e}\ln n + \mathrm{e}).$$

Using computer[2)], we can verify that $S(n,i,k)$ is not an integer for $2 \leqslant n \leqslant 13542$ except for $S(2,2,1) = 1$ and $S(4,4,2) = 1$. This completes the proof of Theorem 1.1. □

The experiment is conducted mainly using a Windows 10 (64-bit) system with an Intel (R) Core (TM) i5-8600 CPU @ 3.10 GHz and 16 GB of RAM. The experimental programming language is C (gcc version 8.1.0), using GMP library (version 6.3.0), a GNU Multiple Precision arithmetic library, as the big number calculateion framework. The running time of the experiment under single process is about 540 hours.

[1)] The details can be found at `https://github.com/zsben2/esf/tree/main/python`.

[2)] A sample pseudocode can be found in the appendix Appendix A, and the details can be found at `https://github.com/zsben2/esf/tree/main/c`.

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Appendix A. Pseudocode

First, we have

$$T(n,1) = \frac{1}{n} + T(n-1,1), \tag{A.1}$$

for $2 \le n \le N$.

Because there are formulas

$$T(k,k) = \frac{1}{k}T(k-1,k-1),$$

for $2 \le k \le K$,

$$T(n,k) = \frac{1}{n}T(n-1,k-1) + T(n-1,k),$$

for $2 \le k \le K$, $k < n \le N$, and when initializing variable `T1`, there is

$$T(k-1,k) = 0$$

for $2 \le k \le K$, i.e., $T(1,2) = T(2,3) = \cdots = T(K-1,K) = 0$. So we have

$$T(n,k) = \frac{1}{n}T(n-1,k-1) + T(n-1,k) \tag{A.2}$$

for $2 \le k \le K$, $k \le n \le N$.

In addition, the following formulas hold:

$$S(n,i,1) = S(n-1,i,1) + \frac{1}{n} \tag{A.3}$$

for $2 \le n \le N$, $1 \le i < n$;

$$S(n,i,k) = T(n,k) - \frac{1}{i}S(n,i,k-1) \tag{A.4}$$

for $2 \le n \le N$, $1 \le i < n$, $2 \le k < \min\{n, \mathrm{e}\ln n + \mathrm{e}\}$;

$$S(n,n,k) = T(n-1,k). \tag{A.5}$$

for $2 \le n \le N$, $1 \le k < \min\{n, \mathrm{e}\ln n + \mathrm{e}\}$.

In the following pseudocode, `r[n]` represents $\frac{1}{n}$ for $1 \le n \le N$. `S1_1[i]` and `Snik` represent $S(n-1,i,1)$ and $S(n,i,k)$ respectively, for $2 \le n \le N$, $1 \le i \le n$, $1 \le k < \min\{n, \mathrm{e}\ln n + \mathrm{e}\}$. `T1[k]` and `T2[k]` represent $T(n-1,k)$ and $T(n,k)$ respectively, for $1 \le n \le N$, $1 \le k \le \min(n,K)$.

```
void check_Snik() {
    r = zeros(N + 1); // reciprocal
    S1_1 = zeros(N + 1); // save S(n-1, i, 1)
    T1 = zeros(K + 1); // save T(n-1, k)
    T2 = zeros(K + 1); // save T(n, k)

    r[1] = 1;
    T1[1] = 1;
    for (n = 2; n <= N; n++) {
        r[n] = 1 / n;
```

```
            // k = 1
eq1:        T2[1] = r[n] + T1[1];
            // k >= 2
            for (k = 2; k <= min(n, K); k++)
eq2:            T2[k] = T1[k-1] * r[n] + T1[k];

            max_k = (int) min(n - 1, e * log_e(n) + e); // <= K
            // 1 <= i < n
            for (i = 1; i < n; i++) {
                // k = 1
eq3:            Snik = S1_1[i] + r[n];
                print_if_integer(Snik);
                S1_1[i] = Snik; // save S(n, i, 1)

                // 2 <= k <= max_k
                for (k = 2; k <= max_k; k++) {
eq4:                Snik = T2[k] - Snik * r[i];
                    print_if_integer(Snik);
                }
            }
            // i = n, 1 <= k <= max_k
            for (k = 1; k <= max_k; k++) {
eq5:            Snik = T1[k];
                print_if_integer(Snik);
            }
            S1_1[n] = T1[1]; // save S(n, n, 1)

            // Rolling to use T1, T2 memory
            T3 = T1;
            T1 = T2;
            T2 = T3;
        }
    }
```

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