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Some remarks on "Fixed point theorems for enriched nonexpansive mappings in geodesic spaces" [Filomat 37(11): 3403-3409, 2023]

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Abstract. In this paper, we address several critical points regarding the work of J. Ali and M. Jubair (Filomat 37(11): 3403-3409, 2023). We have identified mathematically incorrect results and provide counterexamples to support our findings. Additionally, we observe that many of the results presented by J. Ali and M. Jubair [1] seem to be direct corollaries of the earlier work by R. Shukla and R. Panicker (J. Funct. Spaces 2022, Article ID 6161839, 8 pages, 2022).

1. Main Results

J. Ali and M. Jubair [1] have claimed to extend the class of enriched non-expansive mappings in CAT(0) spaces. However, R. Shukla and R. Panicker [4] had already considered enriched non-expansive mappings in hyperbolic metric spaces. As Leustean [3] demonstrated, CAT(0) spaces are indeed complete uniformly convex hyperbolic metric spaces. Furthermore, it is noteworthy that the paper by R. Shukla and R. Panicker [4] was published on May 3, 2022, while the submission date for the paper by J. Ali and M. Jubair [1] is June 6, 2022.

J. Ali and M. Jubair [1] presented Lemma 2.4, which establishes the existence of a fixed point.
Lemma 2.4. Let $\mathcal{M}$ be a nonempty closed convex subset of a complete CAT(0) space X satisfying Opial's condition. Let $\mathcal{G} : \mathcal{M} \to \mathcal{M}$ be an enriched non-expansive map. Then, $\mathcal{G}x = x$.

It should be noted that every complete CAT(0) space satisfies Opial's property, see [2]. Therefore, the assumption of Opial's condition in the statement of Lemma 2.4 is redundant. In the proof of Lemma 2.4, the authors assumed the existence of a sequence $\{\tau_n\}$ that Δ-converges to a point $x \in \mathcal{M}$ and satisfies $\lim_{n\to\infty} d(\tau_n, \mathcal{G}\tau_n) = 0$. However, it is unclear from the assumptions of the lemma or the arguments provided in its proof how such a sequence exists with the properties that it Δ-converges to a point $x \in \mathcal{M}$ and satisfies $\lim_{n\to\infty} d(\tau_n, \mathcal{G}\tau_n) = 0$.

J. Ali and M. Jubair established Theorem 3.1 in their paper [1], which constitutes the main result of their study. For the convenience of the readers, the theorem is stated as follows:

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Theorem 3.1. Let $\mathcal{M}$ be a nonempty bounded closed convex subset of a complete CAT(0) space X and $\mathcal{G} : \mathcal{M} \to \mathcal{M}$ be enriched non-expansive mapping. Then the set $F(\mathcal{G})$ is nonempty.

It can be seen that Theorem 3.1 in [1] is essentially a corollary of [4, Theorem 22].

Authors in [1] considered the modified/simplified Mann iteration process as follows:
Let $\mathcal{M}$ be a convex subset of a CAT(0) space X, τ_0 be an arbitrary point in $\mathcal{M}$, $b \in [0, \infty)$ and the sequence $\{\tau_n\}$ defined as follows:

$$\tau_{n+1} = \frac{b}{b+1}\tau_n \oplus \frac{1}{b+1}\mathcal{G}(\tau_n), \quad n \in \mathbb{Z}_+ \tag{1}$$

The authors in [1] proved Lemma 3.3, which is subsequently employed to establish Theorem 3.4, Theorem 3.5, and Theorem 3.6. For the convenience of the readers, we present Lemma 3.3 as follows:

Lemma 3.3. Let $\{\tau_n\}$ be a sequence developed by the iteration process (1) and $F(\mathcal{G}) \neq \emptyset$ Then $\lim_{n\to\infty} d(\tau_n, \mathcal{G}\tau_n) = 0$.

The above Lemma is not true in general, we consider the following example.

Example 1.1. *Let $X = \mathbb{R}$, $\mathcal{M} = [0, 1]$, $\mathcal{G}\tau = 1 - \tau$ for all $\tau \in \mathcal{M}$. Define usual metric in $\mathbb{R}$, that is*

$$d(x, y) = |x - y|, \quad \text{for all } x, y \in \mathbb{R}$$

It can be seen that $\mathcal{M}$ is a closed convex subset of $\mathbb{R}$ (CAT(0) space). Further, $F(\mathcal{G}) = \frac{1}{2} \neq \emptyset$, $\mathcal{G}$ is nonexpansive mappings so $\mathcal{G}$ is enriched non-expansive mapping for $b = 0$. Thus all the assumptions of Lemma 3.3 are satisfied. However, for $\tau_0 \neq \frac{1}{2}$, the sequence $\{\tau_n\}$ becomes:

$$\tau_{n+1} = \mathcal{G}\tau_n = 1 - \tau_n \text{ for all } \quad n \in \mathbb{N} \cup \{0\}$$

that is,

$$\tau_n = \{\tau_0, 1 - \tau_0, \tau_0, 1 - \tau_0, \dots\}$$

Thus,

$$d(\tau_n, \mathcal{G}\tau_n) = |\tau_0 - (1 - \tau_0)| = |1 - 2\tau_0| \neq 0$$

Hence $\lim_{n\to\infty} d(\tau_n, \mathcal{G}\tau_n) \neq 0$.

To prove Lemma 3.3, the authors in [1] used the following argument:
For the same initial guess $\tau_0 \in \mathcal{M}$, the sequence generated by Mann type iteration process (1) using $\mathcal{G}$ is the same as that generated by the Mann iteration process using $\mathcal{G}_\alpha$. However, this argument is not mathematically correct because $\mathcal{G} \neq \mathcal{G}_\alpha$ (in general). Therefore,

$$\tau_{n+1} = \frac{b}{b+1}\tau_n \oplus \frac{1}{b+1}\mathcal{G}(\tau_n) \neq \frac{b}{b+1}\tau_n \oplus \frac{1}{b+1}\mathcal{G}_\alpha(\tau_n) \text{ for all } n \in \mathbb{N} \cup \{0\}.$$

The essence of the correct version of Lemma 3.3 is discussed in the proof of [4, Theorem 22].

One of the main results in [1] is Theorem 3.4, which is stated below:

Theorem 3.4. Presume that X satisfies Opial's property, then the sequence $\{\tau_n\}$ developed by modified Mann iteration process (1) Δ-converges to a fixed point of the mapping $\mathcal{G}$.

It is worth noting that the assumption of Opial's property in the statement of Theorem 3.4 appears redundant. Additionally, the proof of Theorem 3.4 is flawed as it relies on Lemma 3.3.
We construct an explicit example where all hypotheses of Theorem 3.4 are satisfied, yet its conclusion fails to hold.

Example 1.2. *Consider $X = \mathbb{R}$ with the standard metric $d(x, y) = |x - y|$ for all $x, y \in \mathbb{R}$. Let $\mathcal{M} = [0, 1]$ and define the mapping $\mathcal{G} : \mathcal{M} \to \mathcal{M}$ by $\mathcal{G}\tau = 1 - \tau$ for all $\tau \in \mathcal{M}$.*
We observe that:

- *$\mathcal{M}$ is a closed convex subset of the CAT(0) space $\mathbb{R}$ (with Opial's property).*
- *Both strong convergence and Δ-convergence coincide in $\mathbb{R}$.*
- *The fixed point set $F(\mathcal{G}) = \{\frac{1}{2}\} \neq \emptyset$.*
- *$\mathcal{G}$ is nonexpansive (and thus enriched nonexpansive for $b = 0$).*

Hence, all hypotheses of Lemma 3.3 and Theorem 3.4 are satisfied. However, for any initial point $\tau_0 \neq \frac{1}{2}$, the iterative sequence $\{\tau_n\}$ defined by

$$\tau_{n+1} = \mathcal{G}\tau_n = 1 - \tau_n \quad \text{for all } n \in \mathbb{N} \cup \{0\}$$

yields the oscillating sequence

$$\tau_n = \begin{cases} \tau_0 & \text{if } n \text{ is even} \\ 1 - \tau_0 & \text{if } n \text{ is odd} \end{cases}$$

which clearly does not converge to the fixed point $\frac{1}{2}$.

The correct version of Theorem 3.4 is a corollary of [4, Theorem 22].

Finally, the authors in [1] presented two theorems, namely Theorem 3.5 and Theorem 3.6. However, the proofs of both theorems rely on Lemma 3.3, which has been demonstrated to be incorrect. Therefore, the validity of these theorems is questionable from a mathematical standpoint.

Theorem 3.5. The sequence $\{\tau_n\}$ developed by the iteration process (1) converges strongly to a fixed point of $\mathcal{G}$ if and only if $\lim_{n\to\infty} \inf d(\tau_n, F(\mathcal{G}) = 0$.

Theorem 3.6. Presume that the mapping $\mathcal{G}$ satisfies condition (I). Then the sequence $\{\tau_n\}$ developed by the iteration process (1) converges strongly to a fixed point of $\mathcal{G}$.

We provide a clear counterexample that satisfies all hypotheses of Theorem 3.6 but explicitly violates its conclusion, demonstrating the theorem's invalidity.

Example 1.3. *Let (X, d) be a CAT(0) space with $\mathcal{M} \subseteq X$ and $\mathcal{G} : \mathcal{M} \to \mathcal{M}$ identical to those in Example 1.1. Properties of $\mathcal{G}$:*

1. *Nonexpansiveness: For all $x, y \in \mathcal{M}$,*

   $$d(\mathcal{G}x, \mathcal{G}y) = |(1 - x) - (1 - y)| = |x - y| = d(x, y).$$

   Thus, $\mathcal{G}$ is nonexpansive (and hence enriched nonexpansive for $b = 0$).
2. *Condition (I): Let $\psi : [0, \infty) \to [0, \infty)$ be defined by $\psi(z) = 2z$. Then:*
   - *ψ is nondecreasing, $\psi(0) = 0$, and $\psi(z) > 0$ for $z > 0$.*
   - *For any $x \in \mathcal{M}$, we have:*

     $$d(x, \mathcal{G}x) = |x - (1 - x)| = |1 - 2x| = 2\left|x - \frac{1}{2}\right| = \psi\left(d(x, F(\mathcal{G}))\right).$$

   Thus, $\mathcal{G}$ satisfies Condition (I).

Hence, all hypotheses of Lemma 3.3 and Theorem 3.6 are satisfied. However, for any initial point $\tau_0 \neq \frac{1}{2}$*, the iterative sequence* $\{\tau_n\}$ *defined by*

$$\tau_{n+1} = \mathcal{G}(\tau_n) = 1 - \tau_n \quad \text{for all } n \in \mathbb{N} \cup \{0\}$$

yields the oscillating sequence

$$\tau_n = \begin{cases} \tau_0 & \text{if } n \text{ is even} \\ 1 - \tau_0 & \text{if } n \text{ is odd} \end{cases}$$

which clearly does not converge to the fixed point $\frac{1}{2}$.

The correct versions of Theorems 3.5 and 3.6 can be derived as corollaries from [4, Theorem 22].

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